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Averages over annuli or tubes on the moving planes

By

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Abstract

We consider the L^2 bounds of the three types of the classical maximal averages over (i) annuli on the plane, (ii) tubes on the plane, and (iii) tubes along the cones. We study those averages over tubes and annuli embedded on the planes of $\mathbb{R}^3$ or $\mathbb{R}^4$ varying on the space variables x. In particular, the planes in $\mathbb{R}^3$ containing tubes and annuli have their normal vectors $(A(x), -1)$ where A is 2×2 matrix. The model is the Heisenberg group plane. For this case the average of f given by $\frac{1}{|R|}\int_R f\left(x-y, x_3 - \langle E(x), y\rangle\right) dy$ where E is the skew symmetric 2×2 matrix. In this paper, we introduce an author's recent classification of L^2 norm of the annulus or Nikodym maximal functions according to the rank condition of the two different types of matrices $AE + (AE)^T$ or $A + A^T$. Finally, we obtain the bound for the cone-type Nikodym maximal operator associated with the moving planes.

§ 1. Introduction

Given $\delta > 0$ and $t > 0$, let $C_{t,\delta} = \{y \subset \mathbb{R}^2 : 1-\delta \le |y/t| \le 1+\delta\}$ be an annulus with a radius t and a width $t\delta$, whose centers are at the origin. We also let $R_{\theta,\delta} = \{y \subset \mathbb{R}^2 : |y\cdot e(\theta)| < 1, |y\cdot e^{\perp}(\theta)| < \delta\}$ be a rectangle with dimensions $1\times\delta$. For an integrable function f in $\mathbb{R}^2$, we define the annulus maximal average of f on the plane by

$$M_\delta f(x) = \sup_{t>0} \frac{1}{|C_{t,\delta}|}\int_{y\in C_{t,\delta}} |f(x-y)|dy,$$

and the Nikodym maximal average of f over tubes on the plane by

$$N_\delta f(x) = \sup_{\theta\in[0,2\pi]} \frac{1}{|R_{\theta,\delta}|}\int_{y\in R_{\theta,\delta}} |f(x-y)|dy.$$

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First of all, we investigate the norms of these classical maximal operators.

§ 1.1. Annulus Maximal Theorem

In 1986, Bourgain [2] and later Schlag [16] obtained the $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$-norm of the annuli maximal operator.

Theorem 1.1 (Annulus Maximal Function in $\mathbb{R}^2$, [2],[16]). *For all $f \in L^2(\mathbb{R}^2)$,*

$$\|M_\delta f\|_{L^2(\mathbb{R}^2)} \lesssim \log(1/\delta)^{1/2}\|f\|_{L^2(\mathbb{R}^2)}.$$

As $\delta \to 0$, the above $M_\delta f(x)$ approaches to the circular maximal function. To obtain the boundedness of the circular maximal operator, Bourgain treated efficiently both of the transversal and tangential intersections of two annuli. Indeed, the area of any transverse intersection of the two annuli with width δ is as small as $O(\delta^2)$, which leads a good L^2 bound, whereas the number of the tangential intersections are small enough to give a good L^1 bound. This estimate on the dual side combined with the suitable interpolation implies the L^p boundedness of the circular maximal function for $p > 2$.

§ 1.2. Nikodym Maximal Theorem

In 1977, Cordoba [3], by summing the area of the overlapping rectangles, had obtained the $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$-norm of the Nikodym maximal operator N_δ as

Theorem 1.2 (Nikodym Maximal Function in $\mathbb{R}^2$, [3]). *For all $f \in L^2(\mathbb{R}^2)$,*

$$\|N_\delta f\|_{L^2(\mathbb{R}^2)} \lesssim \log(1/\delta)^{1/2}\|f\|_{L^2(\mathbb{R}^2)}.$$

The re-scaled rectangular average of $\frac{1}{|R_{\theta,\delta}|}\int_{y\in R_{\theta,\delta}} f(x-y)dy$ can be regarded as one wave packet of f_R given by

$$f_R(x) = \int_R e^{2\pi i\xi\cdot x}\widehat{f}(\xi)d\xi$$

whose Fourier support is $\delta^{1/2}\times\delta$ rectangle R (dual to the re-scaled $R_{\theta,\delta}$ in the definition of N_δ above). The square sum of f_R over all rectangle R located along unit circle $S^1 = \{\xi : |\xi| = 1\}$ is controlled by f in L^4 as

$$\Big\|\Big(\sum_{R \text{ along } S^1} |f_R|^2\Big)^{1/2}\Big\|_{L^4(\mathbb{R}^2)} \lesssim \delta^{-\epsilon}\|f\|_{L^4(\mathbb{R}^2)}.$$

This estimate follows from the direct application of Theorem 1.2 combined with the Littlewood-Paley inequality of the same sized intervals. This square sum estimate combined with the reverse square sum estimate [1]

$$\Big\| \sum_{R \text{ along } S^1} f_R \Big\|_{L^4(\mathbb{R}^2)} \lesssim \Big\| \Big(\sum_{R \text{ along } S^1} |f_R|^2 \Big)^{1/2} \Big\|_{L^4(\mathbb{R}^2)} \tag{1.1}$$

leads the full range $4/3 < p < 4$ for the $L^p(\mathbb{R}^2)$ boundness of the Bochner Riesz means with an arbitrary small index $\epsilon > 0$. This also implies the estimate of the Fourier restriction along S^1 in $\mathbb{R}^2$.

§ 1.3. Cone type Nikodym Maximal Theorem

Given $e(\theta) = (\cos\theta, \sin\theta)$ and $\delta > 0$, we set a $\delta \times \delta \times 1$ tube in $\mathbb{R}^3$

$$T_{\theta,\delta} = \{(se(\theta) + y, s) : y = O(\delta) \text{ and } s \in [0,1]\}$$

which is thinner than $1 \times \delta^{1/2} \times \delta$ tube located along the translated light cones. For this case, the Nikodym maximal average over $(x,0) + T_{\theta,\delta}$ is given by

$$N_\delta^{\text{cone}} f(x) = \sup_{\theta \in [0,2\pi]} Ave_{(x,0)+T_{\theta,\delta}}(f)$$

where

$$Ave_{(x,0)+T_{\theta,\delta}}(f) = \frac{1}{\delta^2} \int_{(y,s)\in[-\delta,\delta]^2\times[0,1]} f(x - y - se(\theta), s) dy ds.$$

In 1992, Mockenhaupt, Seeger and Sogge in [12] obtain that

Theorem 1.3 (Cone type Maximal Function, [12]). *For all $f \in L^2(\mathbb{R}^4)$,*

$$\|N_\delta^{cone} f\|_{L^2(\mathbb{R}^4)} \lesssim \log(1/\delta)^{3/2} \|f\|_{L^2(\mathbb{R}^4)}.$$

