# Generalized Coordination of Multi-robot Systems

**Suggested Citation:** Kazunori Sakurama and Toshiharu Sugie (2021), "Generalized Coordination of Multi-robot Systems", : Vol. xx, No. xx, pp 1–18. DOI: 10.1561/XXXXXXXXX.

**Kazunori Sakurama**
Kyoto University
sakurama@i.kyoto-u.ac.jp

**Toshiharu Sugie**
Osaka University
sugie@jrl.eng.osaka-u.ac.jp



now
the essence of knowledge
Boston — Delft

# Contents


**1 Introduction** **3**

1.1 Background . . . . . 3

1.2 Research trends in coordination control . . . . . 5

1.3 Focus of the monograph . . . . . 6

1.4 Organization of the monograph . . . . . 8

1.5 Notation . . . . . 8

**2 Overview** **12**

2.1 Overall picture . . . . . 12

2.2 Control objectives . . . . . 13

2.3 Control with relative measurements . . . . . 24

2.4 Problem formulation . . . . . 29

2.5 Notes and references . . . . . 31

**I Mathematical Preliminaries** **33**

**3 Group Theory** **34**

3.1 Basics . . . . . 34

3.2 Group actions . . . . . 36

3.3 Semidirect products . . . . . 37

3.4 Group orbits . . . . . 38

3.5 Invariant subsets . . . 40
3.6 Invariant functions . . . 42
3.7 Free group actions . . . 45
3.8 Free action numbers . . . 46
3.9 Notes and references . . . 51

**4 Graph Theory** **52**
4.1 Basics . . . 52
4.2 Cliques and maximal cliques . . . 54
4.3 Conventional rigidity . . . 55
4.4 Clique rigidity . . . 57
4.5 Intersection graphs . . . 62
4.6 Notes and references . . . 63

**5 Stability Theory for Gradient-flow Systems** **64**
5.1 Terminology . . . 64
5.2 Lagrange stability . . . 66
5.3 Asymptotic stability . . . 71
5.4 Remarks on non-differentiable functions . . . 74
5.5 Notes and references . . . 75

**II Multi-robot Coordination Problems** **76**

**6 Pairwise Coordination** **77**
6.1 Problem formulation . . . 77
6.2 Controller design . . . 78
6.3 Stability analysis . . . 81
6.4 Examples . . . 82
6.5 Notes and references . . . 86

**7 Generalized Coordination with "Absolute" Measurements** **88**
7.1 Problem formulation . . . 88
7.2 Characterization of the best approximate indicators . . . 91
7.3 Controller design . . . 95
7.4 Stability analysis . . . 97
7.5 Existence of indicators . . . 98

7.6 Notes and references . . . . . . . . . . . . . . . . . . . . . 99

**8 Generalized Coordination with "Relative" Measurements 101**

8.1 Problem formulation . . . . . . . . . . . . . . . . . . . . . 102
8.2 Characterization of indicators . . . . . . . . . . . . . . . . . 104
8.3 Controller design . . . . . . . . . . . . . . . . . . . . . . . 109
8.4 Stability analysis . . . . . . . . . . . . . . . . . . . . . . . 111
8.5 Relations between coordination, measurement, and networks 114
8.6 Notes and references . . . . . . . . . . . . . . . . . . . . . 119

**9 Application Examples 120**

9.1 Formation selection . . . . . . . . . . . . . . . . . . . . . 120
9.2 Scaling reflection-free formation . . . . . . . . . . . . . . . 122
9.3 Position assignment with local indices . . . . . . . . . . . . 125
9.4 Formation control of non-holonomic robots . . . . . . . . . 128
9.5 Notes and references . . . . . . . . . . . . . . . . . . . . . 131

**10 Concluding Remarks 133**

**Acknowledgements 135**

**Appendices 136**

**A Examples of Frame Transformation Sets 137**

**B Real Analytic Functions 141**

**C Gradients of Squared Distance Functions 144**

**D Partial Difference 147**

D.1 Partial difference and high-order partial difference . . . . . 147
D.2 Verification of dependency of functions . . . . . . . . . . . 148
D.3 Relations to integrals and partial derivatives . . . . . . . . 152
D.4 Decomposition of functions . . . . . . . . . . . . . . . . . . 153

**E Procrustes Problems 158**

# Generalized Coordination of Multi-robot Systems

Kazunori Sakurama[1] and Toshiharu Sugie[2]

[1] *Kyoto University; sakurama@i.kyoto-u.ac.jp*
[2] *Osaka University; sugie@jrl.eng.osaka-u.ac.jp*

---

ABSTRACT

Multi-robot systems have huge potential for practical applications, which include sensor networks, area surveillance, environment mapping, and so forth. In many applications, cooperative coordination of the robots plays a central role. There are various types of coordination tasks such as consensus, formation, coverage, and pursuit. Most developments of control methods have been taken place for each task individually so far. The purpose of this monograph is to provide a systematic design method applicable to a wide range of coordination tasks for multi-robot systems. The features of the monograph are two-fold: (i) The coordination problem is described in a unified way instead of handling various problems individually, and (ii) a complete solution to this problem is provided in a compact way by using the tools of "group" and "graph" theories efficiently. As for item (i), it is shown that various coordination tasks can be formulated as a generalized coordination problem, where each robot should converge to some desired configuration set under the given information network topology among robots. In this problem, the solvability (i.e., whether robots can achieve the given coordination task or not) fully depends on the characteristics of both the desired configuration set and the network topology. Therefore, concerning item (ii), it

---

Kazunori Sakurama and Toshiharu Sugie (2021), "Generalized Coordination of Multi-robot Systems", : Vol. xx, No. xx, pp 1–18. DOI: 10.1561/XXXXXXXXXX.

is clarified when the generalized coordination problem can be solved in terms of the desired configuration set and the network topology. Furthermore, it is shown how to design a controller which achieves the given configuration task. In particular, the case where each robot can get only local information (e.g., relative position between two robots) is discussed.

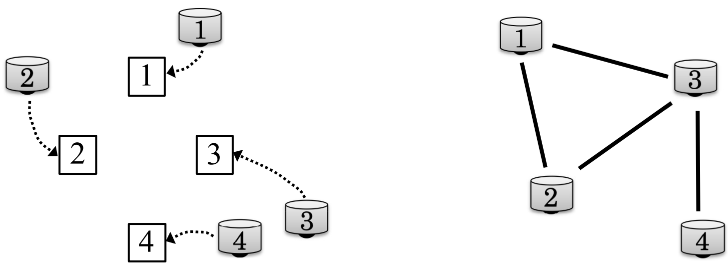
# 1

# Introduction

## 1.1 Background

Multi-robot systems have attracted a lot of attention because of its potential to various applications, which include sensor networks, area surveillance, object transport, environment mapping, building health monitoring, air pollution monitoring, search and rescue after disasters (Lima and Custodio, 2005; Darmanin and Bugeja, 2017). In many applications, cooperative coordination of multi-robots plays a central role. It enhances efficiency, robustness, and reliability in many application tasks, and is required to reduce human operations. Therefore, this topic has been extensively studied in many disciplines, such as robotics, control and measurement engineering, informatics, and so forth. In this monograph, we focus on cooperative coordination problems of multi-robot systems in (mainly two-dimensional or three-dimensional) space from a viewpoint of control theory as Bullo *et al.*, 2009; Cortes and Egerstedt, 2017, which are also investigated through the terminology of multi-agent systems (Martínez *et al.*, 2007; Shamma, 2008; Mesbahi and Egerstedt, 2010; Cao *et al.*, 2012).

At this point, let us look at a simple example of coordination problems, which may help us to understand some basic concepts that




**(a)** Desired configuration and expected motion **(b)** Network topology

**Figure 1.1:** Example of a coordination task.

will be necessary in this section. Suppose that the given task is to shape a formation by four robots in a plane as shown in Fig. 1.1a. In Fig. 1.1a, the numbered squares describe a "desired configuration" that the robots are expected to form. In Fig. 1.1b, each line represents the interaction of robots, defining the "network topology" to determine the information structure of the multi-robot system. The robots directly connected by lines are called "neighbors". For example, robots 1 and 3 are neighbors of robot 2, but robot 4 is not. While, we say the network is "connected", when there exists a sequence of interaction lines between every pair of nodes. The robots mainly interact with each other through sensing. We say that "absolute" measurement is available if each robot can measure the absolute positions (i.e., the positions in a global coordinate frame) of its neighbors. When only the positions in its local coordinate frame associated with each robot can be measured, we say that "relative" measurement is available.

For cooperative coordination of multi-robots, distributed control with relative measurement plays the key role. Distributed control is a methodology to control each robot based on local information of its neighboring robots, which is important because we can apply the control method for any number of robots. This property is called scalability. On the other hand, control with relative measurement is critical to the autonomy of robots, because any external system to obtain absolute measurement is not necessary. Many types of coordination problems have been investigated based on distributed control with absolute/relative measurement so far, e.g., consensus (Olfati-Saber and Murray, 2004; Olfati-Saber *et al.*, 2007), coverage (Cortés *et al.*, 2004), flocking (Olfati-Saber, 2006; Tanner *et al.*, 2007), pursuit (Marshall *et al.*, 2004; Kim *et al.*, 2010), attitude synchronization (Igarashi *et al.*, 2009; Ren, 2010), assignment (Ji *et al.*, 2006; Michael *et al.*, 2008; Smith and Bullo, 2009), and formation (Fax and Murray, 2004; Lin *et al.*, 2005; Ren and Beard, 2008; Anderson *et al.*, 2008; Krick *et al.*, 2009; Dörfler and Francis, 2010; Lin and Jia, 2010; Oh *et al.*, 2015; Queiroz *et al.*, 2019).

## 1.2 Research trends in coordination control

Concerning control system synthesis, a fundamental question is whether there exists a controller achieving the given task. In cooperative coordination of multi-robots, the answer fully depends on both the network topology and the sensing capability of the multi-robot system. Once the existence of such a controller is confirmed, we can proceed to the controller design step. According to the surveys of formation control literature (e.g., Oh *et al.*, 2015; Tron *et al.*, 2016; Ahn, 2020), many of the methods can be classified as (a) Position-based control, (b) Displacement-based control, and (c) Distance-based control according to how to specify the desired configuration. Each control method requires different types of network topology and sensing capability.

In the case of (a), the desired configuration is specified with the target position of each robot in the global coordinate. Though there exist no requirements on the network topology, we need absolute position of each robot to achieve the formation task. In the case of (b), the relative target positions of the robots are given to specify the desired configuration. Hence, the absolute position of each robot does not matter, which allows the translation of the shaped formation. Meanwhile, its rotation is not allowed. To achieve this formation task, it turns out that the network must be connected as Fig. 1.1b. Furthermore, though the absolute position measurement is not required, the absolute direction should be measured in each robot. In the case of (c), only the distances between target positions are assigned to specify the desired configuration. So neither the absolute position nor absolute direction is necessary. Instead, more inter-robot interaction is needed. For example, in the case of Fig.

1.1b, one more interaction line which connects robot 2 with robot 4 directly has to be added. In other words, the network must be getting denser compared to the other cases (Anderson *et al.*, 2008; Krick *et al.*, 2009; Queiroz *et al.*, 2019).

In the same way, bearing-based and angle-based formations are considered in Zhao and Zelazo, 2016; Zhao and Zelazo, 2019; Chen *et al.*, 2020, which yield more flexibility in coordination because these formations are scale-free. The scale-free property is involved in different ways in other papers, e.g., Han *et al.*, 2016; Sakurama *et al.*, 2018; Lin *et al.*, 2016; Zhao, 2018. Also, combinations of several constraints are considered in many papers, e.g., Anderson *et al.*, 2017; Sun *et al.*, 2017; Sakurama *et al.*, 2019.

The existing results may be summarized in Table 1.1, from the viewpoints of (D) requirements of the desired coordination, (N) necessary network topology, and (M) necessary sensing capability. The current research trend is to impose laxer requirements of the desired configuration. Accordingly, the network should be denser, and the less sensing capability is required.

## 1.3 Focus of the monograph

As shown in the above, various types of coordination problems have been studied individually. However, there are so many tasks and their variants. It is not efficient to describe all the existing methods one by one. Instead, this monograph focuses on a generalized coordination problem which can cover a wide range of coordination problems and handle them in a unified manner instead of discussing various problems individually. Then, a complete solution to the problem will be provided.

In the problem formulation, the following three components play essential roles. First, coordination problems are generalized by representing the desired configuration appropriately. This representation enables us to describe various coordination tasks in a unified way. Second, related to sensing capability, relative measurements are precisely and explicitly described, which can be done through coordinate transformation between the global and local coordinate frames. Third, the sensing network of multi-robot systems is modeled with a graph, which

**Table 1.1:** Summary of the existing results on coordination problems

| |
|---|
| (a) Position-based formation<br>(D) absolute target positions are fixed<br>(N) none<br>(M) absolute positions are measurable |
| (b) Displacement-based formation<br>(D) relative target positions are fixed<br>(N) connectivity is required<br>(M) relative positions and absolute directions are measurable |
| (c) Distance-bases formation<br>(D) distances between target positions are fixed<br>(N) rigidity (a precise definition will be given later) is required<br>(M) relative positions are measurable |
| (d) Bearing-based formation<br>(D) bearing of the target formation is fixed<br>(N) bearing rigidity is required<br>(M) relative bearings and absolute directions are measurable |
| (e) Angle-based formation<br>(D) angles between the target robots are fixed<br>(N) angle rigidity is required<br>(M) relative bearings are measurable |

will be necessary for compact presentation and rigorous analysis of the system.

The solvability of the generalized coordination problem (i.e., whether robots can achieve the given coordination task or not) fully depends on the characteristics of the desired configuration, the available relative measurements, and the network topology. Hence, we characterize a strict class of desired configurations which can be achieved with available relative measurements over the given network topology. A complete solution to this problem is provided in a compact way by using the tools of "group" and "graph" theories efficiently. Furthermore, a distributed controller is designed to achieve the given configuration with relative measurements.

The approach of the monograph has several advantages. First, systematic tools are provided to design distributed controllers for coordination problems, which are applicable to a wide range of coordination tasks due to its general description. Moreover, various sensing devices can be handled by employing appropriate coordinate transformations. Second, a kind of converse design problem can be discussed. Namely, given the network topology and sensing capability of robots, it is possible to tell what kind of coordination tasks can be achieved by distributed control with the relative measurements.

This monograph is based on the authors' papers, mainly Sakurama, 2021b, with Sakurama *et al.*, 2012; Sakurama *et al.*, 2015; Sakurama, 2016; Sakurama, 2018; Sakurama *et al.*, 2019.

## 1.4 Organization of the monograph

Chapter 2 gives an overview of the coordination problem tackled in this monograph. Part I provides mathematical preliminaries on group theory in Chapter 3, graph theory in Chapter 4, and stability analysis of gradient-flow systems in Chapter 5. Part II addresses the multi-robot coordination problems. Chapter 6 considers a pairwise coordination problem, which gives a basic idea of the conventional approaches. Chapters 7 and 8 give complete solutions to the generalized coordination problems with absolute and relative measurements, respectively. In Chapter 9, the developed methods are applied to various examples. Chapter 10 concludes the monograph.

## 1.5 Notation

For a set $\mathcal{X}$ and a subset $\mathcal{N} = \{1, 2, \ldots, n\}$ of natural numbers, consider $n$ elements $x_1, x_2, \ldots, x_n$ of $\mathcal{X}$ and a subset $\mathcal{C}$ of $\mathcal{N}$ consisting of $c$ distinct natural numbers. Let $x_{\mathcal{C}} \in \mathcal{X}^c$ denote the $c$-tuple consisting of $x_i$ for $i \in \mathcal{C}$ in order, i.e.,

$$x_{\mathcal{C}} = (x_{j_1}, x_{j_2}, \ldots, x_{j_c}) \text{ for } j_1, j_2, \ldots, j_c \in \mathcal{C} \ (j_1 < j_2 < \cdots < j_c).$$

When $\mathcal{X} = \mathbb{R}^d$, the $c$-tuple $x_{\mathcal{C}} \in (\mathbb{R}^d)^c$ is sometimes regarded as the following matrix in $\mathbb{R}^{d\times c}$:

$$x_{\mathcal{C}} = [x_{j_1} \ x_{j_2} \ \cdots \ x_{j_c}] \text{ for } j_1, j_2, \ldots, j_c \in \mathcal{C} \ (j_1 < j_2 < \cdots < j_c).$$

### Binary operations

| | |
|---|---|
| $\times$ | the Cartesian product. |
| $\ltimes$ | the semidirect product (See Section 3.3). |
| $*$ | the binary operation of a group (See Section 3.1). |
| $\bullet$ | the binary operation of a group action (See Section 3.2). |

### Basic sets

| | |
|---|---|
| $\mathbb{R}$ | the set of real numbers. |
| $\mathbb{R}_+$ | the set of nonnegative real numbers. |
| $\mathbb{Z}_+$ | the set of nonnegative integers. |
| $\mathcal{P}_n$ | the set of permutations of $n$ elements. |
| $\mathcal{P}(\mathcal{N}, \mathcal{N}^*)$ | the set of bijective functions from a set $\mathcal{N}$ to $\mathcal{N}^*$. |

### Basic matrices and vectors

| | |
|---|---|
| $I_d$ | the identity matrix of dimension $n$. |
| $e_{di}$ | the $i$th unit vector of dimension $d$, i.e., $i$th column of $I_d$. |
| $\mathbf{1}_d$ | $:= [1 \ \cdots \ 1]^\top \in \mathbb{R}^d$; the vector of dimension $d$ with all components 1. |
| $\mathrm{Rot}(\theta)$ | $:= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$; the two-dimensional rotation matrix parametrized by $\theta \in [-\pi, \pi)$. |
| $\mathrm{Refl}(w)$ | $:= I_d - 2ww^\top$; the reflection matrix with respect to a unit vector $w \in \mathbb{R}^d$. |

### Operations for matrices and functions

| | |
|---|---|
| $\det(M)$ | the determinant of a square matrix $M$. |
| $\mathrm{tr}(M)$ | the trace of a square matrix $M$. |
| $\mathrm{diag}(d_{\mathcal{N}})$ | the (block) diagonal matrix with the $i$th diagonal entry $d_i$ for $i \in \mathcal{N}$. |
| $f^{-1}(0)$ | $:= \{x \in \mathbb{R}^n : f(x) = 0\}$; the zero set of $f : \mathbb{R}^n \to \mathbb{R}^m$. |
| $\mathcal{L}_\rho^-(f)$ | $:= \{x \in \mathbb{R}^n : f(x) \leq \rho\}$; the $\rho$-sublevel set of $f : \mathbb{R}^n \to \mathbb{R}$ for $\rho \in \mathbb{R}$. |

### Operations for sets

| | |
|---|---|
| $\lvert\mathcal{C}\rvert$ | the number of the elements in a finite countable set $\mathcal{C}$. |
| $\mathrm{cl}(\mathcal{S})$ | the closure of a set $\mathcal{S}$. |

| | |
|---|---|
| $\mathrm{int}(\mathcal{S})$ | the interior of a set $\mathcal{S}$. |
| $\mathrm{pow}(\mathcal{S})$ | the power set (the set of all subsets) of a set $\mathcal{S}$. |
| $\mathrm{scaled}(\mathcal{S})$ | $:= \{sS : s > 0, S \in \mathcal{S}\}$; the scaled set of a set $\mathcal{S}$. |
| $\mathrm{orb}_{\mathcal{H}}(\mathcal{S})$ | $:= \{H \bullet x : H \in \mathcal{H}, x \in \mathcal{S}\}$; the $\mathcal{H}$-orbit of $\mathcal{S}$ (See Section 3.4). |

### Operations for $n$-tuples

| | |
|---|---|
| $\langle x, y \rangle$ | the inner product of $x, y \in \mathcal{X}$ for a metric space $\mathcal{X}$. |
| $\langle x_{\mathcal{N}}, y_{\mathcal{N}} \rangle$ | $:= \sum_{i \in \mathcal{N}} \langle x_i, y_i \rangle$; the inner product of $x_{\mathcal{N}}, y_{\mathcal{N}} \in \mathcal{X}^n$. |
| $\|x_{\mathcal{N}}\|$ | $:= \sqrt{\langle x_{\mathcal{N}}, x_{\mathcal{N}} \rangle}$; the norm of $x_{\mathcal{N}} \in \mathcal{X}^n$. |
| $\mathrm{col}_m(x_{\mathcal{N}})$ | $:= x_m$; the $m$th element of $x_{\mathcal{N}} \in \mathcal{X}^n$. |
| $\mathrm{ave}(x_{\mathcal{N}})$ | $:= \frac{1}{n} \sum_{i \in \mathcal{N}} x_i$; the average of the elements of $x_{\mathcal{N}} \in \mathcal{X}^n$. |
| $\mathrm{cen}(x_{\mathcal{N}})$ | $:= x_{\mathcal{N}} - (\mathrm{ave}(x_{\mathcal{N}}), \ldots, \mathrm{ave}(x_{\mathcal{N}}))$; the center of $x_{\mathcal{N}} \in \mathcal{X}^n$. |
| $\mathrm{dist}(x_{\mathcal{N}}, \mathcal{D})$ | $:= \inf_{y_{\mathcal{N}} \in \mathcal{D}} \|x_{\mathcal{N}} - y_{\mathcal{N}}\|$; the distance from $x_{\mathcal{N}} \in \mathcal{X}^n$ to a non-empty set $\mathcal{D} \subset \mathcal{X}^n$. |
| $\mathrm{proj}_{\mathcal{C}}(\mathcal{D})$ | $:= \{x_{\mathcal{C}} \in \mathcal{X}^{\lvert\mathcal{C}\rvert} : \exists x_{\mathcal{N}} \in \mathcal{X}^n \text{ s.t. } x_{\mathcal{N}} \in \mathcal{D}\}$; the projection of a set $\mathcal{D} \subset \mathcal{X}^n$ onto the space $\mathcal{X}^{\lvert\mathcal{C}\rvert}$ for $\mathcal{C} \subset \mathcal{N}$. |

### Matrix sets

| | |
|---|---|
| $\mathrm{GL}(d)$ | $:= \{M \in \mathbb{R}^{d \times d} : \det(M) \neq 0\}$; the general linear group of dimension $d$. |
| $\mathrm{O}(d)$ | $:= \{M \in \mathbb{R}^{d \times d} : M^\top M = I_d\}$; the orthogonal group of dimension $d$. |
| $\mathrm{SO}(d)$ | $:= \{M \in \mathrm{O}(d) : \det(M) = 1\}$; the special orthogonal group of dimension $d$. |
| $\mathrm{Skew}(d)$ | $:= \{M \in \mathbb{R}^{d \times d} : M + M^\top = 0\}$; the set of skew-symmetric matrices. |

### Notation on graphs

| | |
|---|---|
| $G = (\mathcal{N}, \mathcal{E})$ | the graph with a node set $\mathcal{N}$ and an edge set $\mathcal{E} \subset \mathcal{N}^2$. |

| | |
|---|---|
| $\mathrm{clq}(G)$ | the index set of the maximal cliques in graph $G$ (See Section 4.2). |
| $\mathrm{clq}_i(G)$ | the index set of the maximal cliques in $G$ that node $i$ belongs to (See Section 4.2). |
| $\mathcal{N}_i$ | $:= \{j \in \mathcal{N} : \{i,j\} \in \mathcal{E}\} \cup \{i\}$; the neighbor set of node $i$. |
| $\Gamma_r(G)$ | the $r$-intersection graph of the maximal cliques in $G$ (See Section 4.5). |
| $(x_{\mathcal{N}}, G)$ | the framework of $x_{\mathcal{N}} \in (\mathbb{R}^d)^n$ over graph $G$ (See Section 4.3). |
| $(\mathcal{D}, G)$ | the set framework of $\mathcal{D} \subset (\mathbb{R}^d)^n$ over graph $G$ (See Section 4.4). |

**Sets of scalar functions**

| | |
|---|---|
| $\mathcal{V}_{\mathrm{c1}}$ | the set of scalar, continuously differentiable functions. |
| $\mathcal{V}_{\mathrm{ind}}(\mathcal{D})$ | the set of indicators of a set $\mathcal{D}$, defined in (7.6). |
| $\mathcal{V}_{\mathrm{dis}}(G)$ | the set of functions having distributed gradients, defined in (7.8). |
| $\mathcal{V}_{\mathrm{app}}(\mathcal{D})$ | the set of approximate indicators of a set $\mathcal{D}$, defined in (7.10). |
| $\mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$ | the set of functions having relative gradients, defined in (8.8). |

# 2

# Overview

This chapter overviews the issues discussed in this monograph for multi-robot coordination problems. Section 2.1 provides an overall picture of the control problem. In Section 2.2, control objectives are given. In Section 2.3, the control system of each robot is described in detail. Section 2.4 formulates the target problem in a formal way.

## 2.1 Overall picture

Consider a multi-robot coordination system consisting of $n$ robots in a $d$-dimensional space as shown in Fig. 2.1. Let $\mathcal{N} = \{1, 2, \ldots, n\}$ be the index set of the robots. Each robot is governed by the equation

$$\dot{x}_i(t) = f_i(x_i(t), u_i(t), t), \tag{2.1}$$

where $x_i(t) \in \mathbb{R}^d$ and $u_i(t) \in \mathbb{R}^d$ are the state and the input of robot $i \in \mathcal{N}$ and $f_i : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ describes the system dynamics which could be nonlinear. The robots exchange information mainly by sensing each other. The information structure is described by an undirected graph $G = (\mathcal{N}, \mathcal{E})$, where $\mathcal{E} \subset \mathcal{N}^2$ denotes the set of edges. That is, $\{i, j\} \in \mathcal{E}$ implies that two robots $i$ and $j$ exchange information with each other. Also, $\mathcal{N}_i$ denotes the set of the neighbors of robot $i$ which

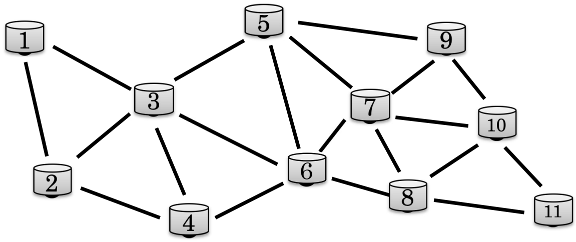



**Figure 2.1:** An example of a multi-robot system with a sensing network.

is defined by

$$\mathcal{N}_i := \{j \in \mathcal{N} : \{i, j\} \in \mathcal{E}\} \cup \{i\}.$$

Robot $i$ detects the relative position $x_j^{[i]}(t) \in \mathbb{R}^d$ of the neighbor robot $j \in \mathcal{N}_i$ in its local coordinate frame, e.g., $x_j^{[i]}(t) = x_j(t) - x_i(t)$. Then, the control input $u_i(t)$ should be generated as

$$u_i(t) = c_i(x_{\mathcal{N}_i}^{[i]}(t)) \tag{2.2}$$

for some function $c_i : (\mathbb{R}^d)^{n_i} \to \mathbb{R}^d$, where $n_i = |\mathcal{N}_i|$ and $x_{\mathcal{N}_i}^{[i]} = (x_{j_1}^{[i]}, x_{j_2}^{[i]}, \ldots, x_{j_{n_i}}^{[i]}) \in (\mathbb{R}^d)^{n_i}$ describes the collection of the relative positions of the neighbors $j_1, j_2, \ldots, j_{n_i} \in \mathcal{N}_i$ such that $j_1 < j_2 < \cdots < j_{n_i}$. The control objective is to converge the collective states $x_{\mathcal{N}} := (x_1, x_2, \ldots, x_n)$ of all the robots to the set $\mathcal{D} \subset (\mathbb{R}^d)^n$ of the desired configurations. We want to find $u_i(t) (i \in \mathcal{N})$ of the form (2.2) which aims to derive

$$\lim_{t \to \infty} x_{\mathcal{N}}(t) \in \mathcal{D}.$$

## 2.2 Control objectives

Two types of coordination tasks will be considered as the control objectives of the multi-robot system. The first one is pairwise coordination, formulated with functions each of which depends only on a pair of robots. This coordination is easily handled and has been widely employed in many papers. The second one is the main target of this monograph, namely, generalized coordination, which is expressed with a distance function from the desired configuration set. This coordination contains a wide range of tasks including pairwise coordination, and leads to a unified solution to multi-robot coordination problems.

### 2.2.1 Pairwise coordination

Let $\psi_{ij} : (\mathbb{R}^d)^2 \to \mathbb{R}_+$ be a non-negative function of $x_i$ and $x_j$, the states of a pair of robots $i, j \in \mathcal{N}$, which determines a desired configuration of the two robots. Then, we say that *pairwise coordination* with respect to functions $(\psi_{ij}(x_i, x_j))_{i,j\in\mathcal{N},i\neq j}$ is achieved if

$$\lim_{t\to\infty} \psi_{ij}(x_i(t), x_j(t)) = 0 \quad \forall i, j \in \mathcal{N}, i \neq j. \tag{2.3}$$

Functions $(\psi_{ij}(x_i, x_j))_{i,j\in\mathcal{N},i\neq j}$ are said to be *realizable* if

$$\exists x^*_{\mathcal{N}} \in (\mathbb{R}^d)^n \quad \text{s.t.} \quad \psi_{ij}(x^*_i, x^*_j) = 0 \quad \forall i, j \in \mathcal{N}, i \neq j. \tag{2.4}$$

In general, there are infinitely many convergent points according to (2.3). This makes the coordination control problem complicated, and it will turn out to be important to clarify DOF (degrees of freedom) of the coordination. Let us see this by typical examples of pairwise coordination as follows.

**Example 2.1.** *Displacement-based formation* is to achieve a configuration prescribed by the desired relative positions $r_{ij} \in \mathbb{R}^d$ between robots as

$$\lim_{t\to\infty} (x_i(t) - x_j(t)) = r_{ij} \quad \forall i, j \in \mathcal{N}, i \neq j. \tag{2.5}$$

This is a pairwise coordination problem (2.3) with respect to the pairwise functions

$$\psi_{ij}(x_i, x_j) = \|x_i - x_j - r_{ij}\|^2. \tag{2.6}$$

The functions in (2.6) are realizable if there exists a collection of vectors $x^*_{\mathcal{N}} \in (\mathbb{R}^d)^n$ satisfying $x^*_i - x^*_j = r_{ij}$ for all $i, j \in \mathcal{N}, i \neq j$. Note that under (2.5), the positions $x_{\mathcal{N}}(t)$ of the robots can be translated from the configuration $x^*_{\mathcal{N}}$ as Fig. 2.2. Hence, this coordination has the DOF of translation.

