Non-Integrability of the reduced planar three-body problem with generalized force

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Abstract

We consider the planer three-body problem with generalized potentials. Some non-integrability results for these systems have been obtained by analyzing the variational equations along the homothetic solutions. But we can not apply it to several exceptional cases. For example, in the case of inverse-square potentials, the variational equations along the homothetic solutions are solvable. We obtain sufficient conditions for non-integrability for these exceptional cases by focusing on some particular solutions that are different from homothetic solutions.

1 Introduction

We consider the planar motion of three mass particles P_1, P_2 and P_3 which interact mutually according to a generalized potential. Let $m_j \in \mathbb{R}$ denote the mass of j-th particle satisfying $m_i \neq 0,\ m_i + m_j \neq 0 (i \neq j),\ m_1 + m_2 + m_3 \neq 0$. The conditions are satisfied if they are positive. But our result includes the case of complex numbers. Let $\boldsymbol{q}_j = (q_{2j-1}, q_{2j})$ denote the inertial Cartesian coordinates of P_j. The distance from P_i to P_j is given by the Euclidian norm

$$r_{ij} := \|\boldsymbol{q}_i - \boldsymbol{q}_j\| = \sqrt{(q_{2i-1} - q_{2j-1})^2 + (q_{2i} - q_{2j})^2}.$$

The configuration space is

$$\mathcal{X} := \left\{ (\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) \in (\mathbb{R}^2)^3 \right\} \setminus \Delta$$

where

$$\Delta := \bigcup_{i<j} \Delta_{ij}\ , \quad \Delta_{ij} := \{ \boldsymbol{q} = (\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) \in (\mathbb{R}^2)^3 \mid \boldsymbol{q}_i = \boldsymbol{q}_j \}.$$

The set Δ_{ij} stands for two-body collisions between P_i and P_j. Let $\boldsymbol{p}_j := (p_{2j-1}, p_{2j}) \in \mathbb{R}^2$ denote the momentum corresponding to $\boldsymbol{q}_j$. The cotangent bundle $\mathcal{M} = T^*\mathcal{X}$ possesses the canonical symplectic form $\omega := \sum_{j=1}^{6} dp_j \wedge dq_j$. The motion of three particles can be represented as the Hamiltonian system with the Hamiltonian

$$H(\boldsymbol{z}) := \sum_{j=1}^{3} \frac{\|\boldsymbol{p}_j\|^2}{2m_j} + \sum_{i<j} m_i\, m_j\, u(r_{ij}(\boldsymbol{q})) \tag{1}$$

where $\boldsymbol{q} = (\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) \in \mathcal{X}$, $\boldsymbol{p} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3) \in (\mathbb{R}^2)^3$, $\boldsymbol{z} = (\boldsymbol{q}, \boldsymbol{p})$, and $u: \mathbb{R}_{>0} \to \mathbb{R}$ is a generalized force potential. The Hamiltonian vector field X_H is defined by $\iota_{X_H}\omega = -dH$ and the Hamiltonian equations are given by $\dot{\boldsymbol{z}} = X_H(\boldsymbol{z})$ where ι is the internal product.

A function F is called the first integral of a Hamiltonian system if F is conserved along each solution of the Hamiltonian equations. Hamiltonian system with k degrees of freedom is called *Liouville integrable* or *integrable* if there are k first integrals $F_1, \dots F_k$ that are pairwise commute with respect to the Poisson bracket, and that are functionally independent. Moreover if $F_1, \dots, F_k$ are meromorphic, the Hamiltonian H is said to be *meromorphically integrable*. The behavior of orbits of integrable Hamiltonian systems is well-known (see [4]) while one of nonintegrable systems is thought to be chaotic. Therefore determining integrability of Hamiltonian systems is an important subject.

The three-body problem has some symmetry, hence the system has corresponding first integrals following Nöther's theorem. In particular, each component of linear momentum $\boldsymbol{K}_m = (K_{m1}, K_{m2})$ and angular momentum K_{am}:

$$K_{m1}(\boldsymbol{p}) := \sum_{j=1}^{3} p_{2j-1}\ , \qquad K_{m2}(\boldsymbol{p}) := \sum_{j=1}^{3} p_{2j}$$

$$K_{am}(\boldsymbol{q}, \boldsymbol{p}) := \sum_{j=1}^{3} \left(p_{2j-1}q_{2j} - p_{2j}q_{2j-1} \right)$$

are first integrals of the system corresponding to invariance under translation and rotation. Moreover, there exists another first integral K_{-2}

$$K_{-2}(\boldsymbol{q}, \boldsymbol{p}) := \langle \boldsymbol{q}, \boldsymbol{p} \rangle^2 - 2\|\boldsymbol{q}\|^2 H(\boldsymbol{q}, \boldsymbol{p})$$

in the case of $u(r) = r^{-2}$.

It has been attempted for proving the non-integrability of the three-body problem applying several methods. As classical results, Bruns [5] proved that there is no additional first integral which is represented by an algebraic function in the case of $u(r) = r^{-1}$. After that, Poincaré [19] proved the non-existence of an analytic first integral depending analytically on a mass parameter.

Another method in this field was originated by Kovalevskaya [12] by focusing on singularities. Ziglin established the theory of the monodromy group for proving the non-integrability, which is based on Kovalevskaya's method [24, 25]. He also proved non-integrability in the case of $u(r) = r^{-1}$ in the framework of the Ziglin theory [26]. By applying the Ziglin analysis, Yoshida provided criteria for the non-integrability of the case of $u(r) = r^{2n} (n \in \mathbb{Z})$ [23].

The Morales-Ramis theory, which is based on differential Galois theory (Picard-Vessiot Theory), is one of the strongest theories for proving non-integrability of Hamiltonian systems. The proof of non-integrability of the three-body problem $u(r) = r^{-1}$ with the differential Galois theory was given by [6, 22], and a simpler proof was obtained in [15] based on ideas of [18]. In addition, the non-integrability in more general cases $u(r) = r^{-n}$ $(n \in \mathbb{N} \setminus \{2\})$ was proved by [16]. These proof are based on analyzing the variational equations along the homothetic solutions which have zero linear momentum and zero angular momentum.

In this paper, we show non-integrability of the three-body problem with the following potential:

$$u(r) = r^{-2}.$$

We note that the Hamiltonian function with these potential are meromorphic function on $\mathcal{M}$.

This system is represented as a homogeneous potential system of -2 degree. It is numerically shown that the system exhibits complicated dynamics [3, 10]. Therefore these systems are thought to be non-integrable. The variational equations along homothetic solutions of those systems has no obstructions to integrability [8]. Julliard Tosel [21] proved the non-integrability in the case that masses are $1, 1, m$ by focusing on an isosceles solution as a particular solution. We will take different particular solutions (collinear solutions) from existing results and show the non-integrability for almost all masses. Our main results are the following :

Theorem 1.1. Suppose $u(r) = r^{-2}$. Then the reduced planer three-body problem with $u(r)$ is meromorphically non-integrable if $\mathrm{Dis}(B_m) \neq 0$.

Here $\mathrm{Dis}(B)$ represents the discriminant of a polynomial $B \in \mathbb{C}[z]$. We state the definition of the polynomial $B_m \in \mathbb{C}[m][x]$ in the proof.

This paper is organized as follows. In the next section, we introduce reductions of the system. In Section 3 we give some important theorems for proving our main results as preliminary. In Section 4 we give the proof of the theorem. In the last section, we examine the obtained results.

2 Reductions

The three-body problem has symmetry with respect to translation and rotation. Hence we can reduce the degrees of freedom. In this section, we introduce symplectic reductions and a change of variables.

2.1 Symplectic reductions and transformations

We first perform complexification of independent and all dependent variables i.e. we set $t \in \mathbb{C}$ and $\boldsymbol{z} = (\boldsymbol{q}, \boldsymbol{p}) \in \mathcal{X}^{\mathbb{C}} \times \mathbb{C}^6$ where

$$\mathcal{X}^{\mathbb{C}} := \left\{(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) \in (\mathbb{C}^2)^3\right\} \setminus \Delta^{\mathbb{C}},$$

and

$$\Delta^{\mathbb{C}} := \bigcup_{i<j} \Delta^{\mathbb{C}}_{ij}\ , \quad \Delta^{\mathbb{C}}_{ij} := \{\boldsymbol{q} = (\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) \in (\mathbb{C}^2)^3 \mid \boldsymbol{q}_i = \boldsymbol{q}_j\}.$$

Hereafter we consider complexified Hamiltonian system.

