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A review of modified scattering for the $1d$ cubic NLS

By

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Abstract

We review the various 'PDE proofs' of sharp L^∞ decay and modified scattering for the one-dimensional cubic nonlinear Schrödinger equation with small initial data in a weighted Sobolev space. We conclude with a discussion of the proof of this result using techniques related to complete integrability.

§ 1. Introduction

In this paper we will discuss the one-dimensional cubic nonlinear Schrödinger equation (NLS)

$$i\partial_t u = -\partial_x^2 u + |u|^2 u, \tag{1.1}$$

where $u = u(t,x)$ is a complex-valued function on $\mathbb{R} \times \mathbb{R}$. This equation is one of the most widely-studied dispersive equations, arising naturally in many physical models. We have chosen the $+$ sign for the nonlinearity, resulting in the defocusing equation. Choosing the $-$ sign leads to the focusing equation, which is particularly important in physical settings as it supports solitary wave solutions (in contrast to the defocusing case). In this paper, the difference between these cases will not be so important, as we will typically focus on small solutions (which excludes solitary waves anyway) and utilize primarily perturbative techniques that do not distinguish between the two cases.

In addition to its physical relevance, the $1d$ cubic NLS has received a lot of attention due to the fact that it is completely integrable. In particular, it can be solved via inverse scattering, and in fact these techniques may be used to describe the long-term behavior of solutions. There are additionally several arguments using PDE methods (i.e.

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arguments that do not rely on complete integrability) that demonstrate the long-time behavior of solutions, although these are all restricted to the case of *small* solutions. In this paper, we will review all of these PDE arguments, in addition to a rough sketch of the approach using inverse scattering.

Solutions to the underlying *linear* Schrödinger equation

$$i\partial_t u = -\partial_x^2 u, \quad u|_{t=0} = \varphi \tag{1.2}$$

have the following asymptotic behavior as $t \to \infty$, obtained by explicit computation or by stationary phase:

$$u(t,x) \sim (2it)^{-1/2} e^{ix^2/4t} \widehat{\varphi}(\tfrac{x}{2t}),$$

where $\widehat{\varphi}$ denotes the Fourier transform of φ. We say that a solution to a nonlinear Schrödinger equation *scatters* if it behaves like a solution to (1.2) as $t \to \infty$. The asymptotic behavior for (1.1) is different than that of the underlying linear equation, although solutions do exhibit the same decay rate of $t^{-1/2}$. To describe the asymptotic behavior, one must incorporate a logarithmic phase correction—we call this *modified scattering*. In particular, one has the following:

Theorem 1.1. *Let φ be a Schwartz function and suppose $u_0 = \varepsilon\varphi$. If $\varepsilon > 0$ is sufficiently small, then the solution to* (1.1) *with initial data $u|_{t=0} = u_0$ exists for all time and obeys the decay estimate*

$$\sup_{x\in\mathbb{R}} |u(t,x)| \lesssim \varepsilon(1+t)^{-\frac{1}{2}} \quad \textit{for all} \quad t > 0.$$

Furthermore, there exists ψ such that

$$u(t,x) = (2it)^{-\frac{1}{2}} e^{\frac{ix^2}{4t} - \frac{i}{2}|\psi(\frac{x}{2t})|^2 \log t} \psi(\tfrac{x}{2t}) + o(t^{-\frac{1}{2}}) \tag{1.3}$$

in L^∞ as $t \to \infty$.

Remark. One can improve the decay/regularity conditions on the initial data above. A natural space for this problem is the weighted Sobolev space $H^{1,1}$, which consists of functions $f \in L^2$ such that $\partial_x f \in L^2$ and $xf \in L^2$. One can also quantify the error in the asymptotic formula above. The proofs we present can obtain $t^{-\frac{1}{2}-\delta}$ for some small δ (e.g. $\delta = \frac{1}{20}$). The optimal error estimate is $\mathcal{O}(t^{-1}\log t)$, cf. [4].

We have stated the result only for small initial data, which is the only case currently within reach without relying on complete integrability. For large data, techniques capitalizing on the complete integrability of (1.1) yield a similar (in fact, more precise) statement concerning the asymptotic behavior of solutions in the defocusing case. These techniques can also recover the long-time behavior (even in the presence of solitons) in the focusing case. We discuss these techniques in Section 8 below.

All of the 'PDE proofs' of Theorem 1.1 rely essentially on a combination of an energy-type estimate and an ODE-type analysis, which together yield the desired decay and asymptotic behavior. The energy-type norm in question is

$$\|J(t)u(t)\|_{L^2(\mathbb{R})}, \quad \text{where} \quad J(t) = x + 2it\partial_x.$$

The operator $J(t)$ is connected to the Galilean symmetry of the equation. For solutions to the linear equation (1.2), this norm is exactly conserved. For solutions as in Theorem 1.1, one shows that the norm is bounded by $t^{C\varepsilon^2}$ by using energy-type estimates, exploiting in a crucial way the gauge-invariance of the nonlinearity $|u|^2u$. The other norm one considers is a dispersive-type norm, namely,

$$\|u(t)\|_{L^\infty(\mathbb{R})},$$

which one aims to prove decays like $t^{-1/2}$. The ODE-type analysis comes into play when estimating this norm; in particular, to establish the decay one needs to use an integrating factor to remove a non-integrable term from the ODE. The use of the integrating factor leads to the phase correction in the asymptotic formula for the solution.

The rest of the paper is organized as follows. In Section 2 we will introduce notation and collect some preliminary lemmas. In Section 3, we discuss the proof of Theorem 1.1 due to Hayashi and Naumkin [13], which is an ODE-type argument for the profile $f(t) = e^{-it\Delta}u(t)$. In Section 4, we discuss the argument of Lindblad and Soffer [18]. Their argument is based on energy estimates for the related function $w(t)$ defined via $u(t,x) = (2it)^{-1/2}e^{ix^2/4t}w(t, \frac{x}{2t})$. In Section 5, we discuss a slight refinement of the analysis of [18] that relaxes the conditions imposed on the initial data. In Section 6, we discuss the proof of Ifrim and Tataru, which is based on a wave packet decomposition for the solution $u(t)$. In the special case of NLS, the variable describing the wave packet coefficients turns out to be the same as the variable used in Section 5. In Section 7 we discuss the proof of Kato and Pusateri, who analyzed the Fourier transform of the profile (i.e. $\hat{f}(t)$) in the framework of the space-time resonance method. Finally, in Section 8, we give a sketch of the argument of Deift and Zhou [4], who utilized the complete integrability of the equation and studied the problem via inverse scattering.

The arguments presented below all have their relative strengths and weaknesses. For example, the arguments of [13, 14] can be applied to establish modified scattering for the long-range NLS in higher dimensions $d \in \{2, 3\}$, whereas the arguments of [17, 18] utilize the polynomial structure of $|u|^2u$ and [4] requires complete integrability (which only holds in $1d$). Similarly, the arguments of [13, 14] work with essentially optimal regularity/decay assumptions for the initial data, while [4, 17] need $H^{1,1}$ data and [18] requires even stronger conditions (although the argument becomes simpler). Additionally, [4, 13, 18] rely strongly on the structure of the equation (using either a

factorization of the free propagator or the complete integrability), whereas the methods of [14, 17] have proven to be fairly flexible and applicable to a wide range of models (see [5, 8–10, 12, 15, 20] for a few such examples). Finally, only [4] seems to be able to handle the case of large initial data, whereas [13, 14, 17, 18] are all restricted to the small-data setting.

§ 2. Preliminaries

We introduce the time-dependent modulation and dilation operators $M(t)$ and $D(t)$, where

$$[M(t)f](x) = e^{ix^2/4t} f(x) \quad \text{and} \quad [D(t)f](x) = (2it)^{-1/2} f(\tfrac{x}{2t}).$$

We write either $U(t)$ or $e^{it\Delta}$ for the free Schrödinger propagator, defined as the Fourier multiplier operator

$$U(t) = \mathcal{F}^{-1} e^{-it\xi^2} \mathcal{F}.$$

Here $\mathcal{F}$ denotes the Fourier transform; we may also denote $\mathcal{F}f$ by $\hat{f}$.

The physical-space representation of this operator is given by convolution with a complex Gaussian; that is, we have the identity

$$e^{it\Delta} f(x) = (4\pi it)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{i|x-y|^2/4t} f(y)\, dy.$$

This explicit formula shows that the propagator admits the factorization

$$U(t) = M(t)D(t)\mathcal{F}M(t), \tag{2.1}$$

which will play a key role throughout this paper.

We next introduce the Galilean operator $J(t) = x + 2it\partial_x$. On one hand, direct computation shows that

$$J(t) = M(t)2it\partial_x M(-t).$$

On the other hand, a simple ODE argument shows that we may write

$$J(t) = U(t)xU(-t). \tag{2.2}$$

Both of these representations of $J(t)$ will be useful throughout this paper.

The focus on this paper is on the long-time behavior of solutions. Thus, our discussion of well-posedness and related issues will be rather brief: Solutions to (1.1) with L^2 initial data may be constructed by a contraction mapping argument utilizing Strichartz estimates; in particular, solutions are constructed to obey the Duhamel formula

$$u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta} (|u|^2 u)(s)\, ds. \tag{2.3}$$

The time of existence depends only on the L^2-norm, which is conserved under (1.1); thus, by iterating the local well-posedness result one can obtain a global-in-time solution, which belongs to $C_t(\mathbb{R}; L^2(\mathbb{R}))$. By persistence of regularity/decay arguments, one also sees that nicer data leads to nicer solutions. For example, data in the weighted Sobolev space $H^{1,1}$ leads to solutions belonging to $C_t(\mathbb{R}; H^{1,1}(\mathbb{R}))$. Note, however, that while the $\dot{H}^1$-norm of solutions remains bounded in time (by conservation of energy), the weighted L^2-norm does not.

For a textbook treatment of well-posedness and related issues, see [2].

We will use just a bit of Littlewood–Paley theory. We denote the Littlewood–Paley projection to frequency $N \in 2^{\mathbb{Z}}$ by P_N. We similarly have the projection operators $P_{\leq N}$ and $P_{>N}$. These operators are bounded on all L^p spaces and obey a set of standard 'Bernstein estimates', e.g.

$$\|P_N f\|_{L^p(\mathbb{R}^d)} \lesssim N^{\frac{d}{q}-\frac{d}{p}} \|f\|_{L^q(\mathbb{R}^d)} \quad \text{for} \quad q \leq p$$

and

$$\|P_N f\|_{L^p(\mathbb{R}^d)} \lesssim N^{-s} \||\nabla|^s f\|_{L^p(\mathbb{R}^d)}.$$

For more details, see (for example) [22].