**Example 2.2.** *Distance-based formation* is to achieve the desired distance $d_{ij} > 0$ between robots as

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = d_{ij} \quad \forall i, j \in \mathcal{N}, i \neq j. \tag{2.7}$$

This is a pairwise coordination problem (2.3) with respect to the pairwise functions

$$\psi_{ij}(x_i, x_j) = (\|x_i - x_j\|^2 - d_{ij}^2)^2. \tag{2.8}$$

The functions in (2.8) are realizable if there exists a collection of vectors $x^*_{\mathcal{N}} \in (\mathbb{R}^d)^n$ satisfying $\|x^*_i - x^*_j\| = d_{ij}$ for all $i, j \in \mathcal{N}, i \neq j$. Under (2.7), $x_{\mathcal{N}}(t)$ can be translated, rotated, and reflected from $x^*_{\mathcal{N}}$ as illustrated in Fig. 2.3. Hence, this coordination has the DOF of translation, rotation, and reflection[a].

[a]Here, not only rotation and translation, but also reflection is included in DOF from the viewpoint of possible transformation.

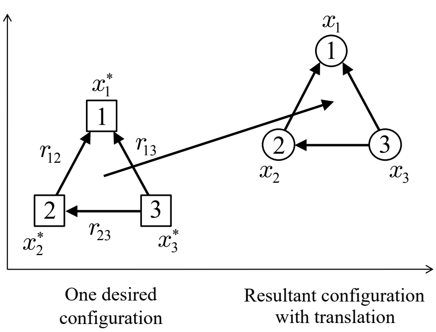



**Figure 2.2:** Displacement-based formation: the resultant configuration $(x_1, x_2, x_3)$ can be translated from $(x_1^*, x_2^*, x_3^*)$.

**Example 2.3.** *Encircling formation* is to achieve the two conditions for a given target: (i) the desired distance from the target is attained by each robot, (ii) the desired angle between each pair of the robots around the target is achieved. Without loss of generality, assume that the target is at the origin. Then, conditions (i) and (ii) are described as follows:

$$\begin{cases} \lim_{t\to\infty} \|x_i(t)\| = d_i \quad \forall i \in \mathcal{N} \\ \lim\limits_{t\to\infty} \cos^{-1} \dfrac{\langle x_i(t), x_j(t)\rangle}{\|x_i(t)\|\|x_j(t)\|} = \theta_{ij} \quad \forall i, j \in \mathcal{N}, i \neq j, \end{cases} \tag{2.9}$$

where $d_i > 0$ is the desired distance of robot $i$ and $\theta_{ij} \in [0, \pi]$ is the desired angle between robots $i$ and $j$. This is a pairwise coordination problem (2.3) with respect to the pairwise functions

$$\begin{aligned} \psi_{ij}(x_i, x_j) = (\|x_i\|^2 - d_i^2)^2 + (\|x_j\|^2 - d_j^2)^2 \\ + (\langle x_i, x_j\rangle - d_i d_j \cos\theta_{ij})^2. \end{aligned} \tag{2.10}$$

The functions in (2.10) are realizable when there exists a collection of vectors $x_{\mathcal{N}}^* \in (\mathbb{R}^d)^n$ satisfying $\|x_i^*\| = d_i$ and $\langle x_i^*, x_j^*\rangle = d_i d_j \cos\theta_{ij}$ for all $i, j \in \mathcal{N}, i \neq j$. Note that under (2.9), $x_{\mathcal{N}}(t)$ can be rotated and reflected from $x_{\mathcal{N}}^*$ as illustrated in Fig. 2.4. Hence, this coordi-

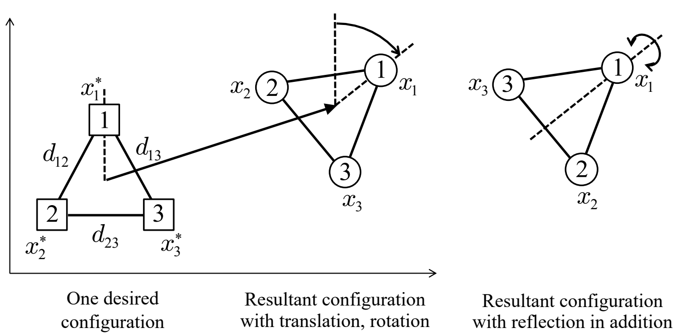



**Figure 2.3:** Distance-based formation: the resultant configuration $(x_1, x_2, x_3)$ can be translated, rotated, and reflected from $(x_1^*, x_2^*, x_3^*)$.

nation has the DOF of rotation and reflection, but does not have that of translation.

**Example 2.4.** We say that the robots reach *consensus* if

$$\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0 \quad \forall i, j \in \mathcal{N}. \tag{2.11}$$

This is a pairwise coordination problem (2.3) with respect to the pairwise functions

$$\psi_{ij}(x_i, x_j) = \|x_i - x_j\|^2. \tag{2.12}$$

The functions in (2.12) are always realizable because (2.4) holds when $x_i^*$ are the same for all $i \in \mathcal{N}$.

Whether functions $(\psi_{ij}(x_i, x_j))_{i,j\in\mathcal{N},i\neq j}$ are realizable or not usually depends on the parameters describing the desired configurations. Appropriate parameters can be easily determined from one desired configuration $x_{\mathcal{N}}^* \in (\mathbb{R}^d)^n$. For example, in distance-based formation of Example 2.2, once one $x_{\mathcal{N}}^*$ is given, realizable functions in (2.8) can be assigned with the desired distances $d_{ij} = \|x_i^* - x_j^*\|$.

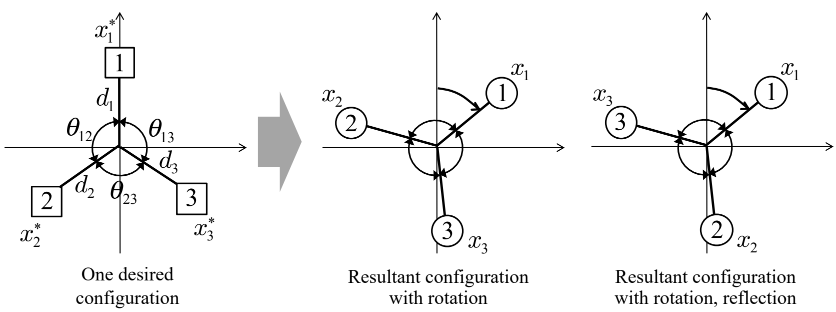



**Figure 2.4:** Encircling formation: the resultant configuration $(x_1, x_2, x_3)$ can be rotated and reflected from $(x_1^*, x_2^*, x_3^*)$.

### 2.2.2 Generalized coordination

Pairwise coordination implicitly assumes that the coordination problem in question is decomposable into coordination problems on pairs of robots. However, this is not the case in general. To overcome this drawback, we introduce generalized coordination. For a set $\mathcal{D} \subset (\mathbb{R}^d)^n$, we say that *generalized coordination* with respect to $\mathcal{D}$ is achieved if

$$\lim_{t\to\infty} \mathrm{dist}(x_{\mathcal{N}}(t), \mathcal{D}) = 0. \tag{2.13}$$

The set $\mathcal{D}$, called a *desired configuration set*, describes desired configurations in a general way. Here, the distance function is defined as

$$\mathrm{dist}(x_{\mathcal{N}}, \mathcal{D}) := \inf_{y_{\mathcal{N}}\in\mathcal{D}} \|x_{\mathcal{N}} - y_{\mathcal{N}}\|$$

for $x_{\mathcal{N}} \in (\mathbb{R}^d)^n$ and $\mathcal{D} \subset (\mathbb{R}^d)^n$, where the norm $\|x_{\mathcal{N}}\|$ of the $n$-tuples $x_{\mathcal{N}} \in (\mathbb{R}^d)^n$ is defined with the Euclidean norms $\|x_i\|$ of vectors $x_i \in \mathbb{R}^d$ $(i \in \mathcal{N})$ as follows:

$$\|x_{\mathcal{N}}\| := \sqrt{\sum_{i\in\mathcal{N}} \|x_i\|^2}.$$

For example, distance-based formation shown in Example 2.2 can be expressed by (2.13) with

$$\mathcal{D} = \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : \|x_i - x_j\| = d_{ij} \ \ \forall i, j \in \mathcal{N}, i \neq j\}. \tag{2.14}$$

Alternatively, we can express the set $\mathcal{D}$ in (2.14) as follows. Let $x^*_{\mathcal{N}} \in \mathcal{D}$ be a desired configuration which is chosen arbitrarily in $\mathcal{D}$, then any $x_{\mathcal{N}} \in \mathcal{D}$ can be described as

$$x_i = Sx^*_i + \tau \ \ (i \in \mathcal{N}) \tag{2.15}$$

with an orthogonal matrix $S \in \mathbb{R}^{d\times d}$ representing transformation of rotation and reflection and a vector $\tau \in \mathbb{R}^d$ representing translation transformation of $x_{\mathcal{N}}$ from $x^*_{\mathcal{N}}$. Both the transformations show DOF (degrees of freedom) of the coordination as shown in Example 2.2. Hence, defining $\mathcal{S} = \mathrm{O}(d)$, $\mathcal{T} = \mathbb{R}^d$,

$$\mathcal{D} = \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : \exists (S, \tau) \in \mathcal{S}\times\mathcal{T} \ \text{ s.t. } \ x_i = Sx^*_i + \tau \ \ \forall i \in \mathcal{N}\} \tag{2.16}$$

is an alternative representation to (2.14), where $\mathrm{O}(d)$ denotes the orthogonal group of dimension $d$.

By extending both sets $\mathcal{S}$ and $\mathcal{T}$ to general ones, much broader class of coordination can be defined by (2.16). The Cartesian set $\mathcal{S} \times \mathcal{T}$, which is named a *coordination freedom set*, directly determines DOF in coordination. Specifically, $\mathcal{S}$ can give the DOF of rotation, scale, and reflection, while $\mathcal{T}$ can give that of translation in coordination.

Various coordination tasks can be expressed by the generalized coordination with respect to the set $\mathcal{D}$ of the form (2.16). Some examples are given as follows.

**Example 2.5.** The examples of the pairwise coordination in Subsection 2.2.1 are expressed by the generalized coordination problem (2.13) with respect to $\mathcal{D}$ in (2.16) through the following $\mathcal{S} \times \mathcal{T}$.

- Displacement-based formation in Example 2.1 is given with $\mathcal{S} \times \mathcal{T} = \{I_d\} \times \mathbb{R}^d$.
- Distance-based formation in Example 2.2 is given with $\mathcal{S} \times \mathcal{T} = \mathrm{O}(d) \times \mathbb{R}^d$.
- Encircling formation in Example 2.3 is given with $\mathcal{S} \times \mathcal{T} = \mathrm{O}(d) \times \{0\}$.

The following are examples which cannot be described by the pairwise coordination (2.3).

**Example 2.6.** *Position-based formation* is an individual regulation problem for the desired position $x_i^* \in \mathbb{R}^d$ as

$$\lim_{t \to \infty} x_i(t) = x_i^* \quad \forall i \in \mathcal{N}.$$

This is a generalized coordination problem (2.13) with respect to the desired configuration set $\mathcal{D}$ in (2.16) for $\mathcal{S} \times \mathcal{T} = \{I_d\} \times \{0\}$.

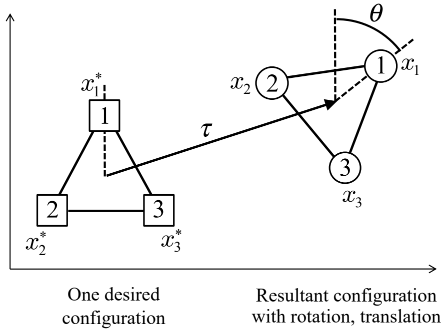



**Figure 2.5:** Reflection-free formation: the resultant configuration $(x_1, x_2, x_3)$ can be rotated and translated from $(x_1^*, x_2^*, x_3^*)$, where $\theta \in [-\pi, \pi)$ and $\tau \in \mathbb{R}^d$ denote rotation angle and translation vector, respectively.

**Example 2.7.** *Reflection-free formation* has the DOF of rotation and translation as the distance-based formation (2.7), but prohibits reflection from the desired configuration. This is characterized with $\mathcal{S} \times \mathcal{T} = \mathrm{SO}(d) \times \mathbb{R}^d$ in (2.16) instead of $\mathcal{S} \times \mathcal{T} = \mathrm{O}(d) \times \mathbb{R}^d$, where $\mathrm{SO}(d)$ denotes the special orthogonal group of dimension $d$. See Fig. 2.5 for an example in a $d = 2$-dimensional space, where $\theta \in [-\pi, \pi)$ and $\tau \in \mathbb{R}^2$ represent a rotation angle and translation vector of $x_\mathcal{N}$ from $x^*_\mathcal{N}$. In this case, $S = \mathrm{Rot}(\theta) \in \mathcal{S}$ and $\tau \in \mathcal{T}$ are assigned in (2.15), where $\mathrm{Rot}(\cdot)$ represents the rotation matrix of dimension 2 as

$$\mathrm{Rot}(\theta) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

**Example 2.8.** *Scaling position-based formation* is to achieve the desired position $x_i^* \in \mathbb{R}^d$ with an arbitrary scale as

$$\exists s > 0 \ \text{ s.t. } \lim_{t\to\infty} x_i(t) = s x_i^* \quad \forall i \in \mathcal{N}.$$

The scale factor $s > 0$ is common to the robots, and the resultant configuration of $x_\mathcal{N}(t)$ is similar to $x^*_\mathcal{N}$ as illustrated in Fig. 2.6. This task is a generalized coordination problem (2.13) with respect

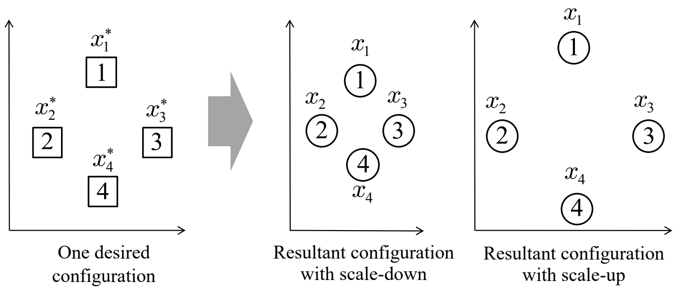



**Figure 2.6:** Scaling position-based formation: the resultant configuration $(x_1, x_2, x_3, x_4)$ can be scaled from $(x_1^*, x_2^*, x_3^*, x_4^*)$.

to $\mathcal{D}$ in (2.16) for $\mathcal{S} \times \mathcal{T} = \text{scaled}(\{I_d\}) \times \{0\}$, where scaled$(\cdot)$ represents the scaled set, i.e., $\text{scaled}(\mathcal{S}) := \{sS : s > 0, S \in \mathcal{S}\}$ for a set $\mathcal{S}$.

**Example 2.9.** *Scaling displacement-based formation* is to achieve the desired configuration with an arbitrary scale in the displacement-based formation (2.5) as

$$\exists s > 0 \ \text{ s.t. } \ \lim_{t\to\infty} (x_i(t) - x_j(t)) = s r_{ij} \quad \forall i, j \in \mathcal{N}, i \neq j.$$

This problem is characterized with $\mathcal{S} \times \mathcal{T} = \text{scaled}(\{I_d\}) \times \mathbb{R}^d$ in (2.16).

**Example 2.10.** *Scaling distance-based formation* is given as

$$\exists s > 0 \ \text{ s.t. } \ \lim_{t\to\infty} \|x_i(t) - x_j(t)\| = s d_{ij} \quad \forall i, j \in \mathcal{N}, i \neq j$$

by adding the DOF of scale to the distance-based formation (2.7). This problem is characterized with $\mathcal{S} \times \mathcal{T} = \text{scaled}(\mathrm{O}(d)) \times \mathbb{R}^d$ in (2.16).

**Example 2.11.** *Scaling reflection-free formation* is a generalized coordination problem (2.13) with respect to $\mathcal{D}$ in (2.16) for $\mathcal{S} \times \mathcal{T} = \mathrm{scaled}(\mathrm{SO}(d)) \times \mathbb{R}^d$.

The following examples are the generalized coordination that cannot be expressed with $\mathcal{D}$ of the form (2.16).

**Example 2.12.** *Formation selection* is a task to select and form one of the prescribed configuration patterns, as illustrated in Fig. 2.7. Let $p \in \mathbb{Z}_+ \backslash \{0\}$ be the number of the patterns, and let $\mathcal{Q} = \{1, 2, \ldots, p\}$ denote the index set of the patterns. The $q$th desired configuration pattern for $q \in \mathcal{Q}$ is prescribed by $x_{\mathcal{N}}^{*q} \in (\mathbb{R}^d)^n$. Then, this task is represented as

$$\exists q \in \mathcal{Q} \;\text{ s.t. } \lim_{t \to \infty} x_i(t) = x_i^{*q} \;\; \forall i \in \mathcal{N}.$$

This task is a generalized coordination problem (2.13) with respect to the desired configuration set

$$\mathcal{D} = \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : \exists q \in \mathcal{Q} \;\text{ s.t. } x_{\mathcal{N}} = x_{\mathcal{N}}^{*q}\} = \bigcup_{q \in \mathcal{Q}} \{x_{\mathcal{N}}^{*q}\}, \tag{2.17}$$

which is a discrete set consisting of multiple points.

**Example 2.13.** *Position assignment* is to achieve the desired configuration described by $x_1^*, x_2^*, \ldots, x_n^* \in \mathbb{R}^d$ with any assignment. That is, $x_i(t)$ can be assigned to any of $x_1^*, x_2^*, \ldots, x_n^*$ as long as the assignments are not overlapped as illustrated in Fig. 2.8. This task is represented as

$$\exists \alpha \in \mathcal{P}_n \;\text{ s.t. } \lim_{t \to \infty} x_i(t) = x_{\alpha(i)}^* \;\; \forall i \in \mathcal{N}, \tag{2.18}$$

where $\mathcal{P}_n$ represents the set of permutations of $n$ elements. Equation (2.18) means that the reference $x_k^*$ for $k = \alpha(i)$ is assigned to robot $i$ through a permutation $\alpha \in \mathcal{P}_n$. This is a generalized coordination

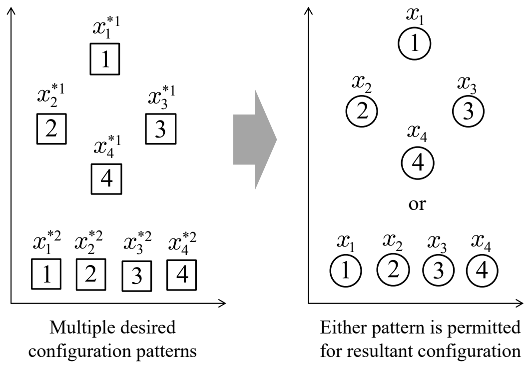



**Figure 2.7:** Formation selection: the resultant configuration $(x_1, x_2, x_3, x_4)$ is expected to form either of the desired patterns $(x_1^{*1}, x_2^{*1}, x_3^{*1}, x_4^{*1})$ or $(x_1^{*2}, x_2^{*2}, x_3^{*2}, x_4^{*2})$.

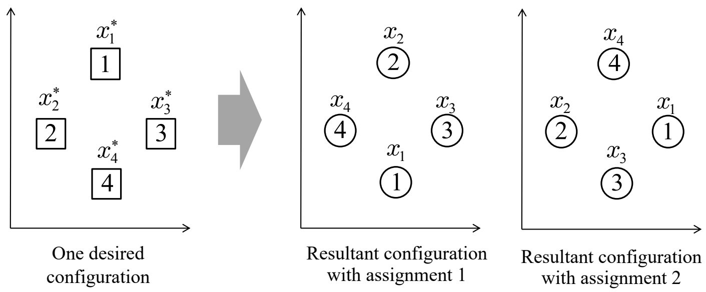



**Figure 2.8:** Position assignment: the resultant configuration $(x_1, x_2, x_3, x_4)$ has to form the desired configuration $(x_1^*, x_2^*, x_3^*, x_4^*)$ with any assignment.

problem (2.13) with respect to the desired configuration set

$$\begin{aligned}\mathcal{D} &= \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : \exists \alpha \in \mathcal{P}_n \ \text{ s.t. } \ x_i = x^*_{\alpha(i)} \quad \forall i \in \mathcal{N}\} \\ &= \bigcup_{\alpha \in \mathcal{P}_n} \{(x^*_{\alpha(1)}, x^*_{\alpha(2)}, \dots, x^*_{\alpha(n)})\},\end{aligned} \tag{2.19}$$

which consists of multiple points and is discrete.

## 2.3 Control with relative measurements

So far, everything is described in a global coordinate frame which is common to all the robots. However, depending on the sensing capability, measurements of each robot are given in its own local coordinate frame. Hence, first, the relation between the global coordinate and the local coordinate is explicitly stated. Second, kinematic models of robots will be given. Third, admissible controllers subject to relative measurements are described.

### 2.3.1 Frame transformation due to sensing capability

Let $\Sigma$ and $\Sigma_i(t)$ denote the global coordinate frame and the local coordinate frame corresponding to robot $i$, respectively. Suppose $p(t) \in \mathbb{R}^d$ represents the position of a moving object at time $t$ in the global coordinate frame $\Sigma$, and $p^{[i]}(t) \in \mathbb{R}^d$ corresponds to its presentation in the local coordinate frame $\Sigma_i(t)$. Then, the following relation holds:

$$p(t) = M_i(t)p^{[i]}(t) + b_i(t) \tag{2.20}$$

with a matrix $M_i(t) \in \mathbb{R}^{d\times d}$ and a vector $b_i(t) \in \mathbb{R}^d$. One simple example in a two-dimensional space is shown in Fig. 2.9, where $M_i(t) = \mathrm{Rot}(\theta_i(t)) \in \mathrm{SO}(2)$ represents the rotation of $\Sigma_i(t)$, where $\theta_i(t) \in [-\pi, \pi)$ is the bearing angle of the front of robot $i$, and $b_i(t) = x_i(t) \in \mathbb{R}^2$ represents the translation of $\Sigma_i(t)$.

In general, $M_i(t)$ and $b_i(t)$ are heterogeneous and cannot be specified by anyone including robot $i$ itself. However, we may assume that they belong to some known sets $\mathcal{M}$ and $\mathcal{B}$, i.e.,

$$M_i(t) \in \mathcal{M}, \quad b_i(t) \in \mathcal{B}$$

holds. In the above example, these sets are given by $\mathcal{M} = \mathrm{SO}(2)$, $\mathcal{B} = \mathbb{R}^2$. The pair $(\mathcal{M}, \mathcal{B})$ will define the possible transformation of the local coordinate frame $\Sigma_i(t)$. Therefore, the Cartesian product $\mathcal{M} \times \mathcal{B}$ is called *the frame transformation set*. Since the frame transformation set depends on the sensing capability of the robots, several examples will be given below.

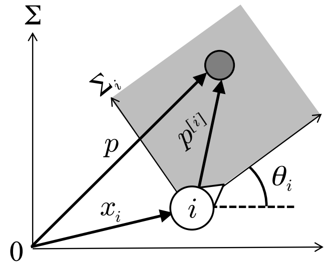
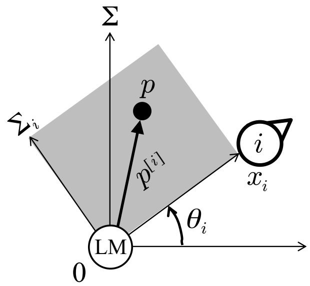



**Figure 2.9:** Global and local coordinate frames $\Sigma$, $\Sigma_i(t)$ in a two-dimensional space.

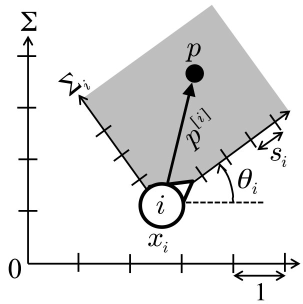
**Example 2.14.** If each robot recognizes its absolute position and the absolute bearing by using a GPS (Global Positioning Sensor), the global coordinate $p(t)$ is directly measurable, implying that $p^{[i]}(t) = p(t)$. Then, (2.20) holds for $M_i(t) = I_d$ and $b_i(t) = 0$, and the global and local coordinate frames $\Sigma, \Sigma_i(t)$ are equivalent. In this case, the frame transformation set is assigned as $\mathcal{M} \times \mathcal{B} = \{I_d\} \times \{0\}$.

**Example 2.15.** If there is a landmark observable by all the robots, but the absolute bearing is unavailable to them, the origins of the local coordinate frames $\Sigma_i(t)$ of the robots can be assigned to the position of the landmark, but the orientations cannot be aligned. Then, $b_i(t) = 0$ is obtained while the difference of the orientations is expressed by a rotation matrix $M_i(t) \in \mathrm{SO}(d)$ in (2.20). In this case, the transformation in rotation can occur, and $\mathcal{M} \times \mathcal{B} = \mathrm{SO}(d) \times \{0\}$ is obtained. See Fig. 2.10a for the illustration of the relation of the coordinate frames in a two-dimensional space.

**Example 2.16.** If the absolute bearing is available to each robot by using a compass while the absolute position is unavailable, the orientations of $\Sigma$ and $\Sigma_i(t)$ can be aligned while the origins cannot be at the same position. By assigning the position $x_i(t)$ of robot $i$ to the origin $b_i(t)$ of $\Sigma_i(t)$, $M_i(t) = I_d$ and $b_i(t) = x_i(t)$ are obtained in (2.20). Because the value of $x_i(t) \in \mathbb{R}^d$ is unknown, transformation




**(a)** $\mathcal{M} \times \mathcal{B} = \mathrm{SO}(2) \times \{0\}$




**(b)** $\mathcal{M} \times \mathcal{B} = \mathrm{scaled}(\mathrm{SO}(2)) \times \mathbb{R}^2$

**Figure 2.10:** Relation between global and local coordinate frames $\Sigma, \Sigma_i(t)$: (a) the origins of the frames correspond to the position of the landmark, while their orientations do not coincide; (b) the origin of $\Sigma_i(t)$ is at the robot position $x_i(t)$, and neither of the orientations nor scales of the frames are not equivalent.

in translation occurs, and $\mathcal{M} \times \mathcal{B} = \{I_d\} \times \mathbb{R}^d$ is achieved. See Example A.2 for more details.

**Example 2.17.** Without the absolute bearing or the absolute positions, neither the orientations nor the origins of $\Sigma$ and $\Sigma_i(t)$ can coincide. In this case, transformation in rotation and translation occurs, and $M_i(t) \in \mathrm{SO}(d)$ and $b_i(t) = x_i(t)$ are obtained. Then, $\mathcal{M} \times \mathcal{B} = \mathrm{SO}(d) \times \mathbb{R}^d$ is achieved, as illustrated by Fig. 2.9 for the case of $d = 2$. Example A.1 describes the transformation in more detail.

**Example 2.18.** In addition to Example 2.17, if the scale factors of distance sensors are incorrect, the scales of the local coordinate frames $\Sigma_i(t)$ can be different among the robots, which causes transformation in scale. Then, (2.20) is satisfied with $M_i(t) = s_i(t)R_i(t)$ for a rotation matrix $R_i(t) \in \mathrm{SO}(d)$ and a scale $s_i(t) > 0$. In this case, $\mathcal{M} \times \mathcal{B} = \mathrm{scaled}(\mathrm{SO}(d)) \times \mathbb{R}^d$ is obtained. The relation of the frames in a two-dimensional space is illustrated in Fig. 2.10b. See Example A.3 for more details.

**Example 2.19.** In addition to Example 2.17, consider the situation that a flip ambiguity occurs from distance-based localization (Kannan *et al.*, 2007), which causes transformation in reflection to $\Sigma_i(t)$. Then, $M_i(t)$ corresponds to either $R_i(t)$ or $R_i(t)\mathrm{Refl}(w)$ for a matrix $R_i(t) \in \mathrm{SO}(d)$ and a unit vector $w \in \mathbb{R}^d$, where $\mathrm{Refl}(w) := I_d - 2ww^\top$ is the reflection matrix with respect to $w$. Therefore, $M_i(t) \in \mathrm{O}(d)$ holds, and $\mathcal{M} \times \mathcal{B} = \mathrm{O}(d) \times \mathbb{R}^d$ is obtained. See Example A.4 for more details.

### 2.3.2 Kinematic models

For simplicity and clarity, kinematic models over the local coordinate frame are employed. Namely, the state $x_i(t) \in \mathbb{R}^d$ of robot $i$ is supposed to be governed by

$$\dot{x}_i(t) = M_i(t)u_i(t), \tag{2.21}$$

where $u_i(t) \in \mathbb{R}^d$ represents the input to determine the local velocity and $M_i(t) \in \mathcal{M}$ is a coordinate transformation matrix as shown in (2.20). This implies that we assume that each robot is locally controlled so as to move along with its velocity command in its local coordinate frame. Note that the translation term $b_i(t)$ does not matter in kinematics, so it does not appear here.

Some examples of the kinematics models according to the sensing capability are given below.

- Corresponding to Examples 2.14 and 2.16, when the absolute bearing is available, $\mathcal{M} = \{I_d\}$ is obtained. Then, for $M_i(t) = I_d$, the model (2.21) is reduced to the single-integrator system $\dot{x}_i(t) = u_i(t)$.

- Corresponding to Examples 2.15 and 2.17, if the absolute bearing is unavailable, $\mathcal{M} = \mathrm{SO}(d)$ is obtained. Then, the kinematic model is given as (2.21) for $M_i(t) = R_i(t)$ with a rotation matrix

$R_i(t) \in \mathrm{SO}(d)$. Note that the input $u_i(t)$ cannot directly determine which direction robot $i$ moves toward because the motion depends on the unknown rotation $R_i(t)$.

- Corresponding to Example 2.18, if the scale factors of distance sensors are incorrect, $\mathcal{M} = \mathrm{scaled}(\mathrm{SO}(d))$ is employed. Then, the kinematic model is provided as (2.21) for $M_i(t) = s_i(t)R_i(t)$ with $R_i(t) \in \mathrm{SO}(d)$ and $s_i(t) > 0$. Note that in this case, the speed of robot $i$ obtained from the velocity command $u_i(t)$ is unknown because the unknown scale factor $s_i(t)$ multiplies the velocity command in (2.21).