The reduction by translational symmetry. We define a linear transformation by

$$\begin{array}{rccc} \pi_{\mathrm{tr}}: & \mathcal{X}^{\mathbb{C}} \times \mathbb{C}^6 & \longrightarrow & M(\mathcal{X}^{\mathbb{C}}) \times \mathbb{C}^6 \\ & \Psi & & \Psi \\ & (\boldsymbol{q}, \boldsymbol{p}) & \longmapsto & (\boldsymbol{Q}, \boldsymbol{P}) := \left(M\,\boldsymbol{q},\ {}^{t}M^{-1}\boldsymbol{p} \right). \end{array} \tag{2}$$

Here $M \in \mathrm{GL}(6, \mathbb{R})$ is defined by the following:

$$M := \begin{pmatrix} \mu_1\,\mathrm{I}_2 & -\mu_1\,\mathrm{I}_2 & \mathrm{O}_2 \\ -\dfrac{m_1}{m_1+m_2}\mu_2\,\mathrm{I}_2 & -\dfrac{m_2}{m_1+m_2}\mu_2\,\mathrm{I}_2 & \mu_2\,\mathrm{I}_2 \\ \dfrac{m_1}{m_1+m_2+m_3}\,\mathrm{I}_2 & \dfrac{m_2}{m_1+m_2+m_3}\,\mathrm{I}_2 & \dfrac{m_3}{m_1+m_2+m_3}\,\mathrm{I}_2 \end{pmatrix}.$$

where $\mu_1 := \sqrt{\frac{m_1 m_2}{m_1+m_2}}$, $\mu_2 := \sqrt{\frac{m_1 m_3 + m_2 m_3}{m_1+m_2+m_3}}$, I_2 is 2×2 identity matrix, and O_2 is 2×2 zero matrix. This transformations is called *the Jacobi transformation* (for example see [13]).

Let $\boldsymbol{Q} := (\boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{Q}_3)$, $\boldsymbol{Q}_j = (Q_{2j-1}, Q_{2j})$ and $\boldsymbol{P} := (\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3)$, $\boldsymbol{P}_j = (P_{2j-1}, P_{2j})$. The symplectic form is conserved i.e. $\sum_{j=1}^{6} dq_j \wedge dp_j = \sum_{j=1}^{6} dQ_j \wedge dP_j$ and the the transformed equations are the canonical equations for the following Hamiltonian:

$$H_1(\boldsymbol{Q}, \boldsymbol{P}) := \frac{1}{2}\sum_{j=1}^{4} P_j^2 + \frac{1}{2(m_1+m_2+m_3)}(P_5^2 + P_6^2) + \sum_{i<j} \frac{m_i m_j}{r_{ij}(\boldsymbol{Q}_1, \boldsymbol{Q}_2)^2} \tag{3}$$

where r_{ij} are given by the following:

$$\begin{aligned} r_{12}(\boldsymbol{Q}_1, \boldsymbol{Q}_2) &:= \left\| \frac{1}{\mu_1}\boldsymbol{Q}_1 \right\| \\ r_{23}(\boldsymbol{Q}_1, \boldsymbol{Q}_2) &:= \left\| -\frac{m_1}{m_1+m_2}\frac{1}{\mu_1}\boldsymbol{Q}_1 - \frac{1}{\mu_2}\boldsymbol{Q}_2 \right\| \\ r_{13}(\boldsymbol{Q}_1, \boldsymbol{Q}_2) &:= \left\| -\frac{m_2}{m_1+m_2}\frac{1}{\mu_1}\boldsymbol{Q}_1 + \frac{1}{\mu_2}\boldsymbol{Q}_2 \right\|. \end{aligned}$$

The vector (Q_5, Q_6) represents the center of mass, and (P_5, P_6) represents the total momentum. The momentum P_5 and P_6 are conserved, and we can assume $P_5 = P_6 = 0$ without loss of generality since the reduced system split into a direct product of two Hamiltonian terms. Thus we obtain reduced system $(\tilde{\boldsymbol{Q}}, \tilde{\boldsymbol{P}}) := (\boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{P}_1, \boldsymbol{P}_2) \in \mathbb{C}^4 \times \mathbb{C}^4$.

The reduction by rotational symmetry. We introduce reduction with respect to rotational symmetry using the Hopf fibration. The reduced Hamiltonian (3) is invariant under $SO(2)$-action:

$$g\,(\boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{P}_1, \boldsymbol{P}_2) = (\,g\,\boldsymbol{Q}_1, g\,\boldsymbol{Q}_2, g\,\boldsymbol{P}_1, g\,\boldsymbol{P}_2), \qquad g \in SO(2). \tag{4}$$

This action also conserves the symplectic form ω. The first integral $\mu:\ \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}$

$$\mu(\tilde{\boldsymbol{Q}}, \tilde{\boldsymbol{P}}) = Q_1P_2 - Q_2P_1 + Q_3P_4 - Q_4P_3$$

corresponds to the $SO(2)$-action.

Fix $c \in \mathbb{C}$ and restrict the system on level set $\mu(\tilde{\boldsymbol{Q}}, \tilde{\boldsymbol{P}}) = c$. We consider the following transformation:

$$\begin{array}{cccc} \pi_{\mathrm{rot}}: & \mathbb{C}^4 \times \mathbb{C}^4 & \longrightarrow & \mathbb{C}^3 \setminus \{\boldsymbol{0}\} \times \mathbb{C}^3 \\ & \Cup & & \Cup \\ & \left(\tilde{\boldsymbol{Q}}, \tilde{\boldsymbol{P}}\right) & \longmapsto & (\boldsymbol{x}, \boldsymbol{y}) := \left(\mathrm{pr}\left(\mathrm{Q}(\tilde{\boldsymbol{Q}})\,\tilde{\boldsymbol{Q}}\right), \mathrm{pr}\left(\dfrac{1}{\|\boldsymbol{Q}\|^2}\mathrm{Q}(\tilde{\boldsymbol{Q}})\tilde{\boldsymbol{P}}\right)\right) \end{array} \tag{5}$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{y} = (y_1, y_2, y_3)$, and $\mathrm{pr}:\ \mathbb{C}^4 \ni (a_1, a_2, a_3, a_4) \mapsto (a_1, a_2, a_3) \in \mathbb{C}^3$ and $\mathrm{Q}(\tilde{\boldsymbol{Q}})$ is 4×4 matrix defined by

$$\mathrm{Q}(\tilde{\boldsymbol{Q}}) := \begin{pmatrix} Q_1 & Q_2 & -Q_3 & -Q_4 \\ Q_3 & Q_4 & Q_1 & Q_2 \\ Q_4 & -Q_3 & -Q_2 & Q_1 \\ -Q_2 & Q_1 & -Q_4 & Q_3 \end{pmatrix}.$$

We remark that $\tilde{\boldsymbol{P}} = {}^t\mathrm{Q}(\tilde{\boldsymbol{Q}})^t(y_1, y_2, y_3, c)$.

The symplectic form $\omega_{2,c}$ is represented by

$$\omega_{2,c} = \frac{1}{2}\left(\sum_{j=1}^{3} dx_j \wedge dy_j - \frac{c}{\|\boldsymbol{x}\|^3}\sum_{cyc} x_1 dx_2 \wedge dx_3\right)$$

and the Hamiltonian is

$$H_2(\boldsymbol{x}, \boldsymbol{y}; c) := \frac{1}{2}\|\boldsymbol{x}\|\,\|\boldsymbol{y}\|^2 + \frac{c^2}{2\,\|\boldsymbol{x}\|^2} + \sum_{i<j}\frac{m_i m_j}{r_{ij}(\boldsymbol{x})^2} \tag{6}$$

where

$$\begin{aligned} r_{12}(\boldsymbol{x}) &= \frac{1}{2{\mu_1}^2}\|\boldsymbol{x}\| + \frac{1}{2{\mu_1}^2}x_1 \\ r_{23}(\boldsymbol{x}) &= \frac{1}{2}\left(\frac{{\mu_1}^2}{{m_2}^2} + \frac{1}{{\mu_2}^2}\right)\|\boldsymbol{x}\| + \frac{1}{2}\left(\frac{{\mu_1}^2}{{m_2}^2} - \frac{1}{{\mu_2}^2}\right)x_1 + \frac{\mu_1}{m_2\mu_2}x_2 \\ r_{13}(\boldsymbol{x}) &= \frac{1}{2}\left(\frac{{\mu_1}^2}{{m_1}^2} + \frac{1}{{\mu_2}^2}\right)\|\boldsymbol{x}\| + \frac{1}{2}\left(\frac{{\mu_1}^2}{{m_1}^2} - \frac{1}{{\mu_2}^2}\right)x_1 - \frac{\mu_1}{m_1\mu_2}x_2 \end{aligned}$$

in reduced space $\mathbb{C}^3 \setminus \{\mathbf{0}\} \times \mathbb{C}^3$. Note that H_2 is not defined on the reduced space, but a double covering of it.

