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Weighted norm inequalities for the Fourier extension operator via the X-ray tomography

By

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Abstract

This note is an announcement of forthcoming paper [4] which is a work with Professor Jonathan Bennett (University of Birmingham) and so the main purpose is to exhibit results in [4] especially related to the weighted norm estimate for the Fourier extension operator known as Stein and Mizohata-Takeuchi conjectures. To these open problems in [4] we apply the approach using the X-ray tomography principle which has its origin in work of Planchon and Vega [17]. We will explain our results with motivations and how to apply the tomography principle to the weighted norm estimate. We will also provide the explicit and detailed proof of Theorem 4.1 in [1] by Barceló-Bennett-Carbery.

§ 1. Introduction and statements of results

§ 1.1. Background: the Fourier restriction conjecture and related conjectures

Our main object is the Fourier extension operator formally defined by

$$\widehat{gd\sigma}(x) = \int_{\mathbb{S}^{n-1}} g(\xi)e^{ix\cdot\xi}\, d\sigma(\xi), \quad x \in \mathbb{R}^n,$$

for $g : \mathbb{S}^{n-1} \to \mathbb{C}$ where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ and $d\sigma$ denotes surface measure on $\mathbb{S}^{n-1}$. The main question we are interested in is the decay rate of $\widehat{gd\sigma}$ in $\mathbb{R}^n$. To this question, the celebrated restriction conjecture states that

$$\|\widehat{gd\sigma}\|_{L^q(\mathbb{R}^n)} \le C\|g\|_{L^p(\mathbb{S}^{n-1})} \tag{1.1}$$

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holds for all $g \in L^p(\mathbb{S}^{n-1})$ if and only if

$$\frac{1}{q} < \frac{n-1}{2n}, \quad \frac{1}{q} \le \frac{n-1}{n+1}\frac{1}{p'}. \tag{1.2}$$

The restriction conjecture has been verified in dimension $n = 2$ (C. Fefferman and Stein [11], [18]; see also Zygmund [24]), and there has been considerable progress in higher dimensions in recent years (see for example [12] and [20] for further discussion and context). We also refer to the lecture note [22] by Tao for detailed discussion of motivation and relation to other topics.

Note that the difficulty of the sufficient part of this conjecture is to push the exponent q as small as possible in (1.1). For instance, (1.1) with $q = \infty$ is an easy consequence of Riemann-Lebesgue's theorem and the case $q = 2(n+1)/(n-1)$ is also known to be true due to Stein-Tomas which also reveals a close link with the nonlinear dispersive equation. On the other hand, the necessity of the conditions (1.2) is straightforward to verify with simple examples. For the purpose of explaining our approach, let us see these examples. The condition $1/q < (n-1)/2n$ can be obtained by applying (1.1) with $g \equiv 1$. In fact, one has the asymptotic behavior:

$$|\widehat{1d\sigma}(x)| = \Big|\int_{\mathbb{S}^{n-1}} e^{ix\cdot\xi} d\sigma(\xi)\Big| \sim (1+|x|)^{-\frac{n-1}{2}} \tag{1.3}$$

on a large portion of $\mathbb{R}^n$ by the simple application of the stationary phase argument – see [23] or [18] for example. Accordingly, it is also conjectured that an endpoint inequality of the form

$$\|\widehat{gd\sigma}\|_{L^{\frac{2n}{n-1}}(B_R)} \lesssim_\varepsilon R^\varepsilon \|g\|_{L^{\frac{2n}{n-1}}(\mathbb{S}^{n-1})} \tag{1.4}$$

holds for all $\varepsilon > 0$; here B_R denotes the ball of radius R centred at the origin. It is well-known that (1.4) for all $\varepsilon > 0$, indeed implies the restriction conjecture; see [21].

Another condition can be obtained by applying (1.1) with $g = 1_{C(e_n,\delta)}$, where in general for $\omega \in \mathbb{S}^{n-1}$, $C(\omega,\delta) = \mathbb{S}^{n-1} \cap B(\omega,\delta)$ is the δ-cap centered at ω and $\delta > 0$. Again it is straightforward to see

$$|\widehat{1_{C(e_n,\delta)}d\sigma}| \gtrsim \delta^{n-1} 1_{T_{\delta^{-1}}}$$

where $T_{\delta^{-1}}$ is $\delta^{-1}\times\cdots\delta^{-1}\times\delta^{-2}$-tube whose direction is parallel to e_n and centered at the origin. Using this lower bound and tending $\delta \to 0$, one can obtain $1/q \le (n-1)/(n+1)p'$. In particular, this example reveals the relation between $\widehat{gd\sigma}$ and geometry of tubes. In this way, one can create any $\delta^{-1}\times\cdots\times\delta^{-1}\times\delta^{-2}$-tube whose direction and center are arbitrary. Namely for arbitrary direction $\omega \in \mathbb{S}^{n-1}$ and center $v \in \mathbb{R}^n$, if we set $g(\xi) = e^{iv\cdot\xi} 1_{C(\omega,\delta)}$, then $\widehat{gd\sigma}$ has a large mass on the $\delta^{-1}\times\cdots\delta^{-1}\times\delta^{-2}$-tube whose direction ω and centered at v. This link with the tube or line reminds us the Kakeya conjecture which states that

$$\|\mathcal{X}f\|_{L^n(\mathbb{S}^{n-1})} \le C_\varepsilon \|(1-\Delta)^\varepsilon f\|_{L^n(\mathbb{R}^n)} \tag{1.5}$$

holds for all $\varepsilon > 0$ where the maximal X-ray transform $\mathcal{X}$ is given by

$$\mathcal{X}f(\omega) = \sup_{v\in\langle\omega\rangle^\perp} Xf(\omega,v), \quad Xf(\omega,v) := \int_{\mathbb{R}} f(t\omega+v)\,dt, \quad \omega\in\mathbb{S}^{n-1}.$$

In fact, there is a direct implication between two conjectures, see [22] for its proof:

Proposition 1.1.

$$\text{Restriction conjecture} \Rightarrow \text{Kakeya conjecture}.$$

At this stage, one may wonder if the reverse implication is true or not, namely these two conjectures are equivalent or not. To this question there is no known explicit reverse implication although there is some implicit one, for example Bourgain [5] first provide the important progress on the restriction conjecture by employing the progress of the Kakeya conjecture. Related to this question, Stein posed the following conjecture about the weighted norm inequality for $\widehat{gd\sigma}$ known as Stein's conjecture in [19]; see also Córdoba [9] and Carbery–Soria–Vargas [8] for variants of this: for all weight function $w : \mathbb{R}^n :\to \mathbb{R}_{\ge 0}$,

$$\int_{\mathbb{R}^n} |\widehat{gd\sigma}(x)|^2 w(x)\,dx \le C \int_{\mathbb{S}^{n-1}} |g(\xi)|^2 \mathcal{X}w(\xi)\,d\sigma(\xi). \tag{1.6}$$

By the duality and bootstrapping argument, one can see that if Stein's conjecture (1.6) is true then the Kakeya conjecture implies the restriction conjecture and hence these two conjectures become equivalent. These relations clarify the importance of Stein's conjecture, but it is also related to another open problem so-called Mizohata-Takeuchi conjecture back to 1970's [16] which states that

$$\int_{\mathbb{R}^n} |\widehat{gd\sigma}(x)|^2 w(x)\,dx \le C\|Xw\|_{L^\infty(\mathcal{M}_{1,n})} \int_{\mathbb{S}^{n-1}} |g(\xi)|^2\,d\sigma(\xi). \tag{1.7}$$

Here, $\mathcal{M}_{1,n}$ denotes the set of all lines in $\mathbb{R}^n$ parametrized by

$$\mathcal{M}_{1,n} = \{l(\omega,v) : \omega\in\mathbb{S}^{n-1},\ v\in\langle\omega\rangle^\perp\}, \quad l(\omega,v) := \{t\omega+v\in\mathbb{R}^n : t\in\mathbb{R}\}$$

and hence the mixed norm is defined by

$$\|Xw\|_{L^p_\omega L^q_v(\mathcal{M}_{1,n})} := \Big(\int_{\mathbb{S}^{n-1}} \Big(\int_{\langle\omega\rangle^\perp} Xw(\omega,v)^q\,d\lambda_\omega(v)\Big)^{p/q} \sigma(\omega)\Big)^{1/p}$$

for general $1 \le p, q < \infty$, and $\|Xw\|_{L^p_\omega L^q_v(\mathcal{M}_{1,n})} := \sup_{\omega\in\mathbb{S}^{n-1}, v\in\langle\omega\rangle^\perp} Xw(\omega,v)$. In this terminology we have

$$Xw(\omega, v) = \int_{l(\omega,v)} w.$$

Clearly, Stein's conjecture implies (1.7). In this article, we are especially interested in conjectures (1.6) and (1.7) that are open problems even when $n = 2$.

