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A remark on bilinear pseudo-differential operators with symbols in the Sjöstrand class

By

Tomoya Kato*

Abstract

In this short note, we consider the bilinear pseudo-differential operators with symbols belonging to the Sjöstrand class. We show that those operators are bounded from the product of the L^2-based Sobolev spaces $H^{s_1} \times H^{s_2}$ to L^r for $s_1, s_2 > 0$, $s_1 + s_2 = n/2$, and $1 \le r \le 2$.

§ 1. Introduction

For a bounded measurable function $\sigma = \sigma(x, \xi_1, \xi_2)$ on $(\mathbb{R}^n)^3$, the bilinear pseudo-differential operator T_σ is defined by

$$T_\sigma(f_1, f_2)(x) = \frac{1}{(2\pi)^{2n}} \int_{(\mathbb{R}^n)^2} e^{ix\cdot(\xi_1+\xi_2)} \sigma(x, \xi_1, \xi_2) \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2)\, d\xi_1 d\xi_2$$

for $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$. The bilinear Hörmander symbol class, $BS^m_{\rho,\delta} = BS^m_{\rho,\delta}(\mathbb{R}^n)$, $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$, consists of all $\sigma(x, \xi_1, \xi_2) \in C^\infty((\mathbb{R}^n)^3)$ such that

$$|\partial_x^\alpha \partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} \sigma(x, \xi_1, \xi_2)| \le C_{\alpha,\beta_1,\beta_2} (1 + |\xi_1| + |\xi_2|)^{m+\delta|\alpha|-\rho(|\beta_1|+|\beta_2|)}$$

for all multi-indices $\alpha, \beta_1, \beta_2 \in \mathbb{N}_0^n = \{0, 1, 2, \dots\}^n$. When we state the boundedness of the bilinear operators T_σ, we will use the following terminology with a slight abuse. Let X_1, X_2, and Y be function spaces on $\mathbb{R}^n$ equipped with norms $\|\cdot\|_{X_1}$, $\|\cdot\|_{X_2}$, and $\|\cdot\|_Y$, respectively. If there exist a constant C such that the estimate

$$\|T_\sigma(f_1, f_2)\|_Y \le C \|f_1\|_{X_1} \|f_2\|_{X_2}$$

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Supported by the association for the advancement of Science and Technology, Gunma University
*Division of Pure and Applied Science, Faculty of Science and Technology, Gunma University, Kiryu, Gunma 376-8515, Japan
e-mail: t.katou@gunma-u.ac.jp

holds for all $f_1 \in \mathcal{S} \cap X_1$ and $f_2 \in \mathcal{S} \cap X_2$, then we simply say that the operator T_σ is bounded from $X_1 \times X_2$ to Y.

The interest of this short note is the boundedness from $L^2 \times L^2$ to L^1 for the bilinear operator T_σ with the symbol σ belonging to the Sjöstrand class. We shall first recall some related boundedness results on the linear case. For a bounded measurable function $\sigma = \sigma(x, \xi)$ on $(\mathbb{R}^n)^2$, the linear pseudo-differential operator $\sigma(X, D)$ is defined by

$$\sigma(X,D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi) \widehat{f}(\xi) d\xi$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. The (linear) Hörmander symbol class, $S^m_{\rho,\delta} = S^m_{\rho,\delta}(\mathbb{R}^n)$, $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$, consists of all functions $\sigma \in C^\infty((\mathbb{R}^n)^2)$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}$$

for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. In [6], Calderón–Vaillancourt proved that if symbols belong to the Hörmander class $S^0_{0,0}$, the linear pseudo-differential operators are bounded on L^2. Then, Sjöstrand [19] introduced a new wider class generating L^2-bounded pseudo-differential operators than the class $S^0_{0,0}$. This new symbol class is today called as the Sjöstrand symbol class, and is also identified as the modulation space $M^{\infty,1}((\mathbb{R}^n)^2)$ (see also Boulkhemair [5]). See Section 2.2 for the definition of modulation spaces.

We shall next consider the bilinear case. Based on the linear case, one may expect the boundedness for the bilinear pseudo-differential operators of the bilinear Hörmander class $BS^0_{0,0}$ and the Sjöstrand class $M^{\infty,1}((\mathbb{R}^n)^3)$. However, Bényi–Torres [4] pointed out that there exists a symbol in $BS^0_{0,0}$ such that T_σ is not bounded from $L^2 \times L^2$ to L^1. (See, e.g., [17] for the boundedness for the bilinear Hörmander class.) Therefore, since $BS^0_{0,0} \hookrightarrow M^{\infty,1}((\mathbb{R}^n)^3)$, the boundedness for the Sjöstrand class also does not hold in general. On the other hand, Bényi–Gröchenig–Heil–Okoudjou [1] proved that if $\sigma \in M^{\infty,1}((\mathbb{R}^n)^3)$, then T_σ is bounded from $L^2 \times L^2$ to the modulation space $M^{1,\infty}$, whose target space is wider than L^1. Then, Bényi–Okoudjou [2, 3] gave that if σ belongs to $M^{1,1}((\mathbb{R}^n)^3)$, embedded into $M^{\infty,1}((\mathbb{R}^n)^3)$, then T_σ is bounded from $L^2 \times L^2$ to L^1. According to these results, in the bilinear case, we are able to have the boundedness on $L^2 \times L^2$ for Sjöstrand symbol class, paying some kind of cost.

Very recently, in [14], it was proved that if $\sigma \in BS^0_{0,0}$, then the operator T_σ is bounded from $H^{s_1} \times H^{s_2}$ to (L^2, ℓ^1) for $s_1, s_2 > 0$, $s_1 + s_2 = n/2$. Here, H^s, $s \in \mathbb{R}$, is the L^2-based Sobolev space and (L^2, ℓ^1) is the L^2-based amalgam space (see Section 2). The aim of this short note is to improve this boundedness for the class $BS^0_{0,0}$ to that for the Sjöstrand class. The main result is the following.

Theorem 1.1. *Let $s_1, s_2 \in (0, \infty)$ satisfy $s_1 + s_2 = n/2$. Then, if $\sigma \in M^{\infty,1}((\mathbb{R}^n)^3)$, the bilinear pseudo-differential operator T_σ is bounded from $H^{s_1} \times H^{s_2}$ to (L^2, ℓ^1). In particular, all those T_σ are bounded from $H^{s_1} \times H^{s_2}$ to L^r for all $r \in (1, 2]$ and to h^1.*

We end this section by explaining the organization of this note. In Section 2, we will give the basic notations which will be used throughout this paper and recall the definitions and properties of some function spaces. In Section 3, we collect some lemmas for the proof of Theorem 1.1. In Section 4, we show Theorem 1.1.