### 2.3.3 Relative measurements

When robot $i \in \mathcal{N}$ observes robot $j \in \mathcal{N}$ in the local coordinate frame $\Sigma_i(t)$, the measured position is expressed from (2.20) as

$$x_j^{[i]}(t) = M_i^{-1}(t)(x_j(t) - b_i(t)) \tag{2.22}$$

for the position $x_j(t) \in \mathbb{R}^d$ in $\Sigma$ of robot $j$ with some $(M_i(t), b_i(t)) \in \mathcal{M} \times \mathcal{B}$. Here, $x_j^{[i]}(t)$ is called the *relative position* of robot $j$ in $\Sigma_i(t)$.

Based on (2.22), various types of relative measurements can be described as follows.

- Corresponding to Example 2.14, when the absolute position and bearing are available to each robot, the absolute positions of the neighbors can be obtained. Actually, for $\mathcal{M} \times \mathcal{B} = \{I_d\} \times \{0\}$, (2.22) with $(M_i(t), b_i(t)) = (I_d, 0) \in \mathcal{M} \times \mathcal{B}$ is reduced to $x_j^{[i]}(t) = x_j(t)$.
- Corresponding to Example 2.16, when the absolute bearing is available while the absolute position is not, the relative positions of the neighbors are obtained in aligned local coordinate frames. Then, for $\mathcal{M} \times \mathcal{B} = \{I_d\} \times \mathbb{R}^d$, (2.22) with $(M_i(t), b_i(t)) = (I_d, x_i(t)) \in \mathcal{M} \times \mathcal{B}$ is reduced to $x_j^{[i]}(t) = x_j(t) - x_i(t)$.
- Corresponding to Example 2.17, when neither the absolute bearing nor the absolute position is available, the relative positions of the neighbors are obtained in misaligned local coordinate frames.

Actually, for $\mathcal{M} \times \mathcal{B} = \mathrm{SO}(d) \times \mathbb{R}^d$, the relative position (2.22) with $(M_i(t), b_i(t)) = (R_i(t), x_i(t)) \in \mathcal{M} \times \mathcal{B}$ is reduced to $x_j^{[i]}(t) = R_i^\top(t)(x_j(t) - x_i(t))$.

- Additionally, if the scale factors of distance sensors are incorrect, $\mathcal{M} \times \mathcal{B} = \mathrm{scaled}(\mathrm{SO}(d)) \times \mathbb{R}^d$ is obtained, corresponding to Example 2.18. Then, the relative position (2.22) with $(M_i(t), b_i(t)) = (s_i(t)R_i(t), x_i(t)) \in \mathcal{M} \times \mathcal{B}$ is reduced to $x_j^{[i]}(t) = s_i^{-1}(t)R_i^\top(t)(x_j(t) - x_i(t))$.

### 2.3.4 Admissible controllers

Over the sensing network $G = (\mathcal{N}, \mathcal{E})$, each robot obtains the information on the relative positions $x_{\mathcal{N}_i}^{[i]}(t)$ of the neighbors, where the neighbor set $\mathcal{N}_i$ is defined as

$$\mathcal{N}_i := \{j \in \mathcal{N} : \{i, j\} \in \mathcal{E}\} \cup \{i\}.$$

Then, the control input $u_i(t)$ has to be generated only with $x_{\mathcal{N}_i}^{[i]}(t)$, and thus a static controller can be implemented if it is of the form

$$u_i(t) = c_i(x_{\mathcal{N}_i}^{[i]}(t)) \tag{2.23}$$

with a function $c_i : (\mathbb{R}^d)^{|\mathcal{N}_i|} \to \mathbb{R}^d$ depending only on $x_{\mathcal{N}_i}^{[i]}(t)$. This function $c_i(x_{\mathcal{N}_i}^{[i]})$ is called a (static) *distributed controller with relative measurements*. Here, we assume that only such a controller is admissible.

Fig. 2.11 illustrates the block diagram of the overall system, consisting of the kinematic model (2.21), the relative positions (2.22) of the neighbors for $b_i(t) = x_i(t)$, and the distributed controller (2.23) with relative measurements.

## 2.4 Problem formulation

The target problem is formulated from the control-theoretic viewpoint in this section. The concepts of stability are defined as follows. Let a closed set $\mathcal{D} \subset (\mathbb{R}^d)^n$ be an equilibrium set of the system (2.21) for $i \in \mathcal{N}$ with some control input $u_i(t) \in \mathbb{R}^d$. The set $\mathcal{D}$ is said to be

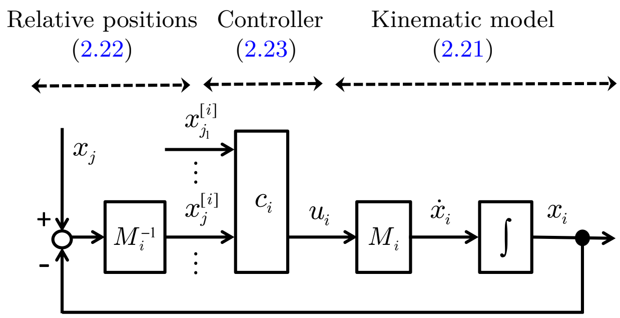



**Figure 2.11:** Block diagram of the overall system, consisting of the kinematic model (2.21), the relative positions (2.22) for $b_i(t) = x_i(t)$, and the distributed controller (2.23) with relative measurements.

*(Lyapunov) stable* if for each $\varepsilon > 0$, there exists an open set $\Delta(\varepsilon) \supset \mathcal{D}$ such that

$$x_{\mathcal{N}}(0) \in \Delta(\varepsilon) \Rightarrow \mathrm{dist}(x_{\mathcal{N}}(t), \mathcal{D}) \le \varepsilon \quad \forall t \ge 0.$$

In addition, $\mathcal{D}$ is said to be *asymptotically stable* if $\mathcal{D}$ is stable and there exists an open set $\Delta \supset \mathcal{D}$ such that

$$x_{\mathcal{N}}(0) \in \Delta \Rightarrow \lim_{t\to\infty} \mathrm{dist}(x_{\mathcal{N}}(t), \mathcal{D}) = 0.$$

To achieve the generalized coordination (2.13) with respect to a desired configuration set $\mathcal{D}$, we want to design a distributed controller with relative measurements such that $\mathcal{D}$ is asymptotically stable. The solvability of this problem fully depends on the characteristics of the triple $(\mathcal{D}, G, \mathcal{M} \times \mathcal{B})$ since the measurements are transformed as (2.22) by unknown $(M_i(t), b_i(t)) \in \mathcal{M} \times \mathcal{B}$ and the available information is limited as (2.23) by $G$. Hence, the condition of the triple to solve this problem has to be specified.

The problem tackled in this monograph is summarized as follows.

**Problem 2.1.** For a desired configuration set $\mathcal{D} \subset (\mathbb{R}^d)^n$, a graph $G$, and a frame transformation set $\mathcal{M} \times \mathcal{B} \subset \mathbb{R}^{d\times d} \times \mathbb{R}^d$, specify the triple $(\mathcal{D}, G, \mathcal{M} \times \mathcal{B})$ such that there exists a distributed controller (2.23) with relative measurements (2.22) for $G$ and $\mathcal{M} \times \mathcal{B}$ such that $\mathcal{D}$ is asymptotically stable for the kinematic model (2.21).

Furthermore, design such a controller when it exists.

## 2.5 Notes and references

There are many reviews and books for multi-robot (multi-agent) coordination problems through a control-theoretic approach such as Bullo *et al.*, 2009; Bai *et al.*, 2011; Ren and Cao, 2011; Oh *et al.*, 2015; Tron *et al.*, 2016; Cortes and Egerstedt, 2017; Sun, 2018; Queiroz *et al.*, 2019; Ahn, 2020. The features of this monograph are two-fold: (i) The description of the coordination tasks is generalized as (2.13) with the desired configuration set $\mathcal{D}$, while individual tasks are treated in the conventional research. (ii) The concept of the local coordinate frame $\Sigma_i(t)$ is generalized with the frame transformation set $\mathcal{M}\times\mathcal{B}$, while only $\mathcal{M}\times\mathcal{B} = \mathrm{SO}(d)\times\mathbb{R}^d$ is employed in the conventional research. As a result, various types of coordination tasks and relative measurements can be expressed in the same manners, and essential connections between them are revealed through the triple $(\mathcal{D}, G, \mathcal{M}\times\mathcal{B})$ in the following chapters. These formulations were firstly introduced in Sakurama *et al.*, 2015 and Sakurama, 2021b, respectively. While the form (2.16) of the desired configuration has been taken for affine formation control as Lin *et al.*, 2016; Zhao, 2018, relative measurements are not considered in these papers.

There are several possible extensions of the problem in this monograph. First, instead of the kinematic model (2.21), we can consider more general dynamics (2.1), e.g., passive systems. The approach of this monograph, the gradient-flow method, is directly applicable to passive systems even if the dynamics of the robots is heterogeneous or includes uncertain parameters. See Bai *et al.*, 2011; Hatanaka *et al.*, 2015 for details on passive systems. Non-holonomic systems are treated in Section 9.4. Second, the static controller (2.23) can be extended to a dynamic one to enhance control performance. For example, in Rozenheck *et al.*, 2015, a PI-type formation controller to remove tracking errors has been proposed. In Sakurama, 2021a, a PI-type formation controller has been employed to remove formation errors caused by uncertain body rotations and input disturbances. Third, the gradient-flow approach is practically useful to time-varying networks although theoretical results are valid only for time-invariant networks. See Section 9.3 for simulation results with a state-dependent time-varying network.

# Part I

# Mathematical Preliminaries

# 3

# Group Theory

A *group* is a set with a binary operation satisfying the four axioms: closure, associativity, identity, and inverse. Group theory is essential to the control theory of multi-robot systems. Actually, the coordination freedom set $\mathcal{S} \times \mathcal{T}$ in Subsection 2.2.2 and the frame transformation set $\mathcal{M} \times \mathcal{B}$ in Subsection 2.3.3 can be handled as a type of group, called a semidirect product. Accordingly, a unified solution to multi-robot coordination problems is provided using tools of group theory.

This chapter provides relevant concepts, including *groups, subgroups, group actions, semidirect products, free group actions, group orbits, invariant subsets, and invariant functions.*

## 3.1 Basics

A set $\mathcal{H}$ is called a *group* with respect to a binary operation $*$ if $\mathcal{H}$ satisfies the following four properties:

- (closure) $H_1 * H_2 \in \mathcal{H}$ for any $H_1, H_2 \in \mathcal{H}$;
- (associativity) $(H_1 * H_2) * H_3 = H_1 * (H_2 * H_3)$ for any $H_1, H_2, H_3 \in \mathcal{H}$;

- (identity) $I_{\mathcal{H}} \in \mathcal{H}$, where $I_{\mathcal{H}}$ is the identity element of $\mathcal{H}$, satisfying $I_{\mathcal{H}} * H = H * I_{\mathcal{H}} = H$ for any $H \in \mathcal{H}$;
- (inverse) $H^{-1} \in \mathcal{H}$ for any $H \in \mathcal{H}$, where $H^{-1}$ is the inverse element of $H$, satisfying $H^{-1} * H = H * H^{-1} = I_{\mathcal{H}}$.

A subset $\check{\mathcal{H}}$ of $\mathcal{H}$ is said to be a *subgroup* of $\mathcal{H}$ if $\check{\mathcal{H}}$ is a group with respect to the same operation as $\mathcal{H}$.

Typical examples of groups are sets of matrices with respect to the binary operations of multiplication and addition as follows.

**Example 3.1.** The following matrix sets are groups with respect to multiplication.

- The general linear group $\mathrm{GL}(d) \subset \mathbb{R}^{d\times d}$, i.e., the set of non-singular matrices.
- The orthogonal group $\mathrm{O}(d) \subset \mathrm{GL}(d)$, i.e., the set of orthogonal matrices.
- The special orthogonal group $\mathrm{SO}(d) \subset \mathrm{O}(d)$, i.e., the set of orthogonal matrices with determinant 1.
- The set $\{I_d\}$, consisting of only the identity matrix $I_d$.
- The set $\{I_d, \mathrm{Refl}(w)\}$, consisting of the identity matrix $I_d$ and the reflection matrix $\mathrm{Refl}(w)$ of a unit vector $w \in \mathbb{R}^d$.

These groups are all subgroups of $\mathrm{GL}(d)$, and $\mathrm{SO}(d)$, $\{I_d\}$, and $\{I_d, \mathrm{Refl}(w)\}$ are subgroups of $\mathrm{O}(d)$.

**Example 3.2.** The following vector sets are groups with respect to addition.

- The Euclidean space $\mathbb{R}^d$.
- Any subspaces of $\mathbb{R}^d$.

- The set $\{0\} \subset \mathbb{R}^d$, consisting of only the zero vector.

Let us define the *scaled set* of a set $\mathcal{H}$ as

$$\mathrm{scaled}(\mathcal{H}) := \{sH : s > 0, H \in \mathcal{H}\}.$$

If $\mathcal{H}$ is a group with respect to a binary operation $*$, scaled($\mathcal{H}$) is a group with respect to the binary operation $*_{\mathrm{s}}$ such that $(s_1 H_1) *_{\mathrm{s}} (s_2 H_2) = (s_1 s_2)(H_1 * H_2)$ for $s_1, s_2 > 0$ and $H_1, H_2 \in \mathcal{H}$. Note that $\mathcal{H}$ is a subgroup of scaled($\mathcal{H}$), and that if $\check{\mathcal{H}}$ is a subgroup of $\mathcal{H}$, scaled($\check{\mathcal{H}}$) is a subgroup of scaled($\mathcal{H}$).

**Example 3.3.** The scaled set of $\mathrm{O}(d)$ is given as

$$\mathrm{scaled}(\mathrm{O}(d)) = \{sW \in \mathbb{R}^{d\times d} : s > 0, W \in \mathrm{O}(d)\},$$

which is a group with respect to production. For $S \in \mathrm{scaled}(\mathrm{O}(d))$, $S^\top S = s^2 W^\top W = s^2 I_d$ holds with some $s > 0, W \in \mathrm{O}(d)$. Taking the determinants of the both sides yields $|\det(S)|^2 = (s^2)^d$. Hence, $s = |\det(S)|^{\frac{1}{d}}$ is obtained, and scaled($\mathrm{O}(d)$) is reduced to

$$\mathrm{scaled}(\mathrm{O}(d)) = \{S \in \mathbb{R}^{d\times d} : S^\top S = |\det(S)|^{\frac{2}{d}} I_d\}. \tag{3.1}$$

## 3.2 Group actions

A group $\mathcal{H}$ is said to *act* on a set $\mathcal{X}$ with respect to a binary operation $\bullet$ if the following three properties are satisfied:

- (closure) $H \bullet x \in \mathcal{X}$ for any $H \in \mathcal{H}$ and $x \in \mathcal{X}$;
- (associativity) $(H_1 * H_2) \bullet x = H_1 \bullet (H_2 \bullet x)$ for any $H_1, H_2 \in \mathcal{H}$, and $x \in \mathcal{X}$;
- (identity) $I_{\mathcal{H}} \bullet x = x$ for any $x \in \mathcal{X}$.

If a group $\mathcal{H}$ acts on $\mathcal{X}$, $\mathcal{H}$ acts on $\mathcal{X}^n$ in the following way:

$$H \bullet x_{\mathcal{N}} := (H \bullet x_1, H \bullet x_2, \ldots, H \bullet x_n) \in \mathcal{X}^n \tag{3.2}$$

for $H \in \mathcal{H}$, $x_1, x_2, \ldots, x_n \in \mathcal{X}$, and $\mathcal{N} = \{1, 2, \ldots, n\}$. Moreover, $\mathcal{H}$ acts on $\mathrm{pow}(\mathcal{X})$ in the following way:

$$H \bullet \mathcal{X}^* := \{H \bullet x \in \mathcal{X} : x \in \mathcal{X}^*\} \in \mathrm{pow}(\mathcal{X}) \tag{3.3}$$

for $H \in \mathcal{H}$ and $\mathcal{X}^* \in \mathrm{pow}(\mathcal{X})$ (i.e., $\mathcal{X}^* \subset \mathcal{X}$), where $\mathrm{pow}(\cdot)$ represents the power set (the set of all subsets) of a set.

## 3.3 Semidirect products

A semidirect product introduces a special transformation into a Cartesian product with a group action, associated with multi-robot coordination problems. Actually, the coordination freedom set and the frame transformation set introduced in Subsections 2.2.2 and 2.3.1 are redefined as semidirect products $\mathcal{S} \ltimes \mathcal{T}$ and $\mathcal{M} \ltimes \mathcal{B}$, as shown below. Then, analysis of the semidirect product plays a key role in revealing an important relation between desired coordination and required measurements through $\mathcal{S} \ltimes \mathcal{T}$ and $\mathcal{M} \ltimes \mathcal{B}$.

Let $\mathcal{S}$ and $\mathcal{T}$ be groups with respect to multiplication and addition, respectively, such that $\mathcal{S}$ acts on $\mathcal{T}$ with respect to multiplication. The *semidirect product* of $\mathcal{S}$ and $\mathcal{T}$, denoted by $\mathcal{S} \ltimes \mathcal{T}$, is the group of the elements of the Cartesian product $\mathcal{S} \times \mathcal{T}$ with respect to the binary operation $*$ defined as

$$(S_1, \tau_1) * (S_2, \tau_2) := (S_1 S_2, \tau_1 + S_1 \tau_2) \in \mathcal{S} \times \mathcal{T} \tag{3.4}$$

for $(S_1, \tau_1), (S_2, \tau_2) \in \mathcal{S} \times \mathcal{T}$. From (3.4), the identity element of $\mathcal{S} \ltimes \mathcal{T}$ is given by $(I_{\mathcal{S}}, 0)$, where $I_{\mathcal{S}}$ and $0$ are the identities of $\mathcal{S}$ and $\mathcal{T}$, respectively, and the inverse element of $(S, \tau) \in \mathcal{S} \ltimes \mathcal{T}$ is derived as

$$(S, \tau)^{-1} = (S^{-1}, -S^{-1}\tau). \tag{3.5}$$

The group action of the semidirect product is defined as follows. Let $\mathcal{X}$ be a set such that $\mathcal{S}$ and $\mathcal{T}$ act on $\mathcal{X}$ with respect to multiplication and addition, respectively. Then, the semidirect product $\mathcal{S} \ltimes \mathcal{T}$ acts on $\mathcal{X}$ in the following way:

$$(S, \tau) \bullet x := Sx + \tau \in \mathcal{X} \tag{3.6}$$

for $(S, \tau) \in \mathcal{S} \ltimes \mathcal{T}$ and $x \in \mathcal{X}$.

Through the semidirect products $\mathcal{S} \ltimes \mathcal{T}$ and $\mathcal{M} \ltimes \mathcal{B}$, the transformations in multi-robot coordination problems can be expressed by the group action (3.6) of the semidirect product together with (3.2) as follows. First, the transformation (2.15) of the desired configuration can be described as

$$x_{\mathcal{N}} = (S, \tau) \bullet x^*_{\mathcal{N}} = (Sx^*_1 + \tau, \ldots, Sx^*_n + \tau) \tag{3.7}$$

at once for all $i \in \mathcal{N}$ with $(S, \tau) \in \mathcal{S} \ltimes \mathcal{T}$. Second, the coordinate transformation (2.20) is described as

$$p(t) = (M_i(t), b_i(t)) \bullet p^{[i]}(t)$$

for $(M_i(t), b_i(t)) \in \mathcal{M} \ltimes \mathcal{B}$. Accordingly, from the inverse (3.5) of the semidirect product, the relative position (2.22) is expressed as

$$x^{[i]}_j(t) = (M_i(t), b_i(t))^{-1} \bullet x_j(t). \tag{3.8}$$

**Example 3.4.** Let $\mathcal{S} \ltimes \mathcal{T}$ be $\mathrm{scaled}(\mathrm{SO}(d)) \ltimes \mathbb{R}^d$, and its element is described as $(S, \tau) = (sR, \tau)$ with $s > 0$, $R \in \mathrm{SO}(d)$, and $\tau \in \mathbb{R}^d$. Then, (3.7) is reduced to

$$x_{\mathcal{N}} = (sR, \tau) \bullet x^*_{\mathcal{N}} = (sRx^*_1 + \tau, \ldots, sRx^*_n + \tau) \tag{3.9}$$

for $(x^*_1, \ldots, x^*_n) \in (\mathbb{R}^d)^n$. Through (3.9), the vectors $x^*_1, \ldots, x^*_n \in \mathbb{R}^d$ are scaled, rotated, and translated according to scale $s$, rotation $R$, and vector $\tau$. By regarding $x^*_{\mathcal{N}}$ as the apexes of a polygon in a plane/space, $x_{\mathcal{N}}$ in (3.9) can be considered as a polygon similar to $x^*_{\mathcal{N}}$.

## 3.4 Group orbits

A group orbit is the set of all resultants of a group action. In multi-robot coordination problems, the desired configuration set (2.16) is compactly expressed as

$$\begin{aligned} \mathcal{D} &= \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : \exists (S, \tau) \in \mathcal{S} \ltimes \mathcal{T} \ \text{ s.t. } \ x_{\mathcal{N}} = (S, \tau) \bullet x^*_{\mathcal{N}}\} \\ &= \mathrm{orb}_{\mathcal{S} \ltimes \mathcal{T}}(x^*_{\mathcal{N}}) \end{aligned} \tag{3.10}$$

with the group orbit defined below. Furthermore, the group orbit helps us characterize the desired configuration set $\mathcal{D}$ achievable with relative measurements under the frame transformation set $\mathcal{M} \ltimes \mathcal{B}$ in Section 8.2.

Consider a set $\mathcal{X}$ and a group $\mathcal{H}$ acting on $\mathcal{X}$. The $\mathcal{H}$-*orbit* of a subset $\mathcal{X}^*$ of $\mathcal{X}$ is defined as

$$\begin{aligned}\mathrm{orb}_{\mathcal{H}}(\mathcal{X}^*) &:= \bigcup_{H \in \mathcal{H}} H \bullet \mathcal{X}^* \\ &= \{H \bullet x \in \mathcal{X} : H \in \mathcal{H}, x \in \mathcal{X}^*\},\end{aligned} \tag{3.11}$$

where the equation follows from (3.3). If $\mathcal{X}^* = \{x^*\}$ is a singleton of $x^* \in \mathcal{X}$, we describe just $\mathrm{orb}_{\mathcal{H}}(x^*)$ for $\mathrm{orb}_{\mathcal{H}}(\{x^*\})$.

**Example 3.5.** The set $\mathcal{D}$ in (3.10) with $\mathcal{S} \ltimes \mathcal{T} = \mathrm{scaled}(\mathrm{SO}(d)) \ltimes \mathbb{R}^d$ corresponds to the desired configuration set $\mathcal{D}$ for scaling reflection-free formation in Example 2.11, described as

$$\begin{aligned}\mathcal{D} &= \{(sRx_1^* + \tau, \ldots, sRx_n^* + \tau) : s > 0, R \in \mathrm{SO}(d), \tau \in \mathbb{R}^d\} \\ &= \mathrm{orb}_{\mathrm{scaled}(\mathrm{SO}(d)) \ltimes \mathbb{R}^d}(x_{\mathcal{N}}^*)\end{aligned} \tag{3.12}$$

from (3.9). This set corresponds to the set of the polygons similar to $x_{\mathcal{N}}^*$ from Example 3.4.

Let $\mathrm{proj}_{\mathcal{C}}(\mathcal{D})$ be the projection of a set $\mathcal{D} \subset \mathcal{X}^n$ onto the $\mathcal{X}^{|\mathcal{C}|}$-space for $\mathcal{C} \subset \mathcal{N} = \{1, 2, \ldots, n\}$, defined as

$$\mathrm{proj}_{\mathcal{C}}(\mathcal{D}) := \{x_{\mathcal{C}} \in \mathcal{X}^{|\mathcal{C}|} : \exists x_{\mathcal{N}} \in \mathcal{X}^n \text{ s.t. } x_{\mathcal{N}} \in \mathcal{D}\}. \tag{3.13}$$

The following lemma shows that the operations of projection and group orbit are commutative.

**Lemma 3.1.** Consider a group $\mathcal{H}$ acting on $\mathbb{R}^d$ and a subset $\mathcal{X}^*$ of $(\mathbb{R}^d)^n$. Then, for $\mathcal{C} \subset \mathcal{N} = \{1, 2, \ldots, n\}$, the following holds:

$$\mathrm{proj}_{\mathcal{C}}(\mathrm{orb}_{\mathcal{H}}(\mathcal{X}^*)) = \mathrm{orb}_{\mathcal{H}}(\mathrm{proj}_{\mathcal{C}}(\mathcal{X}^*)). \tag{3.14}$$

*Proof.* From (3.11) and (3.13),

$$\begin{aligned}\operatorname{proj}_{\mathcal{C}}(\operatorname{orb}_{\mathcal{H}}(\mathcal{X}^*)) &= \operatorname{proj}_{\mathcal{C}}(\{H \bullet x_{\mathcal{N}} : H \in \mathcal{H}, x_{\mathcal{N}} \in \mathcal{X}^*\}) \\ &= \{H \bullet x_{\mathcal{C}} : H \in \mathcal{H}, x_{\mathcal{N}} \in \mathcal{X}^*\} \\ &= \{H \bullet y : H \in \mathcal{H}, y \in \operatorname{proj}_{\mathcal{C}}(\mathcal{X}^*)\} \\ &= \operatorname{orb}_{\mathcal{H}}(\operatorname{proj}_{\mathcal{C}}(\mathcal{X}^*))\end{aligned}$$

holds, and (3.14) is achieved. □

Consider the desired configuration set $\mathcal{D}$ of the orbit as (3.10). The configuration set of a part of the robots, indexed by $\mathcal{C} \subset \mathcal{N}$, is described as $\operatorname{proj}_{\mathcal{C}}(\mathcal{D})$. Lemma 3.1 guarantees that this set is also of the form of an orbit as follows:

$$\begin{aligned}\operatorname{proj}_{\mathcal{C}}(\mathcal{D}) = \operatorname{proj}_{\mathcal{C}}(\operatorname{orb}_{\mathcal{S} \ltimes \mathcal{T}}(x_{\mathcal{N}}^*)) &= \operatorname{orb}_{\mathcal{S} \ltimes \mathcal{T}}(\operatorname{proj}_{\mathcal{C}}(x_{\mathcal{N}}^*)) \\ &= \operatorname{orb}_{\mathcal{S} \ltimes \mathcal{T}}(x_{\mathcal{C}}^*).\end{aligned} \tag{3.15}$$

**Example 3.6.** For the set $\mathcal{D}$ in Example 3.5 and $\mathcal{C} = \{j_1, j_2, \ldots, j_{|\mathcal{C}|}\} \subset \mathcal{N}$,

$$\begin{aligned}\operatorname{proj}_{\mathcal{C}}(\mathcal{D}) &= \operatorname{proj}_{\mathcal{C}}(\operatorname{orb}_{\mathrm{scaled}(\mathrm{SO}(d)) \ltimes \mathbb{R}^d}(x_{\mathcal{N}}^*)) = \operatorname{orb}_{\mathrm{scaled}(\mathrm{SO}(d)) \ltimes \mathbb{R}^d}(x_{\mathcal{C}}^*) \\ &= \{(sRx_{j_1}^* + \tau, \ldots, sRx_{j_{|\mathcal{C}|}}^* + \tau) : s > 0, R \in \mathrm{SO}(d), \tau \in \mathbb{R}^d\}\end{aligned}$$

is obtained from (3.12) and (3.15). This set corresponds to the set of the polygons similar to $x_{\mathcal{C}}^*$.

## 3.5 Invariant subsets

For a set $\mathcal{X}$ and a group $\mathcal{H}$ acting on $\mathcal{X}$, a subset $\mathcal{D}$ of $\mathcal{X}$ is said to be *$\mathcal{H}$-invariant* if $H \bullet x \in \mathcal{D}$ holds for any $H \in \mathcal{H}$ and $x \in \mathcal{D}$, which is expressed as

$$H \bullet \mathcal{D} \subset \mathcal{D} \quad \forall H \in \mathcal{H} \tag{3.16}$$

according to (3.3).

Notably, any $\mathcal{H}$-invariant subset can be characterized by an $\mathcal{H}$-orbit as follows.

**Lemma 3.2.** For a set $\mathcal{X}$ and a group $\mathcal{H}$ acting on $\mathcal{X}$, a subset $\mathcal{D}$ of $\mathcal{X}$ is $\mathcal{H}$-invariant if and only if there exists a subset $\mathcal{X}^*$ of $\mathcal{X}$ such that $\mathcal{D}$ is of the form

$$\mathcal{D} = \mathrm{orb}_{\mathcal{H}}(\mathcal{X}^*). \tag{3.17}$$

*Proof.* (Sufficiency) Consider the set $\mathcal{D}$ in (3.17). From the definition (3.11) of the group orbit and the associativity of the group action, the following holds for any $H \in \mathcal{H}$:

$$H \bullet \mathcal{D} = H \bullet (\bigcup_{\bar{H} \in \mathcal{H}} \bar{H} \bullet \mathcal{X}^*) = \bigcup_{\bar{H} \in \mathcal{H}} (H * \bar{H}) \bullet \mathcal{X}^* \subset \bigcup_{\tilde{H} \in \mathcal{H}} \tilde{H} \bullet \mathcal{X}^* = \mathcal{D},$$

where $\bar{H}$ and $\tilde{H}$ are any elements of $\mathcal{H}$, and the inclusion holds because $H * \bar{H} \in \mathcal{H}$ from the closure of the group operation. The inclusion (3.16) is obtained.

(Necessity) Assume that a subset $\mathcal{D}$ of $\mathcal{X}$ is $\mathcal{H}$-invariant. From (3.11) and (3.16),

$$\mathrm{orb}_{\mathcal{H}}(\mathcal{D}) = \bigcup_{H \in \mathcal{H}} H \bullet \mathcal{D} \subset \bigcup_{H \in \mathcal{H}} \mathcal{D} = \mathcal{D}$$

holds. The inverse inclusion follows from the definition (3.11) of the orbit, and (3.17) holds for $\mathcal{X}^* = \mathcal{D}$. □

Lemma 3.2 indicates that the desired configuration set $\mathcal{D}$ in (3.10), namely $\mathrm{orb}_{\mathcal{S} \ltimes \mathcal{T}}(x_{\mathcal{N}}^*)$, is $(\mathcal{S} \ltimes \mathcal{T})$-invariant. Furthermore, the following lemma shows that the invariance of the desired configuration $\mathrm{proj}_{\mathcal{C}}(\mathcal{D})$ of a part of robots is preserved.

