TITLE:
Variational existence proof for multiple periodic orbits in the planar circular restricted three-body problem

AUTHOR(S):
Kajihara, Yuika; Shibayama, Mitsuru

CITATION:

ISSUE DATE:
2022-03

URL:
http://hdl.handle.net/2433/269308

RIGHT:
This is the Accepted Manuscript version of an article accepted for publication in [Nonlinearity]. IOP Publishing Ltd is not responsible for any errors or omissions in this version of the manuscript or any version derived from it. The Version of Record is available online at https://doi.org/10.1088/1361-6544/ac4c2b. The full-text file will be made open to the public on 4 February 2023 in accordance with publisher's 'Terms and Conditions for Self-Archiving'. This is not the published version. Please cite only the published version. この論文は出版社版ではありません。引用の際には出版社版をご確認ください。
Variational existence proof for multiple periodic orbits in the planar circular restricted three-body problem

Yuika Kajihara¹ and Mitsuru Shibayama¹
¹Graduate School of Informatics, Kyoto University, Yoshida-Honmachi, Sakyō-ku, Kyoto 606-8501, Japan

Abstract

The restricted three-body problem is an important research area that deals with significant issues in celestial mechanics, such as analyzing asteroid movement behavior and orbit design for space probes. We aim to show the existence of periodic orbits in the planar circular restricted three-body problem. To find these orbits, we adapt a variational approach and symmetry.

1 Introduction

The restricted three-body problem has long been studied. It is a special case of the three-body problem and is known to be non-integrable. It deals with significant issues in celestial mechanics, such as analyzing asteroid movement behavior and orbit design for space probes (see [15] for more details). This paper aims to show the existence of multiple periodic orbits in the planar circular restricted three-body problem (R3BP).

Chenciner and Montgomery successfully applied a variational method to the three-body problem. They showed the existence of a remarkable periodic orbit called the figure-eight orbit (see [6]), which has led to a lot of works on the n-body problem. As a recent result in this field, we refer the reader to [17]. Compared with the n-body problem, there are few results on the restricted three-body problem using the variational methods because the technical parts of the level estimates for the restricted three-body problem are more difficult. In [11], Moeckel showed the existence of the transit orbit in the R3BP for regions from around the earth to around the moon. The result in [14] yields the existence of orbits realizing symbolic sequences in the Sitnikov problem. Arioli et al. showed the existence of periodic orbits revolving around Jupiter in [1]. Chen proved the existence of the orbits moving away from the center in [4].

The R3BP is defined by

$$\ddot{x} = \nabla V(x) \quad (x \in \mathbb{C}),$$

where

$$V(x, t; \mu) = \frac{1 - \mu}{|x + \mu e^{it}|} + \frac{\mu}{|x - (1 - \mu)e^{it}|}$$

and \(\mu \in (0, 1)\) is a parameter. Here \(\mathbb{C}\) is regarded as \(\mathbb{R}^2\). In the rotating coordinate system, the equations are represented by

$$\ddot{x} = x + 2\dot{y} + \frac{\partial U}{\partial x},$$
$$\ddot{y} = y - 2\dot{x} + \frac{\partial U}{\partial y},$$

(R3BP\(\mu\))
where

$$U(x; \mu) = \frac{1 - \mu}{\sqrt{(x + \mu)^2 + y^2}} + \frac{\mu}{\sqrt{(x - (1 - \mu))^2 + y^2}}$$

Here $x = x + iy$ and $i = \sqrt{-1}$.

We aim to show the existence of periodic orbits under several boundary conditions in the R3BP. Our proof will use an elementary minimization argument and a level estimate of the action functional in the R3BP. The steps of our proof give a new method for level estimate in the rotating coordinate system.

The result of this paper is organized as follows. Section 2 states our main theorems. Section 3 contains preliminaries for our proof including basic facts on variational methods. Section 4 provides the proofs of the main theorems. In Section 5, we discuss how the obtained periodic orbits behave and state open problems.

## 2 Main results

We define $X^o$, $X^-$, $X^+$, and $Y$ as follows:

$$X^o := \{(x, 0) \mid -\mu \leq x \leq 1 - \mu\},$$
$$X^- := \{(x, 0) \mid x < -\mu\},$$
$$X^+ := \{(x, 0) \mid 1 - \mu < x\}$$
and

$$Y := \{(0, y) \mid -\infty < y < \infty\}.$$

Set $\mathcal{X} = \{X^o, X^-, X^+\}$.

We state our main theorems. Each set $\mathcal{T}(A, B) \subseteq \mathbb{R}$ in the following theorems is defined in Section 4.

**Theorem 2.1.** For any $A, B \in \mathcal{X}$ and $T \in \mathcal{T}(A, B)$, there is a $2T$-periodic orbit $(x(t), y(t))$ of (R3BP) such that $x(2(k - 1)T) \in A, x((2k - 1)T) \in B$ and $\dot{x}(2(k - 1)T) = y(2(k - 1)T) = y((2k - 1)T) = \dot{x}((2k - 1)T) = 0$ for $k \in \mathbb{Z}$.

This theorem shows the existence of periodic orbits that are orthogonal to $x$-axis for $\mu \in (0, 1)$. In the case of $\mu = 1/2$, we can show the existence of more symmetric periodic orbits that are orthogonal to the $x$-axis and $y$-axis.

**Theorem 2.2.** Set $\mu = 1/2$. For any $A \in \mathcal{X}$ and $T \in \mathcal{T}(A, Y)$, there exists a $4T$-periodic orbit $(x(t), y(t))$ of (R3BP) that satisfies $x(2(k - 1)T) \in A$ and $\dot{x}(2(k - 1)T) = y(2(k - 1)T) = y((2k - 1)T) = \dot{x}((2k - 1)T) = x((2k - 1)T) = 0$ for $k \in \mathbb{Z}$.

![Figure 1: Periodic solutions in Theorem 2.1](image)
Remark 1. Figures 1 and 2 may show the outlines of the given periodic orbits obtained from Theorems 2.1 and 2.2 respectively. As a result, the shape of the obtained orbits are symmetric about the \( x \)-axis. Moreover, the orbits from Theorem 2.2 are symmetric about the \( y \)-axis. Note that ‘may’ indicates we do not know their detailed global behavior, as will be discussed in Section 5.