§ 2. Preliminaries

§ 2.1. Basic notations

We collect notations which will be used throughout this paper. We denote by $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$ the sets of real numbers, integers, positive integers, and nonnegative integers, respectively. We denote by Q the n-dimensional unit cube $[-1/2, 1/2)^n$. The cubes $\tau + Q$, $\tau \in \mathbb{Z}^n$, are mutually disjoint and constitute a partition of the Euclidean space $\mathbb{R}^n$. This implies integral of a function on $\mathbb{R}^n$ can be written as

$$\int_{\mathbb{R}^n} f(x)\,dx = \sum_{\tau\in\mathbb{Z}^n} \int_Q f(x+\tau)\,dx. \tag{2.1}$$

We denote by B_R the closed ball in $\mathbb{R}^n$ of radius $R > 0$ centered at the origin. We write the characteristic function on the set Ω as $\mathbf{1}_\Omega$. For $x \in \mathbb{R}^d$, we write $\langle x\rangle = (1+|x|^2)^{1/2}$.

For two nonnegative functions $A(x)$ and $B(x)$ defined on a set X, we write $A(x) \lesssim B(x)$ for $x \in X$ to mean that there exists a positive constant C such that $A(x) \le CB(x)$ for all $x \in X$. We often omit to mention the set X when it is obviously recognized. Also $A(x) \approx B(x)$ means that $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$.

We denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^d$ by $\mathcal{S}(\mathbb{R}^d)$ and its dual, the space of tempered distributions, by $\mathcal{S}'(\mathbb{R}^d)$. The Fourier transform and the inverse Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)$ are given by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x)\,dx,$$
$$\mathcal{F}^{-1}f(x) = \check{f}(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)\,d\xi,$$

respectively. We also use the partial Fourier transform of a Schwartz function $f(x, \xi_1, \xi_2)$, $x, \xi_1, \xi_2 \in \mathbb{R}^n$. In this case, we denote the partial Fourier transform with respect to the x and ξ_j variables by $\mathcal{F}_0$ and $\mathcal{F}_j$, $j = 1, 2$, respectively. We also write the Fourier transform on $(\mathbb{R}^n)^2$ for the ξ_1, ξ_2 variables as $\mathcal{F}_{1,2} = \mathcal{F}_1\mathcal{F}_2$. For $m \in \mathcal{S}'(\mathbb{R}^d)$, the Fourier multiplier operator is defined by

$$m(D)f = \mathcal{F}^{-1}\left[m \cdot \mathcal{F}f\right].$$

We also use the notation $(m(D)f)(x) = m(D_x)f(x)$ when we indicate which variable is considered.

For a measurable subset $E \subset \mathbb{R}^d$, the Lebesgue space $L^p(E)$, $1 \le p \le \infty$, is the set of all those measurable functions f on E such that $\|f\|_{L^p(E)} = \left(\int_E |f(x)|^p\,dx\right)^{1/p} < \infty$ if $1 \le p < \infty$ or $\|f\|_{L^\infty(E)} = \operatorname{ess\,sup}_{x\in E} |f(x)| < \infty$ if $p = \infty$. We also use the notation $\|f\|_{L^p(E)} = \|f(x)\|_{L^p_x(E)}$ when we want to indicate the variable explicitly.

The uniformly local L^2 space, denoted by $L^2_{ul}(\mathbb{R}^d)$, consists of all those measurable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{L^2_{ul}(\mathbb{R}^d)} = \sup_{\nu\in\mathbb{Z}^d} \left(\int_{[-1/2,1/2)^d} |f(x+\nu)|^2\,dx \right)^{1/2} < \infty$$

(this notion can be found in [13, Definition 2.3]).

Let $\mathbb{K}$ be a countable set. We define the sequence spaces $\ell^q(\mathbb{K})$ and $\ell^{q,\infty}(\mathbb{K})$ as follows. The space $\ell^q(\mathbb{K})$, $1 \le q \le \infty$, consists of all those complex sequences $a = \{a_k\}_{k\in\mathbb{K}}$ such that $\|a\|_{\ell^q(\mathbb{K})} = \left(\sum_{k\in\mathbb{K}} |a_k|^q\right)^{1/q} < \infty$ if $1 \le q < \infty$ or $\|a\|_{\ell^\infty(\mathbb{K})} = \sup_{k\in\mathbb{K}} |a_k| < \infty$ if $q = \infty$. For $1 \le q < \infty$, the space $\ell^{q,\infty}(\mathbb{K})$ is the set of all those complex sequences $a = \{a_k\}_{k\in\mathbb{K}}$ such that

$$\|a\|_{\ell^{q,\infty}(\mathbb{K})} = \sup_{t>0} \left\{ t\, \sharp\left(\{k\in\mathbb{K} : |a_k| > t\}\right)^{1/q} \right\} < \infty,$$

where $\sharp$ denotes the cardinality of a set. Sometimes we write $\|a\|_{\ell^q} = \|a_k\|_{\ell^q_k}$ or $\|a\|_{\ell^{q,\infty}} = \|a_k\|_{\ell^{q,\infty}_k}$. If $\mathbb{K} = \mathbb{Z}^n$, we usually write ℓ^q or $\ell^{q,\infty}$ for $\ell^q(\mathbb{Z}^n)$ or $\ell^{q,\infty}(\mathbb{Z}^n)$.

Let X, Y, Z be function spaces. We denote the mixed norm by

$$\|f(x,y,z)\|_{X_x Y_y Z_z} = \Big\| \big\| \|f(x,y,z)\|_{X_x} \big\|_{Y_y} \Big\|_{Z_z}.$$

(Here pay special attention to the order of taking norms.) We shall use these mixed norms for X, Y, Z being L^p or ℓ^p.

§ 2.2. Modulation spaces

We give the definition of modulation spaces which were introduced by Feichtinger [7, 8] (see also Gröchenig [11]). Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ satisfy that $\operatorname{supp}\varphi \subset [-1,1]^d$ and $\sum_{k\in\mathbb{Z}^d} \varphi(\xi - k) = 1$ for any $\xi \in \mathbb{R}^d$. Then, for $1 \le p, q \le \infty$, the modulation space $M^{p,q}(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^{p,q}} = \left\| \varphi(D-k) f(x) \right\|_{L^p_x(\mathbb{R}^d)\ell^q_k(\mathbb{Z}^d)} < \infty.$$

We note that the definition of modulation spaces is independent of the choice of the function φ. If $p_1 \le p_2$ and $q_1 \le q_2$, $M^{p_1,q_1} \hookrightarrow M^{p_2,q_2}$. We have $M^{2,2} = L^2$, and $M^{p,1} \hookrightarrow L^p \hookrightarrow M^{p,\infty}$ for $1 \le p \le \infty$. For more details, see also, e.g., [16, 21].