§ 1.2. Results

Let us focus on the Mizohata-Takeuchi conjecture (1.7). Firstly it is worth to mention that (1.7) is known to be true if the weight w is radial for which case the problem is indeed equivalent to certain uniform eigenvalue estimate involving Bessel function, see [2, 7] or [1]. At the heuristically level there is an easy way to explain how one can obtain the object Xw. In fact from the polar coordinate we know

$$\int_{\mathbb{R}^n} w(x)|x|^{-(n-1)}\, dx = c_n \int_0^\infty w_0(r)\, dr = c_n Xw(e_1, 0)$$

which suggests the line integral of w, Xw, where $w(x) = w_0(|x|)$ since w is radial. However, it is not clear how to deduce the quantity Xw for general weight w and this is one of the difficulties of the Mizohata-Takeuchi conjecture. For this issue, we propose to use the X-ray tomography principle. The basic principle of X-ray tomography is captured by the well-known inversion formula,

$$f = c_n X^*(-\Delta_v)^{\frac{1}{2}} Xf, \tag{1.8}$$

or the closely-related fact that $c_n^{1/2}(-\Delta_v)^{1/4}X$ is an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathcal{M}_{1,n})$ for a certain dimensional constant c_n. We might therefore expect that estimates on $X(|\widehat{gd\sigma}|^2)$, or its variants, may be used to address tackling problems in restriction theory as we have

$$\int_{\mathbb{R}^n} |\widehat{gd\sigma}(x)|^2\, w(x)dx = c_n \int_{\mathcal{M}_{1,n}} X(|\widehat{gd\sigma}|^2)(\omega, v)(-\Delta_v)^{\frac{1}{2}} Xw(\omega, v)\, d\lambda_\omega(v) d\sigma(\omega).$$

A precedent for this approach may be found in the work of Planchon and Vega [17], where certain sharp Strichartz estimates for the Schrödinger equation are obtained from identities involving the Radon transform of $|u(\cdot, t)|^2$, where u is a solution to the free time-dependent Schrödinger equation; see also Beltran and Vega [3], where their X-ray analysis allows them to recover the recent sharp Stein–Tomas restriction theorem of Foschi.

Before going to further detailed argument involving this approach, let us describe the precise problem we will tackle via this X-ray tomography approach. As we mentioned the conjecture (1.7) has been the longstanding open problem and so it would be meaningful to consider the weakened problem as a first attempt. For this purpose we employ the formal Sobolev's embedding which suggests that

$$\|f\|_{L^\infty(\mathbb{R}^{n-1})} \lesssim \big\|(-\Delta)^{\frac{n-1}{2q}} f\big\|_{L^q(\mathbb{R}^{n-1})} \tag{1.9}$$

for all $1 \le q \le \infty$. Strictly speaking this endpoint of Sobolev's embedding is known to be fail. Nevertheless a formal application of (1.9) to the conjecture (1.7) implies

$$\int_{\mathbb{R}^n} |\widehat{gd\sigma}(x)|^2 w(x)\, dx \le C\big\|(-\Delta_v)^{\frac{n-1}{2q}} Xw\big\|_{L^\infty_\omega L^q_v(\mathcal{M}_{1,n})} \int_{\mathbb{S}^{n-1}} |g(\xi)|^2\, d\sigma(\xi) \tag{1.10}$$

for all $1 \le q \le \infty$ and this is what we are interested in. The conjecture (1.7) corresponds to (1.10) with $q = \infty$ and so the problem is to establish (1.10) with q as large as possible. Although these arguments are heuristic our result in below justifies this situation. Rigorously speaking, in order to apply the tomography principle, we instead consider the validity of the local variant

$$\int_{B_R} |\widehat{gd\sigma}|^2\, w \lesssim R^\varepsilon \big\|(-\Delta_v)^{\frac{n-1}{2q}} Xw\big\|_{L^\infty_\omega L^q_v(\mathcal{M}_{1,n})} \int_{\mathbb{S}^{n-1}} |g|^2, \tag{1.11}$$

formulated in the spirit of (1.4). Our main result here states that, for $n = 2$, the case $q = 1$ holds true and moreover the exponent q may be pushed up to 2.

Theorem 1.2 (Bennett, N [4]). *Let $n = 2$. Then* (1.11) *holds true as long as $1 \le q \le 2$. Moreover, we have*

$$\int_{B_R} |\widehat{gd\sigma}|^2\, w \lesssim \log R\big\|(-\Delta_v)^{\frac{1}{4}} Xw\big\|_{L^\infty_\omega L^2_v(\mathcal{M}_{1,2})} \int_{\mathbb{S}^1} |g|^2. \tag{1.12}$$

It is interesting to compare (1.12) with the endpoint restriction estimate (1.4). When $n = 2$, using the isometry of $(-\Delta_v)^{\frac{1}{4}} X$ from $L^2(\mathbb{R}^2)$ to $L^2(\mathcal{M}_{1,2})$, the endpoint estimate (1.4) is equivalent to

$$\int_{B_R} |\widehat{gd\sigma}|^2\, w \lesssim_\varepsilon R^\varepsilon \big\|(-\Delta_v)^{\frac{1}{4}} Xw\big\|_{L^2_{\omega,v}(\mathcal{M}_{1,2})} \|g\|^2_{L^4(\mathbb{S}^1)} \tag{1.13}$$

and this reveals the close link between (1.12) and the original restriction problem.

We can also apply the similar argument to Stein's conjecture and indeed our proof of Theorem 1.2 is based on it. Again the formal application of (1.9) to (1.6) gives

$$\int_{\mathbb{R}^n} |\widehat{gd\sigma}(x)|^2 w(x)\, dx \le C \int_{\mathbb{S}^{n-1}} |g(\xi)|^2 \mathcal{X}_q w(\xi)\, d\sigma(\xi) \tag{1.14}$$

for all $1 \le q \le \infty$ where

$$\mathcal{X}_q w(\xi) := \big\|(-\Delta_v)^{\frac{n-1}{2q}} Xw(\xi, \cdot)\big\|_{L^q_v(\mathcal{M}_{1,n})}.$$

As before (1.14) with $q = \infty$ corresponds to Stein's conjecture and hence the problem is to make q as large as possible.