Remark 2. The word ‘periodic’ in this paper is used in reference to the rotating coordinate system. Therefore, in the stationary coordinate system, ‘\( T \)-periodic’ orbits are periodic if \( T/2\pi \in \mathbb{Q} \) and quasi-periodic if \( T/2\pi \notin \mathbb{Q} \).

3 Preliminaries

3.1 Lagrangians in the stationary and rotating coordinates

We use two different Lagrangian functions in this paper. One is the original Lagrangian which is time periodic:

\[
L(x, \dot{x}, t; \mu) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + V(x, t; \mu),
\]

where \( V \) is introduced in (2) in Section 1. The other Lagrangian is in rotating coordinates and is time-independent:

\[
L_{\text{R3BP}}(x, \dot{x}; \mu) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + x\dot{y} - y\dot{x} + \frac{1}{2}(x^2 + y^2) + U(x; \mu).
\]

We will check that, up to a variable change, the two Lagrangian functions coincide. Indeed, using \( z = x + iy \in \mathbb{C} \), we can write (3) as

\[
L(z, \dot{z}, t; \mu) = \frac{1}{2} \dot{z}^2 + \frac{1 - \mu}{|z + \mu|} + \frac{\mu}{|z - (1 - \mu)|}.
\]

Similarly, using \( w = x + iy \in \mathbb{C} \), we can write (4) as

\[
L_{\text{R3BP}}(w, \dot{w}; \mu) = \frac{1}{2} (w + iw)(\dot{w} - i\dot{w}) + \frac{1 - \mu}{|w - \mu|} + \frac{\mu}{|w - (1 - \mu)|}.
\]
Substituting \( z(t) = w(t)e^{it} \) in the above, we have

\[
\frac{1}{2} \ddot{z} + \frac{1 - \mu}{|z + \mu e^{it}|} + \frac{\mu}{|z - (1 - \mu)e^{it}|} = \frac{1}{2} (\ddot{w}e^{it} + i\dot{w}e^{it})(\ddot{w}e^{it} - i\dot{w}e^{it}) + \frac{1 - \mu}{|we^{it} + \mu e^{it}|} + \frac{\mu}{|we^{it} - (1 - \mu)e^{it}|}
\]

\[
= \frac{1}{2} (\ddot{w} + i\dot{w})(\ddot{w} - i\dot{w}) + \frac{1 - \mu}{|w + \mu|} + \frac{\mu}{|w - (1 - \mu)|}.
\]

Hence, the values of the two Lagrangian functions are the same, so switching between them does not affect our results.

### 3.2 Some well-known facts in variational problems

Let \( \mathcal{D} \) be an open set in \( \mathbb{R}^n \), and \( A, B \subset \mathcal{D} \) be nonempty subsets of affine subspaces in \( \mathbb{R}^n \), for example, a line segment or a half-line. Consider

\[
\Omega(A, B) = \{ x \in H^1([0, T], \mathcal{D}) \mid x(0) \in A, x(T) \in B \}
\]

where the norm is defined by

\[
||x||_{H^1} = \left( \int_0^T |x|^2 dt + \int_0^T |\dot{x}|^2 dt \right)^{1/2}.
\]

The action functional \( A_T(x; \mu) \) is given by

\[
A_T(x; \mu) = \int_0^T L(x, \dot{x}, t; \mu) dt,
\]

where \( T > 0 \) is constant and \( L \) is defined by (3).

We consider a minimizer of \( A_T \), say \( x^* \), i.e., \( x^* \) satisfying

\[
A_T(x^*; \mu) = \inf_{x \in \Omega(A, B)} A_T(x; \mu).
\]

The existence is ensured under some boundary conditions. To show the existence, we can use some useful lemmata. We first define coercivity.

**Definition 3.1 (coercivity).** The action functional \( A_T \) is said to be coercive if it satisfies \( A_T(x; \mu) \rightarrow \infty \) as \( ||x||_{H^1} \rightarrow \infty \).

Hereinafter, \( \Omega(A, B) \) is denoted to \( \Omega \). The following lemma results from Tonelli’s theorem [16].

**Lemma 3.2.** Assume that \( A_T \) is weakly lower semi-continuous. If \( A_T|_{\Omega} \) is coercive, then there exists a minimizer \( x^* \) of \( A_T \) in the weak closure \( \bar{\Omega} \) of \( \Omega \).

It is well-known that action functionals for potential systems are weakly semi-continuous (see for example [7]). We state some sufficient conditions for coercivity in the next three lemmata.

**Lemma 3.3.** Let \( A \) and \( B \) be nonempty sets. If at least one of \( A \) and \( B \) is bounded, then \( A_T|_{\Omega} \) is coercive.

**Lemma 3.4.** Let \( A \) and \( B \) be nonempty sets. Suppose there is a constant \( |C_0| < 1 \) such that for any \( a \in A \) and \( b \in B \), it holds that \( a \cdot b \leq C_0|a||b| \). Then, \( A_T|_{\Omega} \) is coercive.

See [3] for proofs of the above two lemmata.
Lemma 3.5. Let A and B be unbounded sets. Set $A_d := \{a \in A \mid |a| \leq d\}$ and $B_d := \{b \in B \mid |b| \leq d\}$. Suppose there is a constant $M > 0$ such that for any $a \in A \setminus A_d$ and $b \in B \setminus B_d$, it holds that $a \cdot b \leq C_1|a||b|$, where $C_1 \in [-1, 1]$. Then, $A_T^{f, g}$ is coercive.

Proof. Set $\Omega(A, B) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \Omega(A_i, B_j)$ such that $A = \bigcup_{i=1}^{n} A_i$ and $B = \bigcup_{j=1}^{m} B_j$. It is easily seen that if $A_{\Omega(A, B)}$ is coercive for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, then so is $A_{\Omega(A, B)}$. Under the assumption, we divide $\Omega(A, B)$ into the following:

$$\Omega(A, B) = \Omega(A_M, B_M) \cup \Omega(A \setminus A_M, B_M) \cup \Omega(A_M, B \setminus B_M) \cup \Omega(A \setminus A_M, B \setminus B_M).$$

Applying Lemma 3.3 yields that $A_{\Omega(A_M, B_M)}$, $A_{\Omega(A \setminus A_M, B_M)}$, and $A_{\Omega(A_M, B \setminus B_M)}$ are coercive. In the case $C_1 \in (-1, 1)$, Lemma 3.4 gives coercivity of $A_{\Omega(A \setminus A_M, B \setminus B_M)}$. Hence, it suffices to consider the case $C_1 = -1$. Then, $x(T) = -C_2 x(0)$ ($C_2 > 0$). It is clear that $|x(0) - x(T)| = (1 + C_2)|x(0)|$. The rest of the proof is the same as that of Lemma 3.3.