The re-scaled average of $Ave_{(x,0)+T_{\theta,\delta}}(f)$ in [12] can be regarded as one wave packet of f_T whose Fourier support is $1 \times \delta^{-1/2} \times \delta^{-1}$ sized tubes T:

$$f_T(x) = \int_T e^{2\pi i(\xi,\tau)\cdot(x,t)} \widehat{f}(\xi,\tau) d\xi d\tau.$$

This maximal function estimate, combined with the Littlewood-Paley inequality of the same-sized intervals, implies that the estimate for the square sum of f_T over all tubes T located along the light cone $\mathcal{C} = \{(\xi, |\xi|) : |\xi| \approx \delta^{-1}\}$:

$$\Big\| \Big(\sum_{T \text{ along } \mathcal{C}} |f_T|^2 \Big)^{1/2} \Big\|_{L^4(\mathbb{R}^3)} \lesssim \delta^{-\epsilon} \|f\|_{L^4(\mathbb{R}^3)}.$$

[1] The reverse square sum estimate (1.1) follows from the unique overlapping property that $\eta + \eta' = \xi + \xi' + O(\delta)$ for ξ, ξ', η, η' in a 1/10 length arc of a unit circle implies $(\xi, \xi') = (\eta, \eta') + O(\delta)$ in the frequency side of the LHS of (1.1).

This square sum result combined with the reverse square sum estimate [2]

$$\Big\| \sum_{T \text{ along } \mathcal{C}} f_T \Big\|_{L^4(\mathbb{R}^3)} \lesssim \delta^{-1/8} \Big\| \Big(\sum_{T \text{ along } \mathcal{C}} |f_T|^2 \Big)^{1/2} \Big\|_{L^4(\mathbb{R}^3)} \tag{1.2}$$

implies the Bourgain's circular maximal theorem in [12] as well as a nontrivial bound of Riesz means associated with the cone. In particular, the formulation in [17] regarding the gain of regularity in dt integral from the boundedness of the circular maximal average initiated the local smoothing conjecture of the wave propagating operator. For the last almost 30 years, there are great amount of effort for reducing the index $1/8$ to an arbitrary small $\epsilon > 0$. Recently, this conjecture is resolved in [4].

Notation. Let $\rho > 0$. Then, given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^d$, we write $\mathbf{v} = \mathbf{u} + O(\rho)$ if there exists $C > 0$ independent of $\mathbf{u}$, $\mathbf{v}$, and ρ such that

$$|\mathbf{v} - \mathbf{u}| \le C\rho. \tag{1.3}$$

We employ the smooth cutoff functions

(1) ψ supported in $\{u : |u| \le 1\} \subset \mathbb{R}^1$ or $\mathbb{R}^2$ or $\mathbb{R}^3$ with $\psi(u) \equiv 1$ in $|u| < 1/2$ and

(2) χ supported in $\{u : 1/2 \le |u| \le 2\} \subset \mathbb{R}^1$ or $\mathbb{R}^2$ or $\mathbb{R}^3$,

allowing slight line-by-line modifications of χ and ψ. We denote the Euclidean Fourier transform (or the inverse Fourier transform) of f in $\mathcal{S}(\mathbb{R}^d)$ by $\widehat{f}$ (or $f^\vee$). Given two scalars a, b, we write $a \lesssim b$ if $a \le Cb$ for some $C > 0$ independent of a, b. The notation $a \approx b$ denotes that $a \lesssim b$ and $b \lesssim a$. For $\delta > 0$, we write $\|\cdot\| \approx_\epsilon \delta^{-c}$ if $\delta^{-c} \lesssim \|\cdot\| \lesssim \delta^{-\epsilon}\delta^{-c}$ for an arbitrary small $\epsilon > 0$.

§ 1.4. Maximal Functions Along Moving Planes

We now consider some variable coefficient analogues of the above three types of the maximal operators. Let A be 2×2 matrices. We define the maximal annulus average of $f \in L^1_{loc}(\mathbb{R}^3)$ by,

$$\mathcal{M}^A_\delta f(x, x_3) = \sup_{t>0} \frac{1}{|C_{t,\delta}|} \int_{y \in C_{t,\delta}} |f(x - y, x_3 - \langle A(x), y \rangle)| dy \tag{1.4}$$

and the Nikodym maximal function of $f \in L^1_{loc}(\mathbb{R}^3)$ by

$$\mathcal{N}^A_\delta f(x, x_3) = \sup_{\theta \in [0,2\pi]} \frac{1}{|R_{\theta,\delta}|} \int_{y \in T_{\theta,\delta}} |f(x - y, x_3 - \langle A(x), y \rangle)| dy. \tag{1.5}$$

[2]The reverse square sum estimate (1.2) follows from the unique overlapping property that $\eta + \eta' = \xi + \xi' + O(\delta^{-1/2})$ for $\xi, \eta \in r_1 S^1$ and $\xi', \eta' \in r_2 S^1$ with $r_1, r_2 \approx \delta^{-1}$ implies $(\xi, \xi') = (\eta, \eta') + O(\delta^{-1/2})$ in the frequency side of the LHS of (1.1).

The most interesting A is the skew symmetric matrix defined by

$$E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which corresponds to the Heiseberg group planes as the first model example of the rotational curvature of the general Radon transforms. Note that $E(x)$ is the counter-clockwise rotation of x by $\pi/2$. We are interested in the smoothing effects arising from the embedding planes. In Main Theorem of 1 of [10], the author introduced the classification of the $L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ norms of the maximal operators $\mathcal{M}_\delta^A$ according to the skew symmetric rank of A given by $\mathrm{rank}(EA + (EA)^T)$.

Theorem 1.4 (Annulus Maximal Average in [10]). *Suppose that $A \in M_{2\times 2}(\mathbb{R})$.*

- *Let rank$(A) = 2$. Then it holds that*

  (i) if rank$(EA + (EA)^T) = 2$, then $\|\mathcal{M}_\delta^A\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \approx_\epsilon (\frac{1}{\delta})^0$,

  (ii) if rank$(EA + (EA)^T) = 1$, then $\|\mathcal{M}_\delta^A\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \approx_\epsilon (\frac{1}{\delta})^{1/6}$,

  (iii) if rank$(EA + (EA)^T) = 0$, then $\|\mathcal{M}_\delta^A\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \approx_\epsilon (\frac{1}{\delta})^{1/2}$.

- *Let rank$(A) = 1$, then $\|\mathcal{M}_\delta^A\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \approx_\epsilon (\frac{1}{\delta})^0$.*

- *Let rank$(A) = 0$, then $\|\mathcal{M}_\delta^A\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \approx_\epsilon (\frac{1}{\delta})^0$ (Euclidean case in Theorem 1.1).*

In [10], the autor treat the maximal average according to the three cases (i) $\mathrm{rank}(EA + (EA)^T) = 2$, (ii) $\mathrm{rank}(EA + (EA)^T) = 1$ and (iii) $\mathrm{rank}(EA + (EA)^T) = 0$ when A is invertible. Next compare it with the operator norm of the Nikodym maximal function $\mathcal{N}_\delta^A$ determined by the symmetric rank of A given by $\mathrm{rank}(A + A^T)$. In [8, 9], the author had obtained the L^2 boundedness of the corresponding Nikodym maximal operator (1.5) according to the $\mathrm{rank}(A + A^T)$.