Our next theorem provides a variant of (1.14) in the case $n = 2$ and $q = 2$. Its statement naturally involves a bilinear averaging operator BT_δ defined by

$$BT_\delta(g_1, g_2)(\omega) = \int_{\mathbb{S}^{n-1}} \frac{g_1(\xi)\widetilde{g}_2(R_\omega(\xi))}{|\omega \cdot \xi| + \delta} \, d\sigma(\xi) \tag{1.15}$$

for $\delta > 0$ and $g_1, g_2 : \mathbb{S}^{n-1} \to \mathbb{C}$ where $\widetilde{g}(\omega) = \overline{g(-\omega)}$ and $R_\omega(\xi) = \xi - 2(\xi \cdot \omega)\omega$ is the reflection of ξ in the hyperplane $\langle \omega \rangle^\perp$. [1]

Theorem 1.3 (Bennett, N [4]). *Let $n = 2$. Then for all $R \gg 1$,*

$$\begin{aligned} \int_{B_R} |\widehat{gd\sigma}|^2 w \lesssim & \int_{\mathbb{S}^1} BT_{1/R}\big(|g|^2_{1/R}, |g|^2_{1/R}\big)(\omega)^{\frac12} \mathcal{S}w(\omega) \, d\sigma(\omega) \\ & + \int_{\mathbb{S}^1} BT_{1/R}\big(|g|^2_{1/R}, |g|^2_{1/R}\big)(\omega^\perp)^{\frac12} \mathcal{S}w(\omega) \, d\sigma(\omega), \end{aligned} \tag{1.16}$$

where

$$\mathcal{S}w(\omega) := \mathcal{X}_2 w(\omega) = \left(\int_{\langle \omega \rangle^\perp} \big| (-\Delta_v)^{\frac14} X w(\omega, v) \big|^2 \, d\lambda_\omega(v) \right)^{\frac12},$$

and $|g|_{1/R}$ is a suitable mollification of $|g|$ at scale $1/R$, such as that given by convolution with the Poisson kernel on $\mathbb{S}^1$.

The first remark here is about the auxiliary bilinear operator BT_δ. In [4] we observed that BT_δ is very well behaved, satisfying the bounds

$$\big\| BT_\delta(g_1, g_2) \big\|_{L^{\frac12}(\mathbb{S}^1)} \lesssim \log(\delta^{-1})^2 \|g_1\|_{L^1(\mathbb{S}^1)} \|g_2\|_{L^1(\mathbb{S}^1)}, \tag{1.17}$$

$$\big\| BT_\delta(g_1, g_2) \big\|_{L^1(\mathbb{S}^1)} \lesssim \log(\delta^{-1}) \|g_1\|_{L^2(\mathbb{S}^1)} \|g_2\|_{L^2(\mathbb{S}^1)}. \tag{1.18}$$

In particular, our Stein-type inequality (1.16), when combined with (1.17), immediately implies our Mizohata–Takeuchi-type inequality (1.12). Similarly, (1.16) and (1.18), combined with the Cauchy–Schwarz inequality, imply the $n = 2$ endpoint restriction inequality (1.4), thanks to the fact that the controlling operator $\mathcal{S}$, like $\mathcal{X}$, satisfies

[1] One can see that BT_δ is the natural bilinearization on $\mathbb{S}^{n-1}$ of the linear operator T_δ given by

$$T_\delta g(\omega) := \int_{\mathbb{S}^{n-1}} \frac{g(\xi)}{|\xi \cdot \omega| + \delta} \, d\sigma(\xi).$$

In fact for general linear operator T acting on the function on $\mathbb{S}^{n-1}$ we may define its bilinear variant by

$$BT(g_1, g_2)(\omega) := T(g_1(\cdot)\widetilde{g_2}(R_\omega(\cdot)))(\omega).$$

With this definition one can check that, for example, for H, the Hilbert transform on $\mathbb{S}^1$, BH corresponds to the bilinear Hilbert transform, see [14].

suitable bounds on $L^2(\mathbb{R}^2)$ – indeed $\mathcal{S}$ is better behaved than $\mathcal{X}$ in this regard as $\|\mathcal{S}w\|_{L^2(\mathbb{S}^1)} = \sqrt{2\pi}\|w\|_{L^2(\mathbb{R}^2)}$.

Second remark is about the weighted norm inequality for BT_δ. As one can see the difference between (1.14) and (1.16) is the presence of the well-behaved operator BT_δ. From this point of view it is natural to ask what is the natural weight class of v for which the weighted norm estimate

$$(1.19)\quad \Big(\int_{\mathbb{S}^1} |BT_\delta(g_1,g_2)(\omega)|^{\frac{1}{2}} v(\omega)\, d\sigma(\omega)\Big)^2 \lesssim \log(\delta^{-1})^2 \prod_{i=1,2} \Big(\int_{\mathbb{S}^1} |g_i(\omega_i)| v(\omega_i)\, d\sigma(\omega_i)\Big)$$

holds true. For instance if one can ensure that (1.19) holds true with $v = \mathcal{S}w$, then one would obtain (1.14) with $q = 2$ from our result (1.16). For this problem it is worth to provide a way to regard the operator BT_δ as the truncated version of the bilinear Hilbert transform. Perhaps $L^1 \times L^1 \to L^{1/2}$ bound (1.17) reminds us the famous conjecture for the bilinear Hilbert transform, see Lacey-Thiele [14]. Let us explain further details. We assume, as we may, that g_1, g_2 are nonnegative and symmetric. Using the parametrization $\omega = (\cos\theta, \sin\theta)$ and $\xi = (\cos\varphi, \sin\varphi)$ in (1.15), we have

$$BT_\delta(g_1,g_2)(\omega) = \int_0^{2\pi} \frac{G_1(\varphi)G_2(2\theta-\varphi)}{|\cos(\theta-\varphi)|+\delta}\, d\varphi = \int_0^{2\pi} \frac{G_1(\theta-\varphi)G_2(\theta+\varphi)}{|\cos\varphi|+\delta}\, d\varphi,$$

where $G_i(\varphi) = g_i(\cos\varphi, \sin\varphi)$ for $i = 1, 2$. After considering suitable rotations, we see that BT_δ behaves like

$$(1.20)\qquad BH_\delta(h_1,h_2)(\theta) := \int_{-1/100}^{1/100} \frac{h_1(\theta+\varphi)h_2(\theta-\varphi)}{|\varphi|+\delta}\, d\varphi.$$

From this realization the problem (1.19) is more or less equivalent to identifying the weight class of V for which

$$(1.21)\quad \Big(\int_0^{2\pi} |BH_\delta(h_1,h_2)(\theta)|^{\frac{1}{2}} V(\theta)\, d\theta\Big)^2 \lesssim \log(\delta^{-1})^2 \prod_{i=1,2} \Big(\int_0^{2\pi} |h_i(\theta_i)| V(\theta_i)\, d\theta_i\Big)$$

holds true. Alternatively we may also ask the Fefferman-Stein type estimate instead of (1.21). Namely, by using some suitable maximal operator M, for example Hardy-Littlewood maximal operator, the question is the validity of

$$(1.22)\quad \Big(\int_0^{2\pi} |BH_\delta(h_1,h_2)(\theta)|^{\frac{1}{2}} V(\theta)\, d\theta\Big)^2 \lesssim \log(\delta^{-1})^2 \prod_{i=1,2} \Big(\int_0^{2\pi} |h_i(\theta_i)| MV(\theta_i)\, d\theta_i\Big)$$

for all weight V. The positive answer to (1.22) also provides some progress on (1.14). As far as we are aware there seems no investigation about these weighted norm estimates for the bilinear Hilbert transform and it would be an interesting problem. For the weighted norm estimate for the fractional integral operator with exponent below 1 in the place of 1/2, we refer [15].

As a final remark it is interesting to compare Theorem 1.3 with Theorem 4.1 in [1] by Barceló-Bennett-Carbery. We will provide their precise statement in Section 2 and here just mention that their result has a common property, namely it involves L^2-norm in right-hand side. In Section 2 we will provide the detailed proof of their result as well since they have just referred the work of Erdoğan [10] (also [6]) for its proof and no one can find the explicit proof.