From the calculation in Section 3.1, if $A_T(x; \mu)$ is coercive, then so is the action functional corresponding to the Lagrangian (4) instead of (3).

3.3 Reversibility

Consider the following ordinary differential equations:

$$\dot{q} = F(q) \quad (q \in \mathbb{R}^n).$$

Definition 3.6 (Reversibility). Let $R$ be an involutory linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, i.e., $R^2 = \text{Id}$. If (6) satisfies $FR + RF = 0$, then (6) is said to be reversible with respect to $R$.

It is easy to show the following lemma.

Lemma 3.7. Assume that (6) satisfies reversibility with respect to $R$. Then if $q(t)$ is a solution of (6), so is $Rq(-t)$.

Set $\text{Fix}(R) = \{q \in \mathbb{R}^n \mid Rq = q\}$. Assume that (6) is reversible with respect to $R$ and let $q$ be a solution for (6). Then, $q(s) \in \text{Fix}(R) \iff q(s + t) = Rq(s - t)$.

See [13] for a more detailed explanation about reversible systems.

Moreover (R3BP$_\mu$) can be represented by

$$\frac{d}{dt} \begin{pmatrix} x \\ y \\ v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ x + 2v_y + \frac{\partial U}{\partial y} \\ y - 2v_x + \frac{\partial U}{\partial y} \end{pmatrix} = F \begin{pmatrix} x \\ y \\ v_x \\ v_y \end{pmatrix}$$

and the system is reversible with respect to

$$R := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Indeed

$$FR \begin{pmatrix} x \\ y \\ v_x \\ v_y \end{pmatrix} + RF \begin{pmatrix} x \\ y \\ v_x \\ v_y \end{pmatrix} = F \begin{pmatrix} x \\ y \\ -v_x \\ -v_y \end{pmatrix} + R \begin{pmatrix} v_x \\ v_y \\ x + 2v_y + \frac{\partial U}{\partial y} \\ y - 2v_x + \frac{\partial U}{\partial y} \end{pmatrix} = 0.$$

If $\mu = 1/2$, then (R3BP$_\mu$) is reversible with respect to $-R$. By the above remarks, we obtain the following proposition.
Proposition 3.8. The system of \((\text{R3BP}_\mu)\) has reversibility with respect to \(R\) defined by (7). Moreover, it is also reversible with respect to \(-R\) if \(\mu = 1/2\).

### 3.4 Sundman’s estimate

In the case that some particles collide at \(t = 0\) in the \(n\)-body problem, the asymptotic behavior of each particle \(x_i(t)\) is represented by \(x_i(t) \sim c + a_i t^{2/3} + o(t)\) as \(t \to 0^+\). This estimate is called Sundman’s estimate. From [2], there is a wide class in which Sundman’s estimate holds, including the classical \(n\)-body problems with Newtonian, quasi-homogeneous and logarithmic potentials.

We prove that this asymptoticity holds for \((\text{R3BP}_\mu)\). We analyze the singular points in \((\text{R3BP}_\mu)\) using the Levi-Civita regularization. Consider the following transformation to study the singular point at \((1 - \mu, 0)\):

\[
(x - (1 - \mu), y) \mapsto (\xi_1^2 - \xi_2^2, 2\xi_1\xi_2)
\]

To construct canonical transformation, we set

\[
(p_x, p_y) \mapsto \left(\frac{\xi_1\eta_1 - \xi_2\eta_2}{2(\xi_1^2 + \xi_2^2)}, \frac{\xi_2\eta_1 + \xi_1\eta_2}{2(\xi_1^2 + \xi_2^2)}\right).
\]

The potential part becomes

\[
\tilde{U}(\xi_1, \xi_2; \mu) = \frac{1 - \mu}{\sqrt{\xi_1^2 + \xi_2^2 + 2(\xi_1^2 - \xi_2^2) + 2\mu - 1} + \sqrt{\xi_1^2 + \xi_2^2}}
\]

\[
= \frac{1}{\xi_1^2 + \xi_2^2} \left(\frac{(1 - \mu)(\xi_1^2 + \xi_2^2)}{\sqrt{\xi_1^2 + \xi_2^2 + 2(\xi_1^2 - \xi_2^2) + 2\mu - 1}} + \mu\right).
\]

Thus, we obtain

\[
\tilde{H}(\xi_1, \xi_2, \eta_1, \eta_2; \mu) = \frac{||\eta||^2}{8||\xi||^2} - (1 - \mu)\frac{\xi_1\eta_2 + \xi_2\eta_1}{2||\xi||^2} + \frac{1}{2}(\xi_2\eta_1 - \xi_1\eta_2)
\]

\[
+ \frac{1}{||\xi||^2} \left(\mu + \frac{(1 - \mu)||\xi||^2}{\sqrt{||\xi||^4 + 2(\xi_1^2 - \xi_2^2) + 2\mu - 1}}\right),
\]

where \(\xi = (\xi_1, \xi_2)\) and \(\eta = (\eta_1, \eta_2)\). If \((x(t), y(t))\) is a solution of \((\text{R3BP}_\mu)\), \(\tilde{H}\) is conserved along each solution, say \(\tilde{H} = h\). Set \(\Gamma := ||\xi||^2(\tilde{H} - h)\). The canonical equations with respect to \(\Gamma\) become

\[
d\xi_1\tau = ||\xi||^2\tilde{H}_{\eta_1} = \eta_1 - (1 - \mu)\xi_2 + o(||\xi||^2 + ||\eta||^2)
\]

\[
d\xi_2\tau = ||\xi||^2\tilde{H}_{\eta_2} = \eta_2 - (1 - \mu)\xi_1 + o(||\xi||^2 + ||\eta||^2)
\]

\[
d\eta_1\tau = -||\xi||^2\tilde{H}_{\xi_1} = (1 - \mu)\eta_2 - 2(1 - \mu - h)\xi_1 + o(||\xi||^2 + ||\eta||^2)
\]

\[
d\eta_2\tau = -||\xi||^2\tilde{H}_{\xi_2} = (1 - \mu)\eta_1 - 2(1 - \mu - h)\xi_2 + o(||\xi||^2 + ||\eta||^2).
\]

The solutions \(\xi(\tau)\) of (9) imply the solutions \(\xi(\tau(t))\) of the canonical equation for Hamiltonian \(\tilde{H}\) by changing the time variable according to \(\frac{d\tau}{dt} = \frac{1}{||\xi||^2}\). Note that since the right hand sides of (9) are analytic at \((\xi_1, \xi_2) = (0, 0)\), the solutions are also analytic.

