RIMS Kôkyûroku Bessatsu
B88 (2021), 147–150

A note on bilinear estimates in the homogeneous Triebel-Lizorkin spaces

By

Jishan FAN* and Tohru OZAWA**

Abstract

In this note, we derive Kozono-Shimada's bilinear estimates from Chae's bilinear estimates of Hölder type in the homogeneous Triebel-Lizorkin spaces.

§ 1. Introduction

Bilinear estimates of Hölder type are basic in the analysis of nonlinear partial differential equations. Particularly, bilinear estimates in Triebel-Lizorkin and Besov spaces are essential in the study of bilinear interactions such as in Navier-Stokes and Euler equations (see [1, 2, 3, 7] and references therein).

In 2004, Kozono and Shimada [1] showed the following bilinear estimates in $\dot{F}^s_{p,q}$:

Theorem 1.1. *(1) Let* $1 < p < \infty, 1 < q < \infty$ *and let* $s > 0, \alpha > 0, \beta > 0$*. Let* $1 < p_1 < \infty, 1 < p_2 \leq \infty$ *and* $1 < r_1 \leq \infty, 1 < r_2 < \infty$ *satisfy* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$*. Then there is a constant* $C = C(p, q, p_1, q_1, r_1, r_2, s, \alpha, \beta)$ *such that for every* $f \in \dot{F}^{s+\alpha}_{p_1,q} \cap \dot{F}^{-\beta}_{r_1,\infty}$ *and* $g \in \dot{F}^{-\alpha}_{p_2,\infty} \cap \dot{F}^{s+\beta}_{r_2,q}$ *the product satisfies* $fg \in \dot{F}^s_{p,q}$ *with the following bilinear estimate*

$$\|fg\|_{\dot{F}^s_{p,q}} \leq C(\|f\|_{\dot{F}^{s+\alpha}_{p_1,q}} \|g\|_{\dot{F}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{F}^{-\beta}_{r_1,\infty}} \|g\|_{\dot{F}^{s+\beta}_{r_2,q}}). \tag{1.1}$$

Received October 28, 2019. Revised June 2, 2020.
2020 Mathematics Subject Classification(s): 35Q30, 46B70, 46M35.
Key Words: *Bilinear estimate, Triebel-Lizorkin spaces.*
This paper is partially supported by NSFC (No. 11971234).
*Department of Applied Mathematics, Nanjing Forestry University, Nanjing 210037, P.R.China.
e-mail: fanjishan@njfu.edu.cn
**Department of Applied Physics, Waseda University, Tokyo 169-8555, Japan.
e-mail: txozawa@waseda.jp

(2) Let $1 < p \leq \infty$ and let $s > 0, \alpha > 0, \beta > 0$. Let $1 < p_1, p_2, r_1, r_2 \leq \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$. Then there is a constant $C = C(p, p_1, p_2, r_1, r_2, s, \alpha, \beta)$ such that for every $f \in F^{s+\alpha}_{p_1,\infty} \cap \dot{F}^{-\beta}_{r_1,\infty}$ and $g \in \dot{F}^{-\alpha}_{p_2,\infty} \cap \dot{F}^{s+\beta}_{r_2,\infty}$ the product satisfies $fg \in \dot{F}^s_{p,\infty}$ with the following bilinear estimate

$$\|fg\|_{\dot{F}^s_{p,\infty}} \leq C(\|f\|_{\dot{F}^{s+\alpha}_{p_1,\infty}} \|g\|_{\dot{F}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{F}^{-\beta}_{r_1,\infty}} \|g\|_{\dot{F}^{s+\beta}_{r_2,\infty}}). \tag{1.2}$$

In [2] (see also [3]), Chae established the following Moser type inequality in Triebel-Lizorkin spaces:

Theorem 1.2. *Let $s > 0$ and let p, q satisfy $(p, q) \in (1, \infty) \times (1, \infty]$ or $p = q = \infty$. Then there exists a constant C such that*

$$\|fg\|_{\dot{F}^s_{p,q}} \leq C(\|f\|_{L^{p_1}} \|g\|_{\dot{F}^s_{p_2,q}} + \|g\|_{L^{r_1}} \|f\|_{\dot{F}^s_{r_2,q}}), \tag{1.3}$$

where $p_1, r_1 \in [1, \infty]$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$.