§ 3. The argument of [13]

In this section we describe the argument of Hayashi and Naumkin [13] for the proof of Theorem 1.1. Recall that we assume $u_0 = \varepsilon\varphi$ for some Schwartz function φ. For convenience, let us additionally assume that $\|\varphi\|_{H^{1,1}} \leq 1$ and $\|\varphi\|_{L^\infty} \leq 1$.

The argument of Hayashi and Naumkin [13] relies on a bootstrap estimate involving the energy-type norm, which is allowed to grow slowly, and the L^∞-norm, which should decay at a rate of $t^{-1/2}$.

The energy-type norm in question is given by $J(t)u(t)$ in L^2. Because of a chain-rule identity for the operator $J(t)$ (cf. (3.1) below), control over the L^∞-norm can easily be transferred to control over the L^2-norm of Ju.

Lemma 3.1 (Energy estimate). *Suppose that $u : [0,T] \times \mathbb{R} \to \mathbb{C}$ is a solution to* (1.1) *obeying*

$$\|xu_0\|_{L^2} \leq \varepsilon \quad \textit{and} \quad \sup_{t\in[0,T]} (1+t)^{\frac{1}{2}} \|u(t)\|_{L^\infty} \leq A\varepsilon.$$

Then for all $t \in [0,T]$,

$$\|J(t)u(t)\|_{L^2} \leq \varepsilon \cdot (1+t)^{3A^2\varepsilon^2}.$$

Proof. We begin by applying $J(t)$ to the Duhamel formula (2.3) and utilizing the identities

$$J(t)U(t) = U(t)x, \quad J(t)U(t-s) = U(t-s)J(s).$$

This leads to

$$\|J(t)u(t)\|_{L^2} \leq \|xu_0\|_{L^2} + \int_0^t \|J(s)[|u|^2u](s)\|_{L^2}\,ds.$$

Now, a computation shows

$$J(s)[|u|^2u] = 2J(s)u - u^2\overline{J(s)u}. \tag{3.1}$$

Thus, continuing from above,

$$\begin{aligned}\|J(t)u(t)\|_{L^2} &\leq \varepsilon + 3\int_0^t \|u(s)\|_{L^\infty}^2 \|J(s)u(s)\|_{L^2}\,ds \\ &\leq \varepsilon + 3A^2\varepsilon^2 \int_0^t (1+s)^{-1}\|J(s)u(s)\|_{L^2}\,ds.\end{aligned}$$

Therefore by Gronwall's inequality, we have

$$\|J(t)u(t)\|_{L^2} \leq \varepsilon \cdot (1+t)^{3A^2\varepsilon^2},$$

as desired. □

Our next goal is to estimate the L^∞-norm of $u(t)$. We will do so through an ODE argument involving the Fourier transform of the 'profile' of u. We let

$$f(t) := U(-t)u(t) \quad \text{and} \quad \tilde{f}(t) := \mathcal{F}M(t)U(-t)u(t).$$

Then, using (2.1) and writing

$$u(t) = U(t)U(-t)u(t) = M(t)D(t)\tilde{f}(t),$$

we compute

$$i\partial_t f(t) = U(-t)\{|u|^2u\} = (2t)^{-1}[\mathcal{F}M(t)]^{-1}\{|\tilde{f}|^2\tilde{f}\}.$$

Defining

$$\mathcal{I}(t) = \mathcal{F}M(t)\mathcal{F}^{-1}, \quad \overline{\mathcal{I}(t)} = \mathcal{F}\overline{M(t)}\mathcal{F}^{-1},$$

we find

$$\begin{aligned} i\partial_t\hat{f}(t) &= (2t)^{-1}\overline{\mathcal{I}(t)}\{|\tilde{f}|^2\tilde{f}\} \\ &= (2t)^{-1}|\hat{f}|^2\hat{f} + (2t)^{-1}\big[|\tilde{f}|^2\tilde{f} - |\hat{f}|^2\hat{f}\big] + (2t)^{-1}\big[\overline{\mathcal{I}(t)} - 1\big]|\tilde{f}|^2\tilde{f}.\end{aligned} \tag{3.2}$$

We now employ an integrating factor to remove the first term on the right-hand side of this equation. With

$$g(t) = e^{i\Theta(t)}\hat{f}(t), \quad \Theta(t) := \int_1^t (2s)^{-1}|\hat{f}(s)|^2\,ds, \tag{3.3}$$

we get

$$i\partial_t g(t) = (2t)^{-1}e^{i\Theta(t)}\Big\{\big[|\tilde{f}|^2\tilde{f} - |\hat{f}|^2\hat{f}\big] + \big[\overline{\mathcal{I}(t)} - 1\big]|\tilde{f}|^2\tilde{f}\Big\}$$

Lemma 3.2 (Controlling error terms). *For $0 < a < \frac{1}{4}$ and $t \geq 1$,*

$$\big\| \big[|\tilde{f}|^2\tilde{f} - |\hat{f}|^2\hat{f}\big] + \big[\overline{\mathcal{I}(t)} - 1\big]|\tilde{f}|^2\tilde{f}\big\|_{L^\infty} \lesssim_a t^{-a}\big\{\|u(t)\|_{L^2} + \|J(t)u(t)\|_{L^2}\big\}^3.$$

Proof. We begin by observing that by Hausdorff–Young and Cauchy–Schwarz,

$$\begin{aligned}\|\hat{f}(t)\|_{L^\infty} + \|\tilde{f}(t)\|_{L^\infty} &\lesssim \|U(-t)u(t)\|_{L^1}\\ &\lesssim \|u(t)\|_{L^2} + \|xU(-t)u(t)\|_{L^2}\\ &\lesssim \|u(t)\|_{L^2} + \|J(t)u(t)\|_{L^2}.\end{aligned}$$

Next, for $0 < a < \frac{1}{4}$, again by Hausdorff–Young and Cauchy–Schwarz,

$$\begin{aligned}\|\tilde{f}(t) - \hat{f}(t)\|_{L^\infty} &= \|[\mathcal{I}(t) - 1]\hat{f}(t)\|_{L^\infty}\\ &\lesssim \|[M(t) - 1]f\|_{L^1}\\ &\lesssim |t|^{-a}\||x|^{2a}U(-t)u(t)\|_{L^1}\\ &\lesssim |t|^{-a}\big\{\|u(t)\|_{L^2} + \|J(t)u(t)\|_{L^2}\big\}.\end{aligned}$$

Estimating similarly,

$$\begin{aligned}\|[\overline{\mathcal{I}(t)} - 1]|\tilde{f}|^2\tilde{f}\|_{L^\infty} &\lesssim |t|^{-a}\||x|^{2a}\mathcal{F}^{-1}|\tilde{f}|^2\tilde{f}\|_{L^1}\\ &\lesssim t^{-a}\big\{\||\tilde{f}|^2\tilde{f}\|_{L^2} + \|\nabla(|\tilde{f}|^2\tilde{f})\|_{L^2}\big\}\\ &\lesssim t^{-a}\big\{\|\tilde{f}\|_{L^\infty}^2\big[\|u\|_{L^2} + \|\nabla\tilde{f}\|_{L^2}\big]\big\}\\ &\lesssim t^{-a}\big\{\|u(t)\|_{L^2} + \|J(t)u(t)\|_{L^2}\big\}^3.\end{aligned}$$

This completes the proof. □

Corollary 3.3 (Dispersive bound). *Suppose that u is a solution to* (1.1) *obeying*

$$\sup_{t\in[0,T]}(1+t)^{-\delta}\|J(t)u(t)\|_{L^2} \leq B\varepsilon \quad \textit{and} \quad \sup_{t\in[0,1]}\|u(t)\|_{L^\infty} \leq C\varepsilon$$

for some $0 < \delta < \frac{1}{12}$. Then there exists $A > 0$ such that for all $t \in [1, T]$,

$$\|u(t)\|_{L^\infty} \leq A\varepsilon(1+t)^{-\frac{1}{2}}.$$

Proof. We let $t \geq 1$ and write

$$u(t) = U(t)U(-t) = M(t)D(t)\hat{f}(t) + M(t)D(t)[\mathcal{I}(t) - 1]\hat{f}(t). \tag{3.4}$$

Then, estimating as in Lemma 3.2 for the second term, we get have

$$\begin{aligned}\|u(t)\|_{L^\infty} &\lesssim t^{-\frac{1}{2}}\|\hat{f}(t)\|_{L^\infty} + t^{-\frac{1}{2}-a}\{\|u(t)\|_{L^2} + \|J(t)u(t)\|_{L^2}\}\\ &\lesssim t^{-\frac{1}{2}}\|g(t)\|_{L^\infty} + \varepsilon t^{-\frac{1}{2}-a+\delta}\end{aligned}$$

for any $0 < a < \frac{1}{4}$, where g is as in (3.3). Thus it remains to estimate g in L^∞.

By Lemma 3.2 and hypothesis, we have

$$\|\partial_t g\|_{L^\infty} \lesssim \varepsilon^3 t^{-1-a+3\delta},$$

so that choosing $3\delta < a < \frac{1}{4}$, we get

$$\begin{aligned}\|g(t)\|_{L^\infty} &\leq \|\hat{f}(1)\|_{L^\infty} + C\varepsilon^3 \int_1^t s^{-1-a+3\delta}\, ds \\ &\leq B\varepsilon + C\varepsilon^3 \cdot \tfrac{1}{a-3\delta}[1 - t^{-(a-3\delta)}] \lesssim \varepsilon\end{aligned}$$

uniformly in $t \in [1, T]$, as desired. □

Using Lemma 3.1 and Corollary 3.3, a continuity argument leads to the following.

Proposition 3.4 (Bounds). *Let u be the global-in-time solution to* (1.1) *with $u_0 = \varepsilon\varphi$ for some $\varphi \in \mathcal{S}$ with $\|\varphi\|_{H^{1,1}} \leq 1$. If ε is sufficiently small, then $u(t)$ obeys*

$$\sup_{t\in[0,\infty)} \|u(t)\|_{L^\infty} \leq C\varepsilon(1+t)^{-\frac{1}{2}} \quad \textit{and} \quad \sup_{t\in[0,\infty)} \|J(t)u(t)\|_{L^\infty} \leq C\varepsilon(1+t)^{\frac{1}{20}}$$

for some $C > 0$.