Considering \((\xi_1, \xi_2) = (0, 0)\) and the Taylor expansion at \(\tau = 0\), we obtain

\[
\xi_1(\tau) = \sum_{i=1}^{\infty} a_i\tau^i, \quad \xi_2(\tau) = \sum_{i=1}^{\infty} b_i\tau^i, \quad \eta_1(\tau) = \sum_{i=0}^{\infty} c_i\tau^i, \quad \eta_2(\tau) = \sum_{i=0}^{\infty} d_i\tau^i.
\]
Substituting (10) into (9), we can determine the coefficients. The relation between \( t \) and \( \xi \) yields

\[
\begin{align*}
t &= \int \|\xi\|^2 d\tau \\
&= \int (a_1 \tau + a_2 \tau^2 + \cdots)^2 + (b_1 \tau + b_2 \tau^2 + \cdots)^2 d\tau \\
&= (a^2_1 + b^2_1) \int \tau^2 d\tau + o(\tau^4) \\
&= \frac{1}{3} (a^2_1 + b^2_1) \tau^3 + o(\tau^4).
\end{align*}
\]

This implies that \( x(t) \) and \( y(t) \) are represented by the forms

\[
x(t) = (1 - \mu) + \sum_{i=2}^{\infty} \tilde{a}_i t^{i/3} \quad \text{and} \quad y(t) = \sum_{i=2}^{\infty} \tilde{b}_i t^{i/3}.
\]

### 4 Proofs of the main theorems

#### 4.1 Variational setting

As observed in Section 3.2, we can consider the action functional with Lagrangian (4) as having the same property of the one associated with (3). Define the action functional \( B_T \) by

\[
B_T(x; \mu) = \int_0^T L_{R3BP}(x, \dot{x}; \mu) dt,
\]

where \( x = (x, y) \) and \( L_{R3BP} \) is given by (4). The Euler-Lagrange equation of (4) is equivalent to \( (R3BP)^\mu \). We consider six types of boundary conditions. Denote \( \Omega_i \) by \( \Omega_i = \{x \in H^j([0, T], \mathbb{D}) \mid x(0) \in A_i, x(T) \in B_i \} \) for each \( i = 1, \ldots, 6 \), where

- **Case 1**: \( A_1 = X^o \) and \( B_1 = X^o \);
- **Case 2**: \( A_2 = X^o \) and \( B_2 = X^+ \);
- **Case 3**: \( A_3 = X^+ \) and \( B_3 = X^+ \);
- **Case 4**: \( A_4 = X^- \) and \( B_4 = X^+ \);
- **Case 5**: \( A_5 = X^o \) and \( B_5 = Y \);
- **Case 6**: \( A_6 = X^+ \) and \( B_6 = Y \).

We summarize our variational settings in the table below.

<table>
<thead>
<tr>
<th>Case</th>
<th>Boundary conditions ( A \rightarrow B )</th>
<th>( \mu )</th>
<th>Period</th>
<th>Region of time ( T(A, B) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>( X^o \rightarrow X^o )</td>
<td>( \mu \in (0, 1) )</td>
<td>( 4T )</td>
<td>( T &gt; T_{L_1}(\mu) )</td>
</tr>
<tr>
<td>Case 2</td>
<td>( X^o \rightarrow X^+ )</td>
<td>( \mu \in (0, 1) )</td>
<td>( 4T )</td>
<td>( T &gt; 0 )</td>
</tr>
<tr>
<td>Case 3</td>
<td>( X^+ \rightarrow X^+ )</td>
<td>( \mu \in (0, 1) )</td>
<td>( 4T )</td>
<td>( T &gt; T_{L_2}(\mu) ) and ( T \neq 2n\pi ) (( n \in \mathbb{Z}_+ ))</td>
</tr>
<tr>
<td>Case 4</td>
<td>( X^- \rightarrow X^+ )</td>
<td>( \mu \in (0, 1) )</td>
<td>( 4T )</td>
<td>( T &gt; 0 ) and ( T \neq (2n - 1)\pi ) (( n \in \mathbb{Z}_+ ))</td>
</tr>
<tr>
<td>Case 5</td>
<td>( X^o \rightarrow Y )</td>
<td>( \mu = 1/2 )</td>
<td>( 2T )</td>
<td>( 2T &gt; T_{L_1}(1/2) )</td>
</tr>
<tr>
<td>Case 6</td>
<td>( X^+ \rightarrow Y )</td>
<td>( \mu = 1/2 )</td>
<td>( 2T )</td>
<td>( T &gt; 0 ) and ( T \neq (n - \frac{1}{2})\pi ) (( n \in \mathbb{Z}_+ ))</td>
</tr>
</tbody>
</table>

Table 4.1: Variational settings
Here, \( T_{L_i}, i = 1, 2 \), are defined in section 4.2. Now, we can give the definition of the time interval \( T(A, B) \) in Theorems 2.1 and 2.2. For example, following Table 4.1, we have:

\[
T(X^+, X^+) = \{ T \in \mathbb{R} \mid T > T_{L_2}(\mu) \text{ and } T \neq 2n\pi \ (n \in \mathbb{Z}_+) \}
\]

We do not need to consider the case of \((A, B) = (X^-, X^-)\) by replacing \( \mu \) into \( 1 - \mu \), i.e.,

\[
T(X^-, X^-) = \{ T \in \mathbb{R} \mid T > T_{L_2}(1 - \mu) \text{ and } T \neq 2n\pi \ (n \in \mathbb{Z}_+) \}.
\]