§ 2.3. Local Hardy space h^1

We recall the definition of the local Hardy space $h^1(\mathbb{R}^n)$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \varphi(x)\,dx \neq 0$. Then, the local Hardy space $h^1(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|f\|_{h^1} = \|\sup_{0<t<1} |\varphi_t * f|\|_{L^1} < \infty$, where $\varphi_t(x) = t^{-n}\varphi(x/t)$. It is known that $h^1(\mathbb{R}^n)$ does not depend on the choice of the function φ, and that $h^1(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$. See Goldberg [10] for more details about h^1.

§ 2.4. Amalgam spaces

For $1 \le p, q \le \infty$, the amalgam space $(L^p, \ell^q)(\mathbb{R}^n)$ is defined to be the set of all those measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{(L^p,\ell^q)(\mathbb{R}^n)} = \|f(x+\nu)\|_{L^p_x(Q)\ell^q_\nu(\mathbb{Z}^n)} = \left\{ \sum_{\nu\in\mathbb{Z}^n} \left(\int_Q \left|f(x+\nu)\right|^p dx \right)^{q/p} \right\}^{1/q} < \infty$$

with usual modification when p or q is infinity. Obviously, $(L^p, \ell^p) = L^p$ and $(L^2, \ell^\infty) = L^2_{ul}$. If $p_1 \ge p_2$ and $q_1 \le q_2$, then $(L^{p_1}, \ell^{q_1}) \hookrightarrow (L^{p_2}, \ell^{q_2})$. In particular, $(L^2, \ell^r) \hookrightarrow L^r$ for $1 \le r \le 2$. In the case $r = 1$, the stronger embedding $(L^2, \ell^1) \hookrightarrow h^1$ holds (see [15, Section 2.3]). See Fournier–Stewart [9] and Holland [12] for more properties of amalgam spaces. We end this subsection with noting the following, which was proved in [15, Lemma 2.1].

Lemma 2.1. *Let $1 \le p, q \le \infty$. If $L > n/\min(p, q)$ and if g is a measurable function on $\mathbb{R}^n$ such that*

$$c\mathbf{1}_Q(x) \le |g(x)| \le c^{-1}\langle x\rangle^{-L}$$

with some positive constant c, then

$$\|f\|_{(L^p,\ell^q)(\mathbb{R}^n)} \approx \|g(x-\nu)f(x)\|_{L^p_x(\mathbb{R}^n)\ell^q_\nu(\mathbb{Z}^n)}\,.$$

§ 3. Lemmas

In this section, we prepare several lemmas. We denote by S the operator

$$S(f)(x) = \int_{\mathbb{R}^n} \frac{|f(y)|}{\langle x-y\rangle^{n+1}}\,dy.$$

We have the following facts, which was proved in [15, Lemmas 4.1 and 4.3].

Lemma 3.1. *Let $1 \le p \le \infty$. The following (1)–(3) hold for all nonnegative measurable functions f, g on $\mathbb{R}^n$.*

(1) $S(f*g)(x) = \big(S(f)*g\big)(x) = \big(f*S(g)\big)(x)$.

(2) $S(f)(x) \approx S(f)(y)$ *for* $x, y \in \mathbb{R}^n$ *such that* $|x-y| \lesssim 1$.

(3) $\|S(f)(\nu)\|_{\ell^p_\nu} \approx \|S(f)(x)\|_{L^p_x}$.

(4) *Let* φ *be a function in* $\mathcal{S}(\mathbb{R}^n)$ *with compact support. Then,* $|\varphi(D-\nu)f(x)|^2 \lesssim S(|\varphi(D-\nu)f|^2)(x)$ *for any* $f \in \mathcal{S}(\mathbb{R}^n)$, $\nu \in \mathbb{Z}^n$, *and* $x \in \mathbb{R}^n$.

The following is proved in [15, Proposition 3.4] (see also [14, Proposition 3.3]).

Lemma 3.2. *Let* $2 < p_1, p_2 < \infty$, $1/p_1 + 1/p_2 = 1/2$, *and let* $f_j \in \ell^{p_j,\infty}(\mathbb{Z}^n)$ *be nonnegative sequences for* $j = 1, 2$. *Then,*

$$\sum_{\nu_1,\nu_2\in\mathbb{Z}^n} f_1(\nu_1) f_2(\nu_2) A_0(\nu_1+\nu_2) \prod_{j=1,2} A_j(\nu_j) \lesssim \|f_1\|_{\ell^{p_1,\infty}} \|f_2\|_{\ell^{p_2,\infty}} \prod_{j=0,1,2} \|A_j\|_{\ell^2}.$$

We end this section by mentioning a lemma which can be found in Sugimoto [18, Lemma 2.2.1]. The explicit proof is given in [15, Lemma 4.4].

Lemma 3.3. *There exist functions* $\kappa \in \mathcal{S}(\mathbb{R}^n)$ *and* $\chi \in \mathcal{S}(\mathbb{R}^n)$ *satisfying that* $\operatorname{supp}\kappa \subset [-1,1]^n$, $\operatorname{supp}\widehat{\chi} \subset B_1$, $|\chi| \ge c > 0$ *on* $[-1,1]^n$ *and*

$$\sum_{\nu\in\mathbb{Z}^n} \kappa(\xi-\nu)\chi(\xi-\nu) = 1, \quad \xi \in \mathbb{R}^n.$$

§ 4. Main results

In this section, we will prove Theorem 1.1. Although Theorem 1.1 can be easily obtained as a corollary of the boundedness from $H^{s_1} \times H^{s_2}$ to (L^2, ℓ^1) for the class $BS^0_{0,0}$ by [14] (see Remark 2 in Section 4.2), we will give the proof for the sake of self-containedness.

§ 4.1. Key proposition

Proposition 4.1 below plays a crucial role in our argument. The corresponding facts to the boundedness for the class $BS^0_{0,0}$ were given in [14, Proposition 4.1] and [15, Proposition 5.1]. We modify them to be fitted for Theorem 1.1. The proof is almost the same as in the proof of [15], and the essential idea goes back to [5].

Proposition 4.1. *Let* $s_1, s_2 \in (0,\infty)$ *satisfy* $s_1 + s_2 = n/2$, *and let* $R_0, R_1, R_2 \in [1,\infty)$. *Suppose* σ *is a bounded continuous function on* $(\mathbb{R}^n)^3$ *such that* $\operatorname{supp}\mathcal{F}\sigma \subset B_{R_0} \times B_{R_1} \times B_{R_2}$. *Then*

$$\|T_\sigma\|_{H^{s_1}\times H^{s_2}\to(L^2,\ell^1)} \lesssim (R_0R_1R_2)^{n/2} \|\sigma\|_{L^2_{ul}((\mathbb{R}^n)^3)}.$$

Proof. We take a function $\theta \in \mathcal{S}(\mathbb{R}^n)$ satisfying that $|\theta| \geq c > 0$ on $Q = [-1/2, 1/2)^n$ and $\operatorname{supp} \widehat{\theta} \subset B_1$. Then, we have by Lemma 2.1

$$\|T_\sigma(f_1, f_2)\|_{(L^2,\ell^1)} \approx \|\theta(x-\mu)T_\sigma(f_1, f_2)(x)\|_{L^2_x(\mathbb{R}^n)\ell^1_\mu(\mathbb{Z}^n)},$$

and by duality

$$\|T_\sigma(f_1, f_2)\|_{(L^2,\ell^1)} \approx \left\| \sup_{\|g\|_{L^2}=1} \left| \int_{\mathbb{R}^n} \theta(x-\mu)T_\sigma(f_1, f_2)(x)\, g(x)\, dx \right| \right\|_{\ell^1_\mu}. \tag{4.1}$$

Hence, in what follows we consider

$$I = \int_{\mathbb{R}^n} \theta(x-\mu)T_\sigma(f_1, f_2)(x)\, g(x)\, dx$$

for any $\mu \in \mathbb{Z}^n$ and all $g \in L^2(\mathbb{R}^n)$.