In particular, we deduce the desired L^∞ decay for the solution. To complete the proof of Theorem 1.1, it remains to establish the desired asymptotic behavior. For this, we return to the ODE argument given above. The difference is that now we have the desired bounds for $u(t)$ in place. In particular, we find that $\|\partial_t g\|_{L^\infty}$ is integrable, which implies the existence of a limit for $g(t)$, say g_∞ (with an explicit rate of convergence).

Next, using the convergence of $g(t)$, we can deduce that

$$\Theta(t) = \Theta_+ + \tfrac{1}{2}|g_\infty|^2 \log t + o(1) \quad \text{as} \quad t \to \infty$$

for some $\Theta_+ \in L^\infty$, where $\Theta(t)$ is as in (3.3). As $\hat{f} = e^{-i\Theta} g$, we deduce

$$\hat{f}(t) = e^{-\frac{i}{2}|\psi|^2 \log t}\psi + o(1) \quad \text{as} \quad t \to \infty,$$

where $\psi := e^{-i\Theta_+} g_\infty$. Finally, since

$$u(t) = M(t)D(t)\hat{f}(t) + o(t^{-\frac{1}{2}})$$

(cf. (3.4)), this asymptotic behavior for $\hat{f}$ implies the desired asymptotic behavior for the solution $u(t)$. This completes the sketch of the proof of Theorem 1.1 as given in [13].

§ 4. The Argument of [18]

We next discuss the argument due to Lindblad and Soffer [18]. This time, we introduce the variable $w(t)$ given by

$$u(t) = M(t)D(t)w(t).$$

Then the L^∞ decay of u is equivalent to L^∞ boundedness of $w(t)$.

The variable $w(t)$ will play a role similar to that of $\hat{f}(t)$ in the previous section. In fact, this variable also appeared in the previous section, where we encountered $w(t) = \tilde{f}(t) = \mathcal{F}M(t)U(-t)u(t)$.

A direct computation shows that u solving (1.1) is equivalent to w solving the PDE

$$i\partial_t w = -\tfrac{1}{2t^2}\partial_x^2 w + \tfrac{1}{2t}|w|^2 w. \tag{4.1}$$

Using the integrating factor $e^{i\Xi(t)}$, where now

$$\Xi(t) = \int_1^t \tfrac{1}{2s}|w(s)|^2\, ds,$$

we can deduce that

$$\|w(t)\|_{L^\infty} \le \|w(1)\|_{L^\infty} + \int_1^t \tfrac{1}{2s^2}\|\partial_x^2 w(s)\|_{L^\infty}\, ds. \tag{4.2}$$

To control the norm appearing in the integral, the strategy of Lindblad and Soffer is to employ energy estimates. In particular, by Sobolev embedding it suffices to control up to three derivatives of w in L^2.

Differentiating (4.1) leads to

$$(i\partial_t + \tfrac{1}{2t^2}\partial_x^2)\partial_x^k w = \tfrac{1}{2t}\partial_x^k(|w|^2 w), \quad k \in \{0,1,2,3\}. \tag{4.3}$$

In particular, integrating by parts, we get

$$\partial_t\langle \partial_x^k w, \partial_x^k w\rangle = -\tfrac{i}{2t}\langle \partial_x^k w, \partial_x^k(|w|^2 w)\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product. Thus

$$\|\partial_x^k w(t)\|_{L^2} \le \|\partial_x^k w(1)\|_{L^2} + \int_1^t \tfrac{1}{2s}\|\partial_x^k(|w|^2 w)\|_{L^2}\, ds. \tag{4.4}$$

We next need the following chain-rule estimate:

Lemma 4.1. *For $k \in \{0,1,2,3\}$,*

$$\|\partial_x^k(|w|^2 w)\|_{L^2} \lesssim \|w\|_{L^\infty}^2 \|\partial_x^k w\|_{L^2}.$$

Proof. The cases $k \in \{0,1\}$ follow immediately from Hölder's inequality and the chain rule. For the case $k = 2$, we distribute the derivatives and observe that the desired inequality follows from

$$\|\partial_x w\|_{L^2} \lesssim \|w\|_{L^\infty}^{\frac{1}{2}} \|\partial_x^2 w\|_{L^2}^{\frac{1}{2}}, \tag{4.5}$$

which in turn follows from an integration by parts. Similarly, for the case $k = 3$, we distribute the derivatives and need to establish the estimates

$$\|(\partial_x w)^3\|_{L^2} \lesssim \|w\|_{L^\infty}^2 \|\partial_x^3 w\|_{L^2} \quad \text{and} \quad \|w\partial_x w(\partial_x^2 w)\|_{L^2} \lesssim \|w\|_{L^\infty}^2 \|\partial_x^3 w\|_{L^2}.$$

These estimates follow from interpolation, specifically the estimates

$$\|\partial_x w\|_{L^6} \lesssim \|w\|_{L^\infty}^{\frac{2}{3}} \|\partial_x^3 w\|_{L^2}^{\frac{1}{3}} \quad \text{and} \quad \|\partial_x^2 w\|_{L^3} \lesssim \|w\|_{L^\infty}^{\frac{1}{3}} \|\partial_x^3 w\|_{L^2}^{\frac{2}{3}}. \tag{4.6}$$

This completes the proof. □

Returning to (4.4), we deduce

$$\|\partial_x^k w(t)\|_{L^2} \leq \|\partial_x^k w(1)\|_{L^2} + C \int_1^t \tfrac{1}{2s} \|w(s)\|_{L^\infty}^2 \|\partial_x^k w(s)\|_{L^2}\, ds. \tag{4.7}$$

We now claim that we can use (4.7) and (4.2) to run a continuity argument, which (choosing data of size $0 < \varepsilon \ll 1$ in all relevant norms) yields the bounds

$$\|w(t)\|_{L^\infty} \leq C\varepsilon \quad \text{and} \quad \|\partial_x^2 w(t)\|_{L^\infty} \leq C t^{\frac{1}{2}} \tag{4.8}$$

uniformly in $t \geq 1$.

Indeed, if we assume that $\|w(t)\|_{L^\infty} \leq 10\varepsilon$ on some interval $[1, T]$, then (4.7) and Gronwall's inequality imply

$$\|\partial_x^3 w(t)\|_{L^2} \leq \varepsilon t^{100C\varepsilon^2} \quad \text{for} \quad t \in [1, T],$$

and hence (using the $1d$ Sobolev embedding $\|f\|_{L^\infty} \leq \|f\|_{L^2}^{1/2} \|\partial_x f\|_{L^2}^{1/2}$)

$$\|\partial_x^2 w(t)\|_{L^\infty} \leq \varepsilon t^{100C\varepsilon^2} \quad \text{for} \quad t \in [1, T].$$

Plugging this back into (4.2) and choosing $100C\varepsilon^2 < \frac{1}{2}$, we find

$$\|w(t)\|_{L^\infty} \leq 2\varepsilon \quad \text{for} \quad t \in [1, T].$$

In particular, this type of self-improving estimate shows that we can run a continuity argument and obtain the bounds in (4.8). As observed above, the L^∞-boundedness of $w(t)$ immediately implies the sharp L^∞ decay for $u(t)$.

Next, we discuss the asymptotic behavior of the solution. As before, with the bounds in place, the estimates above demonstrate that the L^∞ norm of the time derivative of $e^{i\Xi(t)}w(t)$ is integrable, and hence we obtain a limit. Proceeding just as in the previous section, we can obtain a profile ψ such that

$$w(t) = e^{-i\frac{i}{2}|\psi|^2 \log t}\psi + o(1) \quad \text{as} \quad t \to \infty,$$

which once again leads to the modified asymptotics for $u(t)$. Indeed, this time we have the exact identity $u(t) = M(t)D(t)w(t)$.

§ 5. A slight refinement of [18]

The argument of Lindblad and Soffer is a bit simpler than that of Hayashi and Naumkim; however, it requires stronger conditions on the initial data. In particular, the argument of Lindblad and Soffer requires three derivatives of w to belong to L^∞, which in particular would require $\langle x\rangle^3 u_0$ to belong to L^∞. On the other hand, the argument of Hayashi and Naumkin (as presented) required only $H^{1,1}$ data. In fact, without too many changes, their argument can handle $H^{s,s}$ data for any $s > \frac{1}{2}$. In the argument, one would need to replace the J operator with its fractional power, defined by

$$J^s(t) = U(t)|x|^s U(-t) = M(t)(-4t^2\Delta)^{\frac{s}{2}} M(-t).$$

In place of the pointwise chain-rule identity (3.1), one would need to utilize the second identity for $J^s(t)$ above and use the fractional chain rule of [3]. Otherwise, the rest of the argument goes through.

In this section I will present an argument that utilizes the same variable $w(t)$ as in Lindblad–Soffer, but readily handles data from $H^{1,1}$. As usual, we seek to run a continuity argument that simultaneously establishes the bounds

$$\|J(t)u(t)\|_{L^2} \lesssim \varepsilon t^\delta \quad \text{and} \quad \|u(t)\|_{L^\infty} \lesssim \varepsilon t^{-\frac{1}{2}}$$

for $t \geq 1$ and some $0 < \delta \ll 1$. Given these bounds, we also need to demonstrate the desired asymptotic behavior. As we can readily utilize the energy-type estimate of Lemma 3.1 to control the Ju in L^2, I will focus on estimating the L^∞-norm. As in the previous argument, it suffices to bound the L^∞-norm of $w(t)$, where

$$w(t) = \mathcal{F}M(t)U(-t)u(t), \quad \text{so that} \quad u(t) = M(t)D(t)w(t).$$

Note that by (2.2), we have

$$\|Ju\|_{L^2} = \|\nabla w\|_{L^2} \quad \text{and} \quad \|u\|_{L^2} = \|w\|_{L^2}.$$

In particular, the estimate of Lemma 3.1 shows that

$$\|\nabla w(t)\|_{L^2} \leq C\varepsilon(1+t)^{C\varepsilon^2\|w\|_{L^\infty}} \quad \text{uniformly in} \quad t \geq 1, \tag{5.1}$$

where the space-time L^∞-norm of w is over $[1,t] \times \mathbb{R}$.