In a similar way, the cases of \((A, B) = (X^o, X^+)\) and \((X^+, Y)\) lead to the ones of \((A, B) = (X^o, X^-)\) and \((X^-, Y)\). In addition, the set \( T(A, B) \) satisfies \( T(A, B) = T(B, A) \) for any \( A, B \in \mathcal{X} \) from its construction. Hence, it is sufficient to only consider the six cases in Table 4.1. To prove that (11) attains the minimum at some \( x_1^* \) under each boundary condition, it is sufficient to show that (5) is coercive in \( \Omega \), for each \( i \in \{1, \ldots, 6\} \) from Lemma 3.2. Applying Lemma 3.3 to Cases 1, 2, and 5 yields the coercivity of \( B_{\Omega_1}, B_{\Omega_2}, \) and \( B_{\Omega_3} \). Hence, we obtain the following proposition:

**Proposition 4.1.** For each \( i \in \{1, 2, 5\} \), the action functional \( B_{\Omega_2} \), attains the minimum.

Next, we focus on Case 2. If \( T = (n - \frac{1}{2})\pi, \) \( X^+ \) is a subset of \( Y \) in the original coordinates and is an unbounded set. Set a constant map sequence \( a_n(t) := (n, 0) \). It is easy to check \( A_T > 0 \) and \( \lim_{n \to \infty} A_T(a_n; \mu) = 0 \), so \( A_T|_{\Omega_2} \) does not attain the minimum. Hence, \( A_T|_{\Omega_2} \) does not possess the minimum if \( T = (n - \frac{1}{2})\pi \). By contrast, invoking Lemma 3.4 and 3.5 yields that \( A_T|_{\Omega_2} \) has coercivity if \( T \neq (n - \frac{1}{2})\pi \). Similar considerations apply to Cases 5 and 6 and we get the next proposition.

**Proposition 4.2.** The functionals \( B_{\Omega_3}, B_{\Omega_4}, \) and \( B_{\Omega_6} \) are coercive except for \( T = 2n\pi, T = (2n - 1)\pi \) and \( T = (n - \frac{1}{2})\pi \) respectively.

If the obtained minimizers are not singular, we get periodic orbits from the following proposition.

**Proposition 4.3.** If \( x^* \) is a collision-free critical point of \( B_{\Omega_i} \) \((i = 1, \ldots, 6)\), it connects with the reversed solution smoothly. In addition, it is a periodic orbit that is orthogonal to each boundary condition.

**Proof.** The variational standard argument implies that a critical point of \( B_{\Omega_i} \) satisfies \( \dot{x}(0) = 0 \) and \( \dot{y}(T) = 0 \). Applying similar arguments to the rest of the boundary conditions, we conclude that each critical point of \( B_{\Omega_i} \) \((i = 1, \ldots, 6)\) is orthogonal to each boundary condition. Combining Lemma 3.7 and Proposition 3.8 gives a new solution \( Rx^*_i(-t) \). Moreover, we obtain another solution \( -Rx^*_i(-t) \) in Cases 5 and 6. Connecting these, we obtain a periodic orbit. \( \square \)

### 4.2 Estimate of equilibrium points

The R3BP has three equilibrium points \( L_1, L_2, \) and \( L_3 \) on the \( x \)-axis. It is sufficient to consider \( L_1 \) and \( L_2 \) by symmetry with respect to \( \mu \). It is clear that \( L_1 = (l_1, 0) \in \Omega_3 \) and \( L_2 = (l_2, 0) \in \Omega_5 \). In the case of \( \mu = 1/2 \), \( L_1 = (0, 0) \in \Omega_1 \). We need to study a condition under which a minimizer is not identical with the equilibrium points. To check this, we calculate the second variation \( B''_T \) at \( L_1 \) and \( L_2 \). A simple calculation implies that for \( i = 1, 2 \),

\[
B''_T(L_i; \mu)(\delta_1, \delta_2) = \int_0^T \left( \delta_1^2 + \delta_2^2 + (1 + \frac{\partial U}{\partial x^2}) \delta_1^2 + (1 + \frac{\partial^2 U}{\partial y^2}) \delta_2^2 + 2(\delta_1 \delta_2 - \delta_2 \delta_1) \right) dt \quad (12)
\]

where

\[
\alpha_i(\mu) = \frac{1}{2} \frac{\partial^2 U(x; \mu)}{\partial x^2} \bigg|_{x=L_i} = - \frac{\partial^2 U(x; \mu)}{\partial y^2} \bigg|_{x=L_i} \quad (i = 1, 2).
\]
If (12) at \( L_i \) is negative for some \( \delta_1 \) and \( \delta_2 \), then \( L_i \) is not a minimizer \((i = 1, 2)\). In Cases 1 and 3, we take a test path as \( \delta(t) = (\delta_1(t), \delta_2(t)) = (\nu_1 \cos\left(-\frac{\pi}{T}t\right), \nu_2 \sin\left(-\frac{\pi}{T}t\right)) \). Then the second variation can be estimated by

\[
\mathcal{E}'_i(L_1; \mu)(\delta_1, \delta_2)
= \frac{T}{2} \left\{ (1 + \gamma^2 + 2\alpha_1(\mu)\nu_1^2 + 4\gamma\nu_1\nu_2 + (1 + \gamma^2 - \alpha_1(\mu))\nu_2^2 \right) \\
= \frac{T}{2} \left\{ (1 + \gamma^2 + 2\alpha_1(\mu)) \left( \nu_1 + \frac{2\gamma}{(1 + \gamma + 2\alpha_1(\mu))} \nu_2 \right)^2 + (1 + \gamma^2 - \alpha_1(\mu)) - \frac{4\gamma^2}{(1 + \gamma^2 + 2\alpha_1(\mu))} \nu_2^2 \right\}
\]

where \( \gamma = -\frac{\pi}{T} \). This calculation shows that if \((1 + \gamma^2 - \alpha_1(\mu)) - 4\gamma^2/(1 + \gamma^2 + 2\alpha_1(\mu)) < 0\), that is, \(\gamma^2 + (\alpha_1(\mu) - 2)\gamma^2 + 1 + \alpha_1(\mu) - 2\alpha_1(\mu)^2 < 0\), then there are constants \(\nu_1 > 0\) and \(\nu_2 > 0\) such that \(\mathcal{E}'_i(L_1; \mu)(\delta_1, \delta_2) < 0\). We conclude that for \(i = 1, 2\), the equilibrium point \(L_1\) is not a minimizer if

\[
T > \sqrt{\frac{2}{\alpha_1(\mu) - 2 + \sqrt{\alpha_1(\mu)(9\alpha_1(\mu) - 8)}}} =: T_{L_1}(\mu). \tag{13}
\]

This is the definition of \(T_{L_1}(\mu)\) in Table 4.1.