Now, we rewrite the integral I by the two steps below. Firstly, by using Lemma 3.3, we decompose the symbol σ as

$$\begin{aligned}
\sigma(x, \xi_1, \xi_2) &= \sum_{\boldsymbol{\nu} \in (\mathbb{Z}^n)^2} \sigma(x, \xi_1, \xi_2)\kappa(\xi_1 - \nu_1)\chi(\xi_1 - \nu_1)\kappa(\xi_2 - \nu_2)\chi(\xi_2 - \nu_2) \\
&= \sum_{\boldsymbol{\nu} \in (\mathbb{Z}^n)^2} \sigma_{\boldsymbol{\nu}}(x, \xi_1, \xi_2)\kappa(\xi_1 - \nu_1)\kappa(\xi_2 - \nu_2),
\end{aligned}$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2) \in (\mathbb{Z}^n)^2$ and we set

$$\sigma_{\boldsymbol{\nu}}(x, \xi_1, \xi_2) = \sigma(x, \xi_1, \xi_2)\chi(\xi_1 - \nu_1)\chi(\xi_2 - \nu_2).$$

Denote the Fourier multiplier operators $\kappa(D - \nu_j)$ by $\square_{\nu_j}$, $j = 1, 2$. Then, the integral I is written as

$$I = \sum_{\boldsymbol{\nu} \in (\mathbb{Z}^n)^2} \int_{\mathbb{R}^n} \theta(x-\mu) T_{\sigma_{\boldsymbol{\nu}}}\big(\square_{\nu_1} f_1, \square_{\nu_2} f_2\big)(x) g(x)\, dx. \tag{4.2}$$

The idea of decomposing symbols by such κ and χ goes back to Sugimoto [18].

Secondly, in the integral of (4.2), we transfer the information of the Fourier transform of $\theta(\cdot - \mu)T_{\sigma_{\boldsymbol{\nu}}}(\square_{\nu_1} f_1, \square_{\nu_2} f_2)$ to g. Observe that

$$\begin{aligned}
&\mathcal{F}\left[T_{\sigma_{\boldsymbol{\nu}}}(\square_{\nu_1} f_1, \square_{\nu_2} f_2)\right](\zeta) \\
&= \frac{1}{(2\pi)^{2n}} \int_{(\mathbb{R}^n)^2} \big(\mathcal{F}_0 \sigma_{\boldsymbol{\nu}}\big)\big(\zeta - (\xi_1 + \xi_2), \xi_1, \xi_2\big) \prod_{j=1,2} \kappa(\xi_j - \nu_j) \widehat{f_j}(\xi_j)\, d\xi_1 d\xi_2.
\end{aligned}$$

Then, combining this with the facts that $\operatorname{supp} \mathcal{F}_0 \sigma_{\boldsymbol{\nu}}(\cdot, \xi_1, \xi_2) \subset B_{R_0}$ and $\operatorname{supp} \kappa(\cdot - \nu_j) \subset \nu_j + [-1, 1]^n$, $j = 1, 2$, we see that

$$\operatorname{supp} \mathcal{F}\left[T_{\sigma_{\boldsymbol{\nu}}}(\square_{\nu_1} f_1, \square_{\nu_2} f_2)\right] \subset \big\{\zeta \in \mathbb{R}^n : |\zeta - (\nu_1 + \nu_2)| \lesssim R_0\big\}.$$

Hence, since $\operatorname{supp}\widehat{\theta} \subset B_1$,

$$\operatorname{supp}\mathcal{F}\left[\theta(\cdot-\mu)T_{\sigma_{\boldsymbol{\nu}}}(\square_{\nu_1}f_1,\square_{\nu_2}f_2)\right] \subset \left\{\zeta\in\mathbb{R}^n : |\zeta-(\nu_1+\nu_2)| \lesssim R_0\right\}$$

for any $\mu\in\mathbb{Z}^n$. Taking a function $\varphi\in\mathcal{S}(\mathbb{R}^n)$ satisfying that $\varphi=1$ on $\{\zeta\in\mathbb{R}^n : |\zeta|\lesssim 1\}$, the integral I given in (4.2) can be further rewritten as

$$I = \sum_{\boldsymbol{\nu}\in(\mathbb{Z}^n)^2}\int_{\mathbb{R}^n}\theta(x-\mu)T_{\sigma_{\boldsymbol{\nu}}}(\square_{\nu_1}f_1,\square_{\nu_2}f_2)(x)\,\varphi\left(\frac{D+\nu_1+\nu_2}{R_0}\right)g(x)\,dx. \tag{4.3}$$

Now, we shall actually estimate the newly rewritten integral I in (4.3). Since it holds from the facts $\operatorname{supp}\mathcal{F}_{1,2}\sigma(x,\cdot,\cdot)\subset B_{R_1}\times B_{R_2}$ and $\operatorname{supp}\widehat{\chi}\subset B_1$ that

$$\operatorname{supp}\mathcal{F}_{1,2}\sigma_{\boldsymbol{\nu}}(x,\cdot,\cdot)\subset B_{2R_1}\times B_{2R_2},$$

we have

$$\begin{aligned}
&T_{\sigma_{\boldsymbol{\nu}}}(\square_{\nu_1}f_1,\square_{\nu_2}f_2)(x)\\
&= \frac{1}{(2\pi)^{2n}}\int_{(\mathbb{R}^n)^2}\big(\mathcal{F}_{1,2}\sigma_{\boldsymbol{\nu}}\big)(x,y_1-x,y_2-x)\prod_{j=1,2}\mathbf{1}_{B_{2R_j}}(x-y_j)\square_{\nu_j}f_j(y_j)\,dy_1dy_2.
\end{aligned}$$