Transformation. We introduce a change of variable which is derived by the stereographic projection. We first define a change of variables in configuration space:

$$\begin{array}{cccc} \pi_{\rm sp} : & \mathbb{C}^3 \setminus \{\mathbf{0}\} & \longrightarrow & \mathbb{C}^\times \times \mathbb{C} \times \mathbb{C} \\ & \cup & & \cup \\ & \boldsymbol{x} = (x_1, x_2, x_3) & \longmapsto & (Z, r, s) \end{array}$$

which is defined by:

$$x_1 = -Z\,\frac{r}{r^2+1}\left(s + \frac{1}{s}\right), \quad x_2 = \mathrm{i}Z\,\frac{r}{r^2+1}\left(s - \frac{1}{s}\right), \quad x_3 = Z\,\frac{r^2-1}{r^2+1}.$$

where $\mathrm{i} := \sqrt{-1}, \mathbb{C}^\times = \mathbb{C}\backslash\{0\}$.

We lift the transformation $\pi_{\rm sp}$ to cotangent space and construct canonical transformation $\hat{\pi}_{\rm sp}$ which is given by the following map:

$$\begin{array}{cccc} \pi_{\rm sp} : & \mathbb{C}^3 \setminus \{\mathbf{0}\} \times \mathbb{C}^3 & \longrightarrow & (\mathbb{C}^\times \times \mathbb{C} \times \mathbb{C}) \times \mathbb{C}^3 \\ & \cup & & \cup \\ & (\boldsymbol{x}, \boldsymbol{y}) & \longmapsto & (Z, r, s, p_Z, p_r, p_s) := \left(\pi_{\rm sp}(\boldsymbol{x}), {}^t D_{\boldsymbol{x}} \pi_{\rm sp}(\boldsymbol{x})^{-1} \boldsymbol{y}\right) \end{array}. \tag{7}$$

Reduced symplectic form $\omega_{3,c}$ is given by

$$\omega_{3,c} = \frac{1}{2}\left(dZ \wedge dp_Z + dr \wedge dp_r + ds \wedge dp_s - \frac{4ic\, r^2}{s\,\left(1+r^2\right)^2} dr \wedge ds \right)$$

and the transformed Hamiltonian can be written as the following function:

$$H_3(Z, r, s, p_Z, p_r, p_s) := \frac{1}{2}p_Z^2 + \frac{(r^2+1)^2}{8Z^2}\,p_r^2 - \frac{(r^2+1)^2}{8Z^2\,r^2}\,s^2 p_s^2 + \frac{c^2}{2Z^2} + \frac{1}{Z^2}\sum_{i<j}\frac{m_i m_j}{R_{ij}(r,s)^2} \tag{8}$$

where distances $R_{ij}(r, s)$ are given by

$$R_{ij}(r,s) := \sqrt{\frac{m_i + m_j}{2 m_i m_j}\,\frac{1}{r^2+1}\left(r^2 - \left(\frac{s}{\beta_{ij}} + \frac{\beta_{ij}}{s}\right) r + 1\right)}, \quad (\, i, j = 1, 2, 3, \quad i < j\,)$$

and β_{ij} is defined by

$$\begin{aligned} \beta_{12} &:= 1\,, \\ \beta_{23} &:= -\frac{m_2(m_1+m_2+m_3) - m_3 m_1}{(m_1+m_2)(m_2+m_3)} + \mathrm{i}\frac{2\sqrt{m_1 m_2 m_3 (m_1+m_2+m_3)}}{(m_1+m_2)(m_2+m_3)}\,, \\ \beta_{13} &:= -\frac{m_1(m_1+m_2+m_3) - m_2 m_3}{(m_3+m_1)(m_1+m_2)} - \mathrm{i}\frac{2\sqrt{m_1 m_2 m_3 (m_1+m_2+m_3)}}{(m_3+m_1)(m_1+m_2)}. \end{aligned}$$

We notice that β_{ij} are different from 0 since the some conditions of masses are imposed i.e. $m_i \neq 0,\ m_i + m_j \neq 0 (i \neq j),\ m_1 + m_2 + m_3 \neq 0$

2.2 Particular solutions of the reduced Hamiltonian system

The Hamiltonian equations derived by (8) are the following:

$$
\begin{aligned}
\frac{d}{dt}Z &= p_Z,\\
\frac{d}{dt}p_Z &= \frac{(r^2+1)^2}{4Z^3}\,p_r^2 - \frac{(r^2+1)^2}{4Z^3\,r^2}\,s^2p_s^2 + \frac{c^2}{Z^3} - \frac{1}{Z^3}\sum_{i<j}\frac{m_im_j}{R_{ij}(r,s)^2},\\
\frac{d}{dt}r &= \frac{(r^2+1)^2}{4Z^2}\,p_r,\\
\frac{d}{dt}p_r &= -\frac{r(r^2+1)}{2Z^2}\,p_r^2 + \frac{r^4-1}{4Z^2\,r^3}\,s^2p_s^2 + \frac{c\,s^2}{Z^2}p_s + \frac{2}{Z^2}\sum_{i<j}\frac{m_im_j}{R_{ij}(r,s)^3}\frac{\partial R_{ij}}{\partial r}(r,s),\\
\frac{d}{dt}s &= -\frac{(r^2+1)^2}{4Z^2\,r^2}\,s^2p_s,\\
\frac{d}{dt}p_s &= \frac{(r^2+1)^2}{4Z^2\,r^2}\,sp_s^2 - \frac{c\,r^2}{Z^2}p_r + \frac{2}{Z^2}\sum_{i<j}\frac{m_im_j}{R_{ij}(r,s)^3}\frac{\partial R_{ij}}{\partial s}(r,s).
\end{aligned}
\tag{9}
$$

The system has an invariant surface, and we can find some specific solutions on them. The system on a level set $\{(Z,r,s,p_Z,p_r,p_s)\,|\,H_3(Z,r,s,p_Z,p_r,p_s)=h\}$ is equivalent to the Hamiltonian system governed by the following Hamiltonian:

$$
\hat{H}_3(Z,r,s,p_Z,p_r,p_s) := \frac{1}{2}Z^2p_Z^2 + \frac{(r^2+1)^2}{8}\,p_r^2 - \frac{(r^2+1)^2}{8\,r^2}\,s^2p_s^2 + \frac{c^2}{2} + \sum_{i<j}\frac{m_im_j}{R_{ij}(r,s)^2}.
\tag{10}
$$

using change of independent variable $t\mapsto\tau:=\int Z(t)^{-2}dt$. Therefore Z and p_Z are separable in Hamiltonian $\hat{H}_3$, and we can define a reduced Hamiltonian:

$$
\begin{aligned}
\check{H}_3(s,p_Z,p_r,p_s) :=\ & \frac{(r^2+1)^2}{8}\,p_r^2 - \frac{(r^2+1)^2}{8\,r^2}\,s^2p_s^2 + \frac{c^2}{2} + \sum_{i<j}\frac{m_im_j}{R_{ij}(r,s)^2}.\\
=\ & \frac{(r^2+1)^2}{8}\,p_r^2 - \frac{(r^2+1)^2}{8\,r^2}\,s^2p_s^2 + \frac{c^2}{2} + \sum_{i<j}\frac{2m_i^2m_j^2}{m_i+m_j}\frac{r^2+1}{r^2-\left(\frac{s}{\beta_{ij}}+\frac{\beta_{ij}}{s}\right)r+1}
\end{aligned}
$$

and the reduced system is represented by

$$
\begin{aligned}
\frac{d}{d\tau}r &= \frac{(r^2+1)^2}{4}\,p_r\\
\frac{d}{d\tau}p_r &= -\frac{r(r^2+1)}{2}\,p_r^2+\frac{r^4-1}{4\,r^3}\,s^2p_s^2+\frac{c\,s^2}{r^2}p_s\\
&\qquad +\sum_{i<j}\frac{2m_i^2m_j^2}{m_i+m_j}\left(\frac{s}{\beta_{ij}}+\frac{\beta_{ij}}{s}\right)\frac{r^2-1}{\left(r^2-\left(\frac{s}{\beta_{ij}}+\frac{\beta_{ij}}{s}\right)r+1\right)^2}\\
\frac{d}{d\tau}s &= -\frac{(r^2+1)^2}{4\,r^2}\,s^2p_s\\
\frac{d}{d\tau}p_s &= \frac{(r^2+1)^2}{4\,r^2}\,sp_s^2-c\,p_r-\sum_{i<j}\frac{2m_i^2m_j^2}{m_i+m_j}\frac{\partial}{\partial s}\left(\frac{r^2+1}{r^2-\left(\frac{s}{\beta_{ij}}+\frac{\beta_{ij}}{s}\right)r+1}\right).
\end{aligned}
$$