In Case 5, we take a new test path as \( \delta(t) = (\nu_1 \cos\left(-\frac{\pi}{T}t\right), \nu_2 \sin\left(-\frac{\pi}{T}t\right)) \) and set \( \gamma = -\frac{\pi}{T} \). It immediately follows that if \( T > T_{L_1}(\mu)/2 \), then \( L_1 \) is not a minimizer.

**Remark 3.** One can not precisely calculate the position of \( L_1 \) except for the case of \( \mu = 1/2 \). However, if we have two functions \( g_i : \mathbb{R} \rightarrow \mathbb{R} \) \((i = 1, 2)\) that satisfy the inequality \( g_1(\mu) \leq L_1 \leq g_2(\mu) \), the second derivative can be estimated by

\[
\frac{\partial^2 U}{\partial x^2} = 2 \left\{ 1 - \frac{\mu}{d_1^3} + 2 \frac{\mu}{d_2^3} \right\} \leq 2 \left\{ 1 - \frac{\mu}{(g_1(\mu) + \mu)^3} + 2 \frac{\mu}{(1 - \mu - g_2(\mu))^3} \right\} =: \beta_1(\mu)
\]

\[
\frac{\partial^2 U}{\partial y^2} = - \left\{ 1 - \frac{\mu}{d_1^3} - \frac{\mu}{d_2^3} \right\} \leq - \left\{ 1 - \frac{\mu}{(g_2(\mu) + \mu)^3} - \frac{\mu}{(1 - \mu - g_1(\mu))^3} \right\} =: \beta_2(\mu).
\]

Using \( \beta_1 \) and \( \beta_2 \), the upper bound of the second variation is given by

\[
\mathcal{E}'_i(L_1; \mu)(\delta_1, \delta_2) \leq \int_0^T \left\{ \delta_1^2 + \delta_2^2 + \delta_1^2 + \delta_2^2 + 2\beta_1(\mu)\delta_1^2 + \beta_2(\mu)\delta_2^2 + 2(\delta_1\delta_2 - \delta_2\delta_1) \right\} dt.
\]

**Remark 4.** For sufficiently small \( \mu \), it is known that \( d_1 = 1 + 3^{-1/3}\mu^{1/3} + o(\mu^2) \) and \( d_2 = 3^{-1/3}\mu^{1/3} + o(\mu^2) \) (See [9]). Applying these yields

\[
\lim_{\mu \to 0} \alpha_1(\mu) = \lim_{\mu \to 0} \left( 1 + \frac{(1 - \mu)(d_1^2 - 1)}{d_1^3 d_2^2} \right) = \lim_{\mu \to 0} \left( 1 + \frac{(1 - \mu)(d_1 - 1)(d_1 + 1)}{d_1^3 d_2^2} \right) = \lim_{\mu \to 0} \left( 1 + \frac{(1 - \mu)(d_1^2 + d_1 + 1)}{d_1^4} \right) = 4.
\]

Thus for each \( i = 1, 2 \), \( \lim_{\mu \to 0} T_{L_i}(\mu) = \pi/(-1 + 2\sqrt{7})^{1/2}(= T_0) \). This period is the same as one in [9]. By contrast, \( \lim_{\mu \to 1} T_{L_2}(\mu) = \pi \) and then our theorems imply the existence of \( 2\pi \)-periodic orbits.

### 4.3 Elimination of interior collisions

To guarantee that the obtained minimizers are smooth, it suffices to show that each minimizer has no collision. Marchal’s theorem in [10] states that any minimizer has no collision in \((0, T)\) in the \(n\)-body problem under the fixed-ends constraint \((x(0) = a, x(T) = b)\). We confirm that it holds for \((R3BP_\mu)\).
Proposition 4.4. Let $H^1$ denote the Sobolev space. If
\[
\mathcal{B}_T(x^*; \mu) = \inf_{x \in H^1} \mathcal{B}_T(x; \mu),
\]
then for any $\mu \in (0, 1)$, the point $x^*$ has no collision for any $t \in (0, T)$.

Proof. Suppose a collision occurs at $t = a$. By a time transformation, we can assume $a = 0$ without loss of generality. Let
\[
\tilde{B}_T(x; \mu) = \int_{-T'}^T \tilde{L}_{R3BP}(x, \tilde{x}; \mu) dt,
\]
where $T' + T'' = T$ and
\[
\tilde{L}_{R3BP}(x, \tilde{x}; \mu) := L_{R3BP}(x + (\mu, 0), \tilde{x}; \mu)
\]
\[
= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + xy - \frac{1}{2}(x^2 + y^2) + \mu(y + x) + \frac{\mu}{\sqrt{x^2 + y^2}} + \frac{1 - \mu}{\sqrt{(x + 1)^2 + y^2}}.
\]

There is no loss of generality in considering $\tilde{B}_T$ instead of $B_T$ since we just use a coordinate transformation.

Let $(x_{\text{col}}, y_{\text{col}})$ denote an orbit with a collision at $t = 0$. We take $x_\theta$ and $y_\theta$ as the following:
\[
x_\theta = \begin{cases} x_{\text{col}} + R_0(t) \cos \theta & t \in [-T', 0), \\
x_{\text{col}} + R_1(t) \cos \theta & t \in [0, T''] \end{cases}
\]
and
\[
y_\theta = \begin{cases} y_{\text{col}} + R_0(t) \sin \theta & t \in [-T', 0), \\
y_{\text{col}} + R_1(t) \sin \theta & t \in [0, T''] \end{cases}
\]
where $R_0(t) = \left(1 + \frac{t}{T'}\right)\rho$ and $R_1(t) = \left(1 - \frac{t}{T''}\right)\rho$.