§ 1.3. Idea of how to apply the X-ray tomography

In this subsection we describe the idea of the proof of Theorem 1.2 via the X-ray tomography approach which means the aforementioned fact that $c_n^{1/2}(-\Delta_v)^{1/4}X$ is an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathcal{M}_{1,n})$. By using this isometry we have

$$\begin{aligned}\int_{\mathbb{R}^n} |\widehat{gd\sigma}|^2 w &= c_n \left\langle (-\Delta_v)^{\frac{1}{4}} X(|\widehat{gd\sigma}|^2), (-\Delta_v)^{\frac{1}{4}} Xw \right\rangle_{L^2(\mathcal{M}_{1,n})} \\ &= c_n \left\langle (-\Delta_v)^{\frac{1}{2}(1-\frac{n-1}{q})} X(|\widehat{gd\sigma}|^2), (-\Delta_v)^{\frac{n-1}{2q}} Xw \right\rangle_{L^2(\mathcal{M}_{1,n})}\end{aligned} \tag{1.23}$$

for all $1 \leq q \leq \infty$. An application of Hölder's inequality now leads to the bound

$$\int_{\mathbb{R}^n} |\widehat{gd\sigma}|^2 w \lesssim \big\|(-\Delta_v)^{\frac{n-1}{2q}} Xw\big\|_{L^\infty_\omega L^q_v} \big\|(-\Delta_v)^{\frac{1}{2}(1-\frac{n-1}{q})} X(|\widehat{gd\sigma}|^2)\big\|_{L^1_\omega L^{q'}_v}. \tag{1.24}$$

Therefore our estimate (1.10) may be reduced to the "tomography bounds"

$$\big\|(-\Delta_v)^{\frac{1}{2}(1-\frac{n-1}{q})} X(|\widehat{gd\sigma}|^2)\big\|_{L^1_\omega L^{q'}_v(\mathcal{M}_{1,n})} \lesssim \|g\|^2_{L^2(\mathbb{S}^{n-1})}. \tag{1.25}$$

Corresponding to the local variant (1.11) we also consider

$$\big\|(-\Delta_v)^{\frac{1}{2}(1-\frac{n-1}{q})} X(\gamma_R|\widehat{gd\sigma}|^2)\big\|_{L^1_\omega L^{q'}_v(\mathcal{M}_{1,n})} \lesssim R^\varepsilon \|g\|^2_{L^2(\mathbb{S}^{n-1})}, \tag{1.26}$$

where γ_R is a smooth bump function adapted to B_R (satisfying certain technical conditions). As arguing before (1.26) implies (1.11). So the proof of Theorem 1.2 is reduced to the analysis of the quantity $(-\Delta_v)^\alpha X(|\widehat{gd\sigma}|^2)$. More precisely, it amounts to show

$$\begin{aligned}\big\|X(\gamma_R|\widehat{gd\sigma}|^2)\big\|_{L^1_\omega L^\infty_v(\mathcal{M}_{1,2})} &\lesssim R^\varepsilon \|g\|^2_{L^2(\mathbb{S}^1)}, \\ \big\|(-\Delta_v)^{\frac{1}{4}} X(\gamma_R|\widehat{gd\sigma}|^2)\big\|_{L^1_\omega L^2_v(\mathcal{M}_{1,2})} &\lesssim R^\varepsilon \|g\|^2_{L^2(\mathbb{S}^1)}.\end{aligned}$$

These will be done by using further Fourier analytic argument in [4] and we do not give detailed proof here. Instead we end this section by the remark that (1.26) cannot be true for $2 \leq q \leq \infty$ as the example $g \equiv 1$ shows. So the simple argument like (1.23) and (1.24) are not enough to reach at the conjecture (1.7).

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§ 2. Statement and detailed proof of Theorem 4.1 in [1] by Barceló-Bennett-Carbery.

§ 2.1. Statement

We first correct notations we will use in this section.

- A unit ball is given by $\mathbb{B}^{(2)} = \{\xi \in \mathbb{R}^2 : |\xi| \leq 1\}$.
- For $\alpha < \beta$, $\omega \in \mathbb{S}^1$, and $x \in \mathbb{R}^2$, $T_{\alpha,\beta}(\omega, x)$ denotes the tube whose short length is α, long length β, centered at x, and the long direction parallel to ω.
- For $\delta > 0$, the δ-fattened sphere is denoted by $\mathbb{S}^1 + O(\delta) := \{\xi \in \mathbb{R}^2 : 1 - \delta \leq |\xi| \leq 1 + \delta\}$.
- We will use g for the function on $\mathbb{S}^1$ and G for the function on $\mathbb{S}^1 + O(\delta)$.
- More generally, for $t > 0$, we define the sphere with radius t by

$$\mathbb{S}^1_t := \{\xi \in \mathbb{R}^2 : |\xi| = t\}, \quad d\sigma_t(\xi) := \delta(t^2 - |\xi|^2)d\xi$$

where δ is the standard Dirac delta. Note that if we define

$$\phi_{t,\varepsilon}(\xi) := \varepsilon^{-1} g(\varepsilon^{-1}(t^2 - |\xi|^2)),$$

where g stands for the centered and normalized gaussian, then we have

$$\lim_{\varepsilon \to 0} \phi_{t,\varepsilon}(\xi) = \delta(t^2 - |\xi|^2), \quad \phi_{t,\varepsilon}(\xi) = c\varepsilon^{-1} \int_{t-\varepsilon}^{t+\varepsilon} \delta(s^2 - |\xi|^2)\, ds. \tag{2.1}$$

- We will identify $\mathbb{S}^1$ with $(-\pi, \pi)$ via

$$\varphi(\theta) = (\cos\theta, \sin\theta) \in \mathbb{S}^1, \quad \theta \in (-\pi, \pi).$$

- For a set $E \subset \mathbb{R}^n$, $n = 1, 2$, we denote the smooth cut-off associated with E by $\chi_E \in C_c^\infty(\mathbb{R}^n)$ satisfying $1_{2^{-1}E} \leq \chi \leq 1_{2E}$.

The statement of the result in [1] is as follows. We will use the notation $⨍$ to denote the averaged integral.

Theorem 2.1 (Theorem 4.1 in [1]). *For all $R \gg 1$,*

$$\int_{\mathbb{B}^{(2)}} |\widehat{gd\sigma}(Rx)|^2 w(x)\, dx \lesssim R^{-1}\log(R) \int_{\mathbb{S}^1} |g(\omega)|^2 \mathcal{M}_R w(\omega)\, d\sigma(\omega), \tag{2.2}$$

where

$$\mathcal{M}_R w(\omega) := \sup_{R^{-1}\le\alpha\le R^{-1/2}} \mathcal{M}_{\alpha,R} w(\omega),$$

$$\mathcal{M}_{\alpha,R} w(\omega) := \sup_{v\in\mathbb{R}^2} \Big(⨍_{T_{(\alpha R)^{-1},1}(\omega,v)} \big[⨍_{T_{\alpha,\alpha^2 R}(\omega,x)} w(y)\, dy \big]^2 dx \Big)^{\frac{1}{2}}. \tag{2.3}$$

After rescaling one can see (2.2) is a variant of (1.6) and indeed it is the weakened version. We will provide the detailed proof of this result in below.

§ 2.2. Equivalent R^{-1}-fattened form

It is convenient to use $G : \mathbb{R}^2 \to \mathbb{C}$ supported on $\mathbb{S}^1 + O(R^{-1})$ instead of $g : \mathbb{S}^1 \to \mathbb{C}$ as the input function.

Lemma 2.2. *Consider two estimate: for $t > 0$,*

$$\int_{\mathbb{B}^{(2)}} |\widehat{g_t d\sigma_t}(Rx)|^2 w(x)\, dx \lesssim R^{-1}\log(R) \int_{\mathbb{S}^1_t} |g_t(\omega)|^2 \mathcal{M}_R w(\omega/|\omega|)\, d\sigma_t(\omega), \quad (g_t : \mathbb{S}^1_t \to \mathbb{C}), \tag{2.4}$$

and

$$\int_{\mathbb{B}^{(2)}} |{}^{\vee}G(Rx)|^2 w(x)\, dx \lesssim R^{-2}\log(R) \int_{\mathbb{R}^2} |G(\xi)|^2 \mathcal{M}_R w(\xi/|\xi|)\, d\xi, \quad (G : \mathbb{S}^1 + O(R^{-1}) \to \mathbb{C}). \tag{2.5}$$

1. *Suppose that* (2.4) *holds uniformly in $t \in (1-R^{-1}, 1+R^{-1})$ uniformly in t. Then we have* (2.5).
2. *Conversely assume* (2.5). *Then we have* (2.4) *with $t = 1$.*

Thanks to Lemma 2.2, it suffices to show (2.5).