We call the system "the reduced system" or "the reduced planer three-body problem". If $c=0$ then $r=1$ and $p_r=0$ satisfy the equation, hence the reduced system has the invariant plane $\Sigma := \{(s,p_Z,p_r,p_s)\,|\,r=1,p_r=0\}$.

There exists particular solution $(r,s,p_r,p_s)=(1,s(\tau),0,-s'(\tau)/s(\tau)^2)$ where $(z,w)=(s(\tau),s'(\tau))$ is any solution of the Hamiltonian system governed the following hamiltonian $\mathcal{H}$:

$$
\mathcal{H}(z,w)=\frac{w^2}{2}+\sum_{i<j}\frac{4m_i^2m_j^2}{m_i+m_j}\frac{\beta_{ij}z^3}{(z-\beta_{ij})^2}.
$$

The set of the particular solutions is an invariant set. The equations on the invariant set are integrable.

3 Preliminary

In this section, we briefly survey the Morales-Ramis theory. The Morales-Ramis theory is a powerful tool to prove non-integrability of given Hamiltonian systems based on the differential Galois theory (see [7, 20] for more details).

3.1 Morales-Ramis theory

Let ω be a symplectic form on $\mathbb{C}^{2n}$. Then $(\mathbb{C}^{2n},\omega)$ is a complex symplectic manifold. Let D be a domain in $\mathbb{C}^{2n}$ and $H: D\to\mathbb{C}^{2n}$ holomorphic. Consider the Hamiltonian system governed by a Hamiltonian H:

$$
\frac{d}{dt}x = X_H(x), \qquad x\in D \tag{11}
$$

where X_H is the Hamiltonian vector field defined intrinsically by the formula $i_{X_H}\omega=-dH$.

Let $x = \phi(t),\ t \in \mathbb{C}$ be a non-stationary particular solution of (11) and Γ the phase curve determined by $x = \phi(t)$. The variational equation of (11) along $x = \phi(t)$ is given by

$$\frac{d}{dt}\xi = DX_H(\phi(t))\xi, \qquad \xi \in T\mathbb{C}^{2n}|_\Gamma \tag{12}$$

where $T\mathbb{C}^{2n}|_\Gamma$ is tangent bundle of $\mathbb{C}^{2n}$ restricted to Γ. Assume that the closure $\overline{\Gamma} \subset \mathbb{P}^{2n}$ contains point at infinity and the vector field $X_H(x)$ can be meromorphically extended on $\overline{\Gamma}$. Here we take as coefficient field of (12), the field of meromorphic functions over $\overline{\Gamma}$. The differential Galois group of variational equations is an algebraic groups with identity component, and for Hamiltonian equations it is a subgroup of the symplectic group. Then we have the following result using important arguments given by Morales–Ruiz and Ramis [14, 17].

Theorem 3.1. *Let G_{VE} be the differential Galois group of* (12). *Suppose that the variational equation* (12) *has no irregular singularities at infinity. If the Hamiltonian system* (11) *is Liouville integrable near Γ with the aid of meromorphic functions, then the identity component G^0_{VE} of G_{VE} is abelian.*

From this theorem, we obtain important tool concerning non-integrability of the Hamiltonian system —– namely, if we show the identity component of the differential Galois group of the variational equation along a particular solution is not abelian, then the Hamiltonian system is not Liouville integrable.

If there exists an invariant manifold with respect to the flow of X_H, then it is possible to reduce the variational equation and to obtain the so-called *normal variational equation*.

For example, we consider the case that (11) has $2m$ dimensional invariant plane

$$N = \left\{(x_1, \dots, x_{2m}, x_{2m+1}, \dots, x_{2n}) \in \mathbb{C}^{2n} \mid x_{2m+1} = 0, \dots, x_{2n} = 0\right\}.$$

as described in [1]. Then the variational equation (12) can be represented as the form

$$\frac{d}{dt}\begin{pmatrix} \xi_N \\ \xi_H \end{pmatrix} = \begin{pmatrix} A_N(t) & O \\ A_1(t) & A_2(t) \end{pmatrix}\begin{pmatrix} \xi_N \\ \xi_H \end{pmatrix}, \qquad \xi_N \in \mathbb{C}^{2m}, \quad \xi_H \in \mathbb{C}^{2(n-m)}$$

where A_N, A_1 and A_2 are $2(n-m) \times 2(n-m)$, $2m \times 2(n-m)$ and $2m \times 2m$ matrices. The normal variational equations are given by

$$\frac{d}{dt}\xi_N = A_N(t)\xi_N, \qquad \xi_N \in \mathbb{C}^{2m}.$$

Let G_{NVE} be the differential Galois group of the normal variational equations and G^0_{NVE} the identity component of G_{NVE}. The following theorem holds.

Theorem 3.2. *If G^0_{VE} is abelian, then G^0_{NVE} is also abelian.*

3.2 Kovacic's algorithm

In general, it is difficult to compute a differential Galois group. However, we know concrete procedures for determining solvability of identity component of differential Galois group based on the classification of the algebraic subgroups of a given algebraic group.

Kovacic's algorithm (see [11]) is an effective algorithm to solve the 2nd-order linear differential equation

$$\frac{d^2}{dx^2}\xi + 2a(x)\,\frac{d}{dx}\xi + b(x)\,\xi = 0, \qquad \xi \in \mathbb{C}. \tag{13}$$

where $a, b \in \mathbb{C}(x)$ and both of them are not constant. We note that any equation of the form (13) can be transformed, through the change of variables $\zeta = \xi\, e^{\int a}$, into

$$\frac{d^2}{dx^2}\zeta = r(x)\zeta, \qquad \zeta \in \mathbb{C}. \tag{14}$$

where $r \in \mathbb{C}(x)\backslash\mathbb{C}$ satisfying the Riccati equation $r = -b + a' + a^2$.

Let G be the differential Galois group of (14) over $\mathbb{C}(x)$ and G^0 the identity component of G. Then G is an algebraic subgroups of $\mathrm{SL}(2,\mathbb{C})$. We have the following four cases for G corresponding on classification of the algebraic subgroups of $\mathrm{SL}(2,\mathbb{C})$.

Theorem 3.3. *One of the following four cases occurs:*

(K-i) $G \subset \mathbb{B}$: *triangular unimodular groups;*

(K-ii) $G \not\subset \mathbb{B}$, $G \subset \mathbb{D}_\infty$: *infinite dihedral group*

(K-iii) G i*s finite (not diagonal);*

(K-iv) $G = \mathrm{SL}(2,\mathbb{C})$.

Only for cases **(K-i)**, **(K-ii)** and **(K-iii)** G^0 is solvable hence one is solvable (14) in closed form. In contrast, for case **(K-iv)** G^0 coinsides $\mathrm{SL}(2,\mathbb{C})$ hence (14) is not integrable.

Each case in Kovacic's algorithm is related with each one of these cases **(K-i)**, **(K-ii)**, **(K-iii)** and **(K-iv)**. The algorithm can possibly provide one solution (ζ_1), so the second one (ζ_2) can be got through $\zeta_2 = \zeta_1 \int(\zeta_1^{-2})dx$.

We skip to give the contents of the Kovacic's algorithm here and write the concrete procedures of it in Appendix B.