The projection of an $\mathcal{H}$-invariant subset of $\mathcal{X}^n$ is also an $\mathcal{H}$-invariant subset as follows.

**Lemma 3.3.** For a set $\mathcal{X}$ and a group $\mathcal{H}$ acting on $\mathcal{X}$, assume that $\mathcal{D}$ is an $\mathcal{H}$-invariant subset of $\mathcal{X}^n$. Then, for any $\mathcal{C} \subset \mathcal{N} = \{1, 2, \ldots, n\}$, $\mathrm{proj}_{\mathcal{C}}(\mathcal{D})$ is an $\mathcal{H}$-invariant subset of $\mathcal{X}^{|\mathcal{C}|}$.

*Proof.* From the assumption that $\mathcal{D}$ is an $\mathcal{H}$-invariant subset of $\mathcal{X}^n$, Lemma 3.2 guarantees that $\mathcal{D} = \mathrm{orb}_{\mathcal{H}}(\mathcal{X}^*)$ holds with some $\mathcal{X}^* \subset \mathcal{X}^n$. Then, from Lemma 3.1,

$$\mathrm{proj}_{\mathcal{C}}(\mathcal{D}) = \mathrm{proj}_{\mathcal{C}}(\mathrm{orb}_{\mathcal{H}}(\mathcal{X}^*)) = \mathrm{orb}_{\mathcal{H}}(\mathrm{proj}_{\mathcal{C}}(\mathcal{X}^*))$$

holds. The right-hand side of this equation is of the form (3.17) for $\mathrm{proj}_{\mathcal{C}}(\mathcal{X}^*)$ instead of $\mathcal{X}^*$. Therefore, Lemma 3.2 guarantees that $\mathrm{proj}_{\mathcal{C}}(\mathcal{D})$ is an $\mathcal{H}$-invariant subset. □

## 3.6 Invariant functions

For a set $\mathcal{X}$ and a group $\mathcal{H}$ acting on $\mathcal{X}$, a function $v : \mathcal{X} \to \mathbb{R}$ is said to be $\mathcal{H}$-*invariant* if

$$v(H \bullet x) = v(x) \quad \forall H \in \mathcal{H}, x \in \mathcal{X}. \tag{3.18}$$

A function $v(x)$ is said to be *relatively* $\mathcal{H}$*-invariant of weight* $\mu : \mathcal{H} \to \mathbb{R}$ if

$$v(H \bullet x) = \mu(H)v(x) \quad \forall H \in \mathcal{H}, x \in \mathcal{X}. \tag{3.19}$$

For some semidirect products $\mathcal{S} \ltimes \mathcal{T}$, examples of (relatively) $(\mathcal{S} \ltimes \mathcal{T})$-invariant functions are given as follows.

**Example 3.7.** For $r_{ij} \in \mathbb{R}^d$, the function $v : (\mathbb{R}^d)^n \to \mathbb{R}_+$ given as

$$v(x_{\mathcal{N}}) = \sum_{\{i,j\} \in \mathcal{E}} \|x_i - x_j - r_{ij}\|^2 \tag{3.20}$$

is $(\{I_d\} \ltimes \mathbb{R}^d)$-invariant (i.e., invariant under translation), where $\mathcal{E}$ is a set of pairs of the elements in $\mathcal{N} = \{1, 2, \ldots, n\}$. This function is used to evaluate the achievement of displacement-based formation (2.5). The invariance in (3.18) is verified for $(I_d, \tau) \in \{I_d\} \ltimes \mathbb{R}^d$ as

$$\begin{aligned} v((I_d, \tau) \bullet x_{\mathcal{N}}) &= v(x_1 + \tau, \ldots, x_n + \tau) \\ &= \sum_{\{i,j\} \in \mathcal{E}} \|(x_i + \tau) - (x_j + \tau) - r_{ij}\|^2 \\ &= \sum_{\{i,j\} \in \mathcal{E}} \|x_i - x_j - r_{ij}\|^2 = v(x_{\mathcal{N}}). \end{aligned} \tag{3.21}$$

**Example 3.8.** For $d_{ij} > 0$, the function $v : (\mathbb{R}^d)^n \to \mathbb{R}_+$ given as

$$v(x_{\mathcal{N}}) = \sum_{\{i,j\} \in \mathcal{E}} (\|x_i - x_j\|^2 - d_{ij}^2)^2 \tag{3.22}$$

is $(\mathrm{O}(d) \ltimes \mathbb{R}^d)$-invariant (i.e., invariant under translation, rotation, and reflection). This function is used to evaluate the achievement of distance-based formation (2.7). The invariance in (3.18) is verified for $(S, \tau) \in \mathrm{O}(d) \ltimes \mathbb{R}^d$ as

$$\begin{aligned} v((S,\tau) \bullet x_{\mathcal{N}}) &= v(Sx_1 + \tau, \ldots, Sx_n + \tau) \\ &= \sum_{\{i,j\} \in \mathcal{E}} (\|(Sx_i + \tau) - (Sx_j + \tau)\|^2 - d_{ij}^2)^2 \\ &= \sum_{\{i,j\} \in \mathcal{E}} (\|S(x_i - x_j)\|^2 - d_{ij}^2)^2 = v(x_{\mathcal{N}}). \end{aligned} \tag{3.23}$$

**Example 3.9.** For $d_{ij} > 0$, the function $v : (\mathbb{R}^d)^n \to \mathbb{R}_+$ given as

$$v(x_{\mathcal{N}}) = \inf_{\sigma > 0} \sum_{\{i,j\} \in \mathcal{E}} (\|x_i - x_j\|^2 - \sigma^2 d_{ij}^2)^2 \tag{3.24}$$

is relatively $(\mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d)$-invariant of weight $|\det(S)|^{\frac{4}{d}}$ for $(S, \tau) \in \mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$, which is used to evaluate the achievement of scaling distance-based formation (2.10). The relative invariance in (3.19) is verified with $S = sW \in \mathrm{scaled}(\mathrm{O}(d))$ for $s > 0$ and $W \in \mathrm{O}(d)$, and $\tau \in \mathbb{R}^d$ as

$$\begin{aligned} &v((S,\tau) \bullet x_{\mathcal{N}}) \\ &\quad = \inf_{\sigma > 0} \sum_{\{i,j\} \in \mathcal{E}} (\|(sWx_i + \tau) - (sWx_j + \tau)\|^2 - \sigma^2 d_{ij}^2)^2 \\ &\quad = s^4 \inf_{\sigma > 0} \sum_{\{i,j\} \in \mathcal{E}} (\|x_i - x_j\|^2 - \frac{\sigma^2}{s^2} d_{ij}^2)^2 \\ &\quad = s^4 \inf_{\bar{\sigma} > 0} \sum_{\{i,j\} \in \mathcal{E}} (\|x_i - x_j\|^2 - \bar{\sigma}^2 d_{ij}^2)^2 \\ &\quad = s^4 v(x_{\mathcal{N}}), \end{aligned} \tag{3.25}$$

where $\bar{\sigma} = \sigma/s$ and $s = |\det(S)|^{\frac{1}{d}}$ from (3.1).

For certain $\mathcal{S} \ltimes \mathcal{T}$, the distance functions of $(\mathcal{S} \ltimes \mathcal{T})$-invariant subsets are (relatively) $(\mathcal{S} \ltimes \mathcal{T})$-invariant functions as follows.

**Lemma 3.4.** Assume that $\mathcal{D}$ is a non-empty, $(\mathcal{S} \ltimes \mathcal{T})$-invariant subset of $(\mathbb{R}^d)^n$. Consider the squared distance function $v(x_\mathcal{N}) = (\mathrm{dist}(x_\mathcal{N}, \mathcal{D}))^2$ for $x_\mathcal{N} \in (\mathbb{R}^d)^n$ and $\mathcal{D}$. If $\mathcal{S} \ltimes \mathcal{T}$ is a subgroup of $\mathrm{O}(d) \ltimes \mathbb{R}^d$, $v(x_\mathcal{N})$ is $(\mathcal{S} \ltimes \mathcal{T})$-invariant. If $\mathcal{S} \ltimes \mathcal{T}$ is a subgroup of $\mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$, $v(x_\mathcal{N})$ is relatively $(\mathcal{S} \ltimes \mathcal{T})$-invariant of weight $|\det(S)|^{\frac{2}{d}}$ for $(S, \tau) \in \mathcal{S} \ltimes \mathcal{T}$.

*Proof.* We consider the case that $\mathcal{S} \ltimes \mathcal{T}$ is a subgroup of $\mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$. Then, the case of $\mathrm{O}(d) \ltimes \mathbb{R}^d$ is obvious because $|\det(S)| = 1$ for $(S, \tau) \in \mathcal{S} \ltimes \mathcal{T} \subset \mathrm{O}(d) \ltimes \mathbb{R}^d$.

Let $(S, \tau) \in \mathcal{S} \ltimes \mathcal{T} \subset \mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$, and from (3.1), (3.2), and (3.6),

$$\begin{aligned} v((S,\tau) \bullet x_\mathcal{N}) &= (\mathrm{dist}((S,\tau) \bullet x_\mathcal{N}, \mathcal{D}))^2 = \inf_{y_\mathcal{N} \in \mathcal{D}} \|(S,\tau) \bullet x_\mathcal{N} - y_\mathcal{N}\|^2 \\ &= \inf_{y_\mathcal{N} \in \mathcal{D}} \sum_{i \in \mathcal{N}} \|S x_i + \tau - y_i\|^2 = \inf_{\bar{y}_\mathcal{N} \in \mathcal{D}} \sum_{i \in \mathcal{N}} \|S(x_i - \bar{y}_i)\|^2 \\ &= \inf_{\bar{y}_\mathcal{N} \in \mathcal{D}} \sum_{i \in \mathcal{N}} (x_i - \bar{y}_i)^\top S^\top S (x_i - \bar{y}_i) \\ &= |\det(S)|^{\frac{2}{d}} \inf_{\bar{y}_\mathcal{N} \in \mathcal{D}} \sum_{i \in \mathcal{N}} \|x_i - \bar{y}_i\|^2 = |\det(S)|^{\frac{2}{d}} v(x_\mathcal{N}) \end{aligned}$$

holds, where $\bar{y}_i = S^{-1}(y_i - \tau) = (S,\tau)^{-1} \bullet y_i$. Here, $\bar{y}_\mathcal{N} = (S,\tau)^{-1} \bullet y_\mathcal{N} \in \mathcal{D}$ holds because $\mathcal{D}$ is an $(\mathcal{S} \ltimes \mathcal{T})$-invariant subset and $(S,\tau)^{-1} \in \mathcal{S} \ltimes \mathcal{T}$. Therefore, (3.19) holds, and $v(x_\mathcal{N})$ is relatively $(\mathcal{S} \ltimes \mathcal{T})$-invariant of weight $|\det(S)|^{\frac{2}{d}}$. □

For multi-robot coordination, the squared distance function $v(x_\mathcal{N}) = (\mathrm{dist}(x_\mathcal{N}, \mathcal{D}))^2$ is used to evaluate the task achievement of the generalized coordination (2.13) with respect to the desired configuration set $\mathcal{D}$. The set $\mathcal{D}$ in (3.10) is $(\mathcal{S} \ltimes \mathcal{T})$-invariant from Lemma 3.2. Furthermore, as shown in Section 2.2.2, $\mathcal{S} \ltimes \mathcal{T}$ is given as a subgroup of $\mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$ in many cases. Then, this $v(x_\mathcal{N})$ is relatively $(\mathcal{S} \ltimes \mathcal{T})$-invariant from Lemma 3.4. To evaluate the task achievement with this function, the robots need to distinguish elements in $\mathcal{D}$ with resolution higher than $\mathcal{S} \ltimes \mathcal{T}$. Sensing resolution is determined from the frame transformation set $\mathcal{M} \ltimes \mathcal{B}$. Therefore, to achieve the generalized coordination with respect this $\mathcal{D}$, $\mathcal{S} \ltimes \mathcal{T} \supset \mathcal{M} \ltimes \mathcal{B}$ needs to be satisfied, which is shown in Subsection 8.5.1.

## 3.7 Free group actions

For a set $\mathcal{X}$ and a group $\mathcal{H}$ acting on $\mathcal{X}$, it is said that $\mathcal{H}$ *acts freely* on $\mathcal{X}$ if for each $x \in \mathcal{X}$,

$$H_1 \bullet x = H_2 \bullet x, H_1, H_2 \in \mathcal{H} \Rightarrow H_1 = H_2 \tag{3.26}$$

holds, or, equivalently,

$$H \bullet x \neq x \ \ \forall H \in \mathcal{H} \backslash \{I_{\mathcal{H}}\}. \tag{3.27}$$

In other words, if $y = H \bullet x$ holds for $x, y \in \mathcal{X}$, such an $H \in \mathcal{H}$ is uniquely determined, when $H$ acts freely on $\mathcal{X}$.

For the desired configuration set $\mathcal{D}$ in (3.10), the achievement of the generalized coordination (2.13), i.e., $x_{\mathcal{N}} \in \mathcal{D}$, indicates that $x_{\mathcal{N}} = (S, \tau) \bullet x_{\mathcal{N}}^*$ with some $(S, \tau) \in \mathcal{S} \ltimes \mathcal{T}$. If $\mathcal{S} \ltimes \mathcal{T}$ acts freely, such an $(S, \tau)$ is uniquely determined. The uniqueness of $(S, \tau)$ leads to the feasibility of the coordination by distributed control with relative measurements, which will be discussed in the next section.

Let us give a couple of examples of free and non-free group actions.

**Example 3.10.** The semidirect product $\mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2$ acts freely on $(\mathbb{R}^2)^2 \backslash \{(x_1, x_2) : x_1 = x_2\}$, which is verified as follows. Let $(s_{\mathrm{a}} R_{\mathrm{a}}, \tau_{\mathrm{a}}), (s_{\mathrm{b}} R_{\mathrm{b}}, \tau_{\mathrm{b}}) \in \mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2$ for $s_{\mathrm{a}}, s_{\mathrm{b}} > 0$, $R_{\mathrm{a}}, R_{\mathrm{b}} \in \mathrm{SO}(2)$, and $\tau_{\mathrm{a}}, \tau_{\mathrm{b}} \in \mathbb{R}^2$. For $x_1, x_2 \in \mathbb{R}^2$ satisfying $x_1 \neq x_2$, from (3.9), the assumption in (3.26) is reduced to

$$\begin{aligned} &(s_{\mathrm{a}} R_{\mathrm{a}}, \tau_{\mathrm{a}}) \bullet (x_1, x_2) = (s_{\mathrm{a}} R_{\mathrm{a}} x_1 + \tau_{\mathrm{a}}, s_{\mathrm{a}} R_{\mathrm{a}} x_2 + \tau_{\mathrm{a}}) \\ &= (s_{\mathrm{b}} R_{\mathrm{b}}, \tau_{\mathrm{b}}) \bullet (x_1, x_2) = (s_{\mathrm{b}} R_{\mathrm{b}} x_1 + \tau_{\mathrm{b}}, s_{\mathrm{b}} R_{\mathrm{b}} x_2 + \tau_{\mathrm{b}}). \end{aligned} \tag{3.28}$$

The first element minus the second one in each two-tuple in (3.28) is reduced to

$$s_{\mathrm{a}} R_{\mathrm{a}} (x_1 - x_2) = s_{\mathrm{b}} R_{\mathrm{b}} (x_1 - x_2). \tag{3.29}$$

Take the norms of the both sides of (3.29), and we obtain $s_{\mathrm{a}} = s_{\mathrm{b}}$ from $x_1 - x_2 \neq 0$. Next, $R_{\mathrm{a}} = R_{\mathrm{b}}$ holds because (3.29) is reduced to $\mathrm{Rot}(\theta_\mathrm{a})(x_1 - x_2) = \mathrm{Rot}(\theta_\mathrm{b})(x_1 - x_2)$ by assigning $R_\mathrm{a} = \mathrm{Rot}(\theta_\mathrm{a})$ and $R_\mathrm{b} = \mathrm{Rot}(\theta_\mathrm{b})$ with $\theta_\mathrm{a}, \theta_\mathrm{b} \in [-\pi, \pi)$, and $\theta_\mathrm{a} = \theta_\mathrm{b}$ is obtained from $x_1 - x_2 \neq 0$. Then, $\tau_\mathrm{a} = \tau_\mathrm{b}$ holds from (3.28). Hence, the conclusion of (3.26) is derived, and thus $\mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2$ acts freely on $(\mathbb{R}^2)^2 \backslash \{(x_1, x_2) : x_1 = x_2\}$.

Example 3.10 implies that when two distinct vectors $x_1, x_2 \in \mathbb{R}^2$ are transformed in rotation, translation, and scale into distinct vectors $\bar{x}_1, \bar{x}_2 \in \mathbb{R}^2$ simultaneously, such a transformation is uniquely determined.

**Example 3.11.** The semidirect product $\mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2$ does not act freely on $\mathbb{R}^2 \backslash \{0\}$, which is verified as follows. Let $(s_\mathrm{a} R_\mathrm{a}, \tau_\mathrm{a})$, $(s_\mathrm{b} R_\mathrm{b}, \tau_\mathrm{b}) \in \mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2$ for $s_\mathrm{a}, s_\mathrm{b} > 0$, $R_\mathrm{a}, R_\mathrm{b} \in \mathrm{SO}(2)$, and $\tau_\mathrm{a}, \tau_\mathrm{b} \in \mathbb{R}^2$. For a non-zero $x \in \mathbb{R}^2$, from (3.9), the assumption in (3.26) is reduced to

$$(s_\mathrm{a} R_\mathrm{a}, \tau_\mathrm{a}) \bullet x = s_\mathrm{a} R_\mathrm{a} x + \tau_\mathrm{a} = (s_\mathrm{b} R_\mathrm{b}, \tau_\mathrm{b}) \bullet x = s_\mathrm{b} R_\mathrm{b} x + \tau_\mathrm{b}.$$

This equation holds for any $s_\mathrm{a}, s_\mathrm{b} > 0$, $R_\mathrm{a}, R_\mathrm{b} \in \mathrm{SO}(2)$, $\tau_\mathrm{a} \in \mathbb{R}^2$ by assigning $\tau_\mathrm{b} = (s_\mathrm{a} R_\mathrm{a} - s_\mathrm{b} R_\mathrm{b}) x + \tau_\mathrm{a}$. Hence, $\mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2$ does not act freely on $\mathbb{R}^2 \backslash \{0\}$.

In contrast to Example 3.10, Example 3.11 implies that the transformations of one vector $x \in \mathbb{R}^2$ into $\bar{x} \in \mathbb{R}^2$ are not unique. This difference can be explained by the free action number in the next section.

## 3.8 Free action numbers

For a set $\mathcal{X}$ and a group $\mathcal{H}$ acting on $\mathcal{X}$, the *free action number* of the group action of $\mathcal{H}$ on $\mathcal{X}$ is defined as

$$\mathrm{fanum}_{\mathcal{X}}(\mathcal{H}) := \min\{n \in \mathbb{Z}_+ : \mathcal{H} \text{ acts freely on } \mathcal{X}^n \backslash \mathcal{Z}_n\} \tag{3.30}$$

for a set $\mathcal{Z}_n \subset \mathcal{X}^n$ of measure zero such that $\mathcal{H}$ acts on $\mathcal{X}^n \backslash \mathcal{Z}_n$. According to the definition (3.26) of the free group action, the free action number (3.30) indicates the smallest integer $n$ satisfying

$$H_1 \bullet x_{\mathcal{N}} = H_2 \bullet x_{\mathcal{N}}, H_1, H_2 \in \mathcal{H} \Rightarrow H_1 = H_2 \tag{3.31}$$

**Table 3.1:** Free action numbers of typical semidirect products $\mathcal{S} \ltimes \mathcal{T}$ on $\mathbb{R}^d$

| $\mathcal{S}$ \ $\mathcal{T}$ | $\{0\}$ | $\neq \{0\}$ |
|---|---|---|
| $\{I_d\}$ | $0$ | $1$ |
| $\mathrm{scaled}(\{I_d\})$ | $1$ | $2$ |
| $\{I_d, \mathrm{Refl}(w)\}$ | $1$ | $2$ |
| $\mathrm{scaled}(\{I_d, \mathrm{Refl}(w)\})$ | $1$ | $2$ |
| $\mathrm{SO}(d)$, $d \geq 2$ | $d-1$ | $d$ |
| $\mathrm{scaled}(\mathrm{SO}(d))$, $d \geq 2$ | $d-1$ | $d$ |
| $\mathrm{O}(d)$, $d \geq 2$ | $d$ | $d+1$ |
| $\mathrm{scaled}(\mathrm{O}(d))$, $d \geq 2$ | $d$ | $d+1$ |

for almost each $x_{\mathcal{N}} \in \mathcal{X}^n$, where $\mathcal{N} = \{1, 2, \ldots, n\}$, or, from (3.27),

$$H \bullet x_{\mathcal{N}} \neq x_{\mathcal{N}} \quad \forall H \in \mathcal{H} \backslash \{I_{\mathcal{H}}\}. \tag{3.32}$$

The following example is derived from Examples 3.10 and 3.11.

**Example 3.12.** The free action number of $\mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2$ on $\mathbb{R}^2$ is two, i.e., $\mathrm{fanum}_{\mathbb{R}^2}(\mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2) = 2$.

Table 3.1 shows the free action numbers of typical semidirect products $\mathcal{S} \ltimes \mathcal{T}$ on $\mathbb{R}^d$. It is confirmed that the result of Example 3.12 is extended to any dimension $d$, i.e., $\mathrm{fanum}_{\mathbb{R}^d}(\mathrm{scaled}(\mathrm{SO}(d)) \ltimes \mathbb{R}^d) = d$ holds.

In the multi-robot coordination for the desired configuration set $\mathcal{D}$ in (3.10), the free action number $\mathrm{fanum}_{\mathbb{R}^d}(\mathcal{S} \ltimes \mathcal{T})$ indicates the minimum number of robots to uniquely determine $(S, \tau) \in \mathcal{S} \ltimes \mathcal{T}$ satisfying $x_{\mathcal{N}} = (S, \tau) \bullet x^*_{\mathcal{N}}$. This number indicates a degree of network connections necessary to achieve coordination by distributed control with relative measurements. For example, to achieve scaling reflection-free formation in Example 2.11, the necessary degree of network connections is $d$ from Table 3.1 for $\mathcal{S} \ltimes \mathcal{T} = \mathrm{scaled}(\mathrm{SO}(d)) \ltimes \mathbb{R}^d$. See Subsection 8.5.2 for more details.

The results in Table 3.1 are shown after a preliminary lemma.

**Lemma 3.5.** Let $\mathcal{S}$ and $\mathcal{T}$ be subgroups of $\mathrm{GL}(d)$ and $\mathbb{R}^d$ with respect to multiplication and addition, respectively, and the following relations hold:

$$\mathrm{fanum}_{\mathbb{R}^d}(\mathcal{S}) \le d \tag{3.33}$$

$$\mathrm{fanum}_{\mathbb{R}^d}(\mathcal{T}) = \begin{cases} 0 & \text{if } \mathcal{T} = \{0\} \\ 1 & \text{if } \mathcal{T} \neq \{0\}. \end{cases} \tag{3.34}$$

If $\mathcal{S}$ acts on $\mathcal{T}$, the following equation holds:

$$\mathrm{fanum}_{\mathbb{R}^d}(\mathcal{S} \ltimes \mathcal{T}) = \mathrm{fanum}_{\mathbb{R}^d}(\mathcal{S}) + \mathrm{fanum}_{\mathbb{R}^d}(\mathcal{T}). \tag{3.35}$$

*Proof.* We consider only the case of $\mathcal{T} \neq \{0\}$ because that of $\mathcal{T} = \{0\}$ is trivial. The expressions (3.33), (3.34), and (3.35) are shown in order.

For a subgroup $\mathcal{S}$ of $\mathrm{GL}(d)$ and linearly independent vectors $x_1, \ldots, x_d \in \mathbb{R}^d$, (3.31) holds for $n = d$ from $\det[x_1 \ \cdots \ x_d] \neq 0$ as

$$S_1[x_1 \ \cdots \ x_d] = S_2[x_1 \ \cdots \ x_d], S_1, S_2 \in \mathcal{S} \Rightarrow S_1 = S_2.$$

Hence, $\mathcal{S}$ acts freely on $(\mathbb{R}^d)^d \backslash \mathcal{Z}_d$, where $\mathcal{Z}_d$ is the set of $d$-tuples of linearly dependent vectors. From the definition (3.30) of the free action number, (3.33) holds.

For a subgroup $\mathcal{T} \neq \{0\}$ of $\mathbb{R}^d$ and a vector $x \in \mathbb{R}^d$, (3.31) holds as

$$x + \tau_1 = x + \tau_2, \ \tau_1, \tau_2 \in \mathcal{T} \Rightarrow \tau_1 = \tau_2,$$

and $\mathrm{fanum}_{\mathbb{R}^d}(\mathcal{T}) = 1$ in (3.34) is obtained.

To prove (3.35), consider $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$ for some $n \in \mathbb{Z}_+$. From the definitions (3.2) and (3.6) of the group actions on multiple vectors and semidirect groups, the assumption in (3.31) is reduced to

$$\begin{aligned} &(S_1, \tau_1) \bullet (x_1, \ldots, x_n) = (S_1 x_1 + \tau_1, \ldots, S_1 x_n + \tau_1) \\ = &(S_2, \tau_2) \bullet (x_1, \ldots, x_n) = (S_2 x_1 + \tau_2, \ldots, S_2 x_n + \tau_2) \end{aligned} \tag{3.36}$$

for $(S_1, \tau_1), (S_2, \tau_2) \in \mathcal{S} \ltimes \mathcal{T}$. Note that (3.36) holds if and only if the following two hold:

$$S_1 x_1 + \tau_1 = S_2 x_1 + \tau_2. \tag{3.37}$$

$$S_1[x_2 - x_1 \ \cdots \ x_n - x_1] = S_2[x_2 - x_1 \ \cdots \ x_n - x_1]. \tag{3.38}$$

For almost every $[x_2 - x_1 \ \cdots \ x_n - x_1] \in \mathbb{R}^{d\times(n-1)}$, (3.38) leads to $S_1 = S_2$ if and only if $n - 1 \geq \mathrm{fanum}_{\mathbb{R}^d}(\mathcal{S})$. If $S_1 = S_2$, (3.37) yields $\tau_1 = \tau_2$. Hence, (3.36) implies that $(S_1, \tau_1) = (S_2, \tau_2)$ holds if and only if $n \geq \mathrm{fanum}_{\mathbb{R}^d}(\mathcal{S}) + 1$. Therefore, from the definition (3.30) of the free action number and (3.34), (3.35) is obtained. □

**Theorem 3.6.** The free action numbers of semidirect products $\mathcal{S} \ltimes \mathcal{T}$ are obtained as Table 3.1.

*Proof.* Assume that $\mathcal{T} = \{0\}$. The case of $\mathcal{T} \neq \{0\}$ follows from (3.34) and (3.35) in Lemma 3.5. For $(S, 0) \in \mathcal{S} \ltimes \mathcal{T} = \mathcal{H}$, (3.32) is reduced to

$$Sx_i \neq x_i \ \exists i \in \{1, \ldots, n\} \ \ \forall S \in \mathcal{S}\backslash\{I_d\}. \tag{3.39}$$

Case of $\mathcal{S} = \mathrm{scaled}(\{I_d\})$: $S \in \mathcal{S}\backslash\{I_d\}$ is represented as $S = sI_d$ for $s > 0, s \neq 1$. Then, for a non-zero vector $x_1 \in \mathbb{R}^d$, $Sx_1 = sx_1 \neq x_1$ holds. Hence, (3.39) is obtained for $n = 1$, and $\mathrm{fanum}_{\mathbb{R}^d}(\mathcal{S}) = 1$ is achieved.

Case of $\mathcal{S} = \{I_d, \mathrm{Refl}(w)\}$ for a unit vector $w \in \mathbb{R}^d$: $S \in \mathcal{S}\backslash\{I_d\}$ is satisfied only when $S = \mathrm{Refl}(w)$. Then, for $x_1 \in \mathbb{R}^d$ satisfying $w^\top x_1 \neq 0$, $Sx_1 = \mathrm{Refl}(w)x_1 = (I_d - 2ww^\top)x_1 \neq x_1$ holds. Hence, (3.39) is obtained for $n = 1$, and $\mathrm{fanum}_{\mathbb{R}^d}(\mathcal{S}) = 1$ is achieved.

Case of $\mathcal{S} = \mathrm{scaled}(\{I_d, \mathrm{Refl}(w)\})$: $S \in \mathcal{S}\backslash\{I_d\}$ takes the form of either $S = \mathrm{Refl}(w)$ or $S = s\bar{S}$ for $\bar{S} \in \{I_d, \mathrm{Refl}(w)\}$ with $s > 0$, $s \neq 1$. The former is the same as the previous case. In the latter case, $Sx_1 = s\bar{S}x_1 \neq x_1$ holds with any non-zero $x_1 \in \mathbb{R}^d$ because if $s\bar{S}x_1 = x_1$ holds, taking the norms of both the sides of the equation yields $s = 1$, which contradicts $s \neq 1$. Hence, $\mathrm{fanum}_{\mathbb{R}^d}(\mathcal{S}) = 1$ is achieved.

Case of $\mathcal{S} = \mathrm{O}(d)$: For linearly independent vectors $x_1, \ldots, x_{d-1} \in \mathbb{R}^d$, consider $S = \mathrm{Refl}(w) \in \mathcal{S}\backslash\{I_d\}$ with the unit vector $w \in \mathbb{R}^d$ orthogonal to all $x_i$, $i \in \{1, \ldots, d-1\}$. Then, $Sx_i = \mathrm{Refl}(w)x_i = (I_d - 2ww^\top)x_i = x_i$ holds for each $i \in \{1, \ldots, d-1\}$. Hence, (3.39) is not obtained for $n = d-1$, and $\mathrm{fanum}_{\mathbb{R}^d}(\mathcal{S}) > d-1$ is obtained. From (3.33) in Lemma 3.5, $\mathrm{fanum}_{\mathbb{R}^d}(\mathcal{S}) = d$ is achieved.