Theorem 1.5 (Nikodym maximal functions in [9]). *Let A be 2×2 nonzero matrices. If A is symmetric, then $\left\|\mathcal{N}_\delta^A\right\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \approx_\epsilon \delta^{-0}$. If A is not symmetric, then it holds that*

(i) If rank$(A + A^T) = 2$, then $\left\|\mathcal{N}_\delta^A\right\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \approx_\epsilon \delta^{-0}$.

(ii) If rank$(A + A^T) = 1$, then $\left\|\mathcal{N}_\delta^A\right\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \approx_\epsilon \delta^{-1/6}$

(iii) If rank$(A + A^T) = 0$, then $\left\|\mathcal{N}_\delta^A\right\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \approx_\epsilon \delta^{-1/4}$.

If $A = \mathbf{0}$, then $\mathcal{N}_\delta^A$ is reduced to the Euclidean case N_δ^A in Theorem 1.2.

The annulus maximal average $\mathcal{M}_\delta^A$ has the best bound $\delta^{-\epsilon}$ when the matrix $EA+(EA)^T$ has full rank (for this case, the matrix A, roughly speaking, is close to skew symmetric matrix cE that is the case of the Heisenberg group convolution) and it has the worst bound $\delta^{-1/2}$ when $A = cI$ which is far from skew symmetric. However, the Nikodym maximal average $\mathcal{N}_\delta^A$ has the best bound $\delta^{-\epsilon}$ when A is close to symmetric (as $A = cI$) and it has the worst bound $\delta^{-1/4}$ exactly when $A = E$, the Heisenberg group case.

We remarked that the cases (i)–(iii) of Theorems 1.4 and 1.5 can be described in terms of eigen-values of A and EA respectively.

Finally, we consider the analogue of $N_\delta^{\rm cone}$. We define the Nikodym maximal function of $f \in L^1_{loc}(\mathbb{R}^3)$ over tubes

$$T_{\theta,\delta,x} + (x, x_3, 0) = (x, x_3, 0) + (se(\theta), \langle A(x), se(\theta)\rangle, s)$$

as $\mathcal{N}_\delta^{cone,A} f(x, x_3)$ given by

$$\sup_\theta \frac{1}{\delta^2} \int_{(y,t)\in[-\delta,\delta]^2\times[0,1]} f(x - y - se(\theta), x_3 - \langle A(x), se(\theta)\rangle, s)dyds. \tag{1.6}$$

Note that the dimension of the tube is $\delta \times \delta \times 0 \times 1$, which corresponds to the case of $N_\delta^{\rm cone}$ in Theorem 1.3, where we could give any width to the third variable. In this paper, we obtain the $L^2(\mathbb{R}^4) \to L^3(\mathbb{R}^3)$ norm of the cone type Nikodym maximal average $\mathcal{N}_\delta^{\rm cone,A}$ for some basic model case $A = E$, which is the Heisenberg group case.

Theorem 1.6 (Main Result, Nykodym Maximal Function Along Variable Cones)**.**

$$\left\|\mathcal{N}_\delta^{\rm cone,A}\right\|_{L^2(\mathbb{R}^4)\to L^2(\mathbb{R}^3)} \lesssim \delta^{-1/4} \quad \textit{for } A = E.$$

This matches with the third case of Theorem 1.5.

Organization. In Section 2, we discuss about the rotational curvatures and some historical model examples from the Heisenberg groups. In Section 3, we examine the proof of Theorems 1.1-1.3 regarding the three types of the classical maximal averages. In Section 4, we prove Theorem 1.6.

§ 2. Motivations of the Moving Planes

The average of $f \in L^2_{loc}(\mathbb{R}^3)$ over a ball embedded in the plane

$$\pi_A(x, x_3) = (x, x_3) - \{(y, A(x)\cdot y)) : y \in \mathbb{R}^2\}$$

is given by

$$\mathcal{A}_{\pi_A}(f)(x,x_3,t) = \int_{y\in\mathbb{R}^2} f(x-y, x_3 - \langle A(x), y\rangle)\psi(y/t)dy.$$

Whenever A is invertible, due to the effect of Phong and Stein's rotational curvature,

$$\|\mathcal{A}_{\pi_A}\|_{L^2_\alpha(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \lesssim 1 \text{ for } \alpha \geq -1.$$

We notice that the model case of the moving plane $\pi_A(x,x_3)$ was the Heisenberg group plane with $A = E$. Then for $(x,x_3) = 0$, the Heisenberg group plane $\pi_E(0) = \{(y,0) : y \in \mathbb{R}^2\} \subset \mathbb{H}^1$ is orthogonal to the center $Z(\mathbb{H}^1) = \{(0,0,y_3) : y_3 \in \mathbb{R}\}$ of the group $\mathbb{H}^1$. Even though $\pi_E(0)$ is flat, the left group translate $\pi_E(x,x_3) = (x,x_3)\cdot\pi_E(0) = \{(x,x_3)\cdot(y,0) : y \in \mathbb{R}^2\}$ rotates as x varies to generate the non-vanishing rotational curvature (Monge-Ampere determinant). See the systematic study of D. H. Phong and E. M. Stein [15]. The family of planes $\pi_E(x,x_3)$ is a very useful model of the variable coefficient surfaces of co-dimension 1. But the surfaces concerned in [11, 12, 13] and in this paper are of co-dimension 2 as

$$\mathcal{A}_{S^1(A)}(f)(x,x_{d+1},t) := \int_{y\in S^{d-1}} f\left(x-ty, x_3 - \langle A(x), ty\rangle\right) d\sigma(y) \tag{2.1}$$

The smoothing effect of the average $\mathcal{A}_{S^1(A)}f$ is induced from both the curvature of S^1 and the rotational curvature of $\pi_A(x,x_3)$. It is very interesting to understand the right analogue of Theorems 1.4 and 1.5 for the case that $A(x)$ is the vector valued polynomial.