3.3 Hamiltonian algebrization algorithm

We say that a linear differential equation is *algebrizable* if it is the pull-back of a linear differential equation with rational coefficients. In order to apply Kovacic's algorithm we need to know whether a given second order linear differential equation is algebrizable. We recall a change of variable $x \mapsto z := z(t)$ is

Hamiltonian if and only if $(z, z') = (z(t), \frac{d}{dt}z(t))$ is a solution of the autonomous 1-degree-of-freedom standard Hamiltonian system governed the following Hamiltonian $\mathcal{H}$;

$$\mathcal{H}(z, w) = \frac{w^2}{2} + \mathcal{V}(z), \qquad \mathcal{V} \in \mathbb{C}(z)$$

Proposition 3.4. *The differential equation*

$$\frac{d^2}{dt^2}\xi = r(t)\xi, \quad \xi \in \mathbb{C} \tag{15}$$

($r(t)$ *is not necessarily rational function*) *is algebrizable through a Hamiltonian change of variable* $t \mapsto z := z(t)$ *if and only if, there exist* f, α *such that*

$$\frac{\alpha'}{\alpha}, \quad \frac{f}{\alpha} \in \mathbb{C}(z), \qquad \text{where} \quad f(z(t)) = r(t), \quad \alpha(z) = 2(h - \mathcal{V}(z)) = (\dot{z})^2$$

Furthermore the algebraic form of the equation (15) *is*

$$\frac{d^2}{dx^2}\xi + \frac{1}{2}\frac{\alpha'}{\alpha}\frac{d}{dx}\xi - \frac{f}{\alpha}\xi = 0 \tag{16}$$

where $' = d/dx$.

From the above it was seen that when $r(t)$ belongs to the field of meromorphic functions over a Riemann surface, then the identity component of the Galois group is preserved by the above change of variables $x \mapsto z := z(t)$ (see [2] for the proof and more details)

4 Proof of the main theorem

Assume $u(r) = r^{-2}$. We consider the reduced Hamiltonian $\check{H}_3$ and take a particular solution $(r, s, p_r, p_s) = (1, s(\tau), 0, -s'(\tau)/s(\tau)^2)$ on the invariant surface Σ.

The normal variational equation along the particular solution is

$$\frac{d^2}{d\tau^2}\delta r + \left(\sum_{i<j}\frac{m_i^2 m_j^2}{m_i + m_j}\frac{(z(\tau)^2 - 4\beta_{ij}z(\tau) + \beta_{ij}^2)\beta_{ij}z(\tau)}{(z(\tau) - \beta_{ij})^4}\right)\delta r = 0.$$

Using the Hamiltonian change of independent variable $\tau \mapsto z := z(\tau)$ as the Proposition 3.4, the normal variational equation can be algebrizable i.e. written as the following second order equation which coefficients are rational function in z:

$$\frac{d^2}{dz^2}\delta r + 2p(z)\frac{d}{dz}\delta r + q(z)\delta r = 0 \tag{17}$$

where

$$
\begin{aligned}
p(z) &= \frac{1}{4}\frac{A'(z)}{A(z)}\,, \quad q(z) = \frac{G(z)}{A(z)}\,, \\
A(z) &= -\sum_{i<j} \frac{m_i^2 m_j^2}{m_i + m_j} \frac{2z^3}{(z-\beta_{ij})^2} \\
G(z) &= \sum_{i<j} \frac{m_i^2 m_j^2}{m_i + m_j} \frac{(z^2 - 4\beta_{ij} z + \beta_{ij}^2)\beta_{ij} z}{(z-\beta_{ij})^4}.
\end{aligned}
$$

We can rewrite $A(z)$ and $F(z)$ as the following form by reducing to a common denominator:

$$
A(z) = \frac{z^3 B_m(z)}{(z-\beta_{12})^2(z-\beta_{23})^2(z-\beta_{13})^2}\ , \quad G(z) = \frac{z\, C_m(z)}{(z-\beta_{12})^4(z-\beta_{23})^4(z-\beta_{13})^4}.
$$

where $B_m(z), C_m(z)$ is a polynomial whose coefficients are rational function of m_j of 4 and 10 degree, respectively. Let $\gamma_j\,(j = 1,2,3,4)$ be 4 roots of polynomial $B_m(z)$:

$$
B_m(z) = b\prod_{j=1}^{4}(z-\gamma_j) = b(z-\gamma_1)(z-\gamma_2)(z-\gamma_3)(z-\gamma_4)
$$

where $b \in \mathbb{C}^{\times}$. We note that each γ_j is different from 0 and β_{ij} since 0 and β_{ij} are different. But γ_i and γ_j $(i \neq j)$ may take the same value i.e. $B_m(z)$ can have multiple roots. For these computation, we have

$$
\begin{aligned}
p(z) &= \frac{3}{4\,z} - \sum_{i<j} \frac{1}{2\,(z-\beta_{ij})} + \sum_{j=1}^{4} \frac{1}{4\,(z-\gamma_j)} \\
q(z) &= \frac{C_m(z)}{b\,z^2 \prod_{i<j}(z-\beta_{ij})^2 \prod_{j=1}^{4}(z-\gamma_j)}.
\end{aligned}
$$

It is remarkable that the local exponent differences do not depend on the masses in contrary to the case of homothetic solutions

In addition, applying change of the dependent variable $\delta r \mapsto \zeta := \delta r\, e^{\int p}$, we can rewrite (17) as

$$
\frac{d^2}{dz^2}\delta r = \rho(z)\delta r \tag{18}
$$

where $\rho(z) = -q(z) + p'(z) + p(z)^2$.

Hereafter we assume that discriminant $\mathrm{Dis}(B_m) \neq 0$ i.e. $\gamma_i \neq \gamma_j\,(i \neq j)$. We find that ρ has poles at $0, \beta_{ij}$ and γ_j. We check out the order of poles and calculate the coefficients of the partial fraction expansion (or Laurent series expansion) of ρ for the purpose of applying the Kovacic's algorithm. We can obtain the coefficients of the partial fraction expansion of q and write down main terms:

$$q(z) = -\frac{1}{2\,z^2} + \sum_{i<j} \frac{1}{(z-\beta_{ij})^2} + (h.o.t.)$$

in this context $(h.o.t.)$ means that the terms has finite poles whose order are at the most one. Additionally we have

$$\begin{aligned}\rho(z) = &- \left(-\frac{1}{2\,z^2} + \sum_{i<j} \frac{1}{(z-\beta_{ij})^2} + \cdots \right) \\ &+ \left(-\frac{3}{4\,z^2} + \sum_{i<j} \frac{1}{2\,(z-\beta_{ij})^2} - \sum_{j=1}^{4} \frac{1}{4\,(z-\gamma_j)^2} \right) \\ &+ \left(\frac{9}{16\,z^2} + \sum_{i<j} \frac{1}{4\,(z-\beta_{ij})^2} + \sum_{j=1}^{4} \frac{1}{16\,(z-\gamma_j)^2} + \cdots \right) \\ = &\frac{5}{16\,z^2} - \frac{1}{4} \sum_{i<j} \frac{1}{(z-\beta_{ij})^2} - \frac{3}{16} \sum_{j=1}^{4} \frac{1}{(z-\gamma_j)^2} + (h.o.t.).\end{aligned}$$

Hence the order of all of (finite) poles are two. On the other hand, we obtain the Laurent series expansion of ρ at $z = \infty$ and write down main terms:

$$\rho(z) = \frac{5}{16\,z^2} + \mathcal{O}\left(\frac{1}{z^3}\right).$$

Now we start the analysis of the equation (18) applying Kovacic's algorithm. We reconfirm that the set of (finite) poles $\Pi' = \{0, \beta_{ij}, \gamma_j\}$ and $o(0) = o(\beta_{ij}) = o(\gamma_j) = o(\infty) = 2$. For the case (K-i), we obtain

$$\begin{aligned}&\left[\sqrt{r}\right]_0 = \left[\sqrt{r}\right]_{\beta_{ij}} = \left[\sqrt{r}\right]_{\gamma_j} = \left[\sqrt{r}\right]_\infty = 0, \\ &\alpha_0^+ = \alpha_\infty^+ = \frac{5}{4}, \quad \alpha_0^- = \alpha_\infty^- = -\frac{1}{4}, \quad \alpha_1^\pm = \alpha_{\beta_{ij}}^\pm = \frac{1}{2}, \quad \alpha_{\gamma_j}^+ = \frac{3}{4}, \quad \alpha_{\gamma_j}^- = \frac{1}{4}.\end{aligned}$$

by the Step 1. Then we obtain the following evaluation

$$\alpha_\infty^{\varepsilon(\infty)} - \sum_{c\in\Pi'} \alpha_c^{\varepsilon(c)} < \frac{5}{4} - \left(-\frac{1}{4} + 3 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} \right) < 0$$

where $\varepsilon(p) \in \{+,-\}$ for $p \in \Pi' \cup \{\infty\}$. Hence the set D defined in Step of the algorithm is empty. Thus we consider the case (K-ii). By Step, we obtain

$$E_0 = E_\infty = \{-1, 2, 5\}, \quad E_{\beta_{ij}} = \{2\}, \quad E_{\gamma_j} = \{1, 2, 3\}.$$