Case of $\mathcal{S} = \mathrm{SO}(2)$: $S \in \mathcal{S}\backslash\{I_2\}$ is represented as $S = \mathrm{Rot}(\theta)$ for $\theta \in (0, 2\pi)$. Then, for any non-zero $x_1 \in \mathbb{R}^2$, $Sx_1 = \mathrm{Rot}(\theta)x_1 \neq x_1$ holds. Hence, (3.39) is obtained for $n = 1$, and $\mathrm{fanum}_{\mathbb{R}^d}(\mathcal{S}) = 1$ is achieved.

Case of $\mathcal{S} = \mathrm{SO}(d)$ with an even positive integer $d$: From Reid and Szendroi, 2005, $S \in \mathcal{S}\backslash\{I_d\}$ can be block-diagonalized as

$$S = W^\top \mathrm{diag}(R_1, \ldots, R_{d/2})W \tag{3.40}$$

for some $W \in \mathrm{O}(d)$ and $(R_1, \ldots, R_{d/2}) \in (\mathrm{SO}(2))^{d/2}$ not equal to $(I_2, \ldots, I_2)$. Let $x_1, \ldots, x_n \in \mathbb{R}^d$ be linearly independent vectors, and we regard $x_\mathcal{N}$ as the corresponding matrix, i.e., $x_\mathcal{N} = [x_1 \ \cdots \ x_n] \in \mathbb{R}^{d\times n}$ of full column rank. First, consider the case of $n \leq d-2$. Let $\bar{W} \in \mathrm{O}(d)$ be the matrix satisfying $\bar{W}x_i = c_i e_{di}$ for every $i \in \{1, 2, \ldots, n\}$ with some $c_i \in \mathbb{R}$, where $e_{di} \in \mathbb{R}^d$ is the $i$th unit vector, and then $\bar{W}x_\mathcal{N} = [C \ 0_{n\times(d-n)}]^\top \in \mathbb{R}^{d\times n}$ holds with $C = \mathrm{diag}(c_1, \ldots, c_n)$, where $0_{n\times(d-n)} \in \mathbb{R}^{n\times(d-n)}$ is the zero matrix. Consider $S$ of the form (3.40) with $W = \bar{W}$, $R_k = I_2$ for $k \in \{1, \ldots, d/2-1\}$, and $R_{d/2} \neq I_d$. Then, from (3.40),

$$\begin{aligned} Sx_\mathcal{N} &= \bar{W}^\top \mathrm{diag}(I_2, \ldots, I_2, R_{d/2})\bar{W}x_\mathcal{N} \\ &= \bar{W}^\top \mathrm{diag}(I_{d-2}, R_{d/2})[C \ 0_{n\times(d-n)}]^\top = \bar{W}^\top [C \ 0_{n\times(d-n)}]^\top \\ &= x_\mathcal{N} \end{aligned}$$

holds from $d - n \geq 2$. Hence, (3.39) does not hold, and $\mathrm{fanum}_{\mathbb{R}^d}(\mathcal{S}) > d-2$ is achieved. Next, consider the case of $n = d-1$. We assume that (3.39) does not hold, and derive a contradiction. Then, from (3.40),

$$Sx_\mathcal{N} = W^\top \mathrm{diag}(R_1, \ldots, R_{d/2})Wx_\mathcal{N} = x_\mathcal{N} \tag{3.41}$$

holds with some $W \in \mathrm{O}(d)$ and $(R_1, \ldots, R_{d/2}) \in (\mathrm{SO}(2))^{d/2}$ not equal to $(I_2, \ldots, I_2)$. Without loss of generality, $R_{d/2} \neq I_2$ is assumed. By multiplying (3.41) by $[0_{2\times(d-2)} \ I_2]W \in \mathbb{R}^{2\times d}$ from the left, we obtain

$$\begin{aligned} &[0_{2\times(d-2)} \ I_2]\mathrm{diag}(R_1, \ldots, R_{d/2})Wx_\mathcal{N} \\ &\qquad = R_{d/2}[0_{2\times(d-2)} \ I_2]Wx_\mathcal{N} = [0_{2\times(d-2)} \ I_2]Wx_\mathcal{N}, \end{aligned}$$

which leads to $[0_{2\times(d-2)} \ I_2]Wx_\mathcal{N} = 0$ from the result of the case of $\mathrm{SO}(2)$. Then, the two columns in the matrix $([0_{2\times(d-2)} \ I_2]W)^\top$ are orthogonal to $x_\mathcal{N}$, which contradicts the assumption that $x_\mathcal{N} \in \mathbb{R}^{d\times(d-1)}$ is of full column rank. Therefore, (3.39) holds for $n = d-1$, and $\mathrm{fanum}_{\mathbb{R}^d}(\mathcal{S}) = d-1$ is achieved.

Case of $\mathcal{S} = \mathrm{SO}(d)$ with an odd positive integer $d$: From Reid and Szendroi, 2005, $S \in \mathcal{S} \backslash \{I_d\}$ can be represented as

$$S = W^\top \mathrm{diag}(1, R_1, \dots, R_{(d-1)/2}) W$$

for $W \in \mathrm{O}(d)$ and $(R_1, \dots, R_{(d-1)/2}) \in (\mathrm{SO}(2))^{(d-1)/2}$ not equal to $(I_2, \dots, I_2)$. Then, the same discussion as the case that $d$ is even leads to $\mathrm{fanum}_{\mathbb{R}^d}(\mathcal{S}) = d - 1$.

The cases of $\mathcal{S} = \mathrm{scaled}(\mathrm{O}(d))$ and $\mathcal{S} = \mathrm{scaled}(\mathrm{SO}(d))$ are shown in the same way as the case of $\mathcal{S} = \mathrm{scaled}(\{I_d, \mathrm{Refl}(w)\})$. $\square$

## 3.9 Notes and references

Since group theory is profound, only limited concepts required to multi-robot control theory are introduced in this chapter. The group theory has been utilized in control theory to deal with the motion of rigid bodies (Bullo and D. Lewis, 2004). Its usage is different in this monograph. The group theory is introduced to describe the desired configuration set $\mathcal{D}$ with the orbit of a semidirect product as (3.10), based on the results of Sakurama, 2021b. Then, the invariance of groups will be especially important to design a function evaluating the achievement of the multi-robot coordination in Chapter 8. See Olver, 1995; Olver, 1999 for invariance of groups. The free action number was developed in Sakurama, 2021b from the free group action, a standard concept of graph theory, to describe a graph topological condition for achieving coordination, as seen in Subsection 8.5.2.

# 4

# Graph Theory

In multi-robot systems, a graph is used to describe a topology of the sensing network of robots as Fig. 2.1. Then, distributed controllers are defined in Subsection 2.3.4 as the controllers which use only the information of neighbors over the graph.

This chapter provides graph-theoretical concepts which play important roles in designing distributed controllers, such as *neighbor sets, maximal cliques, intersection graphs, rigidity*, and *clique rigidity*.

## 4.1 Basics

A *graph* $G = (\mathcal{N}, \mathcal{E})$ is a pair of a *node set* $\mathcal{N}$ (a finite countable set) and an *edge set* $\mathcal{E} \subset \mathcal{N}^2$ (a set of pairs of the nodes). Without loss of generality, suppose that the nodes are indexed as $\mathcal{N} = \{1, 2, \ldots, n\}$, and an edge is of the form $\{i, j\}$ for nodes $i, j \in \mathcal{N}$. We assume that $G$ is simple and undirected. Hence, $\{i, i\} \notin \mathcal{E}$, and $\{i, j\} \in \mathcal{E}$ if and only if $\{j, i\} \in \mathcal{E}$.

For node $i \in \mathcal{N}$, a node directly connected by an edge, i.e., $j \in \mathcal{N}$ such that $\{i, j\} \in \mathcal{E}$, is called a *neighbor* of node $i$. The *neighbor set* of

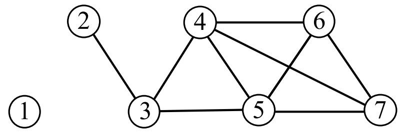



**Figure 4.1:** Example of a graph with 7 nodes.

node $i$ is defined as

$$\mathcal{N}_i := \{j \in \mathcal{N} : \{i, j\} \in \mathcal{E}\} \cup \{i\}. \tag{4.1}$$

For nodes $k, \ell \in \mathcal{N}$, a sequence of nodes $(i_1, i_2, \ldots, i_m) \in \mathcal{N}^m$ such that $i_1 = k$, $i_m = \ell$, and $\{i_h, i_{h+1}\} \in \mathcal{E}$ for all $h \in \{1, 2, \ldots, m-1\}$ is called a *path* between nodes $k, \ell$. Graph $G = (\mathcal{N}, \mathcal{E})$ is said to be *connected* if there is a path between every pair of nodes.

For a node subset $\mathcal{C} \subset \mathcal{N}$, a subgraph $G|_{\mathcal{C}} = (\mathcal{C}, \mathcal{E}|_{\mathcal{C}})$ is said to be *induced* by $\mathcal{C}$ if $\mathcal{E}|_{\mathcal{C}}$ consists of the edges containing the pairs of the nodes in $\mathcal{C}$, that is,

$$\mathcal{E}|_{\mathcal{C}} = \{\{i, j\} \in \mathcal{E} : i, j \in \mathcal{C}\}.$$

**Example 4.1.** Consider the graph $G = (\mathcal{N}, \mathcal{E})$ in Fig. 4.1 for the node set $\mathcal{N} = \{1, 2, 3, 4, 5, 6, 7\}$ and the edge set

$$\mathcal{E} = \{\{2,3\}, \{3,4\}, \{3,5\}, \{4,5\}, \{4,6\}, \{4,7\}, \{5,6\}, \{5,7\}, \{6,7\}\}. \tag{4.2}$$

The neighbor sets are given as

$$\begin{aligned} &\mathcal{N}_1 = \{1\},\ \mathcal{N}_2 = \{2,3\},\ \mathcal{N}_3 = \{2,3,4,5\}, \\ &\mathcal{N}_4 = \mathcal{N}_5 = \{3,4,5,6,7\},\ \mathcal{N}_6 = \mathcal{N}_7 = \{4,5,6,7\}. \end{aligned} \tag{4.3}$$

There is a path between nodes 2 and 6, e.g., $(2, 3, 5, 6)$. This graph is not connected because there is no path between node 1 and the others. The subgroup induced by the node subset $\{2, 3, 4\}$ is given as $G|_{\{2,3,4\}} = (\{2, 3, 4\}, \{\{2, 3\}, \{3, 4\}\})$.

## 4.2 Cliques and maximal cliques

A clique is a node subset which induces a complete subgraph. For multi-robot coordination, edge-based functions are conventionally employed to design distributed controllers. In contrast, by employing clique-based functions, distributed controllers perform the best as shown in Section 7.2. From this viewpoint, cliques are essential for design of distributed controllers.

A node subset $\mathcal{C}$ is called a *clique* in graph $G$ if the subgraph $G|_{\mathcal{C}}$ induced by $\mathcal{C}$ is complete, i.e., $\{i, j\} \in \mathcal{E}$ holds for any $i, j \in \mathcal{C}$, $i \neq j$. The number of the elements in $\mathcal{C}$ is called the *order* of clique $\mathcal{C}$. A clique $\mathcal{C}$ is said to be *maximal* if it is not contained by any other cliques. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q \subset \mathcal{N}$ be the maximal cliques in $G$, and their index set is described as $\mathrm{clq}(G) = \{1, 2, \ldots, q\}$. Let $\mathrm{clq}_i(G)$ be the subset of the indices of the maximal cliques that node $i \in \mathcal{N}$ belongs to, that is

$$\mathrm{clq}_i(G) := \{k \in \mathrm{clq}(G) : i \in \mathcal{C}_k\}. \tag{4.4}$$

**Example 4.2.** Consider the graph $G = (\mathcal{N}, \mathcal{E})$ in Fig. 4.1 with the edge set $\mathcal{E}$ given in (4.2). The cliques of order 1 and 2 are equivalent to the nodes and the edges, respectively. The cliques of order 3 are $\{3, 4, 5\}$, $\{4, 5, 6\}$, $\{4, 5, 7\}$, $\{4, 6, 7\}$, and $\{5, 6, 7\}$. The clique of order 4 is $\{4, 5, 6, 7\}$. The maximal cliques are the following four:

$$\mathcal{C}_1 = \{1\},\ \mathcal{C}_2 = \{2, 3\},\ \mathcal{C}_3 = \{3, 4, 5\},\ \mathcal{C}_4 = \{4, 5, 6, 7\}, \tag{4.5}$$

as illustrated in Fig. 4.2. Corresponding to the maximal cliques $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ in (4.5), the index set of the maximal cliques in $G$ is given by $\mathrm{clq}(G) = \{1, 2, 3, 4\}$. According to (4.4), the index subset of the maximal cliques that each node belongs to is given from (4.5) as follows:

$$\begin{aligned} &\mathrm{clq}_1(G) = \{1\},\ \mathrm{clq}_2(G) = \{2\},\ \mathrm{clq}_3(G) = \{2, 3\}, \\ &\mathrm{clq}_4(G) = \mathrm{clq}_5(G) = \{3, 4\},\ \mathrm{clq}_6(G) = \mathrm{clq}_7(G) = \{4\}. \end{aligned} \tag{4.6}$$

The following lemma gives the equivalence between the neighbor set of each node and the maximal cliques of the node.

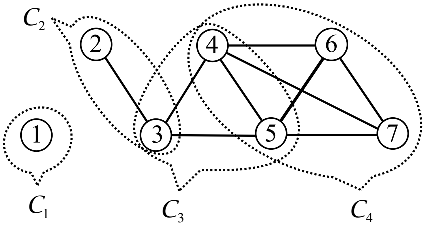



**Figure 4.2:** Maximal cliques in the graph in Fig. 4.1.

**Lemma 4.1.** The following equation holds:

$$\mathcal{N}_i = \bigcup_{k \in \mathrm{clq}_i(G)} \mathcal{C}_k. \tag{4.7}$$

*Proof.* From (4.1), $j \in \mathcal{N}_i$ if and only if $\{i, j\} \in \mathcal{E}$. From the definition of the maximal cliques and (4.4), $j \in \bigcup_{k \in \mathrm{clq}_i(G)} \mathcal{C}_k$, i.e., $j \in \mathcal{C}_k$ for some $k \in \mathrm{clq}_i(G)$, if and only if $\{i, j\} \in \mathcal{E}$. $\square$

**Example 4.3.** Consider the graph $G = (\mathcal{N}, \mathcal{E})$ in Fig. 4.1. From (4.5) and (4.6), the union of $\mathcal{C}_k$ for $k \in \mathrm{clq}_3(G) = \{2, 3\}$ is derived as $\bigcup_{k \in \mathrm{clq}_3(G)} \mathcal{C}_k = \mathcal{C}_2 \cup \mathcal{C}_3 = \{2, 3, 4, 5\}$. This is equivalent to $\mathcal{N}_3$ in (4.3), and the relation (4.7) holds.

## 4.3 Conventional rigidity

Rigidity and global rigidity, introduced from the rigidity theory of bar-and-joint frameworks, provide network-topological conditions to verify the feasibility of the distance-based formation control in Example 2.2. See Subsection 6.4.3 in detail.

For a graph $G = (\mathcal{N}, \mathcal{E})$ and an $n$-tuple of vectors $x^*_{\mathcal{N}} \in (\mathbb{R}^d)^n$, associated with the nodes, the pair $(x^*_{\mathcal{N}}, G)$ is called a *framework* of $x^*_{\mathcal{N}}$ over $G$. The framework $(x^*_{\mathcal{N}}, G)$ is said to be *globally rigid* if the following holds for any $x_{\mathcal{N}} \in (\mathbb{R}^d)^n$:

$$\|x_i - x_j\| = \|x_i^* - x_j^*\| \;\; \forall \{i,j\} \in \mathcal{E} \\ \Rightarrow \|x_i - x_j\| = \|x_i^* - x_j^*\| \;\; \forall i,j \in \mathcal{N}. \quad (4.8)$$

The framework $(x_{\mathcal{N}}^*, G)$ is said to be *rigid* if there exists $\delta > 0$ such that for any $x_{\mathcal{N}} \in (\mathbb{R}^d)^n$,

$$\|x_i - x_j\| = \|x_i^* - x_j^*\| \; \forall \{i,j\} \in \mathcal{E}, \;\; \|x_i - x_i^*\| < \delta \; \forall i \in \mathcal{N} \\ \Rightarrow \|x_i - x_j\| = \|x_i^* - x_j^*\| \;\; \forall i,j \in \mathcal{N}. \quad (4.9)$$

In the condition (4.9) of the rigidity, continuous motion of a part of the framework is not allowed with maintaining the lengths of the edges. In contrast, in the global rigidity (4.8), any motion including discontinuous one is not allowed. Hence, if a framework is globally rigid, it is rigid, while the converse does not hold in general.

Examples of rigid and globally rigid frameworks are given as follows.

**Example 4.4.** Consider frameworks $(x_{\mathcal{N}}^*, G_{\rm a})$, $(x_{\mathcal{N}}^*, G_{\rm b})$, $(x_{\mathcal{N}}^*, G_{\rm c})$, and $(x_{\mathcal{N}}^*, G_{\rm d})$ with 8 nodes in Fig. 4.3, where $x_{\mathcal{N}}^* \in (\mathbb{R}^2)^8$ of the frameworks are all the same.

(a) Framework $(x_{\mathcal{N}}^*, G_{\rm a})$ is not rigid because the left two nodes (the right two nodes) can move continuously while maintaining the lengths of the edges.

(b) Framework $(x_{\mathcal{N}}^*, G_{\rm b})$ is rigid but is not globally rigid because any nodes cannot continuously move but some nodes (e.g., the left lower node) can flip with maintaining the lengths of the edges.

(c) Framework $(x_{\mathcal{N}}^*, G_{\rm c})$ is rigid but is not globally rigid in the same way.

(d) Framework $(x_{\mathcal{N}}^*, G_{\rm d})$ is globally rigid because neither continuous motion nor discontinuous one can occur with maintaining the lengths of the edges.

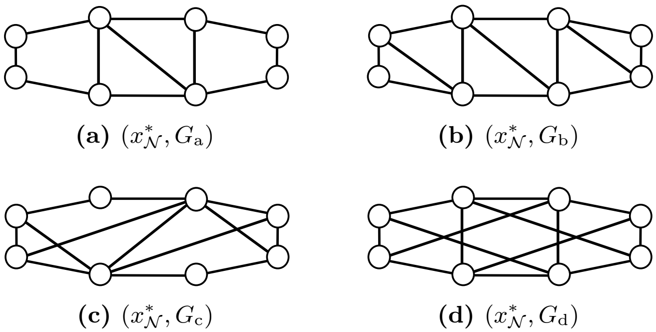



**Figure 4.3:** Examples of frameworks: (a) $(x^*_{\mathcal{N}}, G_{\mathrm{a}})$ is not rigid; (b), (c) $(x^*_{\mathcal{N}}, G_{\mathrm{b}})$ and $(x^*_{\mathcal{N}}, G_{\mathrm{c}})$ are rigid but are not globally rigid; (d) $(x^*_{\mathcal{N}}, G_{\mathrm{d}})$ is globally rigid.

## 4.4 Clique rigidity

Clique rigidity is a network-topological condition generalized from the global rigidity, defined by using the connections between maximal cliques rather than edges. The feasibility of the generalized coordination (2.13) is verified with the clique rigidity of the set framework $(\mathcal{D}, G)$ for a desired configuration set $\mathcal{D}$ and a graph $G$, as discussed in Section 7.5.

For a graph $G$ and a set $\mathcal{D} \subset (\mathbb{R}^d)^n$, the pair $(\mathcal{D}, G)$ is called a *set framework* of $\mathcal{D}$ over $G$. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q$ be the maximal cliques in graph $G$. The set framework $(\mathcal{D}, G)$ is said to be *clique rigid* if

$$x_{\mathcal{C}_k} \in \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D}) \;\; \forall k \in \mathrm{clq}(G) \;\Rightarrow\; x_{\mathcal{N}} \in \mathcal{D}, \tag{4.10}$$

where $\mathrm{proj}_{\cdot}(\cdot)$ is the projection defined in (3.13). The clique rigidity generalizes the global rigidity. Actually, the framework $(x^*_{\mathcal{N}}, G)$ is globally rigid if and only if the set framework $(\mathcal{D}, G)$ is clique rigid for

$$\mathcal{D} = \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : \|x_i - x_j\| = \|x^*_i - x^*_j\| \;\; \forall i, j \in \mathcal{N}, i \neq j\}. \tag{4.11}$$

The relation to the global rigidity is more discussed below.

The following example illustrates how to verify the clique rigidity of the set framework $(\mathcal{D}, G)$ in a concrete case.

**Example 4.5.** Consider the set $\mathcal{D}$ in (3.10) for $\mathcal{S} \ltimes \mathcal{T} = \mathrm{SO}(d) \ltimes \mathbb{R}^d$, that is,

$$\begin{aligned}\mathcal{D} &= \mathrm{orb}_{\mathrm{SO}(d)\ltimes\mathbb{R}^d}(x^*_{\mathcal{N}}) \\ &= \{(Rx^*_1 + \tau, \ldots, Rx^*_n + \tau) : R \in \mathrm{SO}(d), \tau \in \mathbb{R}^d\}, \end{aligned} \tag{4.12}$$

where $n$ points $x^*_1, \ldots, x^*_n$ are transformed in rotation and translation. In contrast, from (3.15),

$$\begin{aligned}\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D}) &= \mathrm{orb}_{\mathrm{SO}(d)\ltimes\mathbb{R}^d}(x^*_{\mathcal{C}_k}) \\ &= \{(R_k x^*_{i_1} + \tau_k, \ldots, R_k x^*_{i_{|\mathcal{C}_k|}} + \tau_k) : R_k \in \mathrm{SO}(d), \tau_k \in \mathbb{R}^d\}\end{aligned} \tag{4.13}$$

is obtained for maximal clique $\mathcal{C}_k = \{i_1, i_2 \ldots, i_{|\mathcal{C}_k|}\}$. Then, the clique rigidity of the set framework $(\mathcal{D}, G)$ can be checked as illustrated by the upper frameworks in Table 4.1 as follows.

(i) Consider a framework $(x^*_{\mathcal{N}}, G)$ for some $x^*_{\mathcal{N}} \in \mathcal{D}$.

(ii) Divide the framework into the frameworks $(x^*_{\mathcal{C}_k}, G|_{\mathcal{C}_k})$ induced by the maximal cliques $\mathcal{C}_k$.

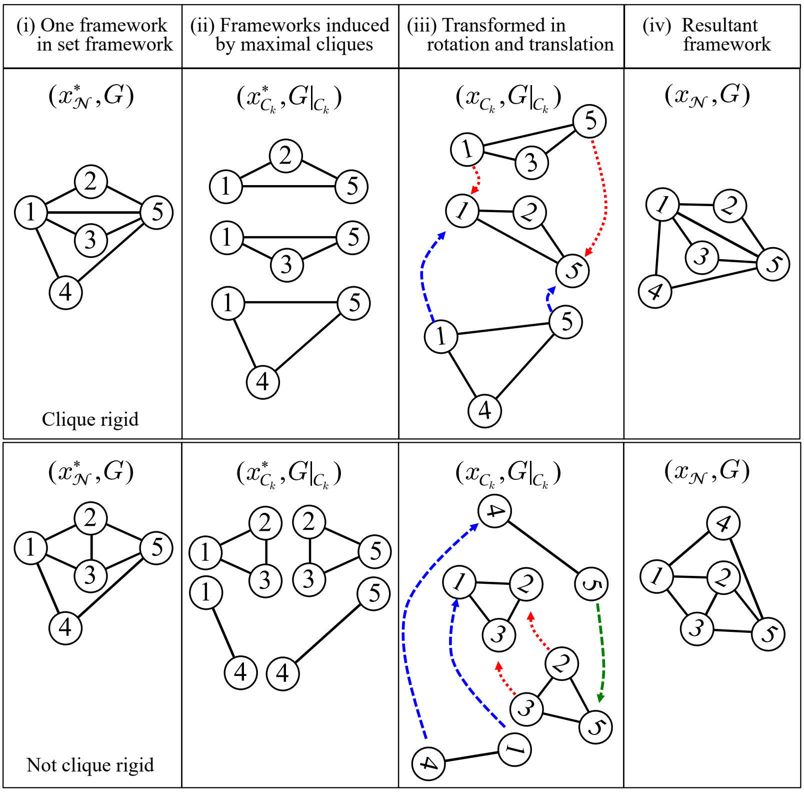
(iii) Derive frameworks $(x_{\mathcal{C}_k}, G|_{\mathcal{C}_k})$ of $x_{\mathcal{C}_k}$ by transforming $x^*_{\mathcal{C}_k}$ in rotation and translation according to $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$ in (4.13). Each node has to be at the same position even if it belongs to different maximal cliques.

(iv) For the resultant framework $(x_{\mathcal{N}}, G)$, check whether $x_{\mathcal{N}}$ always can be obtained from $x^*_{\mathcal{N}}$ through transformation in rotation and translation according to $\mathcal{D}$ in (4.12).

Note that (iii) and (iv) correspond to the assumption and conclusion in the statement (4.10), respectively. On the other hand, the set framework in the lower row of Table 4.1 is not clique rigid because (iv) is not necessarily satisfied.

For specific $\mathcal{D}$ associated with multi-robot coordination problems, the clique rigidity of $(\mathcal{D}, G)$ is equivalent to the connectivity of $G$. We

**Table 4.1:** Illustration of confirming clique rigidity: (upper) clique-rigid set framework and (lower) non-clique-rigid one




will show a couple of examples of such $\mathcal{D}$. The first example is the desired configuration set $\mathcal{D}$ in (2.17) given for the formation selection in Example 2.12.

> **Proposition 4.1.** Consider $\mathcal{D} = \bigcup_{q\in\mathcal{Q}}\{x^{*q}_{\mathcal{N}}\}$ with $x^{*q}_{\mathcal{N}} \in (\mathbb{R}^d)^n$, $q \in \mathcal{Q} = \{1, 2, \ldots, p\}$ for an integer $p \geq 2$, and assume that $x^{*q}_i \neq x^{*\tilde{q}}_i$ holds for any $i \in \mathcal{N}$ and $q, \tilde{q} \in \mathcal{Q}$, $q \neq \tilde{q}$. Then, the set framework $(\mathcal{D}, G)$ is clique rigid if and only if $G$ is connected.

*Proof.* For sufficiency, assume that $G$ is connected and that the assumption part of the statement (4.10) of clique rigidity holds, which is reduced to

$$\forall k \in \mathrm{clq}(G),\ \exists q_k \in \mathcal{Q}\ \text{ s.t. }\ x_{\mathcal{C}_k} = x_{\mathcal{C}_k}^{*q_k}. \tag{4.14}$$

Consider a pair $\mathcal{C}_k, \mathcal{C}_\ell$, $k, \ell \in \mathrm{clq}(G)$, $k \neq \ell$ of maximal cliques, and let $\hat{k} \in \mathcal{C}_k$ and $\hat{\ell} \in \mathcal{C}_\ell$ be contained nodes. From the assumption of the connectivity, there is a path $(i_1(=\hat{k}), i_2, \ldots, i_m(=\hat{\ell}))$ between nodes $\hat{k}$ and $\hat{\ell}$. Because each edge belongs to at least one maximal clique, there exists $k_h \in \mathrm{clq}(G)$ such that $i_h, i_{h+1} \in \mathcal{C}_{k_h}$ for all $h \in \{1, 2, \ldots, m-1\}$. Then, $i_h \in \mathcal{C}_{k_{h-1}} \cap \mathcal{C}_{k_h}$ is satisfied for $h \in \{2, 3, \ldots, m-1\}$, and from (4.14), $x_{i_h} = x_{i_h}^{*q_{k_{h-1}}} = x_{i_h}^{*q_{k_h}}$ holds. Then, $q_{k_{h-1}} = q_{k_h}$ is derived from the assumption. By iterating this process from $h = 2$ to $m$, we obtain $q_{k_1} = q_{k_m}$, yielding $q_k = q_\ell$. In this way, all $q_k$ for $k \in \mathrm{clq}(G)$ coincide with some $q \in \mathcal{Q}$. Then, from (4.14), $x_i = x_i^{*q}$ holds for each $i \in \mathcal{N}$, and the conclusion part of (4.10) is derived.

The necessity is obvious because if $G$ is not connected, the set framework is not clique rigid. □

The second example is the desired configuration set $\mathcal{D}$ for the displacement-based formation in Example 2.5. In fact, the following proposition holds.

**Proposition 4.2.** For $\mathcal{D} = \mathrm{orb}_{\{I_d\} \ltimes \mathbb{R}^d}(x_{\mathcal{N}}^*)$ with $x_{\mathcal{N}}^* \in (\mathbb{R}^d)^n$, the set framework $(\mathcal{D}, G)$ is clique rigid if and only if $G$ is connected.

*Proof.* From the properties (3.2) and (3.6) of the group action, and the definition (3.11) of the group orbit, the assumption part of (4.10) is equivalent to

$$\forall k \in \mathrm{clq}(G),\ \exists \tau_k \in \mathbb{R}^d \text{ s.t. } x_i = x_i^* + \tau_k\ \ \forall i \in \mathcal{C}_k. \tag{4.15}$$

For sufficiency, assume that $G$ is connected and that (4.15) holds. Then, in the same procedure as the proof of Proposition 4.1, from (4.15), $x_{i_h} = x_{i_h}^* + \tau_{k_{h-1}} = x_{i_h}^* + \tau_{k_h}$ is derived, and thus $\tau_{k_{h-1}} = \tau_{k_h}$ holds for $i_h \in \mathcal{C}_{k_{h-1}} \cap \mathcal{C}_{k_h}$. Then, all $\tau_k$ for $k \in \mathrm{clq}(G)$ coincide with some $\tau$, and from (4.15), the conclusion part of (4.10) is derived.

The necessity part holds because if $G$ is not connected, the framework is not clique rigid. □

On the other hand, as for the desired set $\mathcal{D}$ for the distance-based formation of Example 2.5, this set is equivalent to (4.11). Hence, the clique rigidity of $(\mathcal{D}, G)$ is equivalent to the global rigidity of $(x^*_{\mathcal{N}}, G)$. This $\mathcal{D}$ is equivalent to $\mathrm{orb}_{\mathrm{O}(d)\ltimes\mathbb{R}^d}(x^*_{\mathcal{N}})$, and the following is obtained.