§ 2.1. Lie Group

Consider $n\times n$ matrix A, we assign a group $G = G_{n+1}(A)$ identified with $\mathbb{R}^n\times\mathbb{R}$ endowed with the group multiplication $(x,x_{n+1})\cdot(y,y_{n+1}) = (x+y, x_{n+1}+y_{n+1}+\langle A(x),y\rangle)$. Then G is a group with the inverse element of (x,x_{n+1}) given by $(x,x_{n+1})^{-1} = (-x, -x_{n+1} + \langle A(x), x\rangle)$. We can check also that

$$G \text{ is abelian if and only if } A^T = A. \tag{2.2}$$

The Heisenberg group $\mathbb{H}^n$ is a non-abelian case as $\mathbb{H}^n = G_{2n+1}(A)$ with $A^T = -A$ where A is the $2n\times 2n$ skew symmetric matrix E. On this general group $G = G_{n+1}(A)$, we define a convolution of two integrable functions f and g in G by $f *_G g(x,x_{n+1}) = \int f((x,x_3)\cdot(y,y_3)^{-1})g(y,y_3)dydy_3$. Denote $f_A(x,x_{n+1}) = f\left(x, x_{n+1} + \langle A(x), x\rangle\right)$. Then

$$f *_G g(x,x_{n+1}) = \int f\left(x-y, x_{n+1}-y_{n+1} - \langle A(x), y\rangle\right) g_A(y,y_{n+1})dydy_{n+1} \tag{2.3}$$

For simplicity, we redefine $f *_G g$ by replacing g_A with g in (2.3) for the case of no confusion. In view of (2.2), we can interpret that

- non-commutativity helps the regularity of the annulus averages in Theorem 1.4
- non-commutativity becomes an obstacle to the tube averages Theorem 1.5.

§ 2.2. Heisenberg group

The Heisenberg group $\mathbb{H}^n$ is $G_{2n+1}(E)$ as above with $E = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Consider the surface carried measures $[d\sigma]_t = d\sigma(\cdot/t)/t^{2n}$ with $t > 0$ where $d\sigma$ is supported on the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$. We define the convolution of f and a measure $[d\sigma]_t \otimes \delta_{2n+1}$ supported on the Heisenberg group horizontal plane $\mathbb{R}^{2n} \times \{0\}$ and f:

$$\mathcal{A}_{S^{2n-1}(E)}(f)(x, x_{2n+1}, t) = f *_{\mathbb{H}^n} ([d\sigma]_t \otimes \delta_{2n+1})(x, x_{2n+1})$$

where δ_{2n+1} is the Dirac mass at $0 \in \mathbb{R}$ (in the last coordinate). So we can express

$$\mathcal{A}_{S^{2n-1}(E)}(f)(x, x_{2n+1}, t) = \int_{y \in S^{2n-1}} f(x - ty, x_{2n+1} - E(x) \cdot ty)\, d\sigma(y). \tag{2.4}$$

So $\mathcal{A}_{S^1(A)}(f)(x, x_3, t)$ of (2.4) is expressed as the group convolution (2.3) with $G_2(A)$ of f and $g(y, y_3) = d\sigma_t(y) \otimes \delta(y_3)$ with $d\sigma_t$ is a circular measure and δ is the Dirac mass:

$$\mathcal{A}_{S^1(A)}(f)(x, x_3, t) = f *_{G_3(A)} d\sigma_t(\cdot) \otimes \delta(\cdot)$$

Then this is the average of f along the $(2n-1)$ dimensional surface S^{2n-1} embedded in the $2n$-dimensional plane $(x, x_{2n+1}) + \{(y, E(x) \cdot y) : y \in \mathbb{R}^{2n}\}$ whose outer normal vector is $(E(x), -1)$ depending on x. A. Nevo and S. Thangavelu in [14] initiated the study of the maximal average

$$\mathcal{M}_{S^{2n-1}(E)} f(x, x_{2n+1}) = \sup_t \mathcal{A}_{S^{2n-1}(E)}(f)(x, x_{2n+1}, t)$$

In 2004, E.K. Narayanan and S. Thangavelu in [13] and Muller and Seeger in [11] proved that for $n \geq 2$,

$$\|\mathcal{M}_{S^{2n-1}(E)} f\|_{L^p(\mathbb{H}^n)} \lesssim \|f\|_{L^p(\mathbb{H}^n)} \text{ if and only if } p > \frac{2n-1}{2n}.$$

In [11], they study in the step two Nilpotent groups. Recently, Anderson, Cladek, Pramanik and Seeger proved that

$$\|\mathcal{M}_{S^{2n-1}(E)} f\|_{L^p(\mathbb{H}^n)} \lesssim \|f\|_{L^p(\mathbb{H}^n)} \text{ if and only if } p > \frac{2n-1}{2n}$$

for the case

$$\mathcal{A}_t(f)(x, x_{2n+1}) = \int_{y \in S^{2n-1}} f\left(x - ty, x_{2n+1} - E(x) \cdot ty - t^2 \Lambda(y)\right) d\sigma(y)$$

where Λ is a linear functional $\Lambda : \mathbb{R}^{2n} \to \mathbb{R}$.

§ 3. Proof of Theorems 1.1–1.3 based on the Fourier transforms

We give a very simple proof of the corresponding maximal average in the Euclidean space $\mathbb{R}^2$ by utilizing only the decay property of the multiplier on the frequency side. We define a multiplier operator with a symbol a

$$M_a f(x,t) = \int e^{2\pi i x\cdot\xi} a(\xi,t)\widehat{f}(\xi)d\xi \text{ where } t \in \mathbb{R}.$$

§ 3.1. Euclidean Annulus Maximal Function in Theorem 1.1

We shall give a proof of Theorem 1.1 with the bound $\log(1/\delta)$. Let a_j be the symbol for the measure supported on the annulus with width $2^{-j} \ge \delta$ given by

$$a_j(\xi,t) = \chi\left(\frac{|\xi t|}{2^j}\right)\frac{e^{\pm 2\pi i|\xi t|}}{|\xi t|^{1/2}} \text{ and } a_0(\xi,t) = \psi\left(|\xi t|\right)\frac{e^{\pm 2\pi i|\xi t|}}{|\xi t|^{1/2}}. \tag{3.1}$$

Set

$$M^\delta f(x) = \sup_{t>0}\left|\sum_{j=1}^{[\log(1/\delta)]} M_{a_j} f(x,t)\right|.$$

We show that

$$\|M^\delta\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \lesssim \log 1/\delta. \tag{3.2}$$

Proof of (3.2). For $k \in \mathbb{Z}$, let

$$a_j^k(\xi,t) = \chi(2^k t)a_j(\xi,t).$$

Then

$$\begin{aligned}\sup_k\sup_t |M_{a_j^k} f(x,t)|^p &\le \sup_k \int_0^\infty \left|\partial_t\Big(M_{a_j^k} f(x,t) M_{a_j^k} f(x,t)^{p-1}\Big)\right| dt \\ &= \int_0^\infty \sup_k \left|\partial_t M_{a_j^k} f(x,t)\right| \left|M_{a_j^k} f(x,t)\right|^{p-1} dt.\end{aligned} \tag{3.3}$$

Note that $\partial_t M_{a_j^k} f(x,t)$ has the symbol $\partial_t a_j^k(\xi,t)$, where we have the additional factor $|\xi| \approx 2^{j+k}$:

$$\partial_t a_j^k(\xi,t) = 2^{j+k}\chi(t)\tilde{\chi}\left(\frac{|\xi t|}{2^j}\right)\frac{e^{\pm 2\pi i|\xi t|}}{|\xi t|^{1/2}}$$