Then, the Cauchy–Schwarz inequalities and the Plancherel theorem yield that

$$|T_{\sigma_{\boldsymbol{\nu}}}(\square_{\nu_1}f_1,\square_{\nu_2}f_2)(x)| \lesssim \big\|\sigma_{\boldsymbol{\nu}}(x,\xi_1,\xi_2)\big\|_{L^2_{\xi_1,\xi_2}}\prod_{j=1,2}\left\{\left(\mathbf{1}_{B_{2R_j}} * \left|\square_{\nu_j}f_j\right|^2\right)(x)\right\}^{1/2}$$

for any $\boldsymbol{\nu}=(\nu_1,\nu_2)\in(\mathbb{Z}^n)^2$ and $x\in\mathbb{R}^n$. From this, the integral I is estimated as

$$\begin{aligned}
|I| \lesssim &\sum_{\boldsymbol{\nu}\in(\mathbb{Z}^n)^2}\int_{\mathbb{R}^n}|\theta(x-\mu)|\,\big\|\sigma_{\boldsymbol{\nu}}(x,\xi_1,\xi_2)\big\|_{L^2_{\xi_1,\xi_2}}\\
&\times\prod_{j=1,2}\left\{\left(\mathbf{1}_{B_{2R_j}} * \left|\square_{\nu_j}f_j\right|^2\right)(x)\right\}^{1/2}\left|\varphi\left(\frac{D+\nu_1+\nu_2}{R_0}\right)g(x)\right|\,dx.
\end{aligned} \tag{4.4}$$

Next, we decompose the above integral over x by using (2.1). Then the inequality (4.4) coincides with

$$\begin{aligned}
|I| \lesssim &\sum_{\nu_0\in\mathbb{Z}^n}\sum_{\boldsymbol{\nu}\in(\mathbb{Z}^n)^2}\int_{Q}|\theta(x+\nu_0-\mu)|\,\big\|\sigma_{\boldsymbol{\nu}}(x+\nu_0,\xi_1,\xi_2)\big\|_{L^2_{\xi_1,\xi_2}}\\
&\times\prod_{j=1,2}\left\{\left(\mathbf{1}_{B_{2R_j}} * \left|\square_{\nu_j}f_j\right|^2\right)(x+\nu_0)\right\}^{1/2}\left|\varphi\left(\frac{D+\nu_1+\nu_2}{R_0}\right)g(x+\nu_0)\right|\,dx.
\end{aligned}$$

Observe that $|\theta(x+\nu_0-\mu)|\lesssim\langle\nu_0-\mu\rangle^{-L}$ holds for any $x\in Q$ and some constant $L>0$ sufficiently large, and from Lemma 3.1 (4), (1), and (2) that

$$\left(\mathbf{1}_{B_{2R_j}} * \left|\square_{\nu_j}f_j\right|^2\right)(x+\nu_0) \lesssim S\left(\mathbf{1}_{B_{2R_j}} * \left|\square_{\nu_j}f_j\right|^2\right)(\nu_0)$$

for $x \in Q$. Then, by the Cauchy–Schwarz inequality for the integral over x,

$$
\begin{aligned}
|I| &\lesssim \sum_{\nu_0 \in \mathbb{Z}^n} \sum_{\boldsymbol{\nu} \in (\mathbb{Z}^n)^2} \langle \nu_0 - \mu \rangle^{-L} \prod_{j=1,2} \left\{ S\Big(\mathbf{1}_{B_{2R_j}} * \left| \Box_{\nu_j} f_j \right|^2 \Big)(\nu_0) \right\}^{1/2} \\
&\quad \times \int_Q \left\| \sigma_{\boldsymbol{\nu}}(x+\nu_0, \xi_1, \xi_2) \right\|_{L^2_{\xi_1,\xi_2}} \left| \varphi\left(\frac{D+\nu_1+\nu_2}{R_0} \right) g(x+\nu_0) \right| \, dx \\
&\lesssim \sum_{\nu_0 \in \mathbb{Z}^n} \sum_{\boldsymbol{\nu} \in (\mathbb{Z}^n)^2} \langle \nu_0 - \mu \rangle^{-L} \prod_{j=1,2} \left\{ S\Big(\mathbf{1}_{B_{2R_j}} * \left| \Box_{\nu_j} f_j \right|^2 \Big)(\nu_0) \right\}^{1/2} \\
&\quad \times \left\| \sigma_{\boldsymbol{\nu}}(x+\nu_0, \xi_1, \xi_2) \right\|_{L^2_{\xi_1,\xi_2} L^2_x(Q)} \left\| \varphi\left(\frac{D+\nu_1+\nu_2}{R_0} \right) g(x+\nu_0) \right\|_{L^2_x(Q)} .
\end{aligned}
$$

Here, the equivalence

$$
\sup_{\nu_0, \boldsymbol{\nu}} \left\| \sigma_{\boldsymbol{\nu}}(x+\nu_0, \xi_1, \xi_2) \right\|_{L^2_{\xi_1,\xi_2} L^2_x(Q)} \approx \|\sigma\|_{L^2_{ul}((\mathbb{R}^n)^3)} \tag{4.5}
$$

holds. In fact, since $|\chi(x+y)| \lesssim \langle y \rangle^{-L}$ for any $x \in Q$, $y \in \mathbb{R}^n$, and some constant $L > 0$ sufficiently large, we have by (2.1)

$$
\begin{aligned}
&\left\| \sigma_{\boldsymbol{\nu}}(x+\nu_0, \xi_1, \xi_2) \right\|_{L^2_{\xi_1,\xi_2} L^2_x(Q)} \\
&= \left\| \sigma(x+\nu_0, \xi_1+\mu_1, \xi_2+\mu_2) \chi(\xi_1+\mu_1-\nu_1) \chi(\xi_2+\mu_2-\nu_2) \right\|_{L^2_{\xi_1,\xi_2}(Q^2) \ell^2_{\mu_1,\mu_2} L^2_x(Q)} \\
&\lesssim \left\| \sigma(x+\nu_0, \xi_1+\mu_1, \xi_2+\mu_2) \langle \mu_1 - \nu_1 \rangle^{-L} \langle \mu_2 - \nu_2 \rangle^{-L} \right\|_{L^2_{\xi_1,\xi_2}(Q^2) \ell^2_{\mu_1,\mu_2} L^2_x(Q)} \\
&\le \left\| \langle \mu_1 - \nu_1 \rangle^{-L} \langle \mu_2 - \nu_2 \rangle^{-L} \right\|_{\ell^2_{\mu_1,\mu_2}} \sup_{\nu_0, \mu_1 \mu_2} \|\sigma(x+\nu_0, \xi_1+\mu_1, \xi_2+\mu_2)\|_{L^2_{x,\xi_1,\xi_2}(Q^3)} \\
&\approx \|\sigma\|_{L^2_{ul}((\mathbb{R}^n)^3)},
\end{aligned}
$$

which gives the inequality $\lesssim$. On the other hand, since $|\chi| \ge c > 0$ on $Q = [-1/2, 1/2)^n$ (see Lemma 3.3), we have