**Proposition 4.3.** For $\mathcal{D} = \mathrm{orb}_{\mathrm{O}(d)\ltimes\mathbb{R}^d}(x^*_{\mathcal{N}})$ with $x^*_{\mathcal{N}} \in (\mathbb{R}^d)^n$, the set framework $(\mathcal{D}, G)$ is clique rigid if and only if $(x^*_{\mathcal{N}}, G)$ is globally rigid.

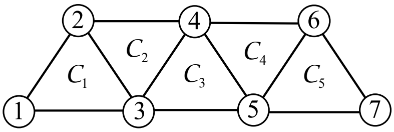
*Proof.* For $\mathcal{D} = \mathrm{orb}_{\mathrm{O}(d)\ltimes\mathbb{R}^d}(x^*_{\mathcal{N}})$, $x_{\mathcal{N}} \in \mathcal{D}$ means that $x_{\mathcal{N}} = (S, \tau) \bullet x^*_{\mathcal{N}}$ for some $(S, \tau) \in \mathrm{O}(d) \ltimes \mathbb{R}^d$, which is equivalent to $x_{\mathcal{N}} \in \mathcal{D}$ for $\mathcal{D}$ in (4.11) from Boutin and Kemper, 2004. Hence, the conclusion parts of (4.8) and (4.10) are equivalent. Their assumption parts can be shown to be equivalent in the same way. $\square$

As for $\mathcal{D} = \mathrm{orb}_{\mathrm{SO}(d)\ltimes\mathbb{R}^d}(x^*_{\mathcal{N}})$, associated with the reflection-free formation in Example 2.7, the set framework $(\mathcal{D}, G)$ is clique rigid only if $(x^*_{\mathcal{N}}, G)$ is rigid but the converse statement does not necessarily hold. Actually, there is no corresponding conventional rigidity. See Sakurama *et al.*, 2019 in detail.

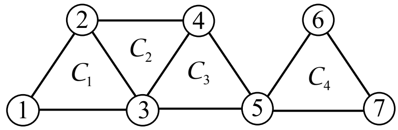
A few examples of clique rigid frameworks can be found in Fig. 4.3 as follows.

**Example 4.6.** Consider the set frameworks $(\mathcal{D}, G_{\mathrm{a}})$, $(\mathcal{D}, G_{\mathrm{b}})$, $(\mathcal{D}, G_{\mathrm{c}})$, and $(\mathcal{D}, G_{\mathrm{d}})$ for the graphs in Fig. 4.3 and $\mathcal{D} = \mathrm{orb}_{\mathcal{S}\ltimes\mathcal{T}}(x^*_{\mathcal{N}})$ with $x^*_{\mathcal{N}}$ in Fig. 4.3 and the following $\mathcal{S} \ltimes \mathcal{T}$.

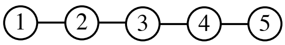
- For $\mathcal{S} \ltimes \mathcal{T} = \{I_2\} \ltimes \mathbb{R}^2$, all the set frameworks are clique rigid from Proposition 4.2 and Example 4.4.
- For $\mathcal{S} \ltimes \mathcal{T} = \mathrm{O}(2) \ltimes \mathbb{R}^2$, only the framework $(x^*_{\mathcal{N}}, G_{\mathrm{d}})$ is clique rigid from Proposition 4.3 and Example 4.4.
- For $\mathcal{S} \ltimes \mathcal{T} = \mathrm{SO}(2) \ltimes \mathbb{R}^2$, only the framework $(x^*_{\mathcal{N}}, G_{\mathrm{b}})$ is clique rigid.



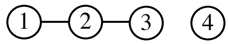

**(a)** Graph $G_{\mathrm{a}}$




**(b)** Graph $G_{\mathrm{b}}$




**(c)** Intersection graph $\Gamma_2(G_{\mathrm{a}})$




**(d)** Intersection graph $\Gamma_2(G_{\mathrm{b}})$

**Figure 4.4:** Examples of 2-intersection graphs: (a) graph $G_{\mathrm{a}}$; (b) graph $G_{\mathrm{b}}$; (c) the 2-intersection graph of $G_{\mathrm{a}}$; (d) that of $G_{\mathrm{b}}$.

## 4.5 Intersection graphs

For a positive integer $r$, the *$r$-intersection graph* of the maximal cliques in $G$, denoted as $\Gamma_r(G)$, is the graph $(\mathrm{clq}(G), \breve{\mathcal{E}}_r)$ with the edge set

$$\breve{\mathcal{E}}_r = \{\{k, \ell\} \in (\mathrm{clq}(G))^2 : |\mathcal{C}_k \cap \mathcal{C}_\ell| \geq r, k \neq \ell\}. \tag{4.16}$$

The intersection graph $\Gamma_r(G)$ represents the topology of the maximal cliques which connect to each other with at least $r$ intersections.

**Example 4.7.** Consider graph $G_{\mathrm{a}}$ in Fig. 4.4a, consisting of 5 maximal cliques of order 3, and graph $G_{\mathrm{b}}$ in Fig. 4.4b, consisting of 4 maximal cliques of order 3. The 2-intersection graphs of $G_{\mathrm{a}}$ and $G_{\mathrm{b}}$, $\Gamma_2(G_{\mathrm{a}})$ and $\Gamma_2(G_{\mathrm{b}})$, are depicted in Figs. 4.4c and 4.4d, respectively.

Intersection graphs are used to verify the clique rigidity for the desired coordination sets $\mathcal{D}$ of multi-robot coordination problems in (3.10). For example, consider the graph $G_{\mathrm{a}}$ in Fig. 4.4a and the set $\mathcal{D}$ for $\mathcal{S} \ltimes \mathcal{T} = \mathrm{SO}(d) \ltimes \mathbb{R}^d$, and $(G_{\mathrm{a}}, \mathcal{D})$ can be guaranteed to be clique-rigid from the connectivity of $\Gamma_2(G_{\mathrm{a}})$ in Fig. 4.4c. See Subsection 8.5.2 for more details.

## 4.6 Notes and references

For graph theory, abundant literature can be found, e.g., Bolloás, 1998. See McKee and McMorris, 1999 for intersection graphs, which are not treated in standard books.

As for how to find maximal cliques $\mathcal{C}_k$ $(k \in \mathrm{clq}(G))$, although it is an NP-complete problem, there are some algorithms. For example, the method of Tomita *et al.*, 2006 requires computation time of $O(3^{n/3})$, where $O(\cdot)$ is the Landau symbol. On the other hand, in multi-robot coordination problems, each robot only needs to know the maximal cliques that it belongs to, i.e., $\mathcal{C}_k$ $(k \in \mathrm{clq}_i(G))$. Lemma 4.1 indicates that the maximal cliques $\mathcal{C}_k$ $(k \in \mathrm{clq}_i(G))$ can be identified from the subgraph $G|_{\mathcal{N}_i}$ of the neighbors. Hence, the computation time necessary to robot $i$ is $O(3^{|\mathcal{N}_i|/3})$, which is not so large if there are not many neighbors.

From the rigidity theory of bar-and-joint frameworks (Sidman and John, 2017), the concepts of rigidity and global rigidity were introduced to verify the feasibility of the distance-based formation as summarized in Anderson *et al.*, 2008; Queiroz *et al.*, 2019. On the other hand, Sakurama, 2021b introduced clique rigidity to verify the feasibility of the generalized coordination with respect to general $\mathcal{D}$. This result will be shown in Section 7.5. Clique rigidity for $\mathcal{D} = \mathrm{orb}_{\mathrm{SO}(d)\ltimes\mathbb{R}^d}(x^*_{\mathcal{N}})$ can be found in Sakurama *et al.*, 2019.

# 5

# Stability Theory for Gradient-flow Systems

In the following chapters, the gradient-flow approach will be employed to design controllers for multi-robot coordination problems. According to this approach, the system of each robot is reduced to the *gradient-flow system* of $v(x_{\mathcal{N}})$ as

$$\dot{x}_i(t) = -\kappa_i \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}(t)),\ x_i(0) = x_i^0 \tag{5.1}$$

with a gain $\kappa_i > 0$ and an initial state $x_i^0 \in \mathbb{R}^d$ for $i \in \mathcal{N}$, where a continuously differentiable function $v : (\mathbb{R}^d)^n \to \mathbb{R}$ is called an *objective function*. Under the system (5.1), the objective function $v(x_{\mathcal{N}}(t))$ is monotonically non-increasing and is expected to converge to a minimum point. Hence, we just have to design a function $v(x_{\mathcal{N}})$ such that a desired configuration is achieved at each minimum point. In the rest of the chapter, we analyze the stability of the gradient-flow system (5.1).

## 5.1 Terminology

Consider a differential equation

$$\dot{x}_{\mathcal{N}}(t) = F(x_{\mathcal{N}}(t)),\ x_{\mathcal{N}}(0) = x_{\mathcal{N}}^0 \in (\mathbb{R}^d)^n \tag{5.2}$$

with a continuous function $F : (\mathbb{R}^d)^n \to (\mathbb{R}^d)^n$. We assume that the solution $x_{\mathcal{N}}(t) \in (\mathbb{R}^d)^n$ of (5.2) uniquely exists for all $t \in [0, \infty)$ for arbitrary $x_{\mathcal{N}}^0 \in (\mathbb{R}^d)^n$. A closed set $\mathcal{A} \subset (\mathbb{R}^d)^n$ is called an *equilibrium set* of (5.2) if

$$x_{\mathcal{N}} \in \mathcal{A} \Rightarrow F(x_{\mathcal{N}}) = 0. \tag{5.3}$$

Let $x_{\mathcal{N}}(t) \in (\mathbb{R}^d)^n$ be the solution of (5.2) for $x_{\mathcal{N}}^0 \in (\mathbb{R}^d)^n$, and some properties related to the stability of the system (5.2) are defined, which are valid even when $\mathcal{A}$ is unbounded.

- The system (5.2) is said to be *Lagrange stable* if for any $x_{\mathcal{N}}^0 \in (\mathbb{R}^d)^n$, there exists $\eta = \eta(x_{\mathcal{N}}^0) > 0$ such that
$$\|x_{\mathcal{N}}(t)\| \le \eta \quad \forall t \ge 0. \tag{5.4}$$
- An equilibrium set $\mathcal{A}$ is said to be *(Lyapunov) stable* if for each $\varepsilon > 0$, there exists an open set $\Delta(\varepsilon) \supset \mathcal{A}$ such that
$$x_{\mathcal{N}}^0 \in \Delta(\varepsilon) \Rightarrow \operatorname{dist}(x_{\mathcal{N}}(t), \mathcal{A}) \le \varepsilon \quad \forall t \ge 0. \tag{5.5}$$
- An equilibrium set $\mathcal{A}$ is said to be *attractive* if there exists an open set $\Delta \supset \mathcal{A}$ such that
$$x_{\mathcal{N}}^0 \in \Delta \Rightarrow \lim_{t\to\infty} \operatorname{dist}(x_{\mathcal{N}}(t), \mathcal{A}) = 0. \tag{5.6}$$
- An equilibrium set $\mathcal{A}$ is said to be *asymptotically stable* if $\mathcal{A}$ is stable and attractive.
- An equilibrium set $\mathcal{A}$ is said to be *globally attractive* if $\mathcal{A}$ is attractive for $\Delta = (\mathbb{R}^d)^n$ in (5.6).
- An equilibrium set $\mathcal{A}$ is said to be *globally asymptotically stable* if $\mathcal{A}$ is stable and globally attractive.

The other terminology is given as follows.

- A *positive orbit* through $x_{\mathcal{N}}^0 \in (\mathbb{R}^d)^n$ is defined as
$$\begin{aligned}\mathcal{O}^+(x_{\mathcal{N}}^0) := \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : \exists t \in [0, \infty)& \\ \text{s.t.}\ \ x_{\mathcal{N}} = x_{\mathcal{N}}(t), x_{\mathcal{N}}(0) = x_{\mathcal{N}}^0&\}.\end{aligned}$$

- A set $\mathcal{I} \subset (\mathbb{R}^d)^n$ is said to be *positively invariant* if

$$x_{\mathcal{N}}^0 \in \mathcal{I} \Rightarrow \mathcal{O}^+(x_{\mathcal{N}}^0) \subset \mathcal{I}.$$

- A function $v : (\mathbb{R}^d)^n \to \mathbb{R}$ is said to be *radially unbounded* if $v(x_{\mathcal{N}}) \to \infty$ as $\|x_{\mathcal{N}}\| \to \infty$.

## 5.2 Lagrange stability

Consider the gradient-flow system (5.1) with an objective function $v(x_{\mathcal{N}})$. From now on, we assume that the solution $x_{\mathcal{N}}(t) \in (\mathbb{R}^d)^n$ of this system uniquely exists for all $t \in [0, \infty)$ for arbitrary $x_{\mathcal{N}}^0 \in (\mathbb{R}^d)^n$. Although we assume that $v(x_{\mathcal{N}})$ is continuously differentiable here, non-differentiable functions can be treated as shown in Section 5.4.

First, three conditions of $v(x_{\mathcal{N}})$ are provided for the Lagrange stability of this system and the global attractiveness of the zero set $(\partial v/\partial x_{\mathcal{N}})^{-1}(0)$, where

$$\frac{\partial v}{\partial x_{\mathcal{N}}}(x_{\mathcal{N}}) = \left(\frac{\partial v}{\partial x_1}(x_{\mathcal{N}}), \ldots, \frac{\partial v}{\partial x_n}(x_{\mathcal{N}})\right),$$
$$\left(\frac{\partial v}{\partial x_{\mathcal{N}}}\right)^{-1}(0) = \left\{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : \frac{\partial v}{\partial x_{\mathcal{N}}}(x_{\mathcal{N}}) = 0\right\}.$$

For preliminaries, two lemmas are given, where $\mathrm{cl}(\cdot)$ represents the closure of a set.

**Lemma 5.1.** [Proposition 2.32 in Haddad and Chellaboina, 2008] For the system (5.2), if a set $\mathcal{I} \subset (\mathbb{R}^d)^n$ is positively invariant, $\mathrm{cl}(\mathcal{I})$ is positively invariant.

**Lemma 5.2.** (LaSalle's invariance principle) [Theorem 3.3 in Haddad and Chellaboina, 2008] For the system (5.2), assume that $\mathcal{D}_c \subset (\mathbb{R}^d)^n$ is a positively invariant, compact set, and that there exists a continuously differentiable function $v : \mathcal{D}_c \to \mathbb{R}$ such that $\dot{v}(x_{\mathcal{N}}) \leq 0$ for any $x_{\mathcal{N}} \in \mathcal{D}_c$, where $\dot{v}(x_{\mathcal{N}}) = \langle \partial v/\partial x_{\mathcal{N}}(x_{\mathcal{N}}), F(x_{\mathcal{N}})\rangle$. Let $\mathcal{I} \subset \mathcal{D}_c$ be the largest positively invariant set contained in

$\{x_{\mathcal{N}} \in \mathcal{D}_c : \dot{v}(x_{\mathcal{N}}) = 0\}$. Then, if $x_{\mathcal{N}}^0 \in \mathcal{D}_c$, $\lim_{t\to\infty} \mathrm{dist}(x_{\mathcal{N}}(t), \mathcal{I}) = 0$ holds.

Under the gradient-flow system (5.1), $v(x_{\mathcal{N}}(t))$ is monotonically non-increasing with respect to $t$ as

$$\begin{aligned}\dot{v}(x_{\mathcal{N}}(t)) &= \left\langle \frac{\partial v}{\partial x_{\mathcal{N}}}(x_{\mathcal{N}}(t)), \dot{x}_{\mathcal{N}}(t) \right\rangle = \sum_{i=1}^{n} \left\langle \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}(t)), \dot{x}_i(t) \right\rangle \\ &= -\sum_{i=1}^{n} \kappa_i \left\| \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}(t)) \right\|^2 \le 0. \end{aligned} \tag{5.7}$$

From this inequality, the stability properties are guaranteed as follows.

**Theorem 5.3.** If a continuously differentiable function $v : (\mathbb{R}^d)^n \to \mathbb{R}$ is radially unbounded, then the system (5.1) is Lagrange stable and $(\partial v/\partial x_{\mathcal{N}})^{-1}(0)$ is globally attractive.

*Proof.* From the radial unboundedness of $v(x_{\mathcal{N}})$, if $v(x_{\mathcal{N}}) \le \lambda_1$ for $\lambda_1 > 0$, $\|x_{\mathcal{N}}\| \le \lambda_2$ holds for some $\lambda_2 = \lambda_2(\lambda_1) > 0$. From (5.7), $v(x_{\mathcal{N}}(t)) \le v(x_{\mathcal{N}}^0)$ holds, and thus $\|x_{\mathcal{N}}(t)\| \le \lambda_2(v(x_{\mathcal{N}}^0))$ holds for all $t \ge 0$. According to (5.4), the system is Lagrange stable.

Let $\hat{x}_{\mathcal{N}}(t) \in (\mathbb{R}^d)^n$ be the solution of (5.1) for $\hat{x}_{\mathcal{N}}^0 \in \mathcal{O}^+(x_{\mathcal{N}}^0)$ instead of $x_{\mathcal{N}}^0$. Because the positive orbit $\mathcal{O}^+(x_{\mathcal{N}}^0)$ is positively invariant under (5.1), $\hat{x}_{\mathcal{N}}(t) \in \mathcal{O}^+(x_{\mathcal{N}}^0)$ holds for all $t \ge 0$. From Lemma 5.1, $\mathrm{cl}(\mathcal{O}^+(x_{\mathcal{N}}^0))$ is positively invariant, which is compact from the Lagrange stability. By applying Lemma 5.2 for $\mathcal{D}_c = \mathrm{cl}(\mathcal{O}^+(x_{\mathcal{N}}^0))$, from (5.7), the solution $x_{\mathcal{N}}(t)$ of (5.1) is ensured to converge to the the largest positively invariant set contained in the set where $\dot{v}(x_{\mathcal{N}}) = 0$. From (5.7), $\{x_{\mathcal{N}} \in \mathcal{D}_c : \dot{v}(x_{\mathcal{N}}) = 0\} = (\partial v/\partial x_{\mathcal{N}})^{-1}(0) \cap \mathcal{D}_c$ holds, and thus the following is obtained:

$$x_{\mathcal{N}}^0 \in \mathcal{D}_c \Rightarrow \lim_{t\to\infty} \mathrm{dist}(x_{\mathcal{N}}(t), \left(\frac{\partial v}{\partial x_{\mathcal{N}}}\right)^{-1}(0) \cap \mathcal{D}_c) = 0.$$

This discussion is valid for each $x_{\mathcal{N}}^0 \in (\mathbb{R}^d)^n$. Hence, $(\partial v/\partial x_{\mathcal{N}})^{-1}(0)$ is globally attractive. □

Theorem 5.3 assumes the radial unboundedness of the objective function $v(x_{\mathcal{N}})$. In contrast, the following two theorems do not assume it, and assume some invariance conditions of $v(x_{\mathcal{N}})$ instead. The first theorem requires invariance under translation, namely, $(\{I_d\} \ltimes \mathbb{R}^d)$-invariance.

**Theorem 5.4.** If a continuously differentiable function $v : (\mathbb{R}^d)^n \to \mathbb{R}$ satisfies the following two conditions, then the system (5.1) is Lagrange stable and $(\partial v/\partial x_{\mathcal{N}})^{-1}(0)$ is globally attractive:

(i) $v(x_{\mathcal{N}})$ is $(\{I_d\} \ltimes \mathbb{R}^d)$-invariant.

(ii) There exists an $n$-tuple $(i_1, i_2, \ldots, i_n)$ of the distinct elements in $\mathcal{N}$ such that for each $\ell \in \{2, 3, \ldots, n\}$ there exists $\hat{\ell} < \ell$ satisfying $\|x_{i_\ell} - x_{i_{\hat{\ell}}}\| \leq \zeta_\ell(v(x_{\mathcal{N}}))$ with a monotonically non-decreasing, non-negative function $\zeta_\ell : \mathbb{R} \to \mathbb{R}_+$.

*Proof.* From (3.2), (3.6), and (3.18), condition (i) indicates that

$$v((I_d, \tau) \bullet x_{\mathcal{N}}) = v(x_1 + \tau, \ldots, x_n + \tau) = v(x_{\mathcal{N}})$$

holds for any $\tau \in \mathbb{R}^d$. By partially differentiating this equation with respect to $\tau$ and substituting $\tau$ with 0,

$$\begin{aligned} 0 &= \left.\frac{\partial v(x_{\mathcal{N}})}{\partial \tau}\right|_{\tau=0} = \left.\frac{\partial v(x_1 + \tau, \ldots, x_n + \tau)}{\partial \tau}\right|_{\tau=0} \\ &= \sum_{i \in \mathcal{N}} \left.\frac{\partial v}{\partial x_i}(x_1 + \tau, \ldots, x_n + \tau)\right|_{\tau=0} = \sum_{i \in \mathcal{N}} \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}) \end{aligned}$$

is achieved. Hence, from (5.1), we obtain

$$\sum_{i \in \mathcal{N}} \frac{\dot{x}_i(t)}{\kappa_i} = -\sum_{i \in \mathcal{N}} \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}(t)) = 0,$$

which leads to

$$\sum_{i \in \mathcal{N}} \frac{x_i(t)}{\kappa_i} = \sum_{i \in \mathcal{N}} \frac{x_i^0}{\kappa_i}. \tag{5.8}$$

Without loss of generality, we assume that $i_\ell = \ell$ in the $n$-tuple in condition (ii). Then, because $\zeta_i(\cdot)$ is monotonically non-decreasing for any $i \in \{2, 3, \ldots, n\}$, from (5.7),

$$\sum_{i=2}^{n} \|x_i(t) - x_{\hat{i}}(t)\|^2 \leq \sum_{i=2}^{n} \zeta_i(v(x_{\mathcal{N}}(t))) \leq \sum_{i=2}^{n} \zeta_i(v(x_{\mathcal{N}}^0)) \tag{5.9}$$

is obtained with some $\hat{i} < i$. On the other hand,

$$\sum_{i=2}^{n} \|x_i - x_{\hat{i}}\|^2 + \left\| \sum_{i \in \mathcal{N}} \frac{x_i}{\kappa_i} \right\|^2 = \|x_{\mathcal{N}} P\|^2 \geq (\sigma_{\min}(P))^2 \|x_{\mathcal{N}}\|^2 \tag{5.10}$$

holds with the matrix $P \in \mathbb{R}^{n \times n}$ defined as

$$P = \begin{bmatrix} \frac{1}{\kappa_1} & -1 & * & \cdots & * \\ \frac{1}{\kappa_2} & 1 & * & \ddots & \vdots \\ \frac{1}{\kappa_3} & 0 & 1 & \ddots & * \\ \vdots & \vdots & \ddots & \ddots & * \\ \frac{1}{\kappa_n} & 0 & \cdots & 0 & 1 \end{bmatrix},$$

where $*$ takes 0 or $-1$ corresponding to $\hat{i}$. In (5.10), $x_{\mathcal{N}}$ is regarded as the corresponding matrix $[x_1 \ \cdots \ x_n] \in \mathbb{R}^{d \times n}$ and $\sigma_{\min}(\cdot)$ represents the smallest singular value of a matrix. Because $\kappa_i > 0$ from the assumption, $P$ is non-singular, and thus $\sigma_{\min}(P) > 0$. From (5.8), (5.9), and (5.10),

$$\|x_{\mathcal{N}}(t)\| \leq \frac{1}{\sigma_{\min}(P)} \sqrt{\sum_{i=2}^{n} \zeta_i(v(x_{\mathcal{N}}^0)) + \left\| \sum_{i=1}^{n} \frac{x_i^0}{\kappa_i} \right\|^2}$$

is obtained for any $t \geq 0$, which yields (5.4). Hence, the system is Lagrange stable.

The global attractiveness of $(\partial v / \partial x_{\mathcal{N}})^{-1}(0)$ is shown in the same way as Theorem 5.3. □

**Example 5.1.** The functions in (3.20) and (3.22) in Examples 3.7 and 3.8 satisfy the conditions in Theorem 5.4. Regard $\mathcal{E}$ as the edge set of a graph $G = (\mathcal{N}, \mathcal{E})$, and without loss of generality, we assume that $G$ is connected. Otherwise, we just have to consider each connected component of $G$. Condition (i) is satisfied as shown in (3.21) and (3.23). As for condition (ii), from the connectivity of $G$, there exists an $n$-tuple $(i_1, i_2, \ldots, i_n)$ of the distinct elements in $\mathcal{N}$ such that for each $\ell \in \{2, 3, \ldots, n\}$, $\{i_{\hat{\ell}}, i_\ell\} \in \mathcal{E}$ holds for some $\hat{\ell} < \ell$. Then, the function in (3.20) satisfies condition (ii) with

$\zeta_\ell(v) = \sqrt{v} + \|r_{i_\ell i_{\hat{\ell}}}\|$ because

$$\sqrt{v(x_{\mathcal{N}})} \geq \|x_{i_\ell} - x_{i_{\hat{\ell}}} - r_{i_\ell i_{\hat{\ell}}}\| \geq \|x_{i_\ell} - x_{i_{\hat{\ell}}}\| - \|r_{i_\ell i_{\hat{\ell}}}\|.$$

Similarly, the function in (3.22) satisfies condition (ii). Note that the functions in (3.20) and (3.22) are not radially unbounded because they are invariant under translation, which means that by assigning $x_i = x_i^* + \tau$ with some $x_i^* \in \mathbb{R}^d$, $v(x_{\mathcal{N}}) = v(x_{\mathcal{N}}^*)$ holds for arbitrary $\tau \in \mathbb{R}^d$.

The next theorem assumes the relative invariance under scale, namely, relative $(\mathrm{scaled}(\{I_d\}) \ltimes \mathbb{R}^d)$-invariance of functions.

**Theorem 5.5.** If a non-negative, continuously differentiable function $v : (\mathbb{R}^d)^n \to \mathbb{R}_+$ is relatively $(\mathrm{scaled}(\{I_d\}) \ltimes \{0\})$-invariant of weight $\mu(s)$ for $(sI_d, 0) \in \mathrm{scaled}(\{I_d\}) \ltimes \{0\}$ with $s > 0$ satisfying $\mathrm{d}\mu/\mathrm{d}s(1) \geq 0$, then the system (5.1) is Lagrange stable and $(\partial v/\partial x_{\mathcal{N}})^{-1}(0)$ is globally attractive.

*Proof.* From (3.2), (3.6), and (3.19), the condition of the relative invariance indicates that

$$v((sI_d, 0) \bullet x_{\mathcal{N}}) = v(sx_{\mathcal{N}}) = \mu(s) v(x_{\mathcal{N}}). \tag{5.11}$$

From (5.11) and the chain rule, we obtain

$$\begin{aligned}\frac{\mathrm{d}\mu}{\mathrm{d}s}(1) v(x_{\mathcal{N}}) &= \left.\frac{\partial \mu(s) v(x_{\mathcal{N}})}{\partial s}\right|_{s=1} = \left.\frac{\partial v(sx_{\mathcal{N}})}{\partial s}\right|_{s=1} \\ &= \left.\left\langle \frac{\partial s x_{\mathcal{N}}}{\partial s}, \frac{\partial v}{\partial x_{\mathcal{N}}}(sx_{\mathcal{N}}) \right\rangle\right|_{s=1} = \left\langle x_{\mathcal{N}}, \frac{\partial v}{\partial x_{\mathcal{N}}}(x_{\mathcal{N}}) \right\rangle. \end{aligned} \tag{5.12}$$

From (5.1), (5.12), and the non-negativeness of $\mathrm{d}\mu/\mathrm{d}s(1)$ and $v(x_{\mathcal{N}})$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|x_{\mathcal{N}}(t) K^{-\frac{1}{2}}\|^2 &= 2\langle x_{\mathcal{N}}(t), \dot{x}_{\mathcal{N}}(t) K^{-1}\rangle = -2\left\langle x_{\mathcal{N}}, \frac{\partial v}{\partial x_{\mathcal{N}}}(x_{\mathcal{N}})\right\rangle \\ &= -2\frac{\mathrm{d}\mu}{\mathrm{d}s}(1) v(x_{\mathcal{N}}) \leq 0 \end{aligned} \tag{5.13}$$

is achieved, where $K = \mathrm{diag}(\kappa_{\mathcal{N}})$ and $x_{\mathcal{N}}$ is regarded as the corresponding matrix $[x_1 \ \cdots \ x_n] \in \mathbb{R}^{d\times n}$. From (5.13), we obtain

$$\|x_{\mathcal{N}}(t)\| \leq \max_{i\in\mathcal{N}} \sqrt{\kappa_i}\|x_{\mathcal{N}}(t) K^{-\frac{1}{2}}\| \leq \max_{i\in\mathcal{N}} \sqrt{\kappa_i}\|x_{\mathcal{N}}^0 K^{-\frac{1}{2}}\|$$

for any $t \geq 0$, which yields the Lagrange stability from (5.4).

The global attractiveness of $(\partial v/\partial x_{\mathcal{N}})^{-1}(0)$ is shown in the same way as Theorem 5.3. □

**Example 5.2.** The function $v(x_{\mathcal{N}})$ in (3.24) in Example 3.9 satisfies the condition in Theorem 5.5 for $\mu(s) = s^4$ from (3.25). This function is not radially unbounded because it is invariant under translation.

## 5.3 Asymptotic stability

Under some assumptions including the Lagrange stability, the zero set $v^{-1}(0)$ of the objective function is asymptotically stable as follows.

**Theorem 5.6.** Let $v : (\mathbb{R}^d)^n \to \mathbb{R}_+$ be a non-negative, continuously differentiable function such that $v^{-1}(0)$ is non-empty and $v(x_{\mathcal{N}})$ is real analytic in an open set containing $v^{-1}(0)$. Assume that the system (5.1) is Lagrange stable. Then, $v^{-1}(0)$ is asymptotically stable.