We begin with the following frequency localized estimate, which is a consequence of Bernstein estimates:

$$\|P_N w(t)\|_{L^\infty_x} \lesssim N^{-\frac{1}{2}} \|\nabla w(t)\|_{L^2_x}. \tag{5.2}$$

In particular, using (5.2) and summing, we get

$$\|P_{>\sqrt{t}} w(t)\|_{L^\infty} \lesssim t^{-\frac{1}{4}} \|\nabla w(t)\|_{L^2},$$

which (under the bootstrap assumption that the weighted norm of u grows like t^δ) is already an acceptable bound.

We therefore turn to the low frequency piece and consider the equation satisfied by

$$w_{lo}(t) := P_{\leq\sqrt{t}} w(t).$$

Note that frequency projections commute with spatial derivatives; however, since the projection is time-dependent, the time derivative may land on the projection. In particular, recalling (4.1), we find that w_{lo} solves

$$\begin{aligned} i\partial_t w_{lo} - \tfrac{1}{2t}|w_{lo}|^2 w_{lo} &= -\tfrac{1}{2t^2}\Delta w_{lo} + \tfrac{1}{2t^{3/2}}\tilde{P}_{\sqrt{t}}\nabla w + \tfrac{1}{2t}\big[P_{\leq\sqrt{t}}(|w|^2 w) - |w_{lo}|^2 w_{lo}\big] \\ &= -\tfrac{1}{2t^2}\Delta w_{lo} + \tfrac{1}{2t^{3/2}}\tilde{P}_{\sqrt{t}}\nabla w \\ &\quad + \tfrac{1}{2t}\big[-P_{>\sqrt{t}}(|w|^2 w) + |w|^2 w - |w_{lo}|^2 w_{lo}\big], \end{aligned}$$

where the Fourier multiplier of $\tilde{P}_1$ is the derivative of the multiplier for P_1, and is in particular supported near $|\xi| \sim 1$.

We now employ an integrating factor and set $v(t) = e^{i\Phi(t)} w_{lo}(t)$, with

$$\Phi(t) = \int_1^t |w_{lo}(s)|^2 \tfrac{ds}{2s},$$

which solves

$$i\partial_t v(t) = e^{i\Phi(t)}\big\{-\tfrac{1}{2t^2}\Delta w_{lo} + \tfrac{1}{2t^{3/2}}\tilde{P}_{\sqrt{t}}\nabla w + \tfrac{1}{2t}\big[-P_{>\sqrt{t}}(|w|^2 w) + |w|^2 w - |w_{lo}|^2 w_{lo}\big]\big\}.$$

We estimate the Laplacian term via Bernstein estimates. Note that as the low frequency projection is present, we can estimate this term without assuming additional regularity on w (in contrast to the approach of [18]). In particular, we estimate

$$t^{-2}\|\Delta w_{lo}\|_{L^\infty} + t^{-\frac{3}{2}}\|\tilde{P}_{\sqrt{t}}\nabla w\|_{L^\infty} \lesssim t^{-\frac{5}{4}}\|\nabla w(t)\|_{L^2}.$$

Next, arguing via Bernstein as above and using the chain rule, we have

$$\|P_{>\sqrt{t}}(|w|^2 w)\|_{L^\infty} \lesssim t^{-\frac{1}{4}} \|\nabla(|w|^2 w)\|_{L^2_x} \lesssim t^{-\frac{1}{4}} \|w\|^2_{L^\infty} \|\nabla w\|_{L^2}.$$

In the remaining term, there is always a copy of $P_{>\sqrt{t}} w$ (or its complex conjugate); thus this term is controlled by

$$\|w\|^2_{L^\infty_x} \|P_{>\sqrt{t}} w\|_{L^\infty_x} \lesssim t^{-\frac{1}{4}} \|w\|^2_{L^\infty} \|\nabla w\|_{L^2}.$$

Collecting the estimates above, we deduce

$$\begin{aligned} \|w(t)\|_{L^\infty_x} &\lesssim \|w(1)\|_{L^\infty_x} + \|P_{>\sqrt{t}} w(t)\|_{L^\infty_x} \\ &\quad + \int_1^t s^{-\frac{5}{4}} \|\nabla w(s)\|_{L^2} + s^{-\frac{5}{4}} \|w(s)\|^2_{L^\infty} \|\nabla w(s)\|_{L^2}\, ds. \\ &\lesssim \varepsilon + t^{-\frac{1}{4}} \|\nabla w(t)\|_{L^2} + \int_1^t s^{-\frac{5}{4}} \|\nabla w(s)\|_{L^2} [1 + \|w(s)\|^2_{L^\infty}]\, ds \end{aligned} \tag{5.3}$$

Combining the estimates in (5.1) and (5.3), we find that by choosing $\varepsilon > 0$ sufficiently small we can close a continuity argument and establish the desired bounds, as before.

Once the estimates are in place, we can argue further to obtain the asymptotic behavior. As before, the estimates suffice to extract a limit for $e^{i\Phi} w_{lo}$ and then to describe the behavior of $e^{i\Phi}$ itself. Due to the explicit decay of $P_{>\sqrt{t}} w$, this is enough to describe the asymptotic behavior of $w(t)$. In particular, we can once again deduce the existence of a profile ψ so that

$$w(t) = e^{-\frac{i}{2}|\psi|^2 \log t} \psi + o(1)$$

as $t \to \infty$. Recalling $u(t) = M(t)D(t)w(t)$, we recover the result.

§ 6. The Argument of [14]

In this section, we discuss the argument of Ifrim and Tataru [14], which relies on the idea of a wave packet decomposition for the solution to (1.1). In this setting, wave packets refer to approximate solutions to the underlying linear equation associated with a given velocity and scale. In the special case of the linear Schrödinger equation, the factorization $U(t) = M(t)D(t)\mathcal{F}M(t)$ and corresponding approximation $U(t) \approx M(t)D(t)\mathcal{F}$ makes it straightforward to identify a good wave packet approximation. In particular, we may choose initial data of the form $e^{ivx}\phi(\frac{x}{\lambda})$, representing a bump centered at the origin with a physical scale λ, concentrated in frequency around v. Then the solution to the linear Schrödinger equation may be approximated as

$$e^{it\Delta}[e^{ivx}\phi(\tfrac{x}{\lambda})] \approx \lambda t^{-\frac{1}{2}} e^{ix^2/4t} \hat{\phi}\big[\tfrac{\lambda}{t}(x - 2tv)\big]. \tag{6.1}$$

In particular, if we are interested in a wave packet decomposition up to time t, we should use wave packets that remain coherent up until this time, which suggests that we use the scale $\lambda \sim t^{\frac{1}{2}}$. With this in mind, we define the family of wave packets to be

$$\Psi_v(t,x) = e^{ix^2/4t}\chi(\tfrac{x-2vt}{\sqrt{t}}), \quad v \in \mathbb{R},$$

where χ is some Schwartz function that is normalized to obey $\int \chi = 1$. Indeed, one can check that this choice of Ψ_v approximately solves the linear Schrödinger equation, with $\mathcal{O}(1/t)$ errors obeying the same localization properties as Ψ_v itself.

The wave packet decomposition of a solution u to (1.1) is defined via the coefficients

$$\gamma(t,v) := \langle u(t,\cdot), \Psi_v(t,\cdot)\rangle_{L^2}.$$

In fact, $\gamma(t,v)$ is an object we have already studied, albeit in a different form.

Lemma 6.1. *Recalling the variable $w(t) = \mathcal{F}M(t)U(-t)u(t)$ from the previous sections, we may write*

$$\gamma(t,v) = w_{lo}(t,v) = [P_{\leq\sqrt{t}}w](t,v).$$

Proof. We perform the change of variables $y = \frac{x}{2t}$ to compute

$$\begin{aligned}\gamma(t,v) &= \langle M(t)D(t)w(t), \Psi_v(t,\cdot)\rangle = c\big\langle w(t,\cdot), \sqrt{t}\chi(\sqrt{t}(\cdot - v)\big\rangle \\ &= \{w(t,\cdot) * [\sqrt{t}\chi(\sqrt{t}\cdot)]\}(v) = w_{lo}(t,v),\end{aligned}$$

as desired. In particular, here we have chosen χ to be the inverse Fourier transform of the standard Littlewood–Paley multiplier. □

As in the previous section we now see that the decay and asymptotic behavior of u may be understood through the behavior of γ via $u(t) \approx M(t)D(t)\gamma(t)$.

In particular, the authors of [14] show that $\|\partial_v\gamma\|_{L^2}$ is controlled by the weighted norm, which is controlled through an energy estimate and Gronwall's inequality under the assumption of sharp L^∞ decay for u. Additionally, they show that the difference between $u(t)$ and $M(t)D(t)\gamma(t)$ is quantitatively small, with the error measured in terms of the weighted norm. Then, as in many of the arguments above, the key boils down to controlling the L^∞ norm of γ and subsequently deducing the long-time behavior, which is achieved through analysis of the ODE

$$i\partial_t\gamma = (2t)^{-1}|\gamma|^2\gamma + \mathcal{O}(t^{-1-}).$$

As we have carried out the details in the previous section (albeit from a slightly different perspective, e.g. quoting 'Bernstein estimates' in place of directly estimating convolutions), we will omit the details here.

§ 7. The argument of [17]

The final 'PDE proof' of Theorem 1.1 that we will present is due to Kato and Pusateri [17]. It is based off of the analysis of $\hat{f}(t)$, where as before f is the profile $f(t) = U(-t)u(t)$. In these variables, we are interested in proving the estimates

$$\|\hat{f}(t)\|_{L^\infty} \lesssim \varepsilon \quad \text{and} \quad \|\partial_\xi \hat{f}(t)\|_{L^2} = \|J(t)u(t)\|_{L^2} \lesssim t^\delta$$

for some small $\delta > 0$, uniformly over $t \geq 1$ (cf. (2.2)). With these estimates in place and the decomposition (3.4), we can establish sharp L^∞ decay for u, as desired.