Proof. First let us show the part *(1)*. With (2.1) in mind,

$$\begin{aligned}
{}^{\vee}G(x) &= \int_{\mathbb{S}^1+O(R^{-1})} G(\xi) e^{-ix\cdot\xi}\, d\xi \\
&= \int_{\mathbb{R}^2} R^{-1}\phi_{1,R^{-1}}(\xi) G(\xi) e^{-ix\cdot\xi}\, d\xi \\
&= \int_{\mathbb{R}^2}\int_{1-R^{-1}}^{1+R^{-1}} G(\xi) e^{-ix\cdot\xi} \delta(s^2 - |\xi|^2)\, ds d\xi \\
&= \int_{1-R^{-1}}^{1+R^{-1}} \widehat{g_s d\sigma_s}(x)\, ds,
\end{aligned}$$

where $g_s := G|_{\mathbb{S}^1_s}$ is the restriction of G to $\mathbb{S}^1_s$. So simple applications of the Minkowski's inequality, (2.4), and the Cauchy-Schwarz inequality reveal that

$$\begin{aligned}
&\int_{\mathbb{B}^{(2)}} |{}^{\vee}G(Rx)|^2 w(x)\, dx \\
\leq&\bigg(\int_{1-R^{-1}}^{1+R^{-1}} \big(\int_{\mathbb{B}^{(2)}} |\widehat{g_s d\sigma_s}(Rx)|^2 w(x)\, dx \big)^{\frac12}\, ds\bigg)^2 \\
\lesssim& R^{-1}\log(R) \bigg(\int_{1-R^{-1}}^{1+R^{-1}} \big(\int_{\mathbb{S}^1_s} |g_s(\omega)|^2 \mathcal{M}_R w(\omega/|\omega|)\, d\sigma_s(\omega) \big)^{\frac12}\, ds\bigg)^2 \\
\lesssim& R^{-1}(R^{-1/2})^2 \log(R) \int_{1-R^{-1}}^{1+R^{-1}} \int_{\mathbb{S}^1_s} |g_s(\omega)|^2 \mathcal{M}_R w(\omega/|\omega|)\, d\sigma_s(\omega) ds \\
=& R^{-2}\log(R) \int_{1-R^{-1}}^{1+R^{-1}} \int_{\mathbb{R}^2} |G(\omega)|^2 \mathcal{M}_R w(\omega/|\omega|) \delta(s^2-|\omega|^2)\, d\omega ds \\
\sim& R^{-2}\log(R) \int_{\mathbb{R}^2} |G(\omega)|^2 \mathcal{M}_R w(\omega/|\omega|)|\omega|^{-1}\, d\omega \\
\sim& R^{-2}\log(R) \int_{\mathbb{R}^2} |G(\omega)|^2 \mathcal{M}_R w(\omega/|\omega|)\, d\omega
\end{aligned}$$

where we also used the fact that $\int_{1-R^{-1}}^{1+R^{-1}} \delta(s^2-|\omega|^2) ds \sim |\omega|^{-1}$ and that the integral is indeed over $\mathbb{S}^1 + O(R^{-1})$, in particular $|\omega| \sim 1$.

Next we show the part *(2)*. We first note that by denoting $\gamma_R(x) = e^{-|x/R|^2}$ we have

$$\int_{\mathbb{B}^{(2)}} |\widehat{gd\sigma}(Rx)|^2 w(x)\, dx \lesssim \int_{\mathbb{B}^{(2)}} |{}^{\vee}\big[(gd\sigma) * \hat{\gamma_R}\big](Rx)|^2 w(x)\, dx.$$

If $\xi \notin \mathbb{S}^1 + O(R^{-1})$, then $(gd\sigma) * \hat{\gamma_R}(\xi)$ decays arbitrary fast as the Gaussian $\hat{\gamma_R}$ does. So, we may regard the support of $(gd\sigma) * \hat{\gamma_R}$ is contained in $\mathbb{S}^1 + O(R^{-1})$ and hence set $G = (gd\sigma) * \hat{\gamma_R}$. Applying (2.5), we see

$$\int_{\mathbb{B}^{(2)}} |\widehat{gd\sigma}(Rx)|^2 w(x)\, dx \lesssim R^{-2}\log(R) \int_{\mathbb{R}^2} |(gd\sigma) * \hat{\gamma_R}(\xi)|^2 \mathcal{M}_R w(\xi/|\xi|)\, d\xi.$$

We complete the proof once we can prove (especially for $p = 2$)

$$\int_{\mathbb{R}^2} |(gd\sigma) * \hat{\gamma_R}(\xi)|^p \mathcal{M}_R w(\xi/|\xi|)\, d\xi \lesssim R^{\frac{p}{p'}} \int_{\mathbb{S}^1} |g(\xi)|^p \mathcal{M}_R w(\xi)\, d\sigma(\xi). \tag{2.6}$$

We will do this by the interpolation. When $p = \infty$, (2.6) follows from the simple observation that

$$\sup_{\xi\in\mathbb{R}^2} \int_{\mathbb{S}^1} |\hat{\gamma_R}(\xi - y)|\, d\sigma(y) \lesssim R$$

since we easily have

$$\sup_{\xi\in\mathbb{R}^2} |(gd\sigma) * \hat{\gamma_R}(\xi)| \leq \|g\|_\infty \sup_{\xi\in\mathbb{R}^2} \int_{\mathbb{S}^1} |\hat{\gamma_R}(\xi - y)|\, d\sigma(y).$$

On the other hand, for $p = 1$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} |(gd\sigma) * \hat{\gamma_R}(\xi)|\mathcal{M}_R w(\xi/|\xi|)\, d\xi \\
\leq &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |g(\xi-\eta)|\delta(1-|\xi-\eta|^2)|\hat{\gamma_R}(\eta)|\mathcal{M}_R w(\xi/|\xi|)\, d\xi d\eta \\
= &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |g(\omega)|\delta(1-|\omega|^2)|\hat{\gamma_R}(\eta)|\mathcal{M}_R w((\omega+\eta)/|\omega+\eta|)\, d\xi d\eta.
\end{aligned}$$

As we will prove in Lemma 2.3 the maximal operator $\mathcal{M}_R$ has a local constant property at scale $R^{-1/2}$ and in particular at scale R^{-1} in the sense that if $\omega, \omega_0 \in \mathbb{S}^1$ and $|\omega - \omega_0| \leq R^{-1/2}$, then

$$\mathcal{M}_R w(\omega) \sim \mathcal{M}_R w(\omega_0).$$

So, for $|\eta| \leq R^{-1}$, we have $|(\omega+\eta)/|\omega+\eta| - \omega/|\omega|| \lesssim R^{-1}$ and hence

$$\mathcal{M}_R w((\omega+\eta)/|\omega+\eta|) \sim \mathcal{M}_R w(\omega/|\omega|) = \mathcal{M}_R w(\omega).$$

This shows

$$\begin{aligned}
\int_{\mathbb{R}^2} |(gd\sigma) * \hat{\gamma_R}(\xi)|\mathcal{M}_R w(\xi/|\xi|)\, d\xi &\lesssim \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |g(\omega)|\delta(1-|\omega|^2)|\hat{\gamma_R}(\eta)|\mathcal{M}_R w(\omega)\, d\sigma(\omega) d\eta \\
&\sim \int_{\mathbb{S}^1} |g(\omega)|\mathcal{M}_R w(\omega)\, d\sigma(\omega)
\end{aligned}$$

which is (2.6) with $p = 1$. Therefore we conclude (2.4) with $t = 1$. □

§ 2.3. Basic properties of maximal operators $\mathcal{M}_R$ and $\mathcal{M}_{\alpha,R}$

First of all, we begin this subsection with one simple geometrical observation which implies the locally constant property of $\mathcal{M}_{\alpha,R}$ and $\mathcal{M}_R$ as well. Recall that for $\alpha < \beta$, $\omega \in \mathbb{S}^1$, and $v \in \mathbb{R}^2$, $T_{\alpha,\beta}(\omega, v)$ is the tube centered at v whose short length is α, long length is β, and the direction is ω. Then one can see that there exists $C, c > 0$ such that

$$cT_{\alpha,\beta}(\omega_0, v) \subset T_{\alpha,\beta}(\omega, v) \subset CT_{\alpha,\beta}(\omega_0, v) \tag{2.7}$$

holds as long as $|\omega - \omega_0| \leq \alpha/(100\beta)$. Using this property, we can show the following.