$$
\begin{aligned}
&\left\| \sigma(x+\nu_0, \xi_1+\nu_1, \xi_2+\nu_2) \right\|_{L^2_{x,\xi_1,\xi_2}(Q^3)} \\
&\lesssim \left\| \sigma(x+\nu_0, \xi_1+\nu_1, \xi_2+\nu_2) \chi(\xi_1) \chi(\xi_2) \right\|_{L^2_{x,\xi_1,\xi_2}(Q^3)} \\
&\le \left\| \sigma(x+\nu_0, \xi_1, \xi_2) \chi(\xi_1-\nu_1) \chi(\xi_2-\nu_2) \right\|_{L^2_{\xi_1,\xi_2}(\mathbb{R}^n \times \mathbb{R}^n) L^2_x(Q)},
\end{aligned}
$$

which gives the inequality $\gtrsim$. (See also the proof of [15, Lemma 2.1].) Hence, we have by (4.5)

$$
\begin{aligned}
|I| &\lesssim \|\sigma\|_{L^2_{ul}} \sum_{\nu_0 \in \mathbb{Z}^n} \sum_{\boldsymbol{\nu} \in (\mathbb{Z}^n)^2} \langle \nu_0 - \mu \rangle^{-L} \\
&\quad \times \prod_{j=1,2} \left\{ S\Big(\mathbf{1}_{B_{2R_j}} * \left| \Box_{\nu_j} f_j \right|^2 \Big)(\nu_0) \right\}^{1/2} \left\| \varphi\left(\frac{D+\nu_1+\nu_2}{R_0} \right) g(x+\nu_0) \right\|_{L^2_x(Q)} .
\end{aligned} \tag{4.6}
$$

In what follows, we write each summand above by

$$A_j(\nu_j,\nu_0) = \left\{ S\left(\mathbf{1}_{B_{2R_j}} * \left|\Box_{\nu_j} f_j\right|^2 \right)(\nu_0) \right\}^{1/2}, \quad j=1,2,$$
$$A_0(\tau,\nu_0) = \left\| \varphi\left(\frac{D+\tau}{R_0}\right) g(x+\nu_0) \right\|_{L^2_x(Q)}.$$

Then, the inequality (4.6) is written by

$$|I| \lesssim \|\sigma\|_{L^2_{ul}}\, I\!I \tag{4.7}$$

with

$$I\!I = \sum_{\nu_0\in\mathbb{Z}^n} \langle \nu_0-\mu\rangle^{-L} \sum_{\boldsymbol{\nu}\in(\mathbb{Z}^n)^2} A_0(\nu_1+\nu_2,\nu_0) \prod_{j=1,2} A_j(\nu_j,\nu_0).$$

Now, we shall estimate $I\!I$. By applying Lemma 3.2 with $s_1+s_2=n/2$ to the sum over $\boldsymbol{\nu}$, we have

$$\begin{aligned} I\!I &= \sum_{\nu_0\in\mathbb{Z}^n} \langle \nu_0-\mu\rangle^{-L} \sum_{\boldsymbol{\nu}\in(\mathbb{Z}^n)^2} \langle\nu_1\rangle^{-s_1}\langle\nu_2\rangle^{-s_2} A_0(\nu_1+\nu_2,\nu_0) \prod_{j=1,2} \langle\nu_j\rangle^{s_j} A_j(\nu_j,\nu_0) \\ &\lesssim \sum_{\nu_0\in\mathbb{Z}^n} \langle \nu_0-\mu\rangle^{-L} \|A_0(\tau,\nu_0)\|_{\ell^2_\tau} \prod_{j=1,2} \|\langle\nu_j\rangle^{s_j} A_j(\nu_j,\nu_0)\|_{\ell^2_{\nu_j}}, \end{aligned}$$

since $\langle\nu_j\rangle^{-s_j} \in \ell^{n/s_j,\infty}(\mathbb{Z}^n)$, $j=1,2$. Then, we use the Hölder inequality to the sum over ν_0 to have

$$I\!I \lesssim \|A_0(\tau,\nu_0)\|_{\ell^2_\tau \ell^2_{\nu_0}} \prod_{j=1,2} \|\langle \nu_0-\mu\rangle^{-L/2} \langle\nu_j\rangle^{s_j} A_j(\nu_j,\nu_0)\|_{\ell^2_{\nu_j}\ell^4_{\nu_0}}. \tag{4.8}$$

Here, the norm of A_0 in (4.8) is estimated by the Plancherel theorem as follows:

$$\begin{aligned} \|A_0(\tau,\nu_0)\|_{\ell^2_\tau\ell^2_{\nu_0}} &= \left\| \varphi\left(\frac{D+\tau}{R_0}\right) g(x+\nu_0) \right\|_{L^2_x(Q)\ell^2_\tau\ell^2_{\nu_0}} \\ &= \left\| \varphi\left(\frac{D+\tau}{R_0}\right) g(x) \right\|_{L^2_x(\mathbb{R}^n)\ell^2_\tau} \approx \left\| \varphi\left(\frac{\zeta+\tau}{R_0}\right) \widehat{g}(\zeta) \right\|_{L^2_\zeta(\mathbb{R}^n)\ell^2_\tau} \approx R_0^{n/2}\|g\|_{L^2}, \end{aligned} \tag{4.9}$$

where we used that $\|\varphi(\frac{\zeta+\tau}{R_0})\|_{\ell^2_\tau} \approx R_0^{n/2}$ for any $\zeta\in\mathbb{R}^n$. Hence, by collecting (4.7), (4.8), and (4.9) we have

$$|I| \lesssim R_0^{n/2}\|\sigma\|_{L^2_{ul}}\,\|g\|_{L^2} \prod_{j=1,2} \|\langle \nu_0-\mu\rangle^{-L/2} \langle\nu_j\rangle^{s_j} A_j(\nu_j,\nu_0)\|_{\ell^2_{\nu_j}\ell^4_{\nu_0}}. \tag{4.10}$$

We substitute (4.10) into (4.1), and then use the Cauchy–Schwarz inequality to ℓ^1_μ. Then,

$$\|T_\sigma(f_1,f_2)\|_{(L^2,\ell^1)} \lesssim R_0^{n/2}\|\sigma\|_{L^2_{ul}} \prod_{j=1,2} \left\| \langle \nu_0-\mu\rangle^{-L/2} \langle\nu_j\rangle^{s_j} A_j(\nu_j,\nu_0) \right\|_{\ell^2_{\nu_j}\ell^4_{\nu_0}\ell^2_\mu}. \tag{4.11}$$