As usual, we can deal with the weighted norm by an energy estimate, and so we will focus on how to control the L^∞ norm and derive the asymptotic behavior. For this, we write the Duhamel formula (2.3) (starting at $t = 1$, say) and apply $\mathcal{F}U(-t)$ to both sides. This leads to

$$\hat{f}(t) = \hat{f}(1) - i \int_1^t e^{is\xi^2} \mathcal{F}\big(|u|^2 u\big)(s)\, ds.$$

Now, observing $\hat{u}(s,\xi) = e^{-is\xi^2} \hat{f}(s,\xi)$, we write

$$\mathcal{F}\big(|u|^2 u\big)(s,\xi) = \tfrac{1}{2\pi} \iint e^{-is(\xi-\eta)^2 + is(\eta-\sigma)^2 - is\sigma^2} \hat{f}(\xi-\eta)\hat{\bar{f}}(\eta-\sigma)\hat{f}(\sigma)\, d\sigma\, d\eta.$$

Inserting this into the formula above and simplifying the phase, we deduce

$$\begin{aligned} \hat{f}(t,\xi) &= \hat{f}(1,\xi) - \tfrac{i}{2\pi} \int_1^t \iint e^{2is\eta[\xi-\sigma]} \hat{f}(\xi-\eta)\hat{\bar{f}}(\eta-\sigma)\hat{f}(\sigma)\, d\sigma\, d\eta\, ds \\ &= \hat{f}(1,\xi) - \tfrac{i}{2\pi} \int_1^t \iint e^{2is\eta\sigma} \hat{f}(\xi-\eta)\hat{\bar{f}}(\eta-\xi+\sigma)\hat{f}(\xi-\sigma)\, d\sigma\, d\eta\, ds. \end{aligned}$$

At this point, we have arrived at an integral that is amenable to a stationary phase type analysis. In particular, we expect a main contribution from any stationary phase points, while oscillation away from these points should yield decay through integration by parts. This strategy for the analysis of dispersive PDE has been developed in recent years and is known as the method of space-time resonances (as introduced in [11]). It has proven to be a robust technique for establishing decay estimates for small solutions for many different models, including even some quasilinear systems (see e.g. [5]).

In the present setting, the phase is simple enough that one can utilize Plancherel's theorem and compute a bit more explicitly. To this end, we introduce

$$F(s,\xi,\eta,\sigma) = \hat{f}(\xi-\eta)\hat{\bar{f}}(\eta-\xi+\sigma)\hat{f}(\xi-\sigma) \tag{7.1}$$

and denote by $\mathcal{F}_{\eta,\sigma}$ the two-dimensional Fourier transform in the variables η, σ. Then we may rewrite the integral as

$$\int_1^t \iint \mathcal{F}_{\eta,\sigma}[e^{2is\eta\sigma}]\mathcal{F}^{-1}_{\eta,\sigma}[F](s,\xi,\eta,\sigma)\, d\eta\, d\sigma\, ds.$$

Now, on the one hand, we have by a direct computation

$$\mathcal{F}_{\eta,\sigma}[e^{2is\eta\sigma}] = \tfrac{1}{2s} e^{-i\frac{\eta\sigma}{2s}}.$$

On the other hand,

$$\tfrac{1}{2\pi} \iint \mathcal{F}^{-1}_{\eta,\sigma}[F](s,\xi,\eta,\sigma)\, d\eta\, d\sigma = F(s,\xi,0,0) = |\hat{f}(s,\xi)|^2 \hat{f}(s,\xi).$$

Thus, we arrive at

$$\begin{aligned}\hat{f}(t,\xi) = \hat{f}(1,\xi) - i\int_1^t \tfrac{1}{2s}|\hat{f}(s,\xi)|^2 \hat{f}(s,\xi)\, ds \\ + \int_1^t \tfrac{1}{2s} \iint [e^{-i\frac{\eta\sigma}{2s}} - 1]\mathcal{F}^{-1}_{\eta,\sigma}[F](s,\xi,\eta,\sigma)\, d\eta\, d\sigma\, ds.\end{aligned}$$

Upon taking the time derivative, we are led to

$$i\partial_t \hat{f} = \tfrac{1}{2t}|\hat{f}|^2 \hat{f} + \tfrac{1}{2t} R,$$

with

$$R(t,\xi) = \iint [e^{-i\frac{\eta\sigma}{2t}} - 1]\mathcal{F}^{-1}_{\eta,\sigma}[F](t,\xi,\eta,\sigma)\, d\eta\, d\sigma. \tag{7.2}$$

Evidently, this is the same ODE as the one appearing in (3.2). Now the path to complete the argument is clear. In particular, we employ the same integrating factor as we did for (3.2) to remove the first term, and then endeavor to exhibit time decay in $R(t,\xi)$. With the desired estimates in place, we can derive the asymptotic behavior as before. Therefore we will focus only on the estimation of the remainder term $R(t,\xi)$.

Lemma 7.1. *We have the following estimate on* (7.2)*: Fix* $0 < a < \frac{1}{4}$*. Then*

$$\|R(t)\|_{L^\infty_\xi} \lesssim_a t^{-a}\{\|u(t)\|_{L^2} + \|J(t)u(t)\|_{L^2}\}^3,$$

uniformly in $t \geq 1$*.*

Proof. We begin by estimating

$$|R(t,\xi)| \lesssim t^{-a} \iint |\sigma|^a |\eta|^a |\mathcal{F}^{-1}_{\eta,\sigma}[F](t,\xi,\eta,\sigma)|\, d\eta\, d\sigma$$

for any $0 < a < \frac{1}{2}$. Now, writing

$$\mathcal{F}^{-1}_{\sigma,\eta}[F] = \mathcal{F}^{-1}_{\sigma}\big\{\mathcal{F}^{-1}_{\eta}\big[\hat{f}(\xi-\eta)\overline{\hat{f}(\xi-\eta-\sigma)}\big]\hat{f}(\xi-\sigma)\big\}$$

and computing explicitly, we can deduce the pointwise estimate

$$|\mathcal{F}^{-1}_{\sigma,\eta}[F](\xi,\eta,\sigma)| \lesssim \int |f(x-\eta)|\,|f(x)|\,|f(x-\sigma)|\,dx,$$

and hence (choosing $0 < a < \frac{1}{4}$ and applying Cauchy–Schwarz),

$$\begin{aligned}
&|R(t,\xi)| \\
&\lesssim t^{-a} \iiint (|x-\eta|^a + |x|^a)(|x-\sigma|^a + |x|^a)|f(x-\eta)|\,|f(x)|\,|f(x-\sigma)|\,dx\,d\eta\,d\sigma \\
&\lesssim t^{-a}\|(1+|x|)^{2a} f\|_{L^1}^2 \|f\|_{L^1} \\
&\lesssim t^{-a}\|(1+|x|)f\|_{L^2}^3 \lesssim t^{-a}\{\|u(t)\|_{L^2} + \|J(t)u(t)\|_{L^2}\}^3,
\end{aligned}$$

which is the desired estimate. □

Under an appropriate bootstrap estimate on the growth of $\|Ju\|_{L^2}$, the estimate on the remainder allows us to close the argument, controlling the L^∞ norm of $\hat{f}$ and eventually deducing the asymptotic behavior of solutions, as above. This completes the proof sketch of the argument of [17].

§ 8. Inverse scattering

In this section, we discuss the approach of Deift and Zhou [4], which capitalizes on the complete integrability of the $1d$ cubic NLS through the use of inverse scattering. While the notion of complete integrability does not quite have a standard definition, there are several properties that are commonly associated with completely integrable models, such as the existence of a Lax pair formulation, the existence of infinitely many conserved quantities, and the possibility of explicitly solving the equation through the use of direct/inverse scattering transformations. The cubic NLS has all of these features and has been widely studied as one of the canonical examples of a completely integrable PDE.

We change convention slightly and consider the equation

$$i\partial_t u = -\partial_x^2 u + 2|u|^2 u. \tag{8.1}$$

The formulation of NLS as a completely integrable model was worked out in [1, 24], and derivations of the correct long-time behavior (including the logarithmic phase correction) were originally given in several works [16, 19, 25]. In [4], Deift and Zhou proved that the long-time asymptotic formula holds with precise error estimates by using their technique of nonlinear steepest descent.

We will present a very rough sketch of this approach, first discussing the complete integrability of NLS and then deriving an asymptotic formula for solutions. We will

try to emphasize the main ideas as we have understood them, but we will make few attempts at proofs or complete technical details. Similarly, we will be most interested in deriving an asymptotic formula comparable to (1.3) (i.e. including the logarithmic phase correction), but will not focus on some of the additional phase factors appearing in the precise asymptotic expansion of the solution. For those interested in learning more about these techniques, we hope the present discussion can serve as a rough outline to be filled in by studying other references in this field. In particular, we refer the reader to [6] for a more detailed review article, to [4] for a rigorous proof of the main result, and to [7] for another proof using $\bar{\delta}$-methods. See also the lecture notes [21] and the textbook [23] for pedagogical treatments of this subject. These last two references motivated much of the presentation here, especially [21] for the formulation of NLS as a completely integrable equation and [23] for the section on nonlinear steepest descent.

§ 8.1. Complete integrability

The starting point for our discussion is to recall the classical problem of recovering a potential from the scattering data for the corresponding Schrödinger operator. That is, given knowledge of the spectrum and (generalized) eigenfunctions of $-\partial_x^2 + u(x)$, how can one recover the potential u? In the context of (8.1), the associated scattering problem is given in terms of a first order matrix-valued operator $\mathcal{L} = \mathcal{L}(u)$, namely,

$$\mathcal{L} = i\sigma_3 \tfrac{d}{dx} + U, \quad \text{where} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & -iu \\ i\bar{u} & 0 \end{bmatrix}.$$

The potential $u = u(t,x)$ will be given by the solution to (8.1) at each fixed time, and we consider the generalized eigenvalue problem $\mathcal{L}\Psi = \lambda\Psi$. It turns out that for the defocusing equation, it is enough to consider continuous spectrum. In the focusing case, one must also contend with the presence of discrete spectrum, corresponding to the presence of soliton solutions.

As u will evolve in time according to (8.1), we must also incorporate some time evolution into the eigenvalue problem (in a consistent way). This can be achieved by imposing

$$\partial_t \Psi = [-2i\lambda^2\sigma_3 + 2\lambda U_1 + U_2]\Psi, \quad \text{where} \quad U_1 = \begin{bmatrix} 0 & u \\ \bar{u} & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} -i|u|^2 & i\partial_x u \\ -i\partial_x \bar{u} & i|u|^2 \end{bmatrix}. \tag{8.2}$$

In this case one finds by a direct computation that (8.1) is equivalent to the consistency condition $\partial_{tx}\Psi = \partial_{xt}\Psi$. We refer the reader to the work [1] for a systematic discussion relating time-dependent eigenvalue problems and completely integrable PDE and, in particular, the derivation of the eigenvalue problem above in relation to the cubic NLS.