Lemma 2.3. *Let $\alpha > 0$. Then $\mathcal{M}_{\alpha,R}w$ is the locally constant at scale $(\alpha R)^{-1}$ in the sense that*

$$\mathcal{M}_{\alpha,R}w(\omega) \sim \mathcal{M}_{\alpha,R}w(\omega_0)$$

holds as long as $\omega, \omega_0 \in \mathbb{S}^1$, $|\omega - \omega_0| \leq (\alpha R)^{-1}$, where the implicit constants are independent of α, R.

In particular, $\mathcal{M}_R w$ is the locally constant at scale $R^{-1/2}$.

Proof. It suffices to show

$$\mathcal{M}_{\alpha,R}w(\omega) \lesssim \mathcal{M}_{\alpha,R}w(\omega_0).$$

From the definition of $\mathcal{M}_{\alpha,R}$, for each $\omega \in \mathbb{S}^1$, there exists $v(\omega) \in \mathbb{R}^2$ such that

$$\mathcal{M}_{\alpha,R}w(\omega) \sim \Big(\fint_{T_{(\alpha R)^{-1},1}(\omega,v(\omega))} \big[\fint_{T_{\alpha,\alpha^2 R}(\omega,x)} w(y)\,dy\big]^2\,dx\Big)^{\frac{1}{2}}.$$

Using (2.7) for $T_{(\alpha R)^{-1},1}(\omega,v(\omega))$ and $T_{\alpha,\alpha^2R}(\omega,x)$, one can replace tubes direction ω in right-hand side by the ones direction ω_0 and hence we see

$$\mathcal{M}_{\alpha,R}w(\omega) \lesssim \Big(\fint_{T_{(\alpha R)^{-1},1}(\omega_0,v(\omega))} \big[\fint_{T_{\alpha,\alpha^2 R}(\omega_0,x)} w(y)\,dy\big]^2\,dx\Big)^{\frac{1}{2}} \lesssim \mathcal{M}_{\alpha,R}w(\omega_0)$$

as long as $|\omega-\omega_0| \le (\alpha R)^{-1}$. □

Another property we will need is the relation between $\mathcal{M}_{\alpha,R}$ and the Kakeya type maximal operator.

Lemma 2.4. *Then we have*

$$\mathcal{M}_{R^{-1/2},R}w(\omega) \sim \sup_{v\in\mathbb{R}^2} \fint_{T_{R^{-1/2},1}(\omega,v)} w(x)\,dx. \tag{2.8}$$

Proof. With the definition of $\mathcal{M}_{R^{-1/2},R}$ in mind, we consider

$$\Big(\fint_{T_{R^{-1/2},1}(\omega,v)} \big[\fint_{T_{R^{-1/2},1}(\omega,x)} w(y)\,dy\big]^2\,dx\Big)^{\frac{1}{2}}$$

for arbitrary $v \in \mathbb{R}^2$. Since the two tubes have same size we see that for all $x \in T_{R^{-1/2},1}(\omega,v)$,

$$100T_{R^{-1/2},1}(\omega,x) \supset T_{R^{-1/2},1}(\omega,v)$$

and hence

$$\inf_{x\in T_{R^{-1/2},1}(\omega,v)} \fint_{100T_{R^{-1/2},1}(\omega,x)} w(y)\,dy \gtrsim \fint_{T_{R^{-1/2},1}(\omega,v)} w(y)\,dy.$$

This yields

$$\Big(\fint_{T_{R^{-1/2},1}(\omega,v)} \big[\fint_{100T_{R^{-1/2},1}(\omega,x)} w(y)\,dy\big]^2\,dx\Big)^{\frac{1}{2}} \gtrsim \fint_{T_{R^{-1/2},1}(\omega,v)} w(y)\,dy$$

and hence by taking the supremum over all v it concludes

$$\mathcal{M}_{R^{-1/2},R}w(\omega) \gtrsim \sup_{v\in\mathbb{R}^2} \fint_{T_{R^{-1/2},1}(\omega,v)} w(x)\,dx.$$

We similarly have for all $x \in T_{R^{-1/2},1}(\omega, v)$,

$$100T_{R^{-1/2},1}(\omega, v) \supset T_{R^{-1/2},1}(\omega, x)$$

and hence

$$\sup_{x \in T_{R^{-1/2},1}(\omega,v)} \fint_{T_{R^{-1/2},1}(\omega,x)} w(y)\, dy \lesssim \fint_{100T_{R^{-1/2},1}(\omega,v)} w(y)\, dy$$

which shows the converse inequality.

□

§ 2.4. Proof of (2.5)

Let us prove (2.5) and hence Theorem 2.1 by following the argument by Erdoğan [10]. Recalling the identification $\varphi(\theta) = (\cos\theta, \sin\theta)$, the support of G is $\varphi((-\pi, \pi)) + O(R^{-1})$. Moreover we may suppose G is supported on $\varphi((-1/2, 1/2)) + O(R^{-1})$ without loss of generality. Now we employ the Whitney decomposition to handle the singularity. For each $j \in \mathbb{N}$, we do the dyadic decomposition $(-1/2, 1/2) = \bigcup_k \tau_k^j$, where τ_k^j is the subdyadic interval of length 2^{-j}. We say $\tau_k^j \sim \tau_{k'}^j$ if τ_k^j and $\tau_{k'}^j$ are not adjacent but have adjacent parent. Then we may decompose the unit cube by

$$(-1/2, 1/2) \times (-1/2, 1/2) = \bigcup_{j=1}^{\log(R^{1/2})} \bigcup_{k,k' : \tau_k^j \sim \tau_{k'}^j} \tau_k^j + \mathcal{D},$$

where $\mathcal{D}$ is more or less the $R^{-1/2}$-neighborhood of the diagonal line: $\{(\theta_1, \theta_2) \in (-1/2, 1/2) \times (-1/2, 1/2) : |x - y| \le R^{-1/2}\}$. In particular, we may decompose $\mathcal{D}$ by

$$\mathcal{D} = \bigcup_{l=1}^{R^{1/2}} I_l \times I_l,$$

where $I_l \subset (-1/2, 1/2)$ is an interval of length $R^{-1/2}$ and the family of intervals $\{I_l\}_l$ are almost disjoint. Therefore we see

$$|{}^\vee G(Rx)|^2 \lesssim \sum_{j=1}^{\log(R^{1/2})} \sum_{k,k' : \tau_k^j \sim \tau_{k'}^j} |{}^\vee G_{\tau_k^j}(Rx) {}^\vee G_{\tau_{k'}^j}(Rx)| + \sum_{l=1}^{R^{1/2}} |{}^\vee G_{I_l}(Rx)|^2,$$

where for the interval $I \subset (-1/2, 1/2)$ we denote $G_I(\xi) = G(\xi)\chi_I \circ \varphi^{-1}(\xi/|\xi|)$. So it suffices to evaluate two quantities

$$I_1 := \int_{\mathbb{B}^{(2)}} \sum_{j=1}^{\log(R^{1/2})} \sum_{k,k':\tau_k^j \sim \tau_{k'}^j} |{}^\vee G_{\tau_k^j}(Rx) {}^\vee G_{\tau_{k'}^j}(Rx)| w(x)\, dx,$$

$$I_2 := \int_{\mathbb{B}^{(2)}} \sum_{l=1}^{R^{1/2}} |{}^\vee G_{I_l}(Rx)|^2 w(x)\, dx.$$