A simple calculation shows that
\[
\int_{S^1} \int_0^T x_\theta \dot{y}_\theta - y_\theta \dot{x}_\theta + \mu(y_\theta + x_\theta) dt d\theta = \int_{S^1} \int_0^T x_{\text{col}} \dot{y}_{\text{col}} - y_{\text{col}} \dot{x}_{\text{col}} + \mu(y_{\text{col}} + x_{\text{col}}) dt d\theta
\]
and
\[
\frac{1}{|S^1|} \int_{S^1} \int_0^T \frac{1}{2}(\dot{x}_\theta^2 + \dot{y}_\theta^2) dt d\theta = \int_0^T \frac{1}{2}(x_{\text{col}}^2 + y_{\text{col}}^2) dt + \int_0^T \frac{1}{2} R_1(t)^2 dt
\]
\[
= \int_0^T \frac{1}{2} x_{\text{col}}^2 + y_{\text{col}}^2 + T \frac{\rho^2}{3}.
\]

The rest of the part is similar to the Kepler problem. By a result in [5], the estimate of interior collisions is given by
\[
\frac{1}{|S^1|} \int_{S^1} \tilde{B}_T(x_\theta; \mu) dt - \tilde{B}_T(x_{\text{col}}; \mu)
\]
\[
\leq \left(\frac{T}{3} + \frac{1}{2T'}\right) \gamma^2 t_0^{4/3} (1 + O(t_0)) + \left(\frac{T}{2} - 3\right) t_0^{1/3} \gamma (1 + O(t_0)) + \left(\frac{T}{2} - 1\right) t_0^{1/3} \gamma + O(t_0^{4/3} \log(1/t_0))
\]
\[
= (\pi - 4) t_0^{1/3} \gamma + O(t_0^{4/3} \log(1/t_0)) \leq 0
\]
where $t_0$ is sufficiently small and $\rho = \gamma t_0^{2/3} + O(t_0^{2/3})$. \qed
4.4 Elimination of boundary collisions

By Proposition 4.4, we only need to consider orbits that have a collision at \( t = 0 \) or \( T \). If a collision occurs at \( t = 0 \), it is shown that the orbit, say \( (x_{\text{col}}, y_{\text{col}}) \), is represented by

\[
x_{\text{col}} = \sum_{j=0}^{\infty} c_{1j} t^{j/3}, \quad y_{\text{col}} = \sum_{j=0}^{\infty} c_{2j} t^{j/3}
\]

We say that an orbit has \( \rho \)-collision if it has a collision at \( t = 0 \) and \( \lim_{t \to +0} \frac{\dot{x}(t)}{\dot{y}(t)} = \tan \rho \). In the case of a double collision in the \( N \)-body problem, Proposition 5.7 of [8] implies that if the collision angle \( \rho \) satisfies \(-\pi < \rho < \pi\), it is not a minimizer. As seen in Section 3, the asymptotic behavior of the R3BP is the same as in the \( N \)-body problem, so we can adapt the approach in [8] to the R3BP. Hence, in the R3BP, it suffices to consider the case \( \rho = \pm \pi \), i.e. the velocity of \( y(0) \) is 0.

As seen in Section 3, Sundman’s estimate is a useful way to study a collision path. Substituting Sundman’s estimate into (R3BP) and applying coefficient comparison, we obtain

\[
x_{\text{col}}(t) = (1 - \mu) + c_1 t^{2/3} + o(t^2), \quad y_{\text{col}}(t) = c_2 t^{5/3} + o(t^2), \quad (t \in [0, \epsilon]),
\]

where \( c_1 = (9/2)^{1/3} \mu^{1/3} \), \( c_2 = -(9/2)^{1/3} \mu^{1/3} \), and \( \epsilon \) is sufficiently small. By a polar coordinate, (11) can be written as

\[
B_T((r, \theta); \mu) = \int_{0}^{T} \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) + r^2 \dot{\theta} + \frac{1}{2} r^2 + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} dt
\]

where \( r_1 \) and \( r_2 \) represent the distance from \((-\mu, 0)\) and from \((1 - \mu, 0)\) respectively.

Proposition 4.5 (Local estimate, The case of \( \rho = \pm \pi \)). Minimizers under our boundary conditions do not have \( \pi \) and \((-\pi)\)-collision.

Proof. Applying a polar transformation to (16), \( \pi \)-collision orbits are represented by the following:

\[
r_{\text{col}}(t) = (1 - \mu) + c_1 t^{2/3} + o(t^2)
\]

\[
\theta_{\text{col}}(t) = \frac{c_2}{1 - \mu} t^{5/3} + o(t^2).
\]

For sufficiently small \( \epsilon > 0 \), we deform \( \theta_{\text{col}}(t) \) to \( \theta_{\text{de}}(t) = \frac{c_2}{1 - \mu} \epsilon^2/3 t + o(t^2) \) in \( t \in [0, \epsilon] \) and do not change \( r_{\text{col}}(t) \), i.e. \( r_{\text{de}} := r_{\text{col}} \). Thus, we obtain

\[
\int_{0}^{\epsilon} K(r_{\text{col}}, \theta_{\text{col}}) - K(r_{\text{col}}, \theta_{\text{de}}) dt = \int_{0}^{\epsilon} \frac{1}{2} r_{\text{col}}^2 (\dot{\theta}_{\text{col}}^2 - \dot{\theta}_{\text{de}}^2) + 2(\dot{\theta}_{\text{col}} - \dot{\theta}_{\text{de}}) dt
\]

Because the main terms of \( r_{\text{col}} \) are \( 1 - \mu \), it is sufficient to calculate

\[
\frac{1}{2} \int_{0}^{\epsilon} (1 - \mu)^2 ((\dot{\theta}_{\text{col}}^2 - \dot{\theta}_{\text{de}}^2) + 2(\dot{\theta}_{\text{col}} - \dot{\theta}_{\text{de}})) dt.
\]