The connection between the solution u to (8.1) (acting as the potential) and the scattering problem for $\mathcal{L} = \mathcal{L}(u)$ is described in terms of two nonlinear maps, namely, the *direct* and *inverse scattering maps*.

To define the direct scattering map, one constructs 2×2 matrix-valued solutions (*Jost solutions*) $\Psi^{\pm}$ to $\mathcal{L}\Psi = \lambda\Psi$ obeying $\lim_{x\to\pm\infty} \Psi^{\pm}(x,\lambda)e^{ix\lambda\sigma_3} = I$ (the identity matrix). Observing that the determinant of any solution is independent of x, and that $\Psi^{\pm}$ must differ by a constant matrix (which must then be independent of x), we deduce that there exists $T = T(\lambda)$ such that $\Psi^{+}(x,\lambda) = \Psi^{-}(x,\lambda)T(\lambda)$. This matrix satisfies $\det T(\lambda) = 1$ and can be shown to have the form

$$T(\lambda) = \begin{bmatrix} a(\lambda) & \overline{b(\lambda)} \\ b(\lambda) & \overline{a(\lambda)} \end{bmatrix}. \tag{8.3}$$

The *direct scattering map* is the map $\mathcal{R}$ sending $u = u(x)$ to the *reflection coefficient* $r = r(\lambda)$, defined by $r(\lambda) = \frac{-b(\lambda)}{\overline{a(\lambda)}}$. By construction, we always have $\|r\|_{L^\infty} < 1$.

We can also describe the direct scattering map a bit more concretely. In particular, if we define the matrix $N = N(x,\lambda)$ via $\Psi^{+} = e^{-i\lambda x\sigma_3}N$, then N satisfies

$$\tfrac{d}{dx}N = \begin{bmatrix} 0 & e^{2i\lambda x}u \\ e^{-2i\lambda x}\bar{u} & 0 \end{bmatrix} N,$$

with $N \to I$ as $x \to \infty$ and $N \to T(\lambda)$ as $x \to -\infty$. In particular, N can be constructed by solving the integral equation

$$N(x,\lambda) = I - \int_x^\infty \begin{bmatrix} 0 & e^{2i\lambda y}u(y) \\ e^{-2i\lambda y}\bar{u}(y) & 0 \end{bmatrix} N(y,\lambda)\,dy.$$

This can be solved via iteration, and in particular we can write series expansions for the entries of N. The function $a(\lambda)$ is obtained via $\lim_{x\to-\infty} N_{11}(x,\lambda)$, while $b(\lambda)$ is obtained via $\lim_{x\to-\infty} N_{21}(x,\lambda)$. These entries can be expressed as

$$N_{11}(x,\lambda) = 1 + \sum_{n=1}^{\infty} A_{2n}(x,\lambda), \quad N_{21}(x,\lambda) = -\sum_{n=0}^{\infty} A_{2n+1}(x,\lambda),$$

where each A_n is an n-dimensional integral over the region $\{x < y_1 < \cdots < y_n\}$ involving n copies of u (or $\bar{u}$) and a suitable phase function. For example, the very first term in the series is given by $A_1(\lambda,x) = \int_x^\infty \bar{u}(y)e^{-2i\lambda y}\,dy$, which we will return to below.

The inverse scattering map aims to recover the potential u from a given reflection coefficient $r = \mathcal{R}(u)$. For this, one again considers the problem $\mathcal{L}\Psi = z\Psi$ (for $z \in \mathbb{C}$), where $\mathcal{L}$ is defined as before in terms of the unknown potential u. If one factors $\Psi(x,z) = M(x,z)e^{-ixz\sigma_3}$, then the equation for Ψ is equivalent to the following equation for M:

$$\tfrac{d}{dx}M = -iz\,\mathrm{ad}\sigma_3(M) + U_1 M, \quad \mathrm{ad}\sigma_3(M) := [\sigma_3, M]. \tag{8.4}$$

One can build special solutions $M_\pm(x,z)$ to this equation (*Beals–Coifman solutions*) that are piecewise analytic on $\mathbb{C}\backslash\mathbb{R}$, converge to the identity as $x \to \infty$, are bounded as $x \to -\infty$, and have distinct boundary values $M_\pm(x,\lambda)$ as z approaches the real line from above/below. The boundary values satisfy a jump condition $M_+(x,\lambda) = M_-(x,\lambda)V(x,\lambda)$, where the jump matrix V depends only on the reflection coefficient, as follows:

$$V(x,\lambda) = \begin{bmatrix} 1-|r(\lambda)|^2 & \overline{-r(\lambda)}e^{-2i\lambda x} \\ r(\lambda)e^{2i\lambda x} & 1 \end{bmatrix}.$$

In particular, the solutions $M_\pm$ satisfy a *Riemann–Hilbert problem* (i.e. the problem of reconstructing a piecewise analytic function in the plane from its jump across a contour) defined purely in terms of r. To define the inverse scattering map $r \mapsto u$, we therefore need to describe (i) why this Riemann–Hilbert problem has a solution and (ii) how the potential u can be recovered from this solution.

(i) By factoring the jump matrix V, we can turn this multiplicative Riemann–Hilbert problem into an additive one, which can then be solved with Cauchy integrals. In particular, we can write $V = (I - w_x^-)^{-1}(I + w_x^+)$, where

$$w_x^+ = \begin{bmatrix} 0 & 0 \\ e^{2i\lambda x}r & 0 \end{bmatrix}, \quad w_x^- = \begin{bmatrix} 0 & -e^{-2i\lambda x}\bar{r} \\ 0 & 0 \end{bmatrix}.$$

The jump condition then becomes

$$M_+(I + w_x^+)^{-1} = M_-(I - w_x^-)^{-1} =: \mu, \tag{8.5}$$

and one can check that $M_+ - M_- = \mu[w_x^+ + w_x^-]$. In particular, using some complex analysis we can write an implicit formula for the solution:

$$M(x,z) = I + \tfrac{1}{2\pi i}\int_{\mathbb{R}} \frac{\mu(x,s)[w_x^+(s) + w_x^-(s)]}{s-z}\,ds. \tag{8.6}$$

By introducing the Cauchy operators $C_\pm f(\lambda) = \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int \frac{f(s)}{s-[\lambda\pm i\varepsilon]}\,ds$, we can use this formula to deduce an implicit formula for μ, namely, the *Beals–Coifman singular integral equation*:

$$\mu = I + \mathcal{C}_w(\mu), \quad \mathcal{C}_w(\cdot) := C_+(\cdot w_x^-) + C_-(\cdot w_x^+).$$

To deduce the existence of a unique solution to this singular integral equation, one basically just needs to observe that $\|\mathcal{C}_w\|_{L^2\to L^2} = \|r\|_{L^\infty} < 1$, so that $I - \mathcal{C}_w$ can be inverted.

(ii) To derive the formula for recovering u from the solution M of the Riemann–Hilbert problem, we consider an expansion of M around $z = \infty$, say $M(x,z) = I +$

$\frac{m_0(x)}{z} + \frac{m_1(x)}{z^2} + \dots$. The equation (8.4) yields a system of equations for the m_j (namely, $i\mathrm{ad}\sigma_3(m_0) = U_1$, with $m_j'(x) = -i\mathrm{ad}\sigma_3(m_{j+1}) + Um_j$ for $j \geq 1$). In particular, we find

$$u(x) = 2im_0^{12}(x) = \lim_{z\to\infty} 2izM_{12}(x,z),$$

which finally defines the *inverse scattering map*. Using (8.6), we may also write $m_j(x) = \frac{1}{2\pi i}\int s^j \mu(s,x)[w_x^-(s) + w_x^+(s)]\,ds$, which leads to another expression for $u(x)$, namely

$$u(x) = -\tfrac{1}{\pi}\int e^{-2ixs}\overline{r(s)}\mu_{11}(x,s)\,ds. \tag{8.7}$$

We have already discussed how to incorporate the time evolution into the generalized eigenvalue problem. However, for the computation of the direct and inverse scattering map, what is most relevant is how the scattering data itself (i.e. the reflection coefficient) evolves. The compete integrability of (8.1) manifests in the fact that the reflection coefficient obeys a simple, unitary evolution. To derive the equation, we recall the Jost solutions $\Psi^\pm = \Psi^\pm(t,x,\lambda)$ associated to the solution $u = u(t)$ to (8.1). At each time, these obey $\Psi^+ \sim e^{-i\lambda x\sigma_3}$ and $\partial_t\Psi^+ \to 0$ as $x \to \infty$, as well as $\Psi^+ \sim e^{-i\lambda x\sigma_3}T(t,\lambda)$ as $x \to -\infty$; here T is as in (8.3), but now $a = a(t,\lambda)$ and $b = b(t,\lambda)$ are defined through the time-dependent $u(t)$. We now consider a second solution Ψ, which necessarily has the form $\Psi = \Psi^+C(t)$ for some matrix $C(t)$. Differentiating this relation with respect to t and recalling (8.2), we deduce

$$\partial_t\Psi^+C + \Psi^+\partial_t C = -2i\lambda^2\sigma_3\Psi^+C + o(1) \quad \text{as} \quad x \to \pm\infty.$$

Sending $x \to \infty$, we can deduce $C'(t) = -2i\lambda^2\sigma_3 C(t)$. Sending instead $x \to -\infty$ and inserting the expression for $C'(t)$, we can then deduce $\partial_t T = -2i\lambda^2\mathrm{ad}\sigma_3(T)$. This implies $\partial_t a = 0$ and $\partial_t b = 4i\lambda^2 b$, which yields $r(t,\lambda) = e^{4i\lambda^2 t}r(0,\lambda)$.