Then we obtain the following evaluation

$$\frac{1}{2}\left(e_\infty - \sum_{c\in\Pi'} e_c \right) < \frac{1}{2}\left(5 - (-1 + 3 \cdot 2 + 4 \cdot 1)\right) < 0$$

where $e_p \in E_p$ for $p \in \Pi' \cup \{\infty\}$. Hence the set D defined in Step of the algorithm is empty. Thus the case (K-ii) does not hold. Finally we look into the case (K-iii). By Step 1, we obtain

$$E_0 = E_\infty = \left\{ 6 + \frac{18k}{n} \,\middle|\, k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2} \right\} \cap \mathbb{Z}, \quad E_{\beta_{ij}} = \{6\},$$
$$E_{\gamma_j} = \left\{ 6 + \frac{6k}{n} \,\middle|\, k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2} \right\} \cap \mathbb{Z}.$$

where $n = 4, 6, 12$. In this case we can obtain the following evaluation for any n:

$$\frac{n}{12}\left(e_\infty - \sum_{c \in \Pi'} e_c \right) < \frac{n}{12}\left(15 - (-3 + 6 \cdot 3 + 3 \cdot 4)\right) < 0,$$

where $e_c \in E_c$. Hence the set D defined in Step is empty. This implies that the case (K-iii) also does not hold.

Consequently, the differential Galois group of the equation (18) is the connected and unsolvable group $\mathrm{SL}(2, \mathbb{C})$. Therefore the differential Galois group of the equation (18) is not commutative. This completes the proof.

5 Discussion and comment

5.1 the condition for discriminant of B_m in the theorem

In the proof of theorem 4.1, we impose the condition $\mathrm{Dis}(B_m) \neq 0$. This condition are fulfilled for almost all mass ratio. Without this condition, it would be difficult to calculate the coefficients of the partial fraction expansion generally because B_m may have multiple roots. Such exceptional cases also seems to be difficult to apply Kovacic's algorithm.

One can expect non-integrability of the reduced system without this condition. In fact, we can apply the same method as the previous proof and conclude non-integrability when some cases satisfying $\mathrm{Dis}(B_m) = 0$. For example, we consider the case of $m_1 : m_2 : m_3 = 1 : 1 : m$ where $m \neq 0, -1, -2$. Then the roots of $\mathrm{Dis}(B_m)$ are $1, -1/8$. By the following calculation for each values of $m = 1, -1/8$, we conclude that the differential Galois group of the equation (18) is the connected and unsolvable in these cases. Hence we can show that the system is also non-integrable in the case of $\mathrm{Dis}(B_m) = 0$ under $m_1 : m_2 : m_3 = 1 : 1 : m$. We will provide the computation in this case in Appendix C.

5.2 Relations between the class of first integrals and transformation

We consider the class of functions for first integrals. In the previous section, we apply the Molares-Ramis theory to the transformed system and prove the non-integrability of the system. However, in this context we only prove the non-integrability in the sense of non-existence of meromorphic (or rational) fist integrals in the transformed coordinates (Z, r, s, p_Z, p_r, p_s). Hence it is not trivial to translate this result into meromorphically non-integrability in the original coordinates $(\boldsymbol{q}, \boldsymbol{p})$.

We used three transformation $\pi_{\mathrm{tr}}, \pi_{\mathrm{rot}}$, and $\hat{\pi}_{\mathrm{sp}}$. Let $F(\boldsymbol{q}, \boldsymbol{p})$ be a first integral which is meromorphic in $(\boldsymbol{q}, \boldsymbol{p})$ and Poisson commutative with other first integrals K_{m1}, K_{m2}, and K_{am}. From homogeneity of the Hamiltonian, we suppose that $F(\boldsymbol{q}, \boldsymbol{p})$ satisfies

$$F(\lambda^{-1}\boldsymbol{q}, \lambda\boldsymbol{p}) = \lambda^{2\sigma} F(\boldsymbol{q}, \boldsymbol{p})$$

for all $\lambda \in \mathbb{C}^\times$ where $\sigma \in \mathbb{Z}$ without loss of generality. The transformation π_{tr} is linear map from $(\boldsymbol{q}, \boldsymbol{p})$ to $(\boldsymbol{Q}, \boldsymbol{P})$, thus $F(\pi_{\mathrm{tr}}(\boldsymbol{q}, \boldsymbol{p}))(= F(M^{-1}\boldsymbol{Q}, {}^tM\boldsymbol{P}))$ is also meromorphic in $(\boldsymbol{Q}, \boldsymbol{P})$.

We rewrite $F(\boldsymbol{Q}, \boldsymbol{P}) = F(\pi_{\mathrm{tr}}(\boldsymbol{q}, \boldsymbol{p}))$. The function $F(\boldsymbol{Q}, \boldsymbol{P})$ satisfies $F(\lambda^{-1}\boldsymbol{Q}, \lambda\boldsymbol{P}) = \lambda^{2\sigma} F(\boldsymbol{Q}, \boldsymbol{P})$ and is Poisson commutative with K_{am}. Hence $F(\boldsymbol{Q}, \boldsymbol{P})$ is invariant with Hamiltonian flows generated by K_{am}. This means that $F(\boldsymbol{Q}, \boldsymbol{P})$ is also invariant under arbitrary rotational action defined by (4). Since the rotating angle is $\theta = -\arctan(Q_2/Q_1)$, we rewrite it into the following form :

$$\begin{aligned}
& F\left(Q_1, Q_2, Q_3, Q_4, P_1, P_2, P_3, P_4\right) \\
&= F\Bigg(\sqrt{Q_1^2 + Q_2^2}, 0, \frac{Q_1}{\sqrt{Q_1^2 + Q_2^2}} Q_3 + \frac{Q_2}{\sqrt{Q_1^2 + Q_2^2}} Q_4, \\
&\qquad -\frac{Q_2}{\sqrt{Q_1^2 + Q_2^2}} Q_3 + \frac{Q_1}{\sqrt{Q_1^2 + Q_2^2}} Q_4, \\
&\qquad \frac{Q_1}{\sqrt{Q_1^2 + Q_2^2}} P_1 + \frac{Q_2}{\sqrt{Q_1^2 + Q_2^2}} P_2, -\frac{Q_2}{\sqrt{Q_1^2 + Q_2^2}} P_1 + \frac{Q_1}{\sqrt{Q_1^2 + Q_2^2}} P_2, \\
&\qquad \frac{Q_1}{\sqrt{Q_1^2 + Q_2^2}} P_3 + \frac{Q_2}{\sqrt{Q_1^2 + Q_2^2}} P_4, -\frac{Q_2}{\sqrt{Q_1^2 + Q_2^2}} P_3 + \frac{Q_1}{\sqrt{Q_1^2 + Q_2^2}} P_4\Bigg) \\
&= \left(\frac{1}{Q_1^2 + Q_2^2}\right)^{\sigma} F\Bigg(1,\, 0,\, \frac{1}{Q_1^2 + Q_2^2}(Q_1Q_3 + Q_2Q_4),\, \frac{1}{Q_1^2 + Q_2^2}(-Q_2Q_3 + Q_1Q_4), \\
&\qquad Q_1P_1 + Q_2P_2,\, -Q_2P_1 + Q_1P_2,\, Q_1P_3 + Q_2P_4,\, -Q_2P_3 + Q_1P_4\Bigg) \\
&= \left(\frac{1}{\|\boldsymbol{x}\| + x_1}\right)^{\sigma} F\Bigg(1,\, 0,\, \frac{x_2}{\|\boldsymbol{x}\| + x_1},\, \frac{x_3}{\|\boldsymbol{x}\| + x_1}, \\
&\qquad (\|\boldsymbol{x}\| + x_1)y_1 + x_2y_2 + x_3y_3, x_3y_2 - x_2y_3 + (\|\boldsymbol{x}\| + x_1)c, \\
&\qquad -x_2y_1 + (\|\boldsymbol{x}\| + x_1)y_2 - x_3c,\, -x_3y_1 + (\|\boldsymbol{x}\| + x_1)y_3 + x_2c\Bigg)
\end{aligned}$$

Therefore a meromorphic first integral $F(\boldsymbol{Q}, \boldsymbol{P})$ can be represented as a meromorphic function for x_j, $y_j (j = 1, 2, 3)$ and $\|\boldsymbol{x}\|$ by transformation π_{rot}.