Hence the above integral is bounded by

$$\text{(RHS) of (3.3)} \lesssim \int_0^\infty \sup_k 2^{j+k}|M_{\tilde{a}_j^k} f(x,t)|^p dt \text{ where } \tilde{a}_j^k \text{ is of the form (3.1).}$$

Let $p=2$ above. Control $\sup_k$ by the summation $\sum_k$. Then

$$\begin{aligned}\int \sup_k \sup_t |M_{a_j^k} f(x,t)|^2 dx &\le \int_0^\infty \int \sum_k 2^{j+k} |M_{\tilde{a}_j} f(x,t)|^2 dxdt \\ &\le \sup_\xi \int \sum_k 2^{j+k} \left| \chi(2^k t) \tilde{\chi}\left(\frac{|\xi t|}{2^j}\right) 2^{-j/2} \right|^2 dt \int |\widehat{f}(\xi)|^2 d\xi \\ &\le \|f\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}$$

We sum these $O(\log(1/\delta))$ pieces to obtain (3.2). □

§ 3.2. Euclidean Nikodym Maximal Functions in Theorem 1.2

For $j=1,\cdots,[1/\delta]$, let $a_0(\xi,\theta)=\psi\left(\xi\cdot e_\theta^\perp\right)\psi\left(\xi\cdot e_\theta\right)$ and let

$$a_j(\xi,\theta) = \chi\left(\frac{\xi\cdot e_\theta^\perp}{2^j}\right)\psi\left(\xi\cdot e_\theta\right) \text{ for } (\xi,\theta)\in\mathbb{R}^2\times[0,2\pi] \tag{3.4}$$

where $e_\theta=(\cos\theta,\sin\theta)$ and $e_\theta^\perp=(\cos(\theta+\pi/2),\sin(\theta+\pi/2))$. Note

$$\partial_\theta e_\theta = e_\theta^\perp \quad\text{and}\quad \partial_\theta e_\theta^\perp = -e_\theta. \tag{3.5}$$

So

$$\begin{aligned}\partial_\theta a_j(x,\theta) &= 2^j \frac{\left(\xi\cdot e_\theta^\perp\right)}{2^j}\chi\left(\frac{\xi\cdot e_\theta^\perp}{2^j}\right)\tilde{\psi}\left(\xi\cdot e_\theta\right) \\ &= 2^j\tilde{\chi}\left(\frac{\xi\cdot e_\theta^\perp}{2^j}\right)\tilde{\psi}\left(\xi\cdot e_\theta\right) + \text{better terms.}\end{aligned}$$

where $\tilde{\chi}$ and $\tilde{\psi}$ are some slight modifications of χ and ψ. Set

$$N^\delta f(x) = \sup_\theta \left| \sum_{j=1}^{[\log(1/\delta)]} M_{a_j} f(x,\theta)\right|. \tag{3.6}$$

We show that

$$\|N^\delta\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \lesssim \log 1/\delta. \tag{3.7}$$

Proof of (3.7). We see that for all θ,

$$M_{a_j} f(x,\theta)^2 \le \int_0^{2\pi} \left|\partial_\theta M_{a_j} f(x,\theta) M_{a_j} f(x,\theta)\right| d\theta + |M_{a_j} f(x,0)|^2.$$

This implies that

$$\int \sup_\theta |M_{a_j} f(x,\theta)|^2 dx \lesssim \int_0^{2\pi} \int 2^j |M_{\tilde{a}_j} f(x,\theta)|^2 dx d\theta + \|f\|^2_{L^2(\mathbb{R}^2)}$$

$$\lesssim \sup_\xi 2^j \int_0^{2\pi} \left| \tilde{\chi}\left(\frac{\xi \cdot e_\theta^\perp}{2^j}\right) \tilde{\psi}(\xi \cdot e_\theta) \right|^2 d\theta \int |\widehat{f}(\xi)|^2 d\xi \leq \|f\|^2_{L^2(\mathbb{R}^2)}. \tag{3.8}$$

where $\tilde{a}_j$ is of the form (3.4). This follows from the sub level set estimate for

$$\int |\tilde{\psi}(\xi \cdot e_\theta)| d\theta \lesssim 2^{-j} \text{ where } \left| \frac{d}{d\theta} \xi \cdot e_\theta \right| = \left| \xi \cdot e_\theta^\perp \right| \approx 2^j.$$

Therefore, (3.8) with (3.6) yields $\|N^\delta\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \lesssim \log(1/\delta)$. □

§ 3.3. Nikodym Maximal Averages in Theorem 1.3 Along Cones

We give a proof of Theorem 1.3 in the general dimension. By parametrization of the S^{d-1}, we set $e(\theta) \in S^{d-1}$ with $d-1$ parameters θ. Let $\delta = 2^{-j/2}$. We can work with the maximal average over the tubes $\{(x,0) + (te(\theta), t) + O(2^{-j/2}) : |t| \approx 1\}$ in θ as

$$N_\delta^{\text{cone}} f(x) = \sup_{e(\theta)\in S^{d-1}} Nf(x,\theta),$$

where for fixed $\delta = 2^{-j/2}$,

$$\begin{aligned} Nf(x,\theta) &:= \psi(x)\psi(\theta) \int_{\mathbb{R}^d\times\mathbb{R}} \varphi\left(\frac{x-y-te(\theta)}{2^{-j/2}}\right) \frac{1}{(2^{-j/2})^d} \chi(t) f(y,t) dy dt \\ &= \psi(x)\psi(\theta) \int e^{2\pi i \xi\cdot(x+te(\theta))} \chi(t) \widehat{\varphi}\left(\frac{\xi}{2^{j/2}}\right) \widehat{f}(\xi,t) d\xi dt. \end{aligned}$$

Lemma 3.1. *The L^2-norm of the Nikodym maximal operator N_δ^{cone} has the following bound:*

$$\|N_\delta^{\text{cone}}\|_{L^2(\mathbb{R}^{d+1})\to L^2(\mathbb{R}^d)} \lesssim 2^{j(d-2)/4} \approx \delta^{-(d-2)/8} \tag{3.9}$$

where the case $d=2$ with the logarithmic factor yields the Theorem 1.3.