Note that \( \theta_{\text{col}} \) and \( \theta_{\text{de}} \) have the same boundary and this implies

\[
\int_{0}^{\epsilon} (\dot{\theta}_{\text{col}} - \dot{\theta}_{\text{de}}) dt = 0.
\]
By the above remarks, (18) is calculated as follows:

\[
\int_0^\epsilon \frac{1}{2} (1 - \mu)(\dot{\theta}_{\text{col}} - \dot{\theta}_{\text{de}})^2 dt + \int_0^\epsilon (1 - \mu)^2 (\dot{\theta}_{\text{col}} - \dot{\theta}_{\text{de}}) dt,
\]

\[
= \frac{1}{2} (1 - \mu)^2 \int_0^\epsilon (\dot{\theta}_{\text{col}} - \dot{\theta}_{\text{de}})(\dot{\theta}_{\text{col}} + \dot{\theta}_{\text{de}}) dt
\]

\[
= \frac{1}{2} (1 - \mu)^2 \int_0^\epsilon \left( \frac{5}{3} \frac{c_2}{1 - \mu} \frac{t^{2/3}}{\epsilon^{2/3}} - \frac{c_2}{1 - \mu} \frac{\epsilon^{2/3}}{\epsilon^{2/3}} \right) \left( \frac{5}{3} \frac{c_2}{1 - \mu} \frac{t^{2/3}}{\epsilon^{2/3}} + \frac{c_2}{1 - \mu} \frac{\epsilon^{2/3}}{\epsilon^{2/3}} + o(t^2) \right) dt
\]

\[
= \frac{1}{2} (1 - \mu)^2 \int_0^\epsilon \left( \frac{5}{3} \frac{c_2}{1 - \mu} \right)^2 t^{4/3} - \left( \frac{c_2}{1 - \mu} \right)^2 \epsilon^{4/3} + o(t^2) dt
\]

\[
= \frac{1}{2} (1 - \mu)^2 \cdot \frac{4}{21} \left( \frac{c_2}{1 - \mu} \right)^2 \epsilon^{7/3} + o(\epsilon^3) = \frac{2}{21} (1 - \mu)^2 \left( \frac{c_2}{1 - \mu} \right)^2 \epsilon^{7/3} + o(\epsilon^3) > 0
\]

Next we consider the value of the potential part. The Taylor expansion shows

\[
r_1^2 = r_{\text{col}}(s)^2 + 2 \mu (1 - \mu) \left( 1 + \frac{c_1 s^2}{1 - \mu} \right) \cos \left( \frac{c_2 s^6}{1 - \mu} \right) + \mu^2
\]

\[
r_2^2 = r_{\text{col}}(s)^2 - (2 - 2 \mu) (1 - \mu) \left( 1 + \frac{c_1 s^2}{1 - \mu} \right) \cos \left( \frac{c_2 s^5}{1 - \mu} \right) + (1 - \mu)^2
\]

where \( s = t^{1/3} \). Substituting (19) into the potential \( U \), we obtain

\[
\int_0^\epsilon U(r_{\text{col}}, \theta_{\text{col}}) - U(r_{\text{col}}, \theta_{\text{de}}) dt = -\frac{\mu c_2^2}{1 - \mu} \epsilon^{4/3} \int_0^\epsilon t^2 + o(t^4) dt
\]

\[
= c_4 \epsilon^{13/3} + o(\epsilon^5) < 0.
\]

Hence, the orbit \((x_{\text{col}}, y_{\text{col}})\) is not a minimizer. In the same manner, we can see that \((-\pi)\)-collision orbits are not minimizers. A similar approach is valid when a collision occurs at \( t = T \).

Moreover, it is clear that collisions with both primaries do not occur since such an orbit can be changed in \( t \in [0, \epsilon] \) similar to the above, and the action value of the deformed orbit is smaller.

5 Global behavior of minimizers and open problems

In this section, we discuss the remaining problems including open problems. Theorems 2.1 and 2.2 show the existence of periodic solutions; however, we do not know how their minimizers behave in time \( t \in (0, T) \). More precisely, we discuss the following:

Q1. Do the obtained periodic solutions in Theorems 2.1 and 2.2 have the same topology as Figures 3 and 4?

Q2. Our main theorems (Theorems 2.1 and 2.2) show the existence of 2\(T\) or 4\(T\)-periodic solutions. Are these periods minimal?

Q3. Are periodic orbits obtained under different boundary conditions different?

Although we do not answer these questions completely, we can partially solve them.
First, we consider Q1. Orbit 5 in Figure 4 satisfies the boundary condition of Case 5. In the R3BP, the integral of the second term \( x\dot{y} - y\dot{x} \) in (4) corresponds to the area and gives a negative value for minimizers. If the direction of the orbits is clockwise, the second term is negative. If it is counter-clockwise, it is positive. We focus on minimizers of (11), so it is sufficient to consider clockwise orbits. By contrast, the integral of the third term \( \frac{x^2 + y^2}{2} \) in (4) is a positive value, so it is difficult to see the behavior of minimizers in the R3BP. The same difficulty occurs for Figures 3 and 4. For example, is orbit 1 in Figure 5 a minimizer of Case 1? Hence, all we can show here is that minimizing orbits are clockwise.

We move on to Q2. If we obtain a \( T \)-periodic solution that follows from one of the main theorems, \( T \) may not be the minimal period of the solution. For any \( n, m \in \mathbb{N} \), \( nT \)-periodic and \( mT \)-periodic
solutions may be the same. We can not show that all periodic solutions are distinct, but we can prove the existence of an infinite number of periodic solutions. The proof is based on Rabinowitz’s idea in [12].

**Proposition 5.1.** Assume that there is a $T_0$-periodic solution that has a minimal period $T_0 > 0$. Then we get infinitely many $T$-periodic solutions satisfying $T \in (0, T_0)$.

**Proof.** We choose $T = T_0/4$. Cases 2, 4, and 6 in Table 4.1 show the existence of a $T_0/2$-periodic solution. Clearly, its minimal period is not $T_0$, so we obtain a new periodic solution and $T'$ is defined by a minimal period of the new solution. We now apply this argument again with $T$ replaced by $T'$. The rest of the proof is simple. \qed

We discuss the final question. Q3 asks, for instance, whether periodic orbits of Theorems 2.1 and 2.2 are different for $\mu = 1/2$. There is another problem. Consider orbit 2 in Figure 5. Is this a minimizer of Case 1 or an orbit consisting of minimizers in Case 2? We guess that if $T - T_{L_1}(\mu)$ is sufficiently small, this problem does not occur because a minimizer of this case may be closed to each equilibrium point. However, it remains an open problem whether a periodic solution satisfies another boundary condition for large $T > 0$.

**Acknowledgement**

Y. K. is supported by the Japan Society for the Promotion of Science (JSPS), Grant-in-Aid for JSPS Fellows No. 20J21214. M. S. is supported by the Japan Society for the Promotion of Science (JSPS), Grant-in-Aid for Scientific Research (C) No. 18K03366.

**References**