The aim of this note is to derive Theorem 1.1 from Theorem 1.2. In the proofs, we will also use the following interpolation inequality [4] (also see [5, 6]):

$$\|f\|_{\dot{F}^{s_3}_{p_3,q_3}} \leq C\|f\|^{1-\theta}_{\dot{F}^{s_1}_{p_1,q_1}} \|f\|^{\theta}_{\dot{F}^{s_2}_{p_2,q_2}} \tag{1.4}$$

with

$$s_3 = (1-\theta)s_1 + \theta s_2, s_1 \neq s_2, 0 < \theta < 1, \tag{1.5}$$

$$\frac{1}{p_3} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}. \tag{1.6}$$

Now we recall the definition of Triebel-Lizorkin spaces. Let $\mathcal{S}$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}$, its Fourier transform $\mathcal{F}(f) = \hat{f}$ is defined by

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \mathrm{d}x.$$

Let $\phi \in \mathcal{S}$ satisfy supp $\hat{\phi} \subset \left\{\xi \in \mathbb{R}^n; \frac{1}{2} \leq |\xi| \leq 2\right\}$, and $\hat{\phi}(\xi) > 0$ if $\frac{1}{2} < |\xi| < 2$. Setting $\hat{\phi}_j = \hat{\phi}(2^{-j}\xi)$ (in other words, $\phi_j(x) = 2^{jn}\phi(2^j x)$), we adjust the normalization constant in front of $\hat{\phi}$ so that

$$\sum_{j\in\mathbb{Z}} \hat{\phi}_j(\xi) = 1 \qquad \text{for all} \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

Given $k \in \mathbb{Z}$, we define the function $S_k \in \mathcal{S}$ by

$$\hat{S}_k(\xi) := 1 - \sum_{j\geq k+1} \hat{\phi}_j(\xi).$$

In particular, we set $\hat{S}_{-1}(\xi) = \hat{\Phi}(\xi)$. We observe that

$$\text{supp } \hat{\phi_j} \cap \text{supp } \hat{\phi_{j'}} = \emptyset \quad \text{if} \quad |j - j'| \geq 2.$$

Let $s \in \mathbb{R}$ and $p, q \in [0, \infty]$. Given $f \in \mathcal{S}'$, we denote $\Delta_j f = \phi_j * f$ and then the homogeneous Triebel-Lizorkin seminorm $\|f\|_{\dot{F}^s_{p,q}}$ is defined by

$$\|f\|_{\dot{F}^s_{p,q}} := \begin{cases} \left\| \left(\sum_{j \in \mathbb{Z}} 2^{jqs} |\Delta_j f(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L^p}, & \text{if} \quad q \in [1, \infty), \\ \| \sup_{j \in \mathbb{Z}} (2^{js} |\Delta_j f(\cdot)|) \|_{L^p}, & \text{if} \quad q = \infty. \end{cases}$$

The homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}$ is a quasi-normed space with the quasi-norm given by $\| \cdot \|_{\dot{F}^s_{p,q}}$.

§ 2. Proof of Theorem 1.1 on the basis of Theorem 1.2

In this section we derive Theorem 1.1 from Theorem 1.2.