To achieve our goal, we shall estimate the norm of A_j in (4.11). We have

$$\begin{aligned}
&\left\| \langle \nu_0 - \mu \rangle^{-L/2} \langle \nu_j \rangle^{s_j} A_j(\nu_j, \nu_0) \right\|_{\ell^2_{\nu_j} \ell^4_{\nu_0} \ell^2_\mu} \\
&= \left\| \langle \nu_0 - \mu \rangle^{-L} S\Big(\mathbf{1}_{B_{2R_j}} * \left| \langle \nu_j \rangle^{s_j} \square_{\nu_j} f_j \right|^2 \Big)(\nu_0) \right\|^{1/2}_{\ell^1_{\nu_j} \ell^2_{\nu_0} \ell^1_\mu} \\
&= \left\| \langle \nu_0 \rangle^{-L} S\Big(\mathbf{1}_{B_{2R_j}} * \left\| \langle \nu_j \rangle^{s_j} \square_{\nu_j} f_j \right\|^2_{\ell^2_{\nu_j}} \Big)(\nu_0 + \mu) \right\|^{1/2}_{\ell^2_{\nu_0} \ell^1_\mu} = (**).
\end{aligned}$$

Then, by using the embedding $\ell^1_{\nu_0} \hookrightarrow \ell^2_{\nu_0}$, Lemma 3.1 (3), and the boundedness of the operator S on L^1, we have

$$(**) \lesssim \left\| \mathbf{1}_{B_{2R_j}} * \left\| \langle \nu_j \rangle^{s_j} \square_{\nu_j} f_j \right\|^2_{\ell^2_{\nu_j}} \right\|^{1/2}_{L^1_x} \lesssim R_j^{n/2} \left\| \langle \nu_j \rangle^{s_j} \square_{\nu_j} f_j \right\|_{\ell^2_{\nu_j} L^2_x},$$

where the first inequality holds if L is suitably large. Here, by the Plancherel theorem

$$\begin{aligned}
\left\| \langle \nu_j \rangle^{s_j} \square_{\nu_j} f_j \right\|_{\ell^2_{\nu_j} L^2_x} &\approx \left\| \langle \nu_j \rangle^{s_j} \kappa(\xi - \nu_j) \widehat{f_j}(\xi) \right\|_{\ell^2_{\nu_j} L^2_\xi} \\
&\approx \left\| \langle \xi \rangle^{s_j} \kappa(\xi - \nu_j) \widehat{f_j}(\xi) \right\|_{\ell^2_{\nu_j} L^2_\xi} \approx \|f_j\|_{H^{s_j}},
\end{aligned}$$

where, we used that $\sum_{\nu_j \in \mathbb{Z}^n} |\kappa(\cdot - \nu_j)|^2 \approx 1$ to have the last equivalence. Therefore,

$$\left\| \langle \nu_0 - \mu \rangle^{-L/2} \langle \nu_j \rangle^{s_j} A_j(\nu_j, \nu_0) \right\|_{\ell^2_{\nu_j} \ell^4_{\nu_0} \ell^2_\mu} \lesssim R_j^{n/2} \|f_j\|_{H^{s_j}}. \tag{4.12}$$

Substituting (4.12) into (4.11), we obtain

$$\|T_\sigma(f_1, f_2)\|_{(L^2, \ell^1)} \lesssim (R_0 R_1 R_2)^{n/2} \|\sigma\|_{L^2_{ul}} \prod_{j=1,2} \|f_j\|_{H^{s_j}},$$

which completes the proof. □

§ 4.2. Proof of Theorem 1.1

From Proposition 4.1, we shall deduce Theorem 1.1.

Proof of Theorem 1.1. We decompose the symbol σ by a partition $\{\varphi(\cdot - k)\}_{k \in \mathbb{Z}^n}$ stated in Section 2.2:

$$\begin{aligned}
\sigma(x, \xi_1, \xi_2) &= \sum_{(k_0, k_1, k_2) \in (\mathbb{Z}^n)^3} \varphi(D_x - k_0) \varphi(D_{\xi_1} - k_1) \varphi(D_{\xi_2} - k_2) \sigma(x, \xi_1, \xi_2) \\
&= \sum_{\boldsymbol{k} \in (\mathbb{Z}^n)^3} \square_{\boldsymbol{k}} \sigma(x, \xi_1, \xi_2)
\end{aligned}$$

with

$$\square_{\boldsymbol{k}}\sigma(x,\xi_1,\xi_2) = \varphi(D_x - k_0)\varphi(D_{\xi_1} - k_1)\varphi(D_{\xi_2} - k_2)\sigma(x,\xi_1,\xi_2),$$

where $\boldsymbol{k} = (k_0, k_1, k_2) \in (\mathbb{Z}^n)^3$. Here, we observe that

$$\square_{\boldsymbol{k}}\sigma(x,\xi_1,\xi_2) = e^{ix\cdot k_0} e^{i\xi_1\cdot k_1} e^{i\xi_2\cdot k_2}\, \square_{(0,0,0)}\left[M_{\boldsymbol{k}}\sigma\right](x,\xi_1,\xi_2),$$

where

$$M_{\boldsymbol{k}}\sigma(x,\xi_1,\xi_2) = e^{-ik_0\cdot x} e^{-ik_1\cdot\xi_1} e^{-ik_2\cdot\xi_2}\sigma(x,\xi_1,\xi_2).$$

Hence,

$$T_{\square_{\boldsymbol{k}}\sigma}(f_1,f_2) = e^{ix\cdot k_0}\, T_{\square_{(0,0,0)}[M_{\boldsymbol{k}}\sigma]}\big(f_1(\cdot + k_1), f_2(\cdot + k_2)\big).$$

Since

$$\operatorname{supp}\mathcal{F}\left[\square_{(0,0,0)}\left[M_{\boldsymbol{k}}\sigma\right]\right] \subset [-1,1]^{3n},$$

we have by Proposition 4.1 and the translation invariance of the Sobolev space

$$\|T_{\square_{\boldsymbol{k}}\sigma}\|_{H^{s_1}\times H^{s_2}\to(L^2,\ell^1)} \lesssim \left\|\square_{(0,0,0)}\left[M_{\boldsymbol{k}}\sigma\right]\right\|_{L^2_{ul}} = \|\square_{\boldsymbol{k}}\sigma\|_{L^2_{ul}}.$$

Furthermore, we have the equivalence

$$\|\square_{\boldsymbol{k}}\sigma\|_{L^2_{ul}} \approx \|\square_{\boldsymbol{k}}\sigma\|_{L^\infty} \tag{4.13}$$

with implicit constants independent of $\boldsymbol{k}$ (see Remark 1 below). Therefore,

$$\begin{aligned}\|T_\sigma\|_{H^{s_1}\times H^{s_2}\to(L^2,\ell^1)} &\le \sum_{\boldsymbol{k}\in(\mathbb{Z}^n)^3} \|T_{\square_{\boldsymbol{k}}\sigma}\|_{H^{s_1}\times H^{s_2}\to(L^2,\ell^1)} \\ &\lesssim \sum_{\boldsymbol{k}\in(\mathbb{Z}^n)^3} \|\square_{\boldsymbol{k}}\sigma\|_{L^2_{ul}} \approx \|\sigma\|_{M^{\infty,1}},\end{aligned}$$

which completes the proof of Theorem 1.1. □

Remark 1. The equivalence (4.13) was already pointed out by Boulkhemair [5, Appendix A.1]. However, for the reader's convenience, we give a proof of (4.13). The way of the proof here is essentially the same as was given in [5, Appendix A.1].