The transformation $\hat{\pi}_{\mathrm{sp}} : \mathbb{C}^3 \setminus \{\boldsymbol{0}\} \times \mathbb{C}^3 \ni (\boldsymbol{x}, \boldsymbol{y}) \longmapsto (Z, r, s, p_Z, p_r, p_s) \in (\mathbb{C}^\times \times \mathbb{C} \times \mathbb{C}) \times \mathbb{C}^3$ is rational and $\hat{\pi}_{\mathrm{sp}}(\|\boldsymbol{x}\|) = Z$, so a meromorphic function in $(\boldsymbol{x}, \|\boldsymbol{x}\|, \boldsymbol{y})$ is mapped to a meromorphic function in (Z, r, s, p_Z, p_r, p_s)

by the transformation $\hat{\pi}_{\mathrm{sp}}$. To summarize the above, the composition of three transformation $\pi_{\mathrm{tr}}, \pi_{\mathrm{rot}}$, and $\hat{\pi}_{\mathrm{sp}}$ maps a meromorphic first integral $F(\boldsymbol{q}, \boldsymbol{p})$ to a meromorphic function in (Z, r, s, p_Z, p_r, p_s) since the composition gives a rational map.

A Kovacic's algorithm

Notation. For the differential equation given by

$$\partial_x^2 \zeta = r\zeta, \quad r = \frac{s}{t}, \quad s, t \in \mathbb{C}[x].$$

we use the following notations.

(1) Denote by Π' be the set of (finite) poles of r, i.e., $\Pi' = \{c \in \mathbb{C} \,|\, t(c) = 0\}$.

(2) Denote by $\Pi = \Pi' \cup \{\infty\}$.

(3) By the order of r at $c \in \Pi'$, $o(r_c)$ we mean the multiplicity of c as a pole of r.

(4) By the order of r at ∞, $o(r_\infty)$, we mean the order of ∞ as a zero of r. That is $o(r_\infty) = \deg(t) - \deg(s)$.

The four cases.

Case (K-i). In this case, $[\sqrt{r}]_c$ and $[\sqrt{r}]_\infty$ stand for the Laurent series of $\sqrt{r}$ at c and at ∞ respectively. Furthermore, we define $\varepsilon(p)$ as follows: if $p \in \Pi$, then $\varepsilon(p) \in \{+, -\}$. Finally, the complex numbers $\alpha_c^+, \alpha_c^-, \alpha_\infty^+, \alpha_\infty^-$ will be defined in the first step. If the differential equation has no poles it only can fall in the Laurent series of this case.

Step 1. Search for each $c \in \Pi'$ and for ∞ the corresponding situation as follows:

(c_0) If $o(r_c) = 0$, then

$$\left[\sqrt{r}\right]_c = 0, \quad \alpha_c^\pm = 0.$$

(c_1) If $o(r_c) = 1$, then

$$\left[\sqrt{r}\right]_c = 0, \quad \alpha_c^\pm = 1.$$

(c_2) If $o(r_c) = 2$, and $r = \cdots + b(x-c)^{-2} + \cdots$, then

$$\left[\sqrt{r}\right]_c = 0, \quad \alpha_c^\pm = \frac{1 \pm \sqrt{1+4b}}{2}.$$

(c_3) If $o(r_c) = 2\nu \geq 4$, and $r = \left(a(x-c)^{-\nu} + \cdots + d(x-c)^{-2}\right)^2 + b(x-c)^{-(\nu+1)} + \cdots$, then

$$\left[\sqrt{r}\right]_c = a(x-c)^{-\nu} + \cdots + d(x-c)^{-2}, \quad \alpha_c^{\pm} = \frac{1}{2}\left(\pm\frac{b}{a} + \nu\right).$$

(∞_1) If $o(r_\infty) > 2$, then

$$\left[\sqrt{r}\right]_\infty = 0, \quad \alpha_\infty^{+} = 1, \quad \alpha_\infty^{-}.$$

(∞_2) If $o(r_\infty) = 2$, and $r = \cdots + bx^2 + \cdots$ then

$$\left[\sqrt{r}\right]_\infty = 0, \quad \alpha_\infty^{\pm} = \frac{1 \pm \sqrt{1+4b}}{2}.$$

(∞_3) If $o(r_\infty) = -2\nu \leq 0$, and $r = (ax^\nu + \cdots + d)^2 + bx^{\nu-1} + \cdots$, then

$$\left[\sqrt{r}\right]_\infty = ax^\nu + \cdots + d, \quad \alpha_\infty^{\pm} = \frac{1}{2}\left(\pm\frac{b}{a} - \nu\right).$$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ m \in \mathbb{Z}_+ \;\middle|\; m = \alpha_\infty^{\varepsilon(\infty)} - \sum_{c\in\Pi'} \alpha_c^{\varepsilon(c)},\ \forall\,(\varepsilon(p))_{p\in\Pi} \right\}.$$

If $D = \emptyset$, then we should start with Case 2. Now, if $\mathrm{Card}(D) > 0$, then for each $n \in D$ we search $\omega \in \mathbb{C}(x)$ defined by

$$\omega = \epsilon(\infty)\left[\sqrt{r}\right]_\infty + \sum_{c\in\Pi'}\left(\varepsilon(c)\left[\sqrt{r}\right]_c + \alpha_c^{\varepsilon(c)}(x-c)^{-1}\right).$$

Step 3. For each $m \in D$, search for a monic polynomial P_m of degree m, such that

$$\partial_x^2 P_m + 2\omega\partial_x P_m + (\partial_x\omega + \omega^2 - r)P_m = 0.$$

If such a polynomial exists, then $\zeta_1 = P_m e^{\int \omega}$ is a solution of the differential equation. Otherwise Case 1 cannot hold.

.

Case (K-ii). Search for each $c \in \Pi'$ and for ∞ the corresponding situation as follows:

Step 1. Search for each $c \in \Pi'$ and ∞ the sets $E_c \neq \emptyset$ and $E_\infty \neq \emptyset$. For each $c \in \Pi'$ and for ∞ we define $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows:

(c_1) If $o(r_c) = 1$, then $E_c = \{12\}$.

(c_2) If $o(r_c) = 2$, and $r = \cdots + b(x-c)^{-2} + \cdots$, then

$$E_c = \left\{ 2 + k\sqrt{1+4b} \;\middle|\; k = 0, \pm 2 \right\} \cap \mathbb{Z}.$$

(c_3) If $o(r_c) = \nu > 2$, then $E_c = \{\nu\}$.
(∞_1) If $o(r_\infty) > 2$, then $E_\infty = \{0, 2, 4\}$.
(∞_2) If $o(r_\infty) = 2$, and $r = \cdots + bx^2 + \cdots$ then

$$E_\infty = \left\{ 2 + k\sqrt{1+4b} \;\middle|\; k = 0, \pm 2 \right\} \cap \mathbb{Z}.$$

(∞_3) If $o(r_\infty) = \nu < 2$, then $E_\infty = \{\nu\}$.