Proof. We take the parametrization $e : U \subset [0,2\pi]^{d-1} \to S^{d-1}$ for the fixed open set U. By the Sobolev embedding inequality,

$$\sup_{\theta\in U} |Nf(\cdot,\theta)| \lesssim \sum_{|\alpha|\leq(d-1)/2} \left(\int_{\theta\in U} |\partial_\theta^\alpha Nf(\cdot,\theta)|^2 d\theta \right)^{1/2},$$

where $e(\theta) \in S^{d-1}$ with $\theta \in U$. Note that $\partial_\theta e(\theta) \in S^{d-1}$. Here, we work with the case that $(d-1)/2$ is an integer. To prove (3.9), it is sufficient to show that

$$\|N_\beta\|_{L^2(\mathbb{R}^{d+1})\to L^2(\mathbb{R}^{d+(d-1)})} \lesssim 2^{j(d-2)/4}, \tag{3.10}$$

where

$$N_\beta f(x,\theta)=\psi(x)\psi(\theta)\int e^{2\pi i\xi\cdot(x+te(\theta))}\left(\prod_{s=1}^{M}\xi\cdot\partial_\theta^{\beta_s}e(\theta)\right)\chi(t)\widehat{\varphi}\left(\frac{\xi}{2^{j/2}}\right)\widehat{f}(\xi,t)d\xi dt,$$

with $\beta_1+\cdots+\beta_M=\beta\in\mathbb{Z}^d, |\beta_i|=1$ and $|\beta|=M\le(d-1)/2$. Note that $(x,\theta)\in\mathbb{R}^d\times U\subset\mathbb{R}^{d+(d-1)}$. By the application of the L^2-invariant scaling $\xi\to 2^{j/2}\xi$, we work with the operator

(3.11)

$$(2^{j/2})^{d/2}2^{Mj/2}\psi(x)\psi(\theta)\int e^{2\pi i2^{j/2}\xi\cdot(x+te(\theta))}\left(\prod_{s=1}^{M}\xi\cdot\partial_\theta^{\beta_s}e(\theta)\right)\chi(t)\widehat{\varphi}\left(\xi\right)\widehat{f}(\xi,t)d\xi dt.$$

Without loss of generality, we assume that $\partial_\theta^{\beta_1}e(\theta)=\partial_{\theta_1}e(\theta)$ and $|\xi|\le 1$ in the above integral. From

$$\begin{aligned}\prod_{s=1}^{M}(\xi\cdot\partial_\theta^{\beta_s}e(\theta))\widehat{\varphi}(\xi)&=\prod_{s=1}^{M}(\xi\cdot\partial_\theta^{\beta_s}e(\theta))\psi\left(\xi\cdot\partial_\theta^{\beta_s}e(\theta)\right)\widehat{\varphi}(\xi)\\&=\sum_{n=0}^{\infty}\chi\left(\frac{\xi\cdot\partial_{\theta_1}e(\theta)}{2^{-n}}\right)(\xi\cdot\partial_{\theta_1}e(\theta))\prod_{s=2}^{M}(\xi\cdot\partial_\theta^{\beta_s}e(\theta))\psi\left(\xi\cdot\partial_\theta^{\beta_s}e(\theta)\right)\widehat{\varphi}(\xi)\\&=\sum_{n=0}^{\infty}2^{-n}\tilde{\chi}\left(\frac{\xi\cdot\partial_{\theta_1}e(\theta)}{2^{-n}}\right)\prod_{s=2}^{M}\tilde{\psi}\left(\xi\cdot\partial_\theta^{\beta_s}e(\theta)\right)\widehat{\varphi}(\xi),\end{aligned}$$

we decompose the above integral in (3.11) as

$$\begin{aligned}\sum_{n=1}^{\infty}(2^{j/2})^{d/2}2^{Mj/2}2^{-n}\psi(x)\psi(\theta)\int e^{2\pi i2^{j/2}\xi\cdot(x+te(\theta))}\chi\left(\frac{\xi\cdot\partial_{\theta_1}e(\theta)}{2^{-n}}\right)\\\times\prod_{s=2}^{M}\psi\left(\xi\cdot\partial_\theta^{\beta_s}e(\theta)\right)\chi(t)\widehat{\varphi}(\xi)\widehat{f}(\xi,t)d\xi dt,\end{aligned}$$

where we can replace $\tilde{\chi},\tilde{\psi}$ with χ,ψ. We apply the change of variables $\xi\to 2^{-n}\xi$ with the L^2-norm invariant scaling and then express the above integral as $\sum_n R_nf(x,\theta)$ where

$$R_nf(x,\theta):=(2^{j/2})^{d/2}2^{Mj/2}2^{-dn/2}2^{-n}\psi(x)\psi(\theta)$$

(3.12)

$$\times\int e^{2\pi i2^{j/2}2^{-n}\xi\cdot(x+te(\theta))}\chi\left(\xi\cdot\partial_{\theta_1}e(\theta)\right)\prod_{s=2}^{M}\psi\left(\frac{\xi\cdot\partial_\theta^{\beta_s}e(\theta)}{2^n}\right)\chi(t)\widehat{\varphi}\left(\frac{\xi}{2^n}\right)\widehat{f}(\xi,t)d\xi dt.$$

We set $B(j,n)=(2^{j/2})^{d/2}2^{Mj/2}2^{-dn/2}2^{-n}$ and $\lambda=2\pi 2^{j/2}2^{-n}$. Then, for $k\in\mathbb{Z}^d$ with $|k|\le 2^n$, we restrict the frequency ξ to $\xi=k+O(1)$ in (3.12) by decomposing

$R_n = \sum R_{n,k}$, where $R_{n,k}f(x,\theta)$ is

$$B(j,n)\psi(x)\psi(\theta)\int e^{i\lambda\xi\cdot(x+te(\theta))}\chi\left(\xi\cdot\partial_{\theta_1}e(\theta)\right)$$

$$\times\prod_{s=2}^{M}\psi\left(\frac{\xi\cdot\partial_\theta^{\beta_s}e(\theta)}{2^n}\right)\chi(t)\widehat{\varphi}\left(\frac{\xi}{2^n}\right)\psi\left(\xi-k\right)\widehat{f}(\xi,t)d\xi dt. \tag{3.13}$$

The mixed Hessian matrix of the above phase function

$$\Phi((x_1,\cdots,x_d,\theta_1,\cdots,\theta_{d-1}),(\xi_1,\cdots,\xi_d,t)) := \xi\cdot(x+te(\theta))$$

for the operator $R_{n,k}$ is

$$\Phi''_{(x,\theta)(\xi,t)}(x,\theta,\xi,t) = \begin{pmatrix} 1\,0\,0 & 0 \\ 0\,1\,0 & 0 \\ 0\,0\,1 & 0 \\ *\,*\,* & \xi\cdot\partial_{\theta_1}e(\theta) \\ *\,*\,* & \vdots \\ *\,*\,* & \xi\cdot\partial_{\theta_{d-1}}e(\theta) \end{pmatrix}.$$