§ 8.2. Nonlinear steepest descent

We now turn to the question of the long-time behavior of solutions. While the direct and inverse scattering maps are fairly complicated nonlinear maps, we might firstly try to approximate these maps by their derivatives at zero (particularly if we are studying small solutions, as above). Returning to the description of these maps above, this leads to the approximations $r(t,\lambda) \sim \int \bar{u}(t,y)e^{-2i\lambda y}\,dy$ and $u(t,x) \sim \int \bar{r}(t,s)e^{-2ixs}\,ds$, which are directly connected to the Fourier and inverse Fourier transforms. Recalling the explicit time evolution of the reflection coefficient, we find that this naive approach leads to the guess that small solutions to (8.1) behave approximately like solutions to the linear Schrödinger equation. This is incorrect, as we know, and so we will need to take a closer look at these maps (particularly the inverse scattering map). Before moving on, however, we remark that the direct/inverse scattering maps can indeed be thought

of as nonlinear versions of the Fourier/inverse Fourier transforms. For example, these maps interchange decay and regularity, and (as we saw above) transform the nonlinear PDE into a simple ODE evolution for the reflection coefficient.

The precise description of the long-time behavior of solutions to (8.1) in works such as [4] is expressed in terms of the reflection coefficient r_0 of the initial condition; in particular, most of the work consists in finding a suitable approximation of the inverse scattering map of $r(t,\lambda) = e^{4i\lambda^2 t} r_0(\lambda)$ for large times t. To this end, we return to the Riemann–Hilbert problem $M_+ = M_- V$, where (incorporating the time dependence into the reflection coefficient) the jump matrix is given by

$$V(t,x,\lambda) = \begin{bmatrix} 1 - |r_0(\lambda)|^2 & \overline{-r_0(\lambda)} e^{-2it\theta} \\ r_0(\lambda) e^{2it\theta} & 1 \end{bmatrix}, \quad \theta = \theta(t,x,\lambda) = 2\lambda^2 + \tfrac{\lambda x}{t}. \tag{8.8}$$

We will also need to extend this matrix analytically into the plane, in which case the $(2,1)$ entry will contain the factor $\overline{-r_0(\bar{z})}$. To extract the leading long-time behavior, the natural approach is to localize around the stationary point for the phase, namely, $z_0 = -\frac{x}{4t}$. However, because both phases $\pm i\theta$ appear in the matrix, it is not immediate how to exhibit decay away from $z = z_0$. Indeed, the $+$ phase decays only when $\operatorname{Im} z \cdot (\operatorname{Re} z - z_0) > 0$, while the $-$ phase decays only when $\operatorname{Im} z \cdot (\operatorname{Re} z - z_0) < 0$. The resolution is to factor the jump matrix, as we now explain.

We first observe that we may factor $V = AB$, with

$$A = \begin{bmatrix} 1 & -\overline{r_0(\bar{z})} e^{-2it\theta} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ r_0(z) e^{2it\theta} & 1 \end{bmatrix}.$$

For simplicity, let us assume $r_0(z)$ decays exponentially as $|z| \to \infty$ in some strip $|\operatorname{Im} z| < 2\eta$ (which holds, for example, provided $u|_{t=0}$ is chosen to be an exponentially decaying Schwartz function). Then the matrix M' defined to be MB^{-1} on $0 < \operatorname{Im} z < \eta$, MA on $-\eta < \operatorname{Im} z < 0$, and M elsewhere, has jumps given by B at $\operatorname{Im} z = \eta$ and A at $\operatorname{Im} z = -\eta$, while the jump across the real axis has been removed. We now observe that the matrix M' has the same asymptotic behavior as M as $|z| \to \infty$ in the strip (which is what we need to recover the potential u), while the jump matrices A and B decay to the identity matrix, at least in the region $\operatorname{Re} z > z_0$. This suggests that for $\operatorname{Re} z > z_0$, we may utilize this factorization in order to localize the Riemann–Hilbert problem around $z = z_0$. In practice, one opens contours at angles of $\pm\frac{\pi}{4}$ starting from $z = z_0$, but we will not be too precise about the implementation of this 'lensing' technique.

To obtain an analogous factorization in the region $\operatorname{Re} z < z_0$, we need the phases to appear in the opposite order. To achieve this, we use the factorization $V = LDU$, with

$$L = \begin{bmatrix} 1 & 0 \\ \frac{r_0(z)}{\tau(z)} e^{2it\theta} & 1 \end{bmatrix}, \quad D = \begin{bmatrix} \tau(z) & 0 \\ 0 & \frac{1}{\tau(z)} \end{bmatrix}, \quad U = \begin{bmatrix} 1 & \frac{-\overline{r_0(\bar{z})}}{\tau(z)} e^{-2it\theta} \\ 0 & 1 \end{bmatrix},$$

where $\tau(z) := 1 - r_0(z)\overline{r_0(\bar{z})}$. Continuing from above, in the region $\operatorname{Re} z < z_0$ we define the matrix M' to be MU^{-1} on $0 < \operatorname{Im} z < \eta$, ML on $-\eta < \operatorname{Im} z < 0$, and M elsewhere. Then the jumps are given by U at $\operatorname{Im} z = \eta$, L at $\operatorname{Im} z = -\eta$, and D along the real axis. Then M' still has the same asymptotic behavior as M when $|z| \to \infty$ in the strip, and the jump matrices L and U decay to the identity away from $z = z_0$. In fact, now the only jump matrix that is not decaying to the identity is the matrix D. As this matrix is diagonal, we can solve the Riemann–Hilbert problem with this jump matrix explicitly. In particular, the solution to $\Delta_+ = D\Delta_-$, where D is as above on $(-\infty, z_0)$ and $D = I$ on (z_0, ∞), is given by $\Delta = \operatorname{diag}\{\delta, \delta^{-1}\}$, where $\delta = \delta(z; z_0) = \exp\{\mathcal{C}_{(-\infty, z_0)} \log \tau(z)\}$, with $\mathcal{C}$ denoting the Cauchy integral $\mathcal{C}_\Gamma f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(s)}{s-z}\, ds$. Noting that $\Delta \to I$ as $z \to \infty$ we see that $M'' = M'\Delta^{-1}(z; z_0)$ still has the same asymptotic behavior as M' (and hence as M) as $|z| \to \infty$, while the jumps for M'' are given by $\Delta(z; z_0) X \Delta^{-1}(z; z_0)$, where $X \in \{A, B, L, U\}$ along the appropriate ray emanating from z_0 (at angles $e^{7i\pi/4}$, $e^{i\pi/4}$, $e^{5i\pi/4}$, and $e^{3i\pi/4}$, respectively).

We will now introduce a localized version of the Riemann–Hilbert problem for M'', where each matrix $\Delta X \Delta^{-1}$ is replaced with a matrix $[X]_\Delta$ that is suitably localized around z_0. We will then isolate as much of the solution as we can (dealing with the parts that are independent of z and the phases by hand), and then essentially reverse the lensing process above to reduce to a Riemann–Hilbert problem containing a version of the original jump matrix (see (8.8)) that is now completely localized at $\lambda = z_0$ and absent the phases. This problem will be explicitly solvable by special functions, and after solving it we will be able to build our approximate solution to the original Riemann–Hilbert problem (and hence our approximation to $u(t, x)$). As the correct scale for localization of the phase around $z = z_0$ is $|z - z_0| \sim t^{-\frac{1}{2}}$, we firstly introduce a new variable $\xi = \sqrt{8t}(z - z_0)$. Our approximation to $M''(z)$ (and hence $M(z)$) will be given by $\Psi(\frac{\xi}{\sqrt{8t}} + z_0)$, where the parametrix Ψ is defined as the solution to $\Psi_+(\xi) = \Psi_-(\xi)[X]_\Delta$ with $\lim_{\xi \to \infty} \Psi(\xi) = I$ and with $[X]_\Delta$ is defined as follows: We take $X \in \{A, B, L, U\}$ along the appropriate ray as above, and define

$$[X]_\Delta(\xi) = [\Delta](z; z_0)[X](z)[\Delta]^{-1}(z; z_0), \quad z = \frac{\xi}{\sqrt{8t}} + z_0,$$

with $[X]$ obtained from X by replacing $r_0(z)$ with $r_0(z_0)$ and $\tau(z)$ with $\tau(z_0)$ (but keeping the phase), and

$$[\Delta](z; z_0) = \operatorname{diag}\{(z - z_0)^{\frac{1}{2\pi i} \log \tau(z_0)}, (z - z_0)^{-\frac{1}{2\pi i} \log \tau(z_0)}\}.$$

This approximation to $\Delta(z; z_0)$ arises from just the boundary term in an integration by parts in the definition of Δ; in particular, it is actually too rough to capture the complete asymptotic formula for u (it ends up missing an additional phase term). In [23], for example, one also appends the factor $\Delta_r(z_0; z_0) := \lim_{z \to z_0} \Delta(z; z_0)[\Delta]^{-1}(z; z_0)$ to the

above approximation of Δ. To simplify matters, we will simply ignore this additional factor for now and return to this point below.

We now need to find an expression for $\lim_{\xi\to\infty}\xi\Psi_{12}(\xi)$. Then (assuming one can control the errors resulting from localization) we should have

$$u(t,x) \sim \lim_{z\to\infty} z\Psi_{12}(z) = \tfrac{1}{\sqrt{8t}}\lim_{\xi\to\infty}\xi\Psi_{12}(\tfrac{\xi}{\sqrt{8t}}+z_0). \tag{8.9}$$

We begin by observing that part of the jump matrices for Ψ have no dependence on ξ (or equivalently on z), and that the part of jump matrices dependent on the phase can accounted for explicitly. In particular, we rewrite the phases $\pm 2it\theta = \mp 4itz_0^2 \pm 4it(z-z_0)^2 = \mp 4itz_0^2 \pm i\xi^2/2$ and note that the entries of $[\Delta]$ along the diagonal are $(\sqrt{8t}\xi)^{\pm\frac{1}{2\pi i}\log\tau(z_0)}$. Considering the parts that are independent of ξ (or equivalently z), we define

$$\Psi'(\xi) = e^{-2itz_0^2\,\mathrm{ad}\sigma_3}[(\sqrt{8t})^{-\frac{1}{2\pi i}\log\tau(z_0)\,\mathrm{ad}\sigma_3}\Psi(\xi)],$$

where $e^{c\,\mathrm{ad}\sigma_3}X = e^{c\sigma_3}Xe^{-c\sigma_3}$. Tracing through the definitions, we firstly observe that

$$\Psi'_{12}(\xi) = (8t)^{-\frac{1}{2\pi i}\log\tau(z_0)}e^{-4itz_0^2}\Psi_{12}(\xi), \tag{8.10}$$

which we will need to recover u. We can also see that Ψ' solves the Riemann–Hilbert problem given by

$$\Psi'_+ = \Psi'_-\xi^{\frac{1}{2\pi i}\log(z_0)\,\mathrm{ad}\sigma_3}[e^{-(i\xi^2/4)\,\mathrm{ad}\sigma_3}W],$$

where W consists of the localized versions of A, B, L, U along appropriate rays, with the phases now removed, and $\Psi' \to I$ as $\xi\to\infty$. Here we are using the fact that if Ψ solves a Riemann–Hilbert problem with jump matrix X, then $e^{\mathrm{ad}A}\Psi$ solves the Riemann–Hilbert problem with jump $e^{\mathrm{ad}A}X$, along with the fact that

$$e^{-(\theta/2)\,\mathrm{ad}\sigma_3}\begin{bmatrix}1 & ce^{\theta}\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & c\\ 0 & 1\end{bmatrix}$$

(and similarly for the other localized matrices).