Let $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$ and let

$$\begin{aligned} &\theta_1 = \theta_2 = \frac{s + \alpha}{s + \alpha + \beta}, \theta_3 = \theta_4 = \frac{\alpha}{s + \alpha + \beta}, \\ &\frac{1}{\tilde{p}_1} = \frac{1 - \theta_1}{p_1} + \frac{\theta_1}{r_1}, \frac{1}{\tilde{r}_1} = \frac{1 - \theta_4}{p_2} + \frac{\theta_4}{r_2}, \\ &\frac{1}{\tilde{p}_2} = \frac{1 - \theta_2}{p_2} + \frac{\theta_2}{r_2}, \frac{1}{\tilde{r}_2} = \frac{1 - \theta_3}{p_1} + \frac{\theta_3}{r_1}. \end{aligned} \tag{2.1}$$

Then

$$\frac{1}{p} = \frac{1}{\tilde{p}_1} + \frac{1}{\tilde{p}_2} = \frac{1}{\tilde{r}_1} + \frac{1}{\tilde{r}_2}$$

and we apply (1.3) and (1.4) to obtain

$$\begin{aligned} \|fg\|_{\dot{F}^s_{p,q}} &\leq C(\|f\|_{L^{\tilde{p}_1}} \|g\|_{\dot{F}^s_{\tilde{p}_2,q}} + \|g\|_{L^{\tilde{r}_1}} \|f\|_{\dot{F}^s_{\tilde{r}_2,g}}) \\ &\leq C(\|f\|^{1-\theta_1}_{\dot{F}^{s+\alpha}_{p_1,q}} \|f\|^{\theta_1}_{\dot{F}^{-\beta}_{r_1,\infty}} \cdot \|g\|^{1-\theta_2}_{\dot{F}^{-\alpha}_{p_2,\infty}} \|g\|^{\theta_2}_{\dot{F}^{s+\beta}_{r_2,q}} \\ &\quad + \|f\|^{1-\theta_3}_{\dot{F}^{s+\alpha}_{p_1,q}} \|f\|^{\theta_3}_{\dot{F}^{-\beta}_{r_1,\infty}} \cdot \|g\|^{1-\theta_4}_{\dot{F}^{-\alpha}_{p_2,\infty}} \|g\|^{\theta_4}_{\dot{F}^{s+\beta}_{r_2,q}}) \\ &\leq C(\|f\|_{\dot{F}^{s+\alpha}_{p_1,q}} \|g\|_{\dot{F}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{F}^{-\beta}_{r_1,\infty}} \|g\|_{\dot{F}^{s+\beta}_{r_2,q}}). \end{aligned} \tag{2.2}$$

This proves (1.1). A similar argument proves (1.2).

□

References

[1] H.Kozono and Y.Shimada, Bilinear estimates in homogeneous Triebel-Lizorkin spaces and the Navier-Stokes equations. Math. Nachr., 276(2004) 63-74.

[2] D.Chae, On the well-posedness of the Euler equations in the Triebel-Lizorkin spaces. Commun. Pure Appl. Math., 55(2002) 0654-0678.

[3] Z.Guo and K.Liu, Remarks on the well-posedness of the Euler equations in the Triebel-Lizorkin spaces. arXiv: 1903.09437 v1.

[4] Y. Meyer, Oscillating patterns in some nonlinear evolution equations, in: Mathematical Foundation of Turbulent Viscous Flows, Lecture Notes in Mathematics Vol. 1871, edited by M. Cannone and T. Miyakawa (Springer-Verlag, 2006), pp. 101-187.

[5] H. Hajaiej, L. Molinet, T. Ozawa, and B. Wang, Necessary and sufficient conditions for the fractional Gagliardo-Nirenberg inequalities and applications to Navier-Stokes and generalized Boson equations, RIMS Kokyuroku Bessatsu, 26 (2011) 159-175.

[6] S. Machihara and T. Ozawa, Interpolation inequalities in Besov spaces, Proc. Am. Math. Soc., 131 (2002) 1553-1556.

[7] K.Kaneko, H.Kozono, and S.Shimizu, Stationary solution to the Navier-Stokes equations in the scaling invariant Besov space and its regularity. Indiana University Mathematics Journal, 68(3)(2019) 857-880.