We first handle the second term which behaves better than the first term. Notice that the support of G_{I_l} is in $\varphi(I_l) + O(R^{-1})$. More precisely, if we denote $\omega_l \in \mathbb{S}^1$ to be the center of $\varphi(I_l)$, then

$$\mathrm{supp}(G_{I_l}) \subset T_{R^{-1},R^{-1/2}}(\omega_l^\perp, \omega_l)$$

and hence the Fourier support of ${}^\vee G_{I_l}(R\cdot)$ is in $T_{1,R^{1/2}}(\omega_l^\perp, R\omega_l)$. So the Fourier support of $|{}^\vee G_{I_l}(R\cdot)|^2$ is contained in $2T_{1,R^{1/2}}({\omega_l}^\perp, R\omega_l)$ which implies

$$\int_{\mathbb{B}^{(2)}} |{}^\vee G_{I_l}(Rx)|^2 w(x)\, dx \lesssim \int_{\mathbb{B}^{(2)}} |{}^\vee G_{I_l}(Rx)|^2 w * |{}^\vee \chi_{T_{1,R^{1/2}}({\omega_l}^\perp, R\omega_l)}|(x)\, dx.$$

Here with the uncertainly principle in mind we have

$$|{}^\vee \chi_{T_{1,R^{1/2}}({\omega_l}^\perp, R\omega_l)}| \sim |T_{1,R^{1/2}}({\omega_l}^\perp, R\omega_l)| \chi_{T_{R^{-1/2},1}(\omega_l,0)} = |T_{R^{-1/2},1}(\omega_l,0)|^{-1} \chi_{T_{R^{-1/2},1}(\omega_l,0)}$$

and hence for any $x \in \mathbb{R}^2$,

$$w * |{}^\vee \chi_{T_{1,R^{1/2}}({\omega_l}^\perp, R\omega_l)}|(x) \lesssim \sup_{v \in \mathbb{R}^2} \fint_{T_{R^{-1/2},1}(\omega_l,v)} w(y)\, dy \sim \mathcal{M}_{R^{-1/2},R} w(\omega_l)$$

thanks to Lemma 2.4. From this, we also have

$$\begin{aligned}
\int_{\mathbb{B}^{(2)}} |{}^\vee G_{I_l}(Rx)|^2 w(x)\, dx &\lesssim \int_{\mathbb{B}^{(2)}} |{}^\vee G_{I_l}(Rx)|^2\, dx \times \mathcal{M}_{R^{-1/2},R} w(\omega_l) \\
&\leq R^{-2} \int_{\mathbb{R}^2} |{}^\vee G_{I_l}(x)|^2\, dx \times \mathcal{M}_{R^{-1/2},R} w(\omega_l) \\
&= R^{-2} \int_{\mathbb{R}^2} |G_{I_l}(\xi)|^2\, d\xi \times \mathcal{M}_{R^{-1/2},R} w(\omega_l).
\end{aligned}$$

Recalling the support of G_{I_l} is in $T_{R^{-1},R^{-1/2}}(\omega_l^\perp, \omega_l)$, especially in $B(\omega_l, R^{-1/2})$ we notice that

$$\int_{\mathbb{R}^2} |G_{I_l}(\xi)|^2\, d\xi \times \mathcal{M}_{R^{-1/2},R} w(\omega_l) \sim \int_{\mathbb{R}^2} |G_{I_l}(\xi)|^2 \mathcal{M}_{R^{-1/2},R} w(\xi/|\xi|)\, d\xi$$

since $\mathcal{M}_{R^{-1/2},R}w$ is locally constant at scale $R^{-1/2}$, see Lemma 2.3. Since $\{I_l\}_l$ has the almost disjointness we can sum up with respect to $l=1,\dots R^{1/2}$ and conclude

$$I_2 \lesssim R^{-2}\int_{\mathbb{R}^2}|G(\xi)|^2\mathcal{M}_{R^{-1/2},R}w(\xi/|\xi|)\,d\xi \le R^{-2}\int_{\mathbb{R}^2}|G(\xi)|^2\mathcal{M}_{R}w(\xi/|\xi|)\,d\xi.$$

To handle I_1, fix j,k,k' and consider $\int_{\mathbb{B}^{(2)}}|{}^\vee G_{\tau_k^j}(Rx){}^\vee G_{\tau_{k'}^j}(Rx)|w(x)\,dx$. Note that the Fourier support of ${}^\vee G_{\tau_k^j}{}^\vee G_{\tau_{k'}^j}$ is the support of $G_{\tau_k^j}*G_{\tau_{k'}^j}\subset[\tau_k^j+O(R^{-1})]+[\tau_{k'}^j+O(R^{-1})]$. If we denote the center of the 2^{-j}-cap $\varphi(\tau_k^j)$ by $\omega_k^j\in\mathbb{S}^1$, then $\tau_k^j+O(R^{-1})\subset T_{2^{-2j},2^{-j}}((\omega_k^j)^\perp,\omega_k^j)$ and hence

$$\operatorname{supp}(G_{\tau_k^j}*G_{\tau_{k'}^j})\subset T_{2^{-2j},2^{-j}}((\omega_k^j)^\perp,\omega_k^j)+T_{2^{-2j},2^{-j}}((\omega_{k'}^j)^\perp,\omega_{k'}^j).$$

Furthermore recalling $\tau_k^j\sim\tau_{k'}^j$ which in particular implies $\tau_{k'}^j\subset 4\tau_k^j$ and hence

$$T_{2^{-2j},2^{-j}}((\omega_k^j)^\perp,\omega_k^j)+T_{2^{-2j},2^{-j}}((\omega_{k'}^j)^\perp,\omega_{k'}^j)\subset 8T_{2^{-2j},2^{-j}}((\omega_k^j)^\perp,\omega_k^j).$$

This shows the Fourier support of ${}^\vee G_{\tau_k^j}(R\cdot){}^\vee G_{\tau_{k'}^j}(R\cdot)$ is contained in $8T_{R2^{-2j},R2^{-j}}((\omega_k^j)^\perp,R\omega_k^j)$ and so

$$\begin{aligned}&\int_{\mathbb{B}^{(2)}}|{}^\vee G_{\tau_k^j}(Rx){}^\vee G_{\tau_{k'}^j}(Rx)|w(x)\,dx\\ \lesssim&\int_{\mathbb{B}^{(2)}}|{}^\vee G_{\tau_k^j}(Rx){}^\vee G_{\tau_{k'}^j}(Rx)|w*|{}^\vee\chi_{T_{R2^{-2j},R2^{-j}}((\omega_k^j)^\perp,R\omega_k^j)}|(x)\,dx.\end{aligned}$$

As before from the uncertainly principle we know

$$|{}^\vee\chi_{T_{R2^{-2j},R2^{-j}}((\omega_k^j)^\perp,R\omega_k^j)}|\sim|T_{R^{-1}2^j,R^{-1}2^{2j}}(\omega_k^j,0)|^{-1}\chi_{T_{R^{-1}2^j,R^{-1}2^{2j}}(\omega_k^j,0)}. \tag{2.9}$$

We further decompose the physical space into rectangles: $\sum_P\chi_P\sim1$, where each P is the rectangle of short length 2^{-j}, long length 1, and the direction is parallel to ω_k^j. Then

$$\begin{aligned}&\int_{\mathbb{B}^{(2)}}|{}^\vee G_{\tau_k^j}(Rx){}^\vee G_{\tau_{k'}^j}(Rx)|w(x)\,dx\\ \lesssim&\sum_P\int_P|{}^\vee G_{\tau_k^j}(Rx){}^\vee G_{\tau_{k'}^j}(Rx)|w*|{}^\vee\chi_{T_{R2^{-2j},R2^{-j}}((\omega_k^j)^\perp,R\omega_k^j)}|(x)\,dx\\ \le&\sum_P\Big(\int_P|{}^\vee G_{\tau_k^j}(Rx){}^\vee G_{\tau_{k'}^j}(Rx)|^2\,dx\Big)^{\frac12}\Big(\int_P w*|{}^\vee\chi_{T_{R2^{-2j},R2^{-j}}((\omega_k^j)^\perp,R\omega_k^j)}|(x)^2\,dx\Big)^{\frac12}.\end{aligned}$$