As the determinant of the first $(d+1)\times(d+1)$ matrix is $|\det(\Phi''_{(x,\theta_1)(\xi,t)})| = |\xi\cdot\partial_{\theta_1}e(\theta)| \approx 1$ in (3.13), the Hörmander theorem implies that

$$\|R_{n,k}\|_{op} = O(B(j,n)\lambda^{-(d+1)/2}) = O(2^{(d-2)j/4}2^{-n/2}), \tag{3.14}$$

where $B(j,n) = (2^{j/2})^{d/2}2^{Mj/2}2^{-dn/2}2^{-n}$ and $\lambda = 2\pi 2^{j/2}2^{-n}$. By applying integration by parts with respect to the x variables of the kernel,

$$\|R^*_{n,k}R_{n,k'}\|_{op} = O(2^{(d-2)j/2}2^{-n}|k-k'|^{-N}). \tag{3.15}$$

By the support condition, $\|R_{n,k}R^*_{n,k'}\| = 0$ if $|k-k'| \geq 10$. The Cotlar–Stein lemma in the chapter 7 of [18] with (3.14) and (3.15) implies that $\|R_n\|_{op} = O(2^{(d-2)j/4}2^{-n/2})$. In turn, this implies (3.10). □

§ 4. Proof of Theorem 1.6

We set the Nikodym-type maximal function $\mathcal{N}_\delta^{\mathrm{cone,A}}$ as

$$\mathcal{N}_j f(x,x_3) = \sup_{\theta\in[0,2\pi]}\int H_j(x,x_3,t,y,y_3,\theta)f(y,y_3,t)dydy_3dt \tag{4.1}$$

where

$$H_j(x,x_3,t,y,y_3,\theta) = \chi(t)\psi\left(x,x_3\right)\varphi\left(\frac{x-(y+te(\theta))}{2^{-j/2}},\frac{x_3-(y_3+tA(x)\cdot e(\theta))}{2^{-j/2}}\right)2^{3j/2}.$$

Suppose that A is the skew-symmetric matrix E. Then, the following boundness holds.

$$\|\mathcal{N}_j\|_{L^2(\mathbb{R}^4)\to L^2(\mathbb{R}^3)} \lesssim 2^{j/8} \approx \delta^{-1/4} \text{ if } A = E. \tag{4.2}$$

Proof of (4.2). By the Fourier inversion formula, we express $\mathcal{N}_j f(x, x_3)$ as

$$\int\int e^{2\pi i[\xi\cdot(x-(y+te(\theta)))+\xi_3(x_3-(y_3+tA(x)\cdot e(\theta)))]}\chi(t)\psi(x,x_3)\widehat{\varphi}\left(\frac{\xi}{2^{j/2}},\frac{\xi_3}{2^{j/2}}\right)d\xi d\xi_3 f(y,y_3,t)dydy_3dt$$
$$= \int e^{2\pi i[\xi\cdot(x+te(\theta))+\xi_3(x_3+tA(x)\cdot e(\theta))]}\chi(t)\psi(x,x_3)\widehat{\varphi}\left(\frac{\xi}{2^{j/2}},\frac{\xi_3}{2^{j/2}}\right)\widehat{f}(\xi,\xi_3,t)d\xi d\xi_3 dt.$$

Then the θ-derivative is given by

$$\int e^{2\pi i[\xi\cdot(x+te(\theta))+\xi_3(x_3+tA(x)\cdot e(\theta))]}2\pi i\left(\xi+\xi_3A(x)\right)\cdot e'(\theta)\chi(t)\psi(x,x_3)\widehat{\varphi}\left(\frac{\xi}{2^{j/2}},\frac{\xi_3}{2^{j/2}}\right)$$
$$\times \widehat{f}(\xi,\xi_3,t)d\xi d\xi_3 dt.$$

We may assume that $|\xi|, |\xi_3| \lesssim 2^{j/2}$ and replace $\widehat{\varphi}$ by the product of ψ's defined on $\mathbb{R}^2$ and $\mathbb{R}^1$. Moreover, from the Sobolev-embedding inequality

$$\sup_{\theta\in[0,2\pi]}|F_\theta(\cdot)| \lesssim \|\partial_\theta^{1/2}F_\theta(\cdot)\|_{L^2([0,2\pi])},$$

it suffices to treat the oscillatory integral operator $Mf(x, x_3, \theta)$ defined by

$$\sum_{2^m\le 2^{j/2}}\int e^{2\pi i[\xi\cdot(x+te(\theta))+\xi_3(x_3+tA(x)\cdot e(\theta))]}\chi(t)\psi(\theta)\psi(x,x_3)\psi\left(\frac{\xi}{2^{j/2}}\right)\psi\left(\frac{\xi_3}{2^{j/2}}\right)$$
$$\times (2^{j/2})^{1/2}\psi\left(\frac{(\xi+\xi_3A(x))\cdot e'(\theta)}{2^j}\right)\widehat{f}(\xi,\xi_3,t)d\xi d\xi_3 dt$$

and prove that $\|M\|_{L^2(\mathbb{R}^4)\to L^2(\mathbb{R}^4)} \lesssim 2^{j/8} \approx \delta^{-1/4}$ for all $A = E$. To this end, we apply the Plancherel theorem on the third variables ξ_3 and x_3. For this it suffices to prove that

$$\|T^{\xi_3}\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \lesssim 2^{j/8} \approx \delta^{-1/4} \text{ for } A = E \text{ uniformly in } \xi_3, \tag{4.3}$$

where

$$T^{\xi_3}f(x,\theta) = (2^{j/2})^{1/2}\psi\left(\frac{\xi_3}{2^{j/2}}\right)\int e^{2\pi i[\xi\cdot(x+te(\theta))+t\xi_3A(x)\cdot e(\theta)]}\chi(t)\psi(\theta)\psi(x)\psi\left(\frac{\xi}{2^{j/2}}\right)$$
$$\times\psi\left(\frac{(\xi+\xi_3A(x))\cdot e'(\theta)}{2^{j/2}}\right)\widehat{f}(\xi,t)d\xi dt. \tag{4.4}$$

It suffices to deal with the case that $|\xi_3| \approx 2^{j/2-k}$ for $k \geq 0$. By applying the change of variable $\xi \to \xi_3\xi$ in $\mathbb{R}^2$ together with $\|\xi_3 f(\xi_3(\cdot,t)\|_{L^2} = \|f(\cdot,t)\|_{L^2}$, we can work with

$$T^{\xi_3}f(x,\theta) = (2^{j/2})^{1/2}\xi_3 \int e^{2\pi i\xi_3[\xi\cdot(x+te(\theta))+tA(x)\cdot e(\theta)]}\chi(t)\psi(\theta)\psi(x)\psi\left(\frac{\xi}{2^k}\right) \times \chi\left(\frac{(\xi+A(x))\cdot e'(\theta)}{2^k}\right)\widehat{f}(\xi,t)d\xi dt.$$