*Proof.* Before proving the stability, some preliminaries are provided. From (5.7), Barbalat's lemma (Khalil, 2002) guarantees that

$$\lim_{t\to\infty} \frac{\partial v}{\partial x_{\mathcal{N}}}(x_{\mathcal{N}}(t)) = 0 \tag{5.14}$$

for the solution $x_{\mathcal{N}}(t)$ of the system (5.1). For $\bar{x}_{\mathcal{N}} \in v^{-1}(0)$, let $\Theta(\bar{x}_{\mathcal{N}})$ be an open bounded set containing $\bar{x}_{\mathcal{N}}$. Under the assumption that the system is Lagrange stable, for each $x^0_{\mathcal{N}} \in (\mathbb{R}^d)^n$, the positive orbit $\mathcal{O}^+(x^0_{\mathcal{N}}) \subset (\mathbb{R}^d)^n$ through $x^0_{\mathcal{N}}$ is bounded. Let

$$\hat{\mathcal{O}}^+(\bar{x}_{\mathcal{N}}) = \bigcup_{x^0_{\mathcal{N}} \in \Theta(\bar{x}_{\mathcal{N}})} \mathcal{O}^+(x^0_{\mathcal{N}}),$$

which is bounded, and

$$x^0_{\mathcal{N}} \in \Theta(\bar{x}_{\mathcal{N}}) \Rightarrow x_{\mathcal{N}}(t) \in \hat{\mathcal{O}}^+(\bar{x}_{\mathcal{N}}) \quad \forall t \geq 0 \tag{5.15}$$

holds from the definition of the positive orbit. Let $\Omega \subset (\mathbb{R}^d)^n$ be an open set containing $v^{-1}(0)$ such that $v(x_\mathcal{N})$ is real analytic in $\Omega$. From the compactness of $\mathrm{cl}(\hat{\mathcal{O}}^+(\bar{x}_\mathcal{N}))$, there exists $\rho(\bar{x}_\mathcal{N}) > 0$ such that

$$\mathcal{L}^-_{\rho(\bar{x}_\mathcal{N})}(v) \cap \mathrm{cl}(\hat{\mathcal{O}}^+(\bar{x}_\mathcal{N})) \subset \Omega, \tag{5.16}$$

where $\mathcal{L}^-_\rho(v) = \{x_\mathcal{N} \in (\mathbb{R}^d)^n : v(x_\mathcal{N}) \le \rho\}$ is the $\rho$-sublevel set of $v(x_\mathcal{N})$ for $\rho \in \mathbb{R}$.

First, we show that $v^{-1}(0)$ is stable. From the real analyticity of $v(x_\mathcal{N})$ in $\Omega$, Lemma B.2 guarantees that for any compact set $\Omega_1(\bar{x}_\mathcal{N}) \subset \Omega$, there exist positive $\beta_1(\Omega_1(\bar{x}_\mathcal{N})), \theta_1(\Omega_1(\bar{x}_\mathcal{N}))$ such that Łojasiewicz's inequality (B.1) holds for $v(x_\mathcal{N})$, which is reduced to

$$\mathrm{dist}(x_\mathcal{N}, v^{-1}(0)) \le \left(\frac{v(x_\mathcal{N})}{\beta_1(\Omega_1(\bar{x}_\mathcal{N}))}\right)^{\frac{1}{\theta_1(\Omega_1(\bar{x}_\mathcal{N}))}} \quad \forall x_\mathcal{N} \in \Omega_1(\bar{x}_\mathcal{N}). \tag{5.17}$$

For a constant $\varepsilon > 0$, let

$$\mathcal{S}_1(\bar{x}_\mathcal{N}, \varepsilon) = \mathrm{int}(\mathcal{L}^-_{\rho_1(\bar{x}_\mathcal{N},\varepsilon)}(v)) \cap \Theta(\bar{x}_\mathcal{N}) \tag{5.18}$$

$$\hat{\mathcal{S}}_1(\bar{x}_\mathcal{N}, \varepsilon) = \mathrm{int}(\mathcal{L}^-_{\rho_1(\bar{x}_\mathcal{N},\varepsilon)}(v)) \cap \hat{\mathcal{O}}^+(\bar{x}_\mathcal{N}) \tag{5.19}$$

with $\rho_1(\bar{x}_\mathcal{N}, \varepsilon) \in (0, \rho(x_\mathcal{N})]$ determined later, where $\mathrm{int}(\cdot)$ represents the interior of a set. Then, from (5.7) and (5.15),

$$x^0_\mathcal{N} \in \mathcal{S}_1(\bar{x}_\mathcal{N}, \varepsilon) \Rightarrow x_\mathcal{N}(t) \in \hat{\mathcal{S}}_1(\bar{x}_\mathcal{N}, \varepsilon) \quad \forall t \ge 0 \tag{5.20}$$

holds. In (5.17), we assign

$$\Omega_1(\bar{x}_\mathcal{N}) = \mathcal{L}^-_{\rho(\bar{x}_\mathcal{N})}(v) \cap \mathrm{cl}(\hat{\mathcal{O}}^+(\bar{x}_\mathcal{N})) \subset \Omega, \tag{5.21}$$

where the inclusion is from (5.16). This $\Omega_1(\bar{x}_\mathcal{N})$ is compact from the compactness of $\mathrm{cl}(\hat{\mathcal{O}}^+(\bar{x}_\mathcal{N}))$. Then, if $x^0_\mathcal{N} \in \mathcal{S}_1(\bar{x}_\mathcal{N}, \varepsilon)$, from (5.19) and (5.20), $v(x_\mathcal{N}(t)) \le \rho_1(\bar{x}_\mathcal{N}, \varepsilon)$ holds, and from this inequality and (5.17),

$$\begin{aligned}\mathrm{dist}(x_\mathcal{N}(t), v^{-1}(0)) &\le \left(\frac{v(x_\mathcal{N}(t))}{\beta_1(\Omega_1(\bar{x}_\mathcal{N}))}\right)^{\frac{1}{\theta_1(\Omega_1(\bar{x}_\mathcal{N}))}} \\ &\le \left(\frac{\rho_1(\bar{x}_\mathcal{N}, \varepsilon)}{\beta_1(\Omega_1(\bar{x}_\mathcal{N}))}\right)^{\frac{1}{\theta_1(\Omega_1(\bar{x}_\mathcal{N}))}} \le \varepsilon \quad \forall t \ge 0\end{aligned} \tag{5.22}$$

holds, where the last inequality holds by assigning $\rho_1(\bar{x}_\mathcal{N}, \varepsilon) > 0$ as

$$\rho_1(\bar{x}_\mathcal{N}, \varepsilon) = \min\{\rho(\bar{x}_\mathcal{N}), \beta_1(\Omega_1(\bar{x}_\mathcal{N}))\varepsilon^{\theta_1(\Omega_1(\bar{x}_\mathcal{N}))}\}.$$

Let

$$\Delta(\varepsilon) = \bigcup_{\bar{x}_{\mathcal{N}} \in v^{-1}(0)} \mathcal{S}_1(\bar{x}_{\mathcal{N}}, \varepsilon),$$

which is an open set containing $v^{-1}(0)$ from (5.18), and if $x^0_{\mathcal{N}} \in \Delta(\varepsilon)$, the solution $x_{\mathcal{N}}(t)$ satisfies (5.22) from the above discussion. This implies that (5.5) holds for $\mathcal{A} = v^{-1}(0)$, and thus $v^{-1}(0)$ is stable.

Second, we show that $v^{-1}(0)$ is attractive. From the real analyticity of $v(x_{\mathcal{N}})$, Łojasiewicz's inequalities guarantee that for any compact set $\Omega_1(\bar{x}) \subset \Omega$ and bounded open set $\Omega_2(\bar{x}) \subset \Omega$, there exist positive $\beta_1(\Omega_1(\bar{x}))$, $\theta_1(\Omega_1(\bar{x}))$, $\beta_2(\Omega_2(\bar{x}))$, $\theta_2(\Omega_2(\bar{x}))$, and $\rho_2(\Omega_2(\bar{x})) \le \rho(\bar{x})$ such that (B.1) and (B.2) hold for $v(x_{\mathcal{N}})$, which are reduced to

$$\beta_1(\Omega_1(\bar{x}))(\mathrm{dist}(x_{\mathcal{N}}, v^{-1}(0)))^{\theta_1(\Omega_1(\bar{x}))} \le \beta_2(\Omega_2(\bar{x})) \left\| \frac{\partial v}{\partial x_{\mathcal{N}}}(x_{\mathcal{N}}) \right\|^{\theta_2(\Omega_2(\bar{x}))}$$
$$\forall x_{\mathcal{N}} \in \Omega_1(\bar{x}) \cap \Omega_2(\bar{x}) \cap \mathrm{int}(\mathcal{L}^-_{\rho_2(\Omega_2(\bar{x}))}(v)). \tag{5.23}$$

We assign $\Omega_1(\bar{x})$ as (5.21) and $\Omega_2(\bar{x})$ as a bound open set satisfying $\Omega_1(\bar{x}) \subset \Omega_2(\bar{x}) \subset \Omega$, which is possible from (5.21). Then,

$$\Omega_1(\bar{x}) \cap \Omega_2(\bar{x}) \cap \mathrm{int}(\mathcal{L}^-_{\rho_2(\Omega_2(\bar{x}))}(v)) = \mathrm{cl}(\hat{\mathcal{O}}^+(\bar{x}_{\mathcal{N}})) \cap \mathrm{int}(\mathcal{L}^-_{\rho_2(\Omega_2(\bar{x}))}(v)) \tag{5.24}$$

holds. Let

$$\mathcal{S}_2(\bar{x}_{\mathcal{N}}) = \mathrm{int}(\mathcal{L}^-_{\rho_2(\Omega_2(\bar{x}))}(v)) \cap \Theta(\bar{x}_{\mathcal{N}}) \tag{5.25}$$

$$\hat{\mathcal{S}}_2(\bar{x}_{\mathcal{N}}) = \mathrm{int}(\mathcal{L}^-_{\rho_2(\Omega_2(\bar{x}))}(v)) \cap \hat{\mathcal{O}}^+(\bar{x}_{\mathcal{N}}), \tag{5.26}$$

and

$$x^0_{\mathcal{N}} \in \mathcal{S}_2(\bar{x}_{\mathcal{N}}) \Rightarrow x_{\mathcal{N}}(t) \in \hat{\mathcal{S}}_2(\bar{x}_{\mathcal{N}}) \quad \forall t \ge 0 \tag{5.27}$$

is achieved from (5.7) and (5.15). For the initial state $x^0_{\mathcal{N}} \in \mathcal{S}_2(\bar{x}_{\mathcal{N}})$, from (5.24), (5.26), and (5.27),

$$x_{\mathcal{N}}(t) \in \Omega_1(\bar{x}) \cap \Omega_2(\bar{x}) \cap \mathrm{int}(\mathcal{L}^-_{\rho_2(\Omega_2(\bar{x}))}(v)) \quad \forall t \ge 0 \tag{5.28}$$

holds. Then, from (5.14), (5.23), and (5.28), $\lim_{t\to\infty} \mathrm{dist}(x_{\mathcal{N}}(t), v^{-1}(0)) = 0$ holds. Let

$$\Delta = \bigcup_{\bar{x}_{\mathcal{N}} \in v^{-1}(0)} \mathcal{S}_2(\bar{x}_{\mathcal{N}}),$$

which is an open set containing $v^{-1}(0)$ from (5.25), and (5.6) holds for $\mathcal{A} = v^{-1}(0)$. Therefore, $v^{-1}(0)$ is attractive. □

## 5.4 Remarks on non-differentiable functions

Assume that an objective function $v : (\mathbb{R}^d)^n \to \mathbb{R}$ is Lipschitz continuous and regular, but not necessarily differentiable. In this case, the discussions in this chapter hold for the gradient-flow system in (5.1) by some extension as follows.

Consider the differential equation (5.2) with a measurable and essentially locally bounded (not necessarily continuous) function $F : (\mathbb{R}^d)^n \to (\mathbb{R}^d)^n$. An absolutely continuous function $x_{\mathcal{N}}(t) \in (\mathbb{R}^d)^n$ is called a *Filippov solution* if $x_{\mathcal{N}}(t)$ satisfies the differential inclusion

$$\dot{x}_{\mathcal{N}}(t) \in \mathcal{K}[F](x_{\mathcal{N}}(t)).$$

Here, $\mathcal{K}[F] : (\mathbb{R}^d)^n \to \mathrm{pow}((\mathbb{R}^d)^n)$ is the set-valued map, defined as

$$\begin{aligned}\mathcal{K}[F](x_{\mathcal{N}}) = \overline{\mathrm{co}}\Big\{ & y_{\mathcal{N}} \in (\mathbb{R}^d)^n : \exists \Xi_k \in (\mathbb{R}^d)^n \backslash \mathcal{Z},\ k = 1, 2, \ldots \\ & \text{s.t. } \lim_{k\to\infty} \Xi_k = x_{\mathcal{N}},\ \lim_{k\to\infty} F(\Xi_k) = y_{\mathcal{N}} \Big\}\end{aligned}$$

with a set $\mathcal{Z} \subset (\mathbb{R}^d)^n$ of measure zero, where $\overline{\mathrm{co}}(\cdot)$ is the closure of the convex hull of a set. Let $F^{-1}(0) \subset (\mathbb{R}^d)^n$ be the zero set of $F(x_{\mathcal{N}})$, defined as

$$F^{-1}(0) = \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : 0 \in \mathcal{K}[F](x_{\mathcal{N}})\}.$$

A closed set $\mathcal{A} \subset (\mathbb{R}^d)^n$ is called an *equilibrium set* of the system (5.2) if $\mathcal{A} \subset F^{-1}(0)$ holds.

Let $\partial_F v : (\mathbb{R}^d)^n \to \mathrm{pow}(\mathbb{R})$ be the *generalized derivative* of $v : (\mathbb{R}^d)^n \to \mathbb{R}$ with respect to $F : (\mathbb{R}^d)^n \to (\mathbb{R}^d)^n$, defined as

$$\begin{aligned}\partial_F v(x_{\mathcal{N}}) = \{ & a \in \mathbb{R} : \exists y_{\mathcal{N}} \in \mathcal{K}[F](x_{\mathcal{N}}) \\ & \text{s.t. } \langle z_{\mathcal{N}}, y_{\mathcal{N}} \rangle = a \quad \forall z_{\mathcal{N}} \in \mathcal{K}\left[\frac{\partial v}{\partial x_{\mathcal{N}}}\right](x_{\mathcal{N}})\}.\end{aligned} \tag{5.29}$$

Then, for the Filippov solution $x_{\mathcal{N}}(t)$ of (5.2), the following holds for almost every $t$:

$$\dot{v}(x_{\mathcal{N}}(t)) \in \partial_F v(x_{\mathcal{N}}(t)). \tag{5.30}$$

Consider the gradient-flow system (5.2) of $v(x_{\mathcal{N}})$. Then, (5.30) holds with $F(x_{\mathcal{N}}) = -\partial v/\partial x_{\mathcal{N}}(x_{\mathcal{N}})\mathrm{diag}(\kappa_{\mathcal{N}})$, which yields

$$\dot{v}(x_{\mathcal{N}}(t)) \leq 0 \tag{5.31}$$

because $\partial_F v(x_{\mathcal{N}}) \subset [0, \infty)$ holds from (5.29). Inequality (5.31) indicates that $v(x_{\mathcal{N}}(t))$ is monotonically non-increasing, and the discussions in Sections 5.2 and 5.3 are valid with the non-smooth version of LaSalle's invariance principle.

## 5.5 Notes and references

The terminology in this chapter mainly follows the standard control theory for nonlinear systems (Khalil, 2002; Haddad and Chellaboina, 2008). However, the contents are not the same because the equilibrium sets are possibly not isolated or compact in this monograph. To handle such equilibrium sets, the Lagrange stability is ensured by using invariance conditions of objective functions in Theorems 5.4 and 5.5. In Theorem 5.6, Łojasiewicz's inequalities are used to guarantee the asymptotic stability. These ideas were taken in Sakurama *et al.*, 2019. Łojasiewicz's inequalities were derived in the original paper (Łojasiewicz, 1965) and the relative book (Łojasiewicz and Zurro, 1999). See Appendix B for details about these inequalities and the real analyticity of functions. As for the contents of Section 5.4, see Clarke, 1983 for the concepts on the differential inclusion, and Shevitz and Paden, 1994; Eren *et al.*, 2004 for the non-smooth version of LaSalle's invariance principle.

# Part II

# Multi-robot Coordination Problems

# 6

# Pairwise Coordination

This chapter deals with the pairwise coordination of multi-robot systems introduced in Subsection 2.2.1, that is,

$$\lim_{t\to\infty} \psi_{ij}(x_i(t), x_j(t)) = 0 \quad \forall i, j \in \mathcal{N}, i \neq j \tag{6.1}$$

with non-negative functions $\psi_{ij} : (\mathbb{R}^d)^2 \to \mathbb{R}_+$ for $i, j \in \mathcal{N}, i \neq j$. The pairwise coordination involves essential design and analysis methods to help us shift smoothly to the rigorous control theory of multi-robot coordination in the following chapters.

## 6.1 Problem formulation

Consider the local coordinate frame $\Sigma_i(t)$ with a transformation matrix $M_i(t) \in \mathcal{M}$, where $\mathcal{M} \subset \mathrm{GL}(d)$ determines the class of transformation matrices. The origin of the local coordinate frame is assumed to be set at the position $x_i(t) \in \mathbb{R}^d$ of robot $i$. Then, as shown in Section 2.3, the kinematic model of robot $i$ and the relative positions of its neighbors $j \in \mathcal{N}_i$ are given as

$$\dot{x}_i(t) = M_i(t)u_i(t), \tag{6.2}$$

$$x_j^{[i]}(t) = M_i^{-1}(t)(x_j(t) - x_i(t)) \tag{6.3}$$

with the input $u_i(t) \in \mathbb{R}^d$. For a graph $G = (\mathcal{N}, \mathcal{E})$, a distributed controller with relative measurements is of the form

$$u_i(t) = c_i(x_{\mathcal{N}_i}^{[i]}(t)) \tag{6.4}$$

with a function $c_i : (\mathbb{R}^d)^{|\mathcal{N}_i|} \to \mathbb{R}^d$.

Consider the pairwise coordination (6.1) with respect to realizable functions $(\psi_{ij}(x_i, x_j))_{i,j\in\mathcal{N},i\neq j}$. This coordination can be expressed by the generalized coordination, introduced in Subsection 2.2.2 as

$$\lim_{t\to\infty} \operatorname{dist}(x_{\mathcal{N}}(t), \mathcal{D}) = 0, \tag{6.5}$$

with respect to the desired configuration set

$$\mathcal{D} = \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : \psi_{ij}(x_i, x_j) = 0 \quad \forall i, j \in \mathcal{N}, i \neq j\}. \tag{6.6}$$

Thanks to the realizability of the functions, this $\mathcal{D}$ is non-empty. The asymptotic stability of $\mathcal{D}$ is the control objective, for which we expect to design a distributed controller with relative measurements.

The problem is formulated as follows.

**Problem 6.1.** For a graph $G = (\mathcal{N}, \mathcal{E})$ and a frame transformation matrix set $\mathcal{M} \subset \mathrm{GL}(d)$, consider the kinematic model (6.2) with the relative positions (6.3) for $M_i(t) \in \mathcal{M}$. The set $\mathcal{D}$ is given as (6.6) for non-negative functions $\psi_{ij} : (\mathbb{R}^d)^2 \to \mathbb{R}^d$, $i, j \in \mathcal{N}$, $i \neq j$ such that $(\psi_{ij}(x_i, x_j))_{i,j\in\mathcal{N},i\neq j}$ are realizable. Then, design a distributed controller of the form (6.4) with relative measurements such that $\mathcal{D}$ is asymptotically stable.

## 6.2 Controller design

The gradient-flow approach is employed. Then, the system is controlled according to the gradient-flow system

$$\dot{x}_i(t) = -\kappa_i \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}(t)) \tag{6.7}$$

with an objective function $v : (\mathbb{R}^d)^n \to \mathbb{R}$ and a positive constant $\kappa_i > 0$. To obtain (6.7) from (6.2), we just have to design a controller (6.4) for

$$c_i(x_{\mathcal{N}_i}^{[i]}) = -\kappa_i M_i^{-1} \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}), \tag{6.8}$$

where $x_j^{[i]} = M_i^{-1}(x_j - x_i)$ for $M_i \in \mathcal{M}$ from (6.3).

From (6.6), we adopt the objective function

$$v(x_{\mathcal{N}}) = \sum_{\{i,j\}\in\mathcal{E}} \psi_{ij}(x_i, x_j) \tag{6.9}$$

because when $v(x_{\mathcal{N}})$ takes the minimum zero, $\psi_{ij}(x_i, x_j) = 0$ holds for every $\{i,j\} \in \mathcal{E}$. Note that (6.9) does not contain $\psi_{ij}(x_i, x_j)$ for $\{i,j\} \notin \mathcal{E}$ to design a distributed controller. Hence, whether the asymptotic stability of $\mathcal{D}$ is achieved or not depends on the topology of graph $G$.

The following theorem guarantees that the controller (6.8) with $v(x_{\mathcal{N}})$ in (6.9) is distributed with relative measurements under some conditions on $\psi_{ij}(x_i, x_j)$.

**Theorem 6.1.** For a graph $G = (\mathcal{N}, \mathcal{E})$, consider the kinematic model (6.2) and the relative positions (6.3) for $M_i(t) \in \mathcal{M}$ with a set $\mathcal{M} \subset \mathrm{GL}(d)$. Let $v(x_{\mathcal{N}})$ be given as (6.9) with non-negative continuously differentiable functions $\psi_{ij} : (\mathbb{R}^d)^2 \to \mathbb{R}_+$ for $\{i,j\} \in \mathcal{E}$.

(i) If $\psi_{ij}(x_i, x_j)$ is given as

$$\psi_{ij}(x_i, x_j) = \frac{1}{4}\|x_i - x_j\|^2, \tag{6.10}$$

the controller (6.8), reduced to

$$c_i(x_{\mathcal{N}_i}^{[i]}) = \kappa_i \sum_{j\in\mathcal{N}_i\setminus\{i\}} x_j^{[i]}, \tag{6.11}$$

is distributed with relative measurements.

(ii) If $\mathcal{M} \subset \mathrm{O}(d)$ and each $\psi_{ij}(x_i, x_j)$ satisfies

$$\psi_{ij}(M^{-1}(x_i - \tau), M^{-1}(x_j - \tau)) = \psi_{ij}(x_i, x_j) \quad \forall x_i, x_j, \tau \in \mathbb{R}^d,\ M \in \mathcal{M}, \tag{6.12}$$

the controller (6.8), reduced to

$$c_i(x^{[i]}_{\mathcal{N}_i}) = -\kappa_i \sum_{j \in \mathcal{N}_i \setminus \{i\}} \left( \frac{\partial \psi_{ij}}{\partial x_i}(0, x^{[i]}_j) + \frac{\partial \psi_{ji}}{\partial x_i}(x^{[i]}_j, 0) \right), \tag{6.13}$$

is distributed with relative measurements.

*Proof.* For $v(x_{\mathcal{N}})$ in (6.9), (6.8) is reduced to

$$c_i(x^{[i]}_{\mathcal{N}_i}) = -\kappa_i M_i^{-1} \sum_{j \in \mathcal{N}_i \setminus \{i\}} \left( \frac{\partial \psi_{ij}}{\partial x_i}(x_i, x_j) + \frac{\partial \psi_{ji}}{\partial x_i}(x_j, x_i) \right). \tag{6.14}$$

Under condition (i), for $\psi_{ij}(x_i, x_j)$ in (6.10),

$$\begin{aligned} \frac{\partial \psi_{ij}}{\partial x_i}(x_i, x_j) + \frac{\partial \psi_{ji}}{\partial x_i}(x_j, x_i) &= \frac{1}{2} \frac{\partial \|x_i - x_j\|^2}{\partial x_i} = x_i - x_j \\ &= -M_i(M_i^{-1}(x_j - x_i)) = -M_i x^{[i]}_j \end{aligned}$$

holds. Then, (6.14) is reduced to (6.11) and is distributed under relative measurements.

Under condition (ii), by partially differentiating (6.12) with respect to $x_i$, from the chain rule and the property of the orthogonal matrix, we obtain

$$\begin{aligned} \frac{\partial \psi_{ij}}{\partial x_i}(x_i, x_j) &= \frac{\partial \psi_{ij}(M^{-1}(x_i - \tau), M^{-1}(x_j - \tau))}{\partial x_i} \\ &= (M^{-1})^\top \frac{\partial \psi_{ij}}{\partial x_i}(M^{-1}(x_i - \tau), M^{-1}(x_j - \tau)) \\ &= M \frac{\partial \psi_{ij}}{\partial x_i}(M^{-1}(x_i - \tau), M^{-1}(x_j - \tau)), \end{aligned} \tag{6.15}$$

which holds for any $M \in \mathcal{M}$ and $\tau \in \mathbb{R}^d$. Assign $M = M_i$ and $\tau = x_i$ in (6.15), and (6.14) with the resultant is reduced to (6.13) and is distributed under relative measurements. □

The functions $\psi_{ij}(x_i, x_j)$ satisfying conditions (i) and (ii) in Theorem 6.1 are used for consensus and other formation problems, respectively, in Section 6.4.

## 6.3 Stability analysis

The stability of the resultant gradient-flow system is guaranteed under some assumptions as follows.

**Theorem 6.2.** For a graph $G = (\mathcal{N}, \mathcal{E})$ and non-negative continuously differentiable functions $\psi_{ij} : (\mathbb{R}^d)^2 \to \mathbb{R}_+$, $\{i, j\} \in \mathcal{E}$, consider the gradient-flow system (6.7) of $v(x_\mathcal{N})$ in (6.9). Assume that functions $\psi_{ij}(x_i, x_j)$ for $\{i, j\} \in \mathcal{E}$ are all radially unbounded, or they all satisfy the following two conditions:

$$\psi_{ij}(x_i + \tau, x_j + \tau) = \psi_{ij}(x_i, x_j) \quad \forall \tau \in \mathbb{R}^d, \tag{6.16}$$

$$\|x_i - x_j\| \le \zeta_{ij}(\psi_{ij}(x_i, x_j)) \tag{6.17}$$

with monotonically non-decreasing functions $\zeta_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$. Then, the system is Lagrange stable, and the following set is globally attractive:

$$\{x_\mathcal{N} \in (\mathbb{R}^d)^n : \sum_{j \in \mathcal{N}_i \setminus \{i\}} \frac{\partial \psi_{ij}}{\partial x_i}(x_i, x_j) = 0 \quad \forall i \in \mathcal{N}\}. \tag{6.18}$$

Additionally, if each $\psi_{ij}(x_i, x_j)$ is real analytic in an open set containing $\psi_{ij}^{-1}(0, 0)$ and $(\psi_{ij}(x_i, x_j))_{\{i,j\} \in \mathcal{E}}$ are realizable, the following set is asymptotically stable:

$$\mathcal{A}(G) = \{x_\mathcal{N} \in (\mathbb{R}^d)^n : \psi_{ij}(x_i, x_j) = 0 \quad \forall \{i, j\} \in \mathcal{E}\}. \tag{6.19}$$

*Proof.* The case that $\psi_{ij}(x_i, x_j)$ are all radially unbounded follows from Theorem 5.3 and is omitted.

Assume that (6.16) and (6.17) hold. Without loss of generality, we assume that $G$ is connected. Otherwise, we just have to consider each connected component of $G$. From (6.9) and (6.16),

$$\begin{aligned} v((I_d, \tau) \bullet x_\mathcal{N}) = v(x_1 + \tau, \ldots, x_n + \tau) &= \sum_{\{i,j\} \in \mathcal{E}} \psi_{ij}(x_i + \tau, x_j + \tau) \\ &= \sum_{\{i,j\} \in \mathcal{E}} \psi_{ij}(x_i, x_j) = v(x_\mathcal{N}) \end{aligned}$$

holds for any $\tau \in \mathbb{R}^d$. Hence, $v(x_\mathcal{N})$ is $(\{I_d\} \ltimes \mathbb{R}^d)$-invariant, and condition (i) in Theorem 5.4 is satisfied. From the connectivity of $G$ and (6.17), condition (ii) in Theorem 5.4 is shown to be satisfied in the same way as Example 5.1. Hence, Theorem 5.4 guarantees that the system is Lagrange stable, and that the zero set $(\partial v/\partial x_\mathcal{N})^{-1}(0)$ is globally attractive. This zero set is reduced to (6.18) from (6.9).

From the realizability of $(\psi_{ij}(x_i, x_j))_{\{i,j\}\in\mathcal{E}}$, $v^{-1}(0)$ is non-empty for $v(x_\mathcal{N})$ in (6.9). From this and the real analyticity of $v(x_\mathcal{N})$, Theorem 5.6 guarantees that the zero set $v^{-1}(0)$ is asymptotically stable. This zero set is reduced to (6.19) from (6.9) and the non-negativeness of $\psi_{ij}(x_i, x_j)$. □

Theorem 6.2 guarantees the asymptotic stability of $\mathcal{A}(G)$ in (6.19), while that of $\mathcal{D}$ in (6.6) is expected as stated in Problem 6.1. To achieve this objective, it is sufficient to ensure $\mathcal{D} = \mathcal{A}(G)$. Whether this equation holds or not depends on the topology of $G$. This is discussed more in the following section for concrete examples.

## 6.4 Examples

In this section, examples of pairwise coordination are given by assigning concrete functions to $\psi_{ij}(x_i, x_j)$.

### 6.4.1 Consensus

For the consensus problem (2.11), consider the function $\psi_{ij}(x_i, x_j)$ in (2.12), which is equivalent to (6.10) in condition (i) of Theorem 6.1. Accordingly, the controller (6.8) is reduced to the distributed controller (6.11) with relative measurements for $\mathcal{M} \subset \mathrm{GL}(d)$. Then, the control input (6.4) is designed as

$$u_i(t) = \kappa_i \sum_{j\in\mathcal{N}_i\setminus\{i\}} x_j^{[i]}(t). \tag{6.20}$$

Notably, $M_i(t) \in \mathcal{M}$ in the relative position (6.3) can be an arbitrary non-singular matrix, and thus various transformations of the local coordinate frame are allowed, which is special to consensus.

Furthermore, based on Theorem 6.2, the global asymptotic stability of $\mathcal{A}(G)$ in (6.19) is guaranteed. For a connected graph $G$, the desired configuration set $\mathcal{D}$ in (6.6) with $\psi_{ij}(x_i, x_j)$ in (6.10), reduced to

$$\mathcal{D} = \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_1 = x_2 = \cdots = x_n\}, \tag{6.21}$$

is globally asymptotically stable. Actually, the following proposition is obtained.

**Proposition 6.1.** For a graph $G = (\mathcal{N}, \mathcal{E})$, consider the kinematic model (6.2), the relative position (6.3) with a non-singular matrix $M_i(t) \in \mathrm{GL}(d)$, and the distributed controller (6.20) with relative measurements for a gain $\kappa_i > 0$. Then, the set $\mathcal{D}$ in (6.21) is globally asymptotically stable if and only if $G$ is connected.