From the bilinear restriction estimate (for example (25) in [10]) we have

$$\int_P|{}^\vee G_{\tau_k^j}(Rx){}^\vee G_{\tau_{k'}^j}(Rx)|^2\,dx\lesssim 2^j\big\|{}^\wedge[\chi_P{}^\vee G_{\tau_k^j}(R\cdot)]\big\|_2^2\big\|{}^\wedge[\chi_P{}^\vee G_{\tau_{k'}^j}(R\cdot)]\big\|_2^2.$$

For the second term we have from (2.9) that

$$\int_P w * |^\vee \chi_{T_{R2^{-2j},R2^{-j}}((\omega_k^j)^\perp, R\omega_k^j)}|(x)^2\,dx \lesssim \int_P [\fint_{T_{R^{-1}2^j,R^{-1}2^{2j}}(\omega_k^j,x)} w(y)\,dy]^2\,dx$$
$$= 2^{-j} \fint_P [\fint_{T_{R^{-1}2^j,R^{-1}2^{2j}}(\omega_k^j,x)} w(y)\,dy]^2\,dx.$$

From these estimates we obtain

$$\begin{aligned}
&\int_{\mathbb{B}^{(2)}} |^\vee G_{\tau_k^j}(Rx)^\vee G_{\tau_{k'}^j}(Rx)|w(x)\,dx \\
&\lesssim \sum_P 2^{j/2} \big\|^\wedge[\chi_P{}^\vee G_{\tau_k^j}(R\cdot)]\big\|_2 \big\|^\wedge[\chi_P{}^\vee G_{\tau_{k'}^j}(R\cdot)]\big\|_2 2^{-j/2} \\
&\quad \times \big(\fint_P [\fint_{T_{R^{-1}2^j,R^{-1}2^{2j}}(\omega_k^j,x)} w(y)\,dy]^2\,dx\big)^{\frac12} \\
&\le \big(\sum_P \big\|^\wedge[\chi_P{}^\vee G_{\tau_k^j}(R\cdot)]\big\|_2^2\big)^{\frac12} \big(\sum_P \big\|^\wedge[\chi_P{}^\vee G_{\tau_{k'}^j}(R\cdot)]\big\|_2^2\big)^{\frac12} \\
&\quad \times \sup_P \big(\fint_P [\fint_{T_{R^{-1}2^j,R^{-1}2^{2j}}(\omega_k^j,x)} w(y)\,dy]^2\,dx\big)^{\frac12} \\
&\sim \big\|^\vee G_{\tau_k^j}(R\cdot)\big\|_2 \big\|^\vee G_{\tau_{k'}^j}(R\cdot)\big\|_2 \sup_P \big(\fint_P [\fint_{T_{R^{-1}2^j,R^{-1}2^{2j}}(\omega_k^j,x)} w(y)\,dy]^2\,dx\big)^{\frac12} \\
&\sim R^{-2} \|G_{\tau_k^j}\|_2 \|G_{\tau_{k'}^j}\|_2 \sup_P \big(\fint_P [\fint_{T_{R^{-1}2^j,R^{-1}2^{2j}}(\omega_k^j,x)} w(y)\,dy]^2\,dx\big)^{\frac12}.
\end{aligned}$$

Since the direction of P is ω_k^j, we notice

$$\sup_P \big(\fint_P [\fint_{T_{R^{-1}2^j,R^{-1}2^{2j}}(\omega_k^j,x)} w(y)\,dy]^2\,dx\big)^{\frac12} = \mathcal{M}_{R^{-1}2^j,R} w(\omega_k^j) \sim \mathcal{M}_{R^{-1}2^j,R} w(\omega_{k'}^j)$$

as $\mathcal{M}_{R^{-1}2^j,R} w$ is locally constant at scale 2^{-j} thank to Lemma 2.3 and $|\omega_k^j - \omega_{k'}^j| \sim 2^{-j}$. Moreover

$$\mathrm{dist}\big(\omega_k^j, \partial\big[\mathrm{supp}(G_{\tau_k^j})\big]\big), \mathrm{dist}\big(\omega_{k'}^j, \partial\big[\mathrm{supp}(G_{\tau_{k'}^j})\big]\big) \sim 2^{-j},$$

and hence again Lemma 2.3 shows

$$\begin{aligned}
&\|G_{\tau_k^j}\|_2 \|G_{\tau_{k'}^j}\|_2 \sup_P \big(\fint_P [\fint_{T_{R^{-1}2^j,R^{-1}2^{2j}}(\omega_k^j,x)} w(y)\,dy]^2\,dx\big)^{\frac12} \\
&\sim \|G_{\tau_k^j}\|_2 \|G_{\tau_{k'}^j}\|_2 \mathcal{M}_{R^{-1}2^j,R} w(\omega_k^j)^{\frac12} \times \mathcal{M}_{R^{-1}2^j,R} w(\omega_{k'}^j)^{\frac12} \\
&\sim \big(\int_{\mathbb{R}^2} |G_{\tau_k^j}(\xi)|^2 \mathcal{M}_{R^{-1}2^j,R} w(\xi/|\xi|)\,d\xi\big)^{\frac12} \big(\int_{\mathbb{R}^2} |G_{\tau_{k'}^j}(\xi)|^2 \mathcal{M}_{R^{-1}2^j,R} w(\xi/|\xi|)\,d\xi\big)^{\frac12} \\
&\le \big(\int_{\mathbb{R}^2} |G_{\tau_k^j}(\xi)|^2 \mathcal{M}_R w(\xi/|\xi|)\,d\xi\big)^{\frac12} \big(\int_{\mathbb{R}^2} |G_{\tau_{k'}^j}(\xi)|^2 \mathcal{M}_R w(\xi/|\xi|)\,d\xi\big)^{\frac12}.
\end{aligned}$$

Now we sum up over k, k'. Note that for each fixed j, $\{\tau_k^j\}_k$ has the almost disjointness. Also for each k, we know $\sharp\{k' : \tau_k^j \sim \tau_{k'}^j\} \leq 4$ and so we may pretend as there is a unique $k' = k'(k)$ such that $\tau_k^j \sim \tau_{k'(k)}^j$. With these in mind we conclude

$$\begin{aligned}
&\sum_{k,k':\tau_k^j\sim\tau_{k'}^j} \int_{\mathbb{B}(2)} |{}^\vee G_{\tau_k^j}(Rx){}^\vee G_{\tau_{k'}^j}(Rx)| w(x)\,dx \\
\lesssim& R^{-2} \sum_k \Big(\int_{\mathbb{R}^2} |G_{\tau_k^j}(\xi)|^2 \mathcal{M}_R w(\xi/|\xi|)\,d\xi \Big)^{\frac12} \Big(\int_{\mathbb{R}^2} |G_{\tau_{k'(k)}^j}(\xi)|^2 \mathcal{M}_R w(\xi/|\xi|)\,d\xi \Big)^{\frac12} \\
\leq& R^{-2} \Big(\sum_k \int_{\mathbb{R}^2} |G_{\tau_k^j}(\xi)|^2 \mathcal{M}_R w(\xi/|\xi|)\,d\xi \Big)^{\frac12} \Big(\sum_k \int_{\mathbb{R}^2} |G_{\tau_{k'(k)}^j}(\xi)|^2 \mathcal{M}_R w(\xi/|\xi|)\,d\xi \Big)^{\frac12} \\
\lesssim& R^{-2} \int_{\mathbb{R}^2} |G(\xi)|^2 \mathcal{M}_R w(\xi/|\xi|)\,d\xi.
\end{aligned}$$

Since this is the uniform estimate with respect to j, we conclude (2.5).

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