Next, we apply the change of variables $x \to 2^k x$, $\xi \to 2^k\xi$, and $t \to 2^k t$. Then, it suffices to treat

$$T^{\xi_3}f(x,\theta) = (2^{j/2})^{1/2}\xi_3 2^{k/2}\psi\left(2^k x\right)\int e^{2\pi i\xi_3 2^{2k}[\xi\cdot(x+te(\theta))+tA(x)\cdot e(\theta)]}\psi(x)\psi(\theta)\psi(\xi)\psi(t) \times \psi\left((\xi+A(x))\cdot e'(\theta)\right)[2^{k/2}\widehat{f}(\xi,2^k t)\chi(2^k t)]d\xi dt, \tag{4.5}$$

where we used $\psi(2^k x) = \psi(2^k x)\psi(x)$ and $\chi(2^k t) = \chi(2^k t)\psi(t)$. Write $2\pi 2^{2k}\xi_3$ in the above exponent as λ and write the compactly supported cutoff function as

$$\psi(x,\theta,\xi,t) = \psi(x)\psi(\theta)\psi(\xi)\psi(t)\chi\left((\xi+A(x))\cdot e'(\theta)\right),$$

and define

$$\mathcal{T}^\lambda f(x,\theta) = \int e^{i\lambda[x\cdot\xi+e(\theta)\cdot\xi t+A(x)\cdot e(\theta)t]}\psi(x,\theta,\xi,t)f(\xi,t)d\xi dt \tag{4.6}$$

to show (4.7) in Lemma 4.1. Assume that Lemma 4.1 hold true.
For $A = E$, it holds that $\|\mathcal{T}^\lambda\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \lesssim \lambda^{-1}\lambda^{-1/4}$, which will be demonstrated in Lemma 4.1 below. Then $\lambda = 2\pi 2^{2k}\xi_3$ with $|\xi_3| \approx 2^{j/2-k}$ in (4.5) implies

$$\begin{aligned}\|T^{\xi_3}\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} &\lesssim \left[(2^{j/2})^{1/2}\xi_3 2^{k/2}\right]\left[(2^{2k}\xi_3)^{-1}(2^{2k}\xi_3)^{-1/4}\right] \\ &= (2^{j/2})^{1/2}2^{j/2}2^{-k/2}(2^{j/2})^{-5/4}(2^k)^{-5/4} = 2^{j/8}2^{-7k/4},\end{aligned}$$

which in turn implies (4.3). Therefore these two estimates yield (4.2). □

Lemma 4.1. *Consider the operators $\mathcal{T}^\lambda$ defined in (4.6) where $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. Then*

$$\|\mathcal{T}^\lambda\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \lesssim \lambda^{-1}\lambda^{-1/4} \text{ for all } A = E. \tag{4.7}$$

Proof of (4.7). The idea for $A = E$ is to apply the one-sided fold singularity result in [5]. Let $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a smooth phase function such that $\text{rank}(\Phi_{xy})(x_0,y_0) = d-1$, and consider the operators

$$T^\lambda f(x) = \int e^{i\lambda\Phi(x,y)}\psi(x,y)f(y)dy \text{ where } \psi \text{ is supported near } (x_0,y_0).$$

Theorem 2.1 of [5] yields that if one of the projections (say π_L) admits a fold singularity at (x_0, y_0), then

$$\|T^\lambda\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = O(\lambda^{(d-1)/2-1/4}). \tag{4.8}$$

See [6] for the two sided fold singularity. Let $h(x,y) = \det\left(\Phi''_{xy}(x,y)\right)$ and $h(x_0,y_0) = 0$. We verify that π_L has a fold singularity at (x_0,y_0) by showing that $|V_L h(x_0,y_0)| \gtrsim 1$ for all left kernel field $V_L = \sum_{j=1}^d b_j \partial_{y_j}$ with $(b_j) \in Ker(\Phi''_{xy}(x_0,y_0))$. For the standard notation, we set the variables $x_3 = \theta$, $y_1 = \xi_1$, $y_2 = \xi_2$ and $y_3 = t$ in (4.6), and rewrite our operator $\mathcal{T}^\lambda$ in (4.6) as

$$\mathcal{T}^\lambda f(x_1,x_2,x_3) = \int_{y\in\mathbb{R}^3} e^{i\lambda[x'\cdot y' + e(x_3)\cdot y' y_3 + A(x')\cdot e(x_3) y_3]}\psi(x,y)f(y)dy$$

where $x = (x_1,x_2,x_3)$ and $y = (y_1,y_2,y_3)$ with $x' = (x_1,x_2)$ and $y' = (y_1,y_2)$. Note that from the support condition (4.5) and $\xi = y'$ and $\theta = x_3$,

$$\psi(x,y) \text{ is supported } |(A(x') + y')\cdot e'(x_3)| \lesssim 1. \tag{4.9}$$

Set $A = (A_1|A_2)$ where A_1, A_2 are columns, and notice

$$Ee(x_3) = e^\perp(x_3) = e'(x_3).$$

Then the mixed Hessian matrix of the phase function Φ for $\mathcal{T}^\lambda$ is given by

$$\Phi''_{xy}(x,y) = \begin{pmatrix} 1 & 0 & A_1 e(x_3) \\ 0 & 1 & A_2 e(x_3) \\ -(\sin x_3)y_3 & (\cos x_3)y_3 & (A(x')+y')\cdot e^\perp(x_3) \end{pmatrix}$$

with the determinant $h(x,y) = \det(\Phi''_{xy})$ given by

$$h(x,y) = (A(x') + y')\cdot e^\perp(x_3) - y_3\langle AEe(x_3), e(x_3)\rangle. \tag{4.10}$$

Along the points $\{h(x,y) = 0\}$ with $\mathrm{rank}(\Phi''_{xy}) = 2$, we find the kernel field

$$\begin{aligned} V_L &= -A_1\cdot e(x_3)\partial_{y_1} + -A_2\cdot e(x_3)\partial_{y_2} + \partial_{y_3} \\ V_R &= (\sin x_3)y_3\partial_{y_1} - (\cos x_3)y_3\partial_{y_2} + \partial_{y_3} \end{aligned}$$

where

$$\begin{aligned} &\nabla_y h(x,y) = (e'(x_3), -\langle Ae'(x_3), e(x_3)\rangle) \\ &\nabla_x h(x,y) = (Ae'(x_3), -(A(x')+y')\cdot e(x_3) + y_3\langle Ae(x_3), e(x_3)\rangle + y_3\langle EAEe(x_3), e(x_3)\rangle) \end{aligned}$$

We can compute

$$|V_L(h)| = |-2\langle AEe(x_3), e(x_3)|\rangle \geq 1$$

This implies that the projection $\pi_L : (x, y) \to (x, \Phi_x(x, y))$ has a fold singularity (namely $\pi_L : \mathcal{C} \to T^*(\mathbb{R}^3)$ is a submersion with a fold) at each point of $\{h(x, y) = 0\}$. Therefore, we can apply (4.8), so that

$$\|\mathcal{T}^\lambda\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \lesssim \lambda^{-(d-1)/2}\lambda^{-1/4}, \text{ which is the desired bound of (4.7) for } d = 3.$$

This implies our fold singularity is only one-sided. □

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