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ m \in \mathbb{Z}_+ \;\middle|\; m = \frac{1}{2}\left(e_\infty - \sum_{c \in \Pi'} e_c \right),\ \forall\, e_p \in E_p,\ p \in \Pi \right\}.$$

If $D = \emptyset$, then we should start with Case 2. Now, if $\mathrm{Card}(D) > 0$, then for each $m \in D$ we search $\theta \in \mathbb{C}(x)$ defined by

$$\theta = \frac{1}{2} \sum_{c \in \Pi'} \frac{e_c}{x - c}.$$

Step 3. For each $m \in D$, search for a monic polynomial P_m of degree m such that

$$\begin{aligned} &\partial_x^3 P_n + 3\theta \partial_x^2 P_n + (3\partial_x\theta + 3\theta^2 - 4r)\partial_x P_n \\ &\quad + \left(\partial_x^2\theta + 3\theta\partial_x\theta + \theta^3 - 4r\theta - 2\partial_x r\right) P_n = 0 \end{aligned}$$

If P_m does not exist, then Case 2 cannot hold. If such a polynomial is found, we can define θ by $\phi = \theta + \partial_x P_m / P_m$ and let ω be a solution of

$$\omega^2 + \phi\omega + \frac{1}{2}\left(\partial_x\phi + \phi^2 - 2r\right)$$

then $\zeta_1 = P_n e^{\int \omega}$ is a solution of the differential equation.

.

Case (K-iii). Search for each $c \in \Pi'$ and for ∞ the corresponding situation as follows:

Step 1. Search for each $c \in \Pi'$ and ∞ the sets $E_c \neq \emptyset$ and $E_\infty \neq \emptyset$. For each $c \in \Pi'$ and for ∞ we define $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows:

(c_1) If $o(r_c) = 1$, then $E_c = \{12\}$.

(c_2) If $o(r_c) = 2$, and $r = \cdots + b(x-c)^{-2} + \cdots$, then

$$E_c = \left\{ 6 + \frac{12k}{n}\sqrt{1+4b} \;\middle|\; k = 0, \pm 1, \ldots, \pm\frac{n}{2},\ n = 4, 6, 12 \right\} \cap \mathbb{Z}.$$

(∞) If $o(r_\infty) = 2$, and $r = \cdots + bx^2 + \cdots$ then

$$E_\infty = \left\{ 6 + \frac{12k}{m}\sqrt{1+4b} \;\middle|\; k = 0, \pm 1, \ldots, \pm\frac{n}{2},\ n = 4, 6, 12 \right\} \cap \mathbb{Z}.$$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ m \in \mathbb{Z}_+ \;\middle|\; m = \frac{n}{12}\left(e_\infty - \sum_{c\in\Pi'} e_c \right),\ \forall\, e_p \in E_p,\ p \in \Pi \right\}.$$

In this case we astt with $n = 4$ to obtain the solution, afterwards $n = 6$ and finally $n = 12$. If $D = \emptyset$, then the differential equation in not integrable because it falls in Case 4. Now, if $\mathrm{Card}(D) > 0$, then for each $m \in D$ with its respective n, we search $\theta \in \mathbb{C}(x)$ and $S \in \mathbb{C}[x]$ defined as

$$\theta = \frac{n}{12}\sum_{c\in\Pi'} \frac{e_c}{x-c}, \qquad S = \prod_{c\in\Pi'} (x-c).$$

Step 3. Search for each $m \in D$, with its respective n, a monic polynomial $P_m = P$ of degree m, such that its coefficients can be determined recursively by

$$\begin{aligned} &P_{-1} = 0, \qquad P_n = -P, \\ &P_{i-1} = -S\partial_x P_i - \left((n-i)\partial_x S - S\theta\right) P_i \\ &\qquad\qquad - (n-i)(i+1)S^2 r P_{i+1}. \quad (i = 0, 1, \ldots n) \end{aligned}$$

If P does not exist (P_{-1} is not identically zero), then the differential equation is not integrable because it falls in Case 4. Now, if ω exists such that

$$\sum_{i=0}^{m} \frac{S^i P}{(n-1)!}\omega^i = 0$$

then $\zeta_1 = P_n e^{\int \omega}$ is a solution of the differential equation where ω is solution of the previous polynomial of degree n.

B Computing in the case of $\mathrm{Dis}(B_m) = 0$ under $m_1 : m_2 : m_3 = 1 : 1 : m$

We compute some exceptional examples $\mathrm{Dis}(B_m) = 0$. We suppose $m_1 = m_2 = 1$ and $m_3 = m$ ($m \neq 0, -1, -2$). Then we obtain

$$\mathrm{Dis}(B_m) = 2147483648\, m^8\, (m+1)^{12}\, (m+2)^6\, (m-1)^2 \left(m + \frac{1}{8} \right).$$

The case of $m=1$. Substituting $m=1$, we obtain

$$\rho(z) = -\frac{25z^6+58z^3+25}{48z^2(z^3-1)}.$$

By changing of independent variable $z \mapsto \tau := z^3$, the equation (18) can be written as a hypergeometric equation:

$$\frac{d^2}{d\tau^2}\delta r + \frac{2}{3\tau}\frac{d}{d\tau}\delta r + \frac{25\tau^2+58\tau+25}{27\tau^2(\tau-1)^2}\delta r \tag{19}$$

which has regular singular points at $0, 1,$and ∞. The difference of characteristic exponents at each singular points in the equation:

$$e_0^+ - e_0^- = \pm\mathrm{i}\frac{\sqrt{97}}{9}\ , \quad e_1^+ - e_1^- = \pm\mathrm{i}\sqrt{15}\ , \quad e_\infty^+ - e_\infty^- = \pm\mathrm{i}\frac{4\sqrt{66}}{9}$$

does not belong to Kimura's list [9], and hence the identity component of differential Galois group of (19) is not commutative.

B.1 The case of $m=-1/8$

Substituting $m=-1/8$, we obtain

$$\begin{aligned}\rho(z) = & \frac{1}{48\,(z-1)^2\,(z+1)^2\,z^2\,(z^2+3\,z+1)^2\,\left(7z+8+\sqrt{15}\right)^2\,\left(7z+8-\sqrt{15}\right)^2} \\ & \times\big(36015\,z^{12}+345744\,z^{11}+1134154\,z^{10}+832608\,z^9 \\ & \quad -4412303\,z^8-14402864\,z^7-19986708\,z^6-14402864\,z^5 \\ & \quad -4412303\,z^4+832608\,z^3+1134154\,z^2+345744\,z+36015\big).\end{aligned}$$

Now we start the analysis of the equation (18) applying Kovacic's algorithm. We can see that the set of singular points of $\rho(z)$, $\Pi' = \{0, \pm 1, \beta^\pm = \frac{-8\pm\sqrt{15}}{7}, \gamma^\pm = \frac{-3\pm\sqrt{5}}{2}\}$ and $o(r_0) = o(r_{\pm 1}) = o(r_{\beta^\pm}) = o(r_{\gamma^\pm}) = o(r_\infty) = 2$. We note that the coefficient of $1/(z+1)^2$ in the partial fraction expansion for ρ is $\kappa_{-1} = -7/12$, so $\sqrt{1+4\kappa_{-1}} = -\frac{2\mathrm{i}}{\sqrt{3}} \notin \mathbb{Q}$. Therefore the condition for Case 3 of the algorithm does not hold. We check only Case 1 and 2.

By Case 1 and Step 1, the conditions (c_2) fails for arbitrary $c \in \Pi'$, and (∞_2) for ∞. In this way we obtain

$$\begin{aligned}&\left[\sqrt{r}\right]_0 = \left[\sqrt{r}\right]_{\pm 1} = \left[\sqrt{r}\right]_{\beta^\pm} = \left[\sqrt{r}\right]_{\gamma^\pm} = \left[\sqrt{r}\right]_\infty = 0,\\ &\alpha_0^+ = \alpha_\infty^+ = \frac{5}{4}, \quad \alpha_0^- = \alpha_\infty^- = -\frac{1}{4}, \quad \alpha_1^\pm = \alpha_{\beta^\pm}^\pm = \frac{1}{2},\\ &\alpha_{-1}^\pm = \frac{1}{2} \pm \frac{\sqrt{3}\,\mathrm{i}}{3}, \quad \alpha_{\gamma^\pm}^+ = \frac{3}{4}, \quad \alpha_{\gamma^\pm}^- = \frac{1}{4}.\end{aligned}$$

By Step 2 we do not obtain d as a non-negative integer, and thus Case 1 does not hold.

We follow Case 2 where the conditions (c_2) fails for arbitrary $c \in \Pi'$, and (∞_2) for ∞. By Step 1, we obtain

$$E_0 = E_\infty = \{-1, 2, 5\}, \quad E_1 = E_{-1} = E_{\beta^\pm} = \{2\}, \quad E_{\gamma^\pm} = \{1, 2, 3\}.$$

By Step 2 we do not obtain d as a non-negative integer, thus Case 2 also does not hold.

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