Let $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}^d}$ be a partition on $\mathbb{R}^d$ stated Section 2.2. We take a function $\phi \in \mathcal{S}(\mathbb{R}^d)$ satisfying that $\phi = 1$ on $[-1,1]^d$. Then,

$$\begin{aligned}\varphi(D-k)f(x) &= \phi(D-k)\varphi(D-k)f(x) \\ &= \int_{\mathbb{R}^d} e^{i(x-y)\cdot k}\check{\phi}(x-y)\,\varphi(D-k)f(y)\,dy.\end{aligned}$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|\varphi(D-k)f(x)| &\lesssim \int_{\mathbb{R}^d} \langle x-y\rangle^{-d-1} |\varphi(D-k)f(y)|\,dy \\ &\lesssim \left(\int_{\mathbb{R}^d} \langle x-y\rangle^{-d-1} |\varphi(D-k)f(y)|^2\,dy\right)^{1/2} = (\dagger)\end{aligned}$$

for any $k \in \mathbb{Z}^d$ and $x \in \mathbb{R}^d$. As in (2.1), we decompose the integral and have

$$\begin{aligned}(\dagger) &\approx \left(\sum_{\nu\in\mathbb{Z}^d} \int_{[-1/2,1/2)^d} \langle x-\nu\rangle^{-d-1} |\varphi(D-k)f(y+\nu)|^2\,dy\right)^{1/2} \\ &\le \|\varphi(D-k)f\|_{L^2_{ul}(\mathbb{R}^d)} \left(\sum_{\nu\in\mathbb{Z}^d} \langle x-\nu\rangle^{-d-1}\right)^{1/2} \approx \|\varphi(D-k)f\|_{L^2_{ul}(\mathbb{R}^d)}\end{aligned}$$

for any $k \in \mathbb{Z}^d$ and $x \in \mathbb{R}^d$. Therefore, we obtain the inequality $\gtrsim$ in (4.13). The opposite inequality is obvious, so that we have the equivalence (4.13).

Remark 2. We derive Theorem 1.1 from the boundedness from $H^{s_1} \times H^{s_2}$ to (L^2, ℓ^1) for the class $BS^0_{0,0}$ obtained in [14]. The proof in this remark was given by the referee of this paper. Before starting the proof, we observe that

$$\left|\partial_x^\alpha \partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} \left(\Box_{(0,0,0)}\left[M_{\boldsymbol{k}}\sigma\right](x,\xi_1,\xi_2)\right)\right| \le C_{\alpha,\beta_1,\beta_2} \left\|\Box_{(0,0,0)}\left[M_{\boldsymbol{k}}\sigma\right]\right\|_{L^\infty}, \tag{4.14}$$

where the notations are the same as above. While this inequality might be well-known (see, e.g., [20, Section 1.3.2]), we give the proof for the reader's convenience. Since

$$\operatorname{supp}\mathcal{F}\left[\Box_{(0,0,0)}\left[M_{\boldsymbol{k}}\sigma\right]\right] \subset [-1,1]^{3n},$$

we have by taking a function $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying that $\phi = 1$ on $[-1,1]^n$

$$\begin{aligned}\Box_{(0,0,0)}\left[M_{\boldsymbol{k}}\sigma\right](x,\xi_1,\xi_2) &= \phi(D_x)\phi(D_{\xi_1})\phi(D_{\xi_2})\left[\Box_{(0,0,0)}\left[M_{\boldsymbol{k}}\sigma\right]\right](x,\xi_1,\xi_2) \\ &= \int_{(\mathbb{R}^n)^3} \check{\phi}(x-y)\check{\phi}(\xi_1-\eta_1)\check{\phi}(\xi_2-\eta_2)\,\Box_{(0,0,0)}\left[M_{\boldsymbol{k}}\sigma\right](y,\eta_1,\eta_2)\,dyd\eta_1d\eta_2.\end{aligned}$$

From this identity, we obtain

$$\begin{aligned}&\left|\partial_x^\alpha \partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} \left(\Box_{(0,0,0)}\left[M_{\boldsymbol{k}}\sigma\right](x,\xi_1,\xi_2)\right)\right| \\ &\le \left\|\Box_{(0,0,0)}\left[M_{\boldsymbol{k}}\sigma\right]\right\|_{L^\infty} \int_{(\mathbb{R}^n)^3} \left|\partial_x^\alpha\check{\phi}(x-y)\right|\left|\partial_{\xi_1}^{\beta_1}\check{\phi}(\xi_1-\eta_1)\right|\left|\partial_{\xi_2}^{\beta_2}\check{\phi}(\xi_2-\eta_2)\right| dyd\eta_1d\eta_2 \\ &\approx \left\|\Box_{(0,0,0)}\left[M_{\boldsymbol{k}}\sigma\right]\right\|_{L^\infty}.\end{aligned}$$

Hence, we have (4.14).

Now, since (4.14) means that $\square_{(0,0,0)}[M_{\boldsymbol{k}}\sigma] \in BS^0_{0,0}$, we see from the boundedness from $H^{s_1} \times H^{s_2}$ to (L^2, ℓ^1) for the class $BS^0_{0,0}$ that

$$\|T_{\square_{(0,0,0)}[M_{\boldsymbol{k}}\sigma]}\|_{H^{s_1}\times H^{s_2}\to(L^2,\ell^1)} \lesssim \left\|\square_{(0,0,0)}[M_{\boldsymbol{k}}\sigma]\right\|_{L^\infty}.$$

Therefore, repeating the same lines as done in the proof of Theorem 1.1, we obtain

$$\begin{aligned}\|T_\sigma\|_{H^{s_1}\times H^{s_2}\to(L^2,\ell^1)} &\le \sum_{\boldsymbol{k}\in(\mathbb{Z}^n)^3} \|T_{\square_{\boldsymbol{k}}\sigma}\|_{H^{s_1}\times H^{s_2}\to(L^2,\ell^1)} \\ &= \sum_{\boldsymbol{k}\in(\mathbb{Z}^n)^3} \|T_{\square_{(0,0,0)}[M_{\boldsymbol{k}}\sigma]}\|_{H^{s_1}\times H^{s_2}\to(L^2,\ell^1)} \\ &\lesssim \sum_{\boldsymbol{k}\in(\mathbb{Z}^n)^3} \left\|\square_{(0,0,0)}[M_{\boldsymbol{k}}\sigma]\right\|_{L^\infty} = \|\sigma\|_{M^{\infty,1}},\end{aligned}$$

which gives the boundedness stated in Theorem 1.1.

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