We are now two transformations away from arriving at a Riemann–Hilbert problem whose jump matrix is simply given by $V(0,0,z_0)$ (cf. (8.8) above) on the real axis. We basically need to undo the lensing process above, now that we have completely localized the jump matrices and removed the phases. The first transformation is to set

$$\Psi'' = \Psi'\xi^{\pm\frac{1}{2\pi i}\log(z_0)\,\mathrm{ad}\sigma_3}[e^{-i(\xi^2/4)}(\pm W)]$$

for $\arg\xi \in (-\frac{\pi}{4},\frac{\pi}{4})\cup(\frac{3\pi}{4},\frac{5\pi}{4})$, W is the appropriate choice of localized matrix (e.g. the localization of B when $\arg\xi\in(0,\frac{\pi}{4})$), and the $\pm$ sign is chosen when $\pm\operatorname{Im}\xi > 0$. This

moves the jumps of Ψ' from the rays to the real axis, and in particular we claim that the jump condition is

$$\Psi''_+ = \Psi''_- e^{-(i\xi^2/4)\mathrm{ad}\sigma_3}[\xi_-^{\frac{1}{2\pi i}\log\tau(z_0)} V(0,0,z_0)\xi_+^{-\frac{1}{2\pi i}\log\tau(z_0)}],$$

with $\Psi'' \to I$ as $\xi \to \infty$. Indeed, in this case the jump across the real axis is computed by the inverse of the limit from below times the limit from above. When $\operatorname{Re}\xi > 0$, we have (using the principal branch of the logarithm) that $\xi^{\frac{1}{2\pi i}\log\tau(z_0)}$ is continuous across the real axis, while for $\operatorname{Re}\xi < 0$ we have $\xi_+^{\frac{1}{2\pi i}\log\tau(z_0)}\xi_-^{-\frac{1}{2\pi i}\log\tau(z_0)} = \tau(z_0)$. This accounts for the removal of the term $\tau(z_0)$ and the recovery of the $1-|r_0(z_0)|^2$ term to obtain the matrix $V(0,0,z_0)$ in the region $\operatorname{Re}\xi < 0$.

Finally, the matrix $\tilde{\Psi} = \Psi''\xi^{\frac{1}{2\pi i}\log\tau(z_0)}e^{-(i\xi^2/4)\sigma_3}$ solves $\tilde{\Psi}_+ = \tilde{\Psi}_- V(0,0,z_0)$. We will construct a solution to this problem with the asymptotic expansion

$$\tilde{\Psi} = \xi^{\frac{1}{2\pi i}\log\tau(z_0)}e^{-(i\xi^2/4)\sigma_3}[I + \tfrac{1}{\xi}\varphi(z_0) + \cdots] \quad \text{as} \quad \xi \to \infty, \tag{8.11}$$

so that $\lim_{\xi\to\infty}\xi\Psi''_{12}(\xi) = \varphi_{12}(z_0)$. As $\Psi'' = \Psi'$ on $i\mathbb{R}_+$, this also yields $\lim_{\xi\to\infty}\xi\Psi'_{12}(\xi) = \varphi_{12}$, which (recalling (8.9) and (8.10)) leads to

$$\begin{aligned} u(t,x) &\sim \tfrac{1}{\sqrt{8t}}(8t)^{\frac{1}{2\pi i}\log\tau(z_0)}e^{4itz_0^2}\varphi_{12}(z_0) \\ &\sim t^{-\frac{1}{2}}e^{ix^2/4t}e^{-\frac{i}{2\pi}\log[1-|r_0(-\frac{x}{4t})|^2]\log 8t}\varphi_{12}(-\tfrac{x}{4t}). \end{aligned} \tag{8.12}$$

We are already getting close to the familiar asymptotic formula (1.3); however, we need to understand the form of the matrix φ. At this point we can also mention what we have lost by failing to use the more precise approximation to $\Delta(z;z_0)$. In particular, the expression above is missing an additional phase factor of the form $\exp\{-\frac{i}{\pi}\int_{-\infty}^{z_0}\log(s-z_0)\frac{d}{ds}[\log(1-|r_0(s)|^2)]\,ds\}$ (see [4, 6, 23]).

We turn to the problem satisfied by $\tilde{\Psi}$ and the behavior of the matrix φ. We first observe that since the jump matrix is independent of ξ, the quantity $\partial_\xi\tilde{\Psi}\cdot\tilde{\Psi}^{-1}$ has no jump on the real axis. On the other hand, using the leading order term of $\tilde{\Psi}$ we see that this quantity is $\mathcal{O}(\xi)$, and hence (by Liouville's theorem) must be a degree one polynomial. Inserting the desired asymptotic expansion of $\tilde{\Psi}$, we deduce $\partial_\xi\tilde{\Psi}\cdot\tilde{\Psi}^{-1} = -\frac{i}{2}\xi\sigma_3 + \frac{i}{2}[\sigma_3,\varphi]$, which leads to a differential equation for $\tilde{\Psi}$, namely

$$\partial_\xi\tilde{\Psi} = \begin{bmatrix} -i\frac{\xi}{2} & i\varphi_{12} \\ -i\varphi_{21} & i\frac{\xi}{2} \end{bmatrix}\tilde{\Psi}.$$

It turns out that one can take $-i\varphi_{21} = \overline{i\varphi_{12}}$, so let us make this replacement now. We next expand the first column and differentiate with respect to ξ once more, leading to decoupled second order differential equations for both $\tilde{\Psi}_{11}$ and $\tilde{\Psi}_{21}$, namely, $y''(\xi) = (-\frac{\xi^2}{4} - \frac{i}{2} \mp |\varphi_{12}|^2)y(\xi)$, respectively. These are solvable with a special function,

namely, the *parabolic cylinder function* $D_\nu(\cdot)$. In particular, $\tilde{\Psi}_{11} = a_{11} D_{i|\varphi_{12}|^2}(e^{i\pi/4}\xi)$ and $\tilde{\Psi}_{21} = a_{21} D_{-1+i|\varphi_{12}|^2}(e^{i\pi/4}\xi)$ for some constants a_{ij}, where D_ν solves $D_\nu'' + (-\xi^2/4 + 1/2 + \nu)D_\nu = 0$. Arguing the same way for the second column, we are led to write $\tilde{\Psi}$ as a matrix product $\tilde{\Psi} = \Lambda(\xi) \cdot S(\xi)$, where Λ is the matrix containing the parabolic cylinder functions and the unknown constants $\{a_{ij}\}_{i,j=1}^2$ (which, in particular, enforces the differential equations) and $S(\xi)$ will be a piecewise constant matrix defined separately on $\pm \operatorname{Im} \xi > 0$ (which will ultimately enforce the jump condition). The key now is to recall that we must also impose the desired asymptotic behavior (8.11). To do this, one can first make suitable choices for a_{ij} and $S(\xi)$ in terms of the unknown φ_{12}. This forces $S(\xi)$ to have a particular form (given in terms of φ_{12}), and compatibility with the jump condition $S^+(\xi) = S^-(\xi)V(0,0,z_0)$ finally determines the form that φ_{12} must take. In particular, one finally deduces that $|\varphi_{12}|^2 = -\frac{1}{2\pi}\log(\tau(z_0))$ and $\arg \varphi_{12} = \frac{\pi}{4} + \arg \Gamma(i|\varphi_{12}|^2) - \arg r_0(z_0)$. We refer the reader to [6, 23] for more details on this point.

Now that we have determined φ_{12}, we can return to (8.12) above and write

$$u(t,x) \sim ct^{-\frac{1}{2}} e^{ix^2/4t} e^{i\alpha(-\frac{x}{4t})} e^{-i|\varphi_{12}(-\frac{x}{4t})|^2 \log t} \varphi_{12}(-\tfrac{x}{4t}),$$

where φ_{12} is defined in terms of r_0 as above and we have collected various additional phases in the factor α. Defining $\psi(y) = \varphi_{12}(-\frac{y}{2})$ and observing (from the explicit expressions appearing in [4, 6, 23]) that $\alpha(y)$ should have some limit as $y \to \infty$, we can see that this formula is compatible with the one established in the small-data setting (see (1.3) and recall we have replaced the nonlinearity $|u|^2$ with $2|u|^2$). This was essentially our goal in this section. Of course, one can also carry out the arguments above in more detail to get the values of c and α (in terms of r_0 and absolute constants) with complete precision; we again refer the reader to [4,6,23]. We can also specialize the formula above to the setting of small initial data to compare with Theorem 1.1 more directly. In this case we observe that $|\varphi_{12}|^2 = -\frac{1}{2\pi}\log(1 - |r_0|^2) \sim \frac{1}{2\pi}|r_0|^2$. Recalling the first order approximation of the reflection coefficient and its connection to the Fourier transform, we can deduce that in this setting we recover $|\varphi_{12}|^2 \sim |\psi|^2 \sim |\hat{u}_0|^2$. This is consistent with the asymptotic formula appearing in (1.3), where the profile is in fact given by the Fourier transform of the data (similar to the linear case), albeit up to some (constant) phase.

The last ingredient missing from the discussion above is the rigorous error analysis, allowing for comparison between the original and the localized Riemann–Hilbert problems. We will not discuss the details here, but will only mention that by recasting the Riemann–Hilbert problems as singular integral equations (see e.g. the connection between M and μ above), the problem can (in part) be reduced to establishing suitable boundedness properties of Cauchy integral operators. We refer the reader to [4] for the details. This concludes our discussion of the argument of [4].

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