*Proof.* Assume that $G$ is connected. From the discussions just before Theorem 6.1 and this proposition, by using the controller (6.20), the system (6.2) is reduced to the gradient-flow system (6.7) with $v(x_{\mathcal{N}})$ in (6.9) for $\psi_{ij}(x_i, x_j)$ in (6.10). Because the function $\psi_{ij}(x_i, x_j)$ satisfies (6.16) and (6.17), Theorem 6.2 guarantees that the set in (6.18) is globally attractive, which is reduced to

$$\{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : \sum_{j \in \mathcal{N}_i \setminus \{i\}} (x_i - x_j) = 0 \ \ \forall i \in \mathcal{N}\}. \tag{6.22}$$

Each $\psi_{ij}(x_i, x_j)$ is real analytic and $(\psi_{ij}(x_i, x_j))_{\{i,j\} \in \mathcal{E}}$ are realizable from Example 2.4. Thus, Theorem 6.2 guarantees that the set $\mathcal{A}(G)$ in (6.19) is asymptotically stable, which is reduced to

$$\mathcal{A}(G) = \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_i = x_j \ \ \forall \{i,j\} \in \mathcal{E}\}. \tag{6.23}$$

The sets in (6.22) and (6.23) are equivalent to the set $\mathcal{D}$ in (6.21) if and only if $G$ is connected (Mesbahi and Egerstedt, 2010). Hence, $\mathcal{D}$ is globally asymptotically stable from the global attractiveness and the asymptotic stability of the sets in (6.22) and (6.23).

Assume that $G$ is not connected. Then, the set $\mathcal{A}(G) \setminus \mathcal{D}$ is non-empty. For an initial state $x_{\mathcal{N}}(0) \in \mathcal{A}(G) \setminus \mathcal{D}$, the state $x_{\mathcal{N}}(t)$ does not move because $\mathcal{A}(G)$ is an equilibrium set. Therefore, $\mathcal{D}$ is not globally asymptotically stable. □

### 6.4.2 Displacement-based formation

For displacement-based formation (2.5), consider the function

$$\psi_{ij}(x_i, x_j) = \frac{1}{4}\|x_i - x_j - r_{ij}\|^2, \tag{6.24}$$

equivalent to (2.6). Assume that $\mathcal{M} = \{I_d\}$, and (6.12) is satisfied. Then, Theorem 6.1 guarantees that the controller (6.8), reduced to (6.13), is a distributed controller with relative measurements. For $\psi_{ij}(x_i, x_j)$ in (6.24),

$$u_i(t) = \kappa_i \sum_{j \in \mathcal{N}_i \setminus \{i\}} (x_j^{[i]}(t) + r_{ij}) \tag{6.25}$$

is obtained under the assumption that $r_{ij} = -r_{ij}$.

Note that for $M_i(t) \in \mathcal{M} = \{I_d\}$, the relative position (6.3) is reduced to

$$x_j^{[i]}(t) = x_j(t) - x_i(t). \tag{6.26}$$

To measure the relative position (6.26), the robots need to obtain the absolute bearing, e.g. by compasses, as discussed in Subsection 2.3.3. This is in contrast to the consensus controller (6.11), which does not require the absolute bearing because any non-singular matrix $M_i(t)$ is allowed in the relative position (6.3).

The desired configuration set $\mathcal{D}$ in (6.6) with $\psi_{ij}(x_i, x_j)$ in (6.24) is given as

$$\mathcal{D} = \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_i - x_j = r_{ij} \quad \forall i, j \in \mathcal{N}, i \neq j\}. \tag{6.27}$$

This set is globally asymptotically stable for a connected graph $G$ as follows.

**Proposition 6.2.** Let $r_{ij} \in \mathbb{R}^d$ be vectors such that there exists $x_{\mathcal{N}}^* \in (\mathbb{R}^d)^n$ satisfying $x_i^* - x_j^* = r_{ij}$ for any $i, j \in \mathcal{N}$, $i \neq j$. For a graph $G = (\mathcal{N}, \mathcal{E})$, consider the kinematic model (6.2), the relative positions (6.26) with $M_i(t) = I_d$, and the distributed controller (6.25) with relative measurements for $\kappa_i > 0$. Then, the set $\mathcal{D}$ in (6.27) is globally asymptotically stable if and only if $G$ is connected.

*Proof.* From the assumption on $r_{ij}$, the functions $\psi_{ij}(x_i, x_j)$ in (6.24) for $i, j \in \mathcal{N}$, $i \neq j$ are realizable. The rest of the proof follows from Proposition 6.1 through the state transformation $\bar{x}_i = x_i - x_i^*$. $\square$

### 6.4.3 Distance-based formation

For distance-based formation (2.7), let us consider the function

$$\psi_{ij}(x_i, x_j) = \frac{1}{8}(\|x_i - x_j\|^2 - d_{ij}^2)^2, \tag{6.28}$$

equivalent to (2.8). For $\mathcal{M} = \mathrm{O}(d)$, (6.12) is satisfied as follows:

$$\begin{aligned}
&\psi_{ij}(M^{-1}(x_i - \tau), M^{-1}(x_j - \tau)) \\
&\quad = \frac{1}{8}(\|M^{-1}(x_i - \tau) - M^{-1}(x_j - \tau)\|^2 - d_{ij}^2)^2 \\
&\quad = \frac{1}{8}(\|x_i - x_j\|^2 - d_{ij}^2)^2 = \psi_{ij}(x_i, x_j)
\end{aligned}$$

for $M \in \mathrm{O}(d), \tau \in \mathbb{R}^d$. Then, from Theorem 6.1, the controller (6.8), reduced to (6.13), is a distributed controller with relative measurements. For $\psi_{ij}(x_i, x_j)$ in (6.28),

$$u_i(t) = \kappa_i \sum_{j \in \mathcal{N}_i \setminus \{i\}} (\|x_j^{[i]}(t)\|^2 - d_{ij}^2) x_j^{[i]}(t) \tag{6.29}$$

is obtained under the assumption that $d_{ij} = d_{ji}$.

From (6.3), the relative position is given as

$$x_j^{[i]}(t) = M_i^\top(t)(x_j(t) - x_i(t)) \tag{6.30}$$

for $M_i(t) \in \mathrm{O}(d)$. Compared with the relative position (6.26) for the displacement-based formation, (6.30) involves the transformation in rotation and reflection. This means that less measurement information is required to the distance-based formation, that is, the absolute bearing is unnecessary.

The desired configuration set $\mathcal{D}$ in (6.6) with $\psi_{ij}(x_i, x_j)$ in (6.28) is reduced to

$$\mathcal{D} = \{x_\mathcal{N} \in (\mathbb{R}^d)^n : \|x_i - x_j\| = d_{ij} \quad \forall i, j \in \mathcal{N}, i \neq j\}. \tag{6.31}$$

This set is asymptotically stable if $(x_\mathcal{N}^*, G)$ is a globally rigid framework for some $x_\mathcal{N}^* \in \mathcal{D}$ as follows.

**Proposition 6.3.** Let $d_{ij}$ be positive numbers such that there exists $x_{\mathcal{N}}^* \in (\mathbb{R}^d)^n$ satisfying $\|x_i^* - x_j^*\| = d_{ij}$ for any $i, j \in \mathcal{N}$, $i \neq j$. For a graph $G = (\mathcal{N}, \mathcal{E})$, consider the kinematic model (6.2), the relative position (6.30) with $M_i(t) \in \mathrm{O}(d)$, and the distributed controller (6.29) with relative measurements for $\kappa_i > 0$. Then, the set $\mathcal{D}$ in (6.31) is asymptotically stable if $(x_{\mathcal{N}}^*, G)$ is globally rigid.

*Proof.* Assume that $(x_{\mathcal{N}}^*, G)$ is globally rigid. From the discussions just before Theorem 6.1 and this proposition, by using the controller (6.29), the system (6.2) is reduced to the gradient-flow system (6.7) with $v(x_{\mathcal{N}})$ in (6.9) for $\psi_{ij}(x_i, x_j)$ in (6.28). The function $\psi_{ij}(x_i, x_j)$ satisfies (6.16) and (6.17), $\psi_{ij}(x_i, x_j)$ is real analytic, and $(\psi_{ij}(x_i, x_j))_{\{i,j\}\in\mathcal{E}}$ are realizable from the assumption on $d_{ij}$, Hence, Theorem 6.2 guarantees that $\mathcal{A}(G)$ in (6.19), reduced to

$$\mathcal{A}(G) = \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : \|x_i - x_j\| = d_{ij} \quad \forall \{i, j\} \in \mathcal{E}\}, \tag{6.32}$$

is asymptotically stable. From the definition (4.8) of the global rigidity, (6.31), and (6.32), $\mathcal{A}(G) \subset \mathcal{D}$ holds. The converse inclusion is obvious, and $\mathcal{A}(G) = \mathcal{D}$ is obtained. Hence, $\mathcal{D}$ in (6.31) is asymptotically stable. □

## 6.5 Notes and references

Conventionally, the gradient-flow approach has been taken to design a distributed controller with an objective function of the form (6.9) consisting of pairwise functions, as summarized in Martínez *et al.*, 2007. Hence, the contents of this chapter are highly relevant to conventional results on multi-robot and multi-agent systems.

As for consensus, the objective function $v(x_{\mathcal{N}})$ in (6.9) with (6.10) is called a Laplacian potential (Olfati-Saber and Murray, 2004), and its gradient-flow system (6.7) is reduced to a linear system with a graph Laplacian matrix. The property of this system can be analyzed with the eigenvalues of the graph Laplacian matrix, associated with the connectivity of the graph as Proposition 6.1. See Mesbahi and Egerstedt, 2010 for multi-robot coordination through a graph-theoretic approach.

As for distance-based formation, graph topology has been investigated under which distance-based formation is achievable in the literature. The key is the rigidity theory of bar-and-joint frameworks as indicated in Section 4.6, which is summarized in Anderson *et al.*, 2008; Queiroz *et al.*, 2019. Correspondingly, Proposition 6.3 indicates that a sufficient condition for the asymptotic stability of $\mathcal{D}$ in (6.31) is the global rigidity of $(x^*_{\mathcal{N}}, G)$. The existing research shows that the necessary and sufficient condition is the rigidity of $(x^*_{\mathcal{N}}, G)$. This gap is caused from obtaining $\mathcal{A}(G) = \mathcal{D}$ in the proof of Proposition 6.3. Actually, we just need the existence of an open set $\Delta \supset \mathcal{D}$ such that $\mathcal{A}(G) \cap \Delta = \mathcal{D}$, which implies the rigidity of $(x^*_{\mathcal{N}}, G)$ defined in (4.9).

# 7

## Generalized Coordination with "Absolute" Measurements

Consider the generalized coordination problem, which is formulated in Subsection 2.2.2 as

$$\lim_{t\to\infty} \mathrm{dist}(x_{\mathcal{N}}(t), \mathcal{D}) = 0, \tag{7.1}$$

with respect to a desired configuration set $\mathcal{D} \subset (\mathbb{R}^d)^n$. In this chapter, a condition of $(\mathcal{D}, G)$ is specified such that there exists a distributed controller to achieve (7.1) over a graph $G$, and such a distributed controller is designed. Moreover, it is shown that even if this condition is not satisfied, the designed controller achieves the generalized coordination (7.1) in the best approximate way.

Here, we assume that the global and local coordinate frames are the same, i.e., the absolute positions of the neighbors are available to each robot. This assumption is removed in the next chapter.

### 7.1 Problem formulation

Under this assumption, the kinematic model is given by the single-integrator system

$$\dot{x}_i(t) = u_i(t) \tag{7.2}$$

as explained in Subsection 2.3.2. For a graph $G = (\mathcal{N}, \mathcal{E})$, the admissible controller is of the form

$$u_i(t) = c_i(x_{\mathcal{N}_i}(t)) \tag{7.3}$$

with a function $c_i : (\mathbb{R}^d)^{|\mathcal{N}_i|} \to \mathbb{R}^d$, where $\mathcal{N}_i \subset \mathcal{N}$ is the neighbor set of robot $i$, defined in (4.1). Note that the function $c_i(x_{\mathcal{N}_i})$ in (7.3) depends on the absolute positions $x_{\mathcal{N}_i}$ of the neighbors. A controller of the form (7.3) is said to be *distributed*.

### 7.1.1 Gradient-flow approach

To achieve the generalized coordination (7.1) with respect to a set $\mathcal{D} \subset (\mathbb{R}^d)^n$, we design a distributed controller with which $\mathcal{D}$ is asymptotically stable. For this purpose, the gradient-flow approach is employed. Consider the gradient-flow system

$$\dot{x}_i(t) = -\kappa_i \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}(t)) \tag{7.4}$$

for a non-negative, continuously differentiable function $v : (\mathbb{R}^d)^n \to \mathbb{R}_+$ and a positive constant $\kappa_i > 0$. Now, the requirements of the objective function $v(x_{\mathcal{N}})$ are listed as follows.

First, according to (7.4), the objective function $v(x_{\mathcal{N}}(t))$ is monotonically non-increasing, and the state $x_{\mathcal{N}}(t)$ locally converges to the zero set $v^{-1}(0)$. Actually Theorem 5.6 guarantees that $v^{-1}(0)$ is asymptotically stable under some assumptions. Hence, for the asymptotic stability of $\mathcal{D}$, $v(x_{\mathcal{N}})$ is expected to satisfy

$$v^{-1}(0) = \mathcal{D}. \tag{7.5}$$

A non-negative function $v(x_{\mathcal{N}})$ satisfying (7.5) is called an *indicator* of $\mathcal{D}$. Let $\mathcal{V}_{\text{ind}}(\mathcal{D})$ be the set of indicators of $\mathcal{D}$, that is,

$$\mathcal{V}_{\text{ind}}(\mathcal{D}) = \{v(x_{\mathcal{N}}) : v^{-1}(0) = \mathcal{D},\ v(x_{\mathcal{N}}) \geq 0 \ \ \forall x_{\mathcal{N}} \in (\mathbb{R}^d)^n\}. \tag{7.6}$$

Note that $\mathcal{V}_{\text{ind}}(\mathcal{D})$ is always non-empty because the squared distance function of $\mathcal{D}$ belongs to $\mathcal{V}_{\text{ind}}(\mathcal{D})$, that is,

$$v(x_{\mathcal{N}}) = (\text{dist}(x_{\mathcal{N}}, \mathcal{D}))^2 \in \mathcal{V}_{\text{ind}}(\mathcal{D}).$$

Next, to obtain the gradient-flow system (7.4) from the single-integrator system (7.2), the distributed controller in (7.3) is designed as

$$c_i(x_{\mathcal{N}_i}) = -\kappa_i \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}) \tag{7.7}$$

for $\kappa_i > 0$. This equation indicates that the gradient of $v(x_{\mathcal{N}})$ with respect to $x_i$ can depend only on the states $x_{\mathcal{N}_i}$ of the neighbors of robot $i$. Such a function $v(x_{\mathcal{N}})$ is said to have a *distributed gradient* for graph $G$. Let $\mathcal{V}_{\mathrm{dis}}(G)$ be the set of the functions having distributed gradients for graph $G$, that is

$$\begin{aligned}\mathcal{V}_{\mathrm{dis}}(G) = \{v(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{c1}} : \forall i \in \mathcal{N},\ \exists \tilde{c}_i : (\mathbb{R}^d)^{|\mathcal{N}_i|} \to \mathbb{R}^d \\ \text{s.t.}\ \ \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}) = \tilde{c}_i(x_{\mathcal{N}_i})\ \ \forall x_{\mathcal{N}} \in (\mathbb{R}^d)^n\},\end{aligned} \tag{7.8}$$

where $\tilde{c}_i(x_{\mathcal{N}_i}) = -c_i(x_{\mathcal{N}_i})/\kappa_i$ and $\mathcal{V}_{\mathrm{c1}}$ is the set of scalar, continuously differentiable functions. Note that $\mathcal{V}_{\mathrm{dis}}(G)$ is non-empty because the zero function $v(x_{\mathcal{N}}) = 0\ \forall x_{\mathcal{N}} \in (\mathbb{R}^d)^n$ is always contained.

### 7.1.2 Best approximate indicators

Now, we are faced with the key issue in this approach: the intersection $\mathcal{V}_{\mathrm{ind}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G)$ is possibly empty, depending on $\mathcal{D}$ and $G$. In other words, there might be no distributed controller of the form (7.7) which asymptotically stabilizes $\mathcal{D}$. Even in such a case, a relaxed condition

$$v^{-1}(0) \supset \mathcal{D} \tag{7.9}$$

can be considered instead of (7.5). A non-negative function $v(x_{\mathcal{N}})$ satisfying (7.9) is called an *approximate indicator* of $\mathcal{D}$. Let $\mathcal{V}_{\mathrm{app}}(\mathcal{D})$ be the set of approximate indicators of $\mathcal{D}$, that is

$$\mathcal{V}_{\mathrm{app}}(\mathcal{D}) = \{v(x_{\mathcal{N}}) : v^{-1}(0) \supset \mathcal{D},\ v(x_{\mathcal{N}}) \geq 0\ \ \forall x_{\mathcal{N}} \in (\mathbb{R}^d)^n\}. \tag{7.10}$$

The intersection $\mathcal{V}_{\mathrm{app}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G)$ is always non-empty unlike $\mathcal{V}_{\mathrm{ind}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G)$ because the zero function belongs to $\mathcal{V}_{\mathrm{app}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G)$.

Note that just finding a function $v(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{app}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G)$ makes no sense because the zero function is contained. Hence, we need to find the most appropriate function $\hat{v}(x_{\mathcal{N}})$ from $\mathcal{V}_{\mathrm{app}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G)$ in terms of achieving the generalized coordination (7.1). Such $\hat{v}(x_{\mathcal{N}})$ can be defined as the function of which zero set $\hat{v}^{-1}(0)$ is the most similar to $\mathcal{D}$ in the following sense:

$$\mathcal{D} \subset \hat{v}^{-1}(0) \subset v^{-1}(0) \quad \forall v(x_{\mathcal{N}}) \in \mathcal{V}_{\text{app}}(\mathcal{D}) \cap \mathcal{V}_{\text{dis}}(G). \tag{7.11}$$

A function $\hat{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\text{app}}(\mathcal{D}) \cap \mathcal{V}_{\text{dis}}(G)$ satisfying (7.11) is called *the best approximate indicator* of $\mathcal{D}$ under $G$. The gradient-flow system (7.4) of the best approximate indicator $\hat{v}(x_{\mathcal{N}})$ can drive the robots to the point nearest to $\mathcal{D}$ among all the functions in $\mathcal{V}_{\text{app}}(\mathcal{D}) \cap \mathcal{V}_{\text{dis}}(G)$. Moreover, if $\mathcal{V}_{\text{ind}}(\mathcal{D}) \cap \mathcal{V}_{\text{dis}}(G)$ is non-empty, $\hat{v}(x_{\mathcal{N}})$ is always an indicator of $\mathcal{D}$.

### 7.1.3 Target problems

The first problem tackled in this chapter is to characterize the best approximate indicator to design a distributed controller as follows.

**Problem 7.1.** For a graph $G = (\mathcal{N}, \mathcal{E})$ and a non-empty set $\mathcal{D} \subset (\mathbb{R}^d)^n$, characterize the best approximate indicators, say $\hat{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\text{app}}(\mathcal{D}) \cap \mathcal{V}_{\text{dis}}(G)$ satisfying (7.11). Moreover, design a distributed controller via the gradient of one of the best approximate indicators, and analyze the stability of the resultant system.

The next problem is to specify $(\mathcal{D}, G)$ such that there exists an indicator having a distributed gradient.

**Problem 7.2.** Derive a necessary and sufficient condition of $(\mathcal{D}, G)$ such that there exists an indicator of $\mathcal{D}$ having a distributed gradient for $G$, that is, $\mathcal{V}_{\text{app}}(\mathcal{D}) \cap \mathcal{V}_{\text{dis}}(G)$ is non-empty.

## 7.2 Characterization of the best approximate indicators

Before addressing Problem 7.1, the functions having distributed gradients, say $v(x_{\mathcal{N}}) \in \mathcal{V}_{\text{dis}}(G)$, are characterized. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q \subset \mathcal{N}$ be the maximal cliques in graph $G$. The key is the decomposability into clique-based functions as follows, where $\text{proj.}(\cdot)$ is the projection defined in (3.13).

**Theorem 7.1.** For a graph $G$, a continuously differential function $v : (\mathbb{R}^d)^n \to \mathbb{R}$ belongs to $\mathcal{V}_{\rm dis}(G)$ if and only if it can be of the form

$$v(x_{\mathcal{N}}) = \sum_{k \in \mathrm{clq}(G)} v_k(x_{\mathcal{C}_k}) \tag{7.12}$$

with some functions $v_k(x_{\mathcal{C}_k})$ for maximal cliques $\mathcal{C}_k$, $k \in \mathrm{clq}(G)$. Moreover, if $v(x_{\mathcal{N}})$ is non-negative and $v^{-1}(0)$ is non-empty, each $v_k(x_{\mathcal{C}_k})$ can be chosen as an indicator of $\mathrm{proj}_{\mathcal{C}_k}(v^{-1}(0))$.

*Proof.* To show sufficiency, assume that a continuously differentiable function $v(x_{\mathcal{N}})$ is of the form (7.12) with some functions $v_k(x_{\mathcal{C}_k})$. Note that $\partial v_k/\partial x_i(x_{\mathcal{C}_k}) = 0$ holds if $k \notin \mathrm{clq}_i(G)$ because $x_{\mathcal{C}_k}$ does not include $x_i$. Then, by partially differentiating $v(x_{\mathcal{N}})$ with respect to $x_i$, we obtain

$$\frac{\partial v}{\partial x_i}(x_{\mathcal{N}}) = \frac{\partial}{\partial x_i} \sum_{k \in \mathrm{clq}(G)} v_k(x_{\mathcal{C}_k}) = \frac{\partial}{\partial x_i} \sum_{k \in \mathrm{clq}_i(G)} v_k(x_{\mathcal{C}_k}) = \tilde{c}_i(x_{\mathcal{N}_i})$$

with some function $\tilde{c}_i(x_{\mathcal{N}_i})$ from the relation (4.7) between the neighbor set $\mathcal{N}_i$ and the maximal cliques $\mathcal{C}_k$ for $k \in \mathrm{clq}_i(G)$. Hence, $v(x_{\mathcal{N}}) \in \mathcal{V}_{\rm dis}(G)$ is satisfied from (7.8).

The necessity part follows from Lemma D.9. The latter part of this theorem follows from Lemma D.7 (a). □

In the characterization (7.12), the functions belonging to $\mathcal{V}_{\rm dis}(G)$ consist of clique-based functions $v_k(x_{\mathcal{C}_k})$, which are parameters to be designed according to control objectives. Because Theorem 7.1 provides a necessary and sufficient condition, the best performance is necessarily obtained for any criterion with an objective function of the form (7.12) by appropriate choice of $v_k(x_{\mathcal{C}_k})$.

The following theorem shows that we just have to assign indicators of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$ to $v_k(x_{\mathcal{C}_k})$ for attaining the best approximate indicators.

**Theorem 7.2.** For a graph $G$ and a non-empty set $\mathcal{D} \subset (\mathbb{R}^d)^n$, a function $\hat{v} : (\mathbb{R}^d)^n \to \mathbb{R}$ is the best approximate indicator of $\mathcal{D}$ having a distributed gradient for $G$, i.e., $\hat{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\rm app}(\mathcal{D}) \cap \mathcal{V}_{\rm dis}(G)$

satisfies (7.11), if and only if it can be of the form

$$\hat{v}(x_{\mathcal{N}}) = \sum_{k \in \mathrm{clq}(G)} \hat{v}_k(x_{\mathcal{C}_k}) \tag{7.13}$$

with indicators $\hat{v}_k(x_{\mathcal{C}_k})$ of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$ for the maximal cliques $\mathcal{C}_k$, $k \in \mathrm{clq}(G)$.

*Proof.* (Sufficiency) Let $\hat{v}(x_{\mathcal{N}})$ be a function of the form (7.13) with indicators $\hat{v}_k(x_{\mathcal{C}_k})$ of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$. First, from Theorem 7.1, $\hat{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{dis}}(G)$ holds. Next,

$$\begin{aligned}
\hat{v}^{-1}(0) &= \bigcap_{k \in \mathrm{clq}(G)} \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{C}_k} \in \hat{v}_k^{-1}(0)\} \\
&= \bigcap_{k \in \mathrm{clq}(G)} \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{C}_k} \in \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})\} \\
&\supset \mathcal{D}
\end{aligned} \tag{7.14}$$

is obtained from (7.13), the indicators $\hat{v}_k(x_{\mathcal{C}_k})$ of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$, and the definition of the projection. Hence, $\hat{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{app}}(\mathcal{D})$ holds from (7.10). Finally, to show (7.11), consider a function $v(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{app}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G)$. From $v(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{dis}}(G)$ and its non-negativeness, Theorem 7.1 guarantees that $v(x_{\mathcal{N}})$ can be of the form (7.12) with indicators $v_k(x_{\mathcal{C}_k})$ of $\mathrm{proj}_{\mathcal{C}_k}(v^{-1}(0))$. From $v(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{app}}(\mathcal{D})$, $\mathcal{D} \subset v^{-1}(0)$ holds, which yields $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D}) \subset \mathrm{proj}_{\mathcal{C}_k}(v^{-1}(0)) = v_k^{-1}(0)$. Take the intersection of these sets in $(\mathbb{R}^d)^n$ for all $k \in \mathrm{clq}(G)$, and we obtain

$$\begin{aligned}
&\bigcap_{k \in \mathrm{clq}(G)} \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{C}_k} \in \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})\} \\
&\qquad \subset \bigcap_{k \in \mathrm{clq}(G)} \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{C}_k} \in v_k^{-1}(0)\} = v^{-1}(0).
\end{aligned} \tag{7.15}$$

From (7.14) and (7.15), (7.11) is obtained.

(Necessity) Assume that a function $\tilde{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{app}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G)$ satisfies (7.11). Here, we show that $\tilde{v}(x_{\mathcal{N}})$ is of the form (7.13) after all. From $\tilde{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{app}}(\mathcal{D})$, $\tilde{v}(x_{\mathcal{N}})$ is non-negative and $\tilde{v}^{-1}(0)$ is non-empty. From these facts and $\tilde{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{dis}}(G)$, Theorem 7.1 guarantees that $\tilde{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{dis}}(G)$ is of the form as (7.12), i.e., the sum of indicators

$\tilde{v}_k(x_{\mathcal{C}_k})$ of $\mathrm{proj}_{\mathcal{C}_k}(\tilde{v}^{-1}(0))$ for $k \in \mathrm{clq}(G)$. Hence, to show that $\tilde{v}(x_{\mathcal{N}})$ is of the form (7.13), it is sufficient to show that

$$\mathrm{proj}_{\mathcal{C}_k}(\tilde{v}^{-1}(0)) = \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D}) \tag{7.16}$$

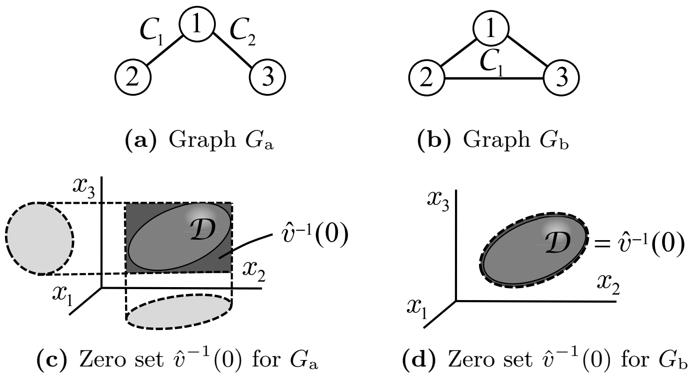
holds for any $k \in \mathrm{clq}(G)$. From $\tilde{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{app}}(\mathcal{D})$, $\mathcal{D} \subset \tilde{v}^{-1}(0)$ holds, which leads to $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D}) \subset \mathrm{proj}_{\mathcal{C}_k}(\tilde{v}^{-1}(0))$. Because $\hat{v}(x_{\mathcal{N}})$ in (7.13) also satisfies (7.11) from the sufficiency part, $\tilde{v}^{-1}(0) = \hat{v}^{-1}(0)$ holds, from which

$$\begin{aligned}
\mathrm{proj}_{\mathcal{C}_\ell}(\tilde{v}^{-1}(0)) &= \mathrm{proj}_{\mathcal{C}_\ell}(\hat{v}^{-1}(0)) \\
&= \mathrm{proj}_{\mathcal{C}_\ell}\Big(\bigcap_{k\in\mathrm{clq}(G)} \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{C}_k} \in \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})\}\Big) \\
&\subset \mathrm{proj}_{\mathcal{C}_\ell}(\{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{C}_\ell} \in \mathrm{proj}_{\mathcal{C}_\ell}(\mathcal{D})\}) \\
&= \mathrm{proj}_{\mathcal{C}_\ell}(\mathcal{D})
\end{aligned}$$

is obtained with (7.14). Hence, (7.16) is achieved. $\square$

Theorem 7.2 shows that the zero set $\hat{v}^{-1}(0)$ of (7.13), given in (7.14), is the most similar to $\mathcal{D}$ in the sense of (7.11). Hence, the zero set $\hat{v}^{-1}(0)$ indicates the control performance achievable under graph $G$ from the viewpoint of how similar a configuration can be obtained by the robots. This is explained through the following example.

**Example 7.1.** Consider the multi-robot system with $n = 3$ robots in $d = 1$-dimensional space. We will compare two graphs $G_{\mathrm{a}}$ and $G_{\mathrm{b}}$ in Figs. 7.1a and 7.1b, respectively, on the control performance in terms of the zero sets $\hat{v}^{-1}(0)$ of (7.13). First, consider $G_{\mathrm{a}}$, which contains two maximal cliques of order two (i.e., edges): $\mathcal{C}_1 = \{1, 2\}$ and $\mathcal{C}_2 = \{1, 3\}$. The desired configuration set $\mathcal{D} \subset (\mathbb{R}^1)^3$ and the zero set $\hat{v}^{-1}(0)$ for $G_{\mathrm{a}}$ are shown in Fig. 7.1c, where each axis corresponds to the position $x_i \in \mathbb{R}$ of each robot. In Fig. 7.1c, $\hat{v}^{-1}(0)$ is described by the dark gray area, that is, the intersection of the elliptical cylinders $\{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{C}_k} \in \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})\}$ for $k = 1, 2$ according to (7.14). Note that $\hat{v}^{-1}(0)$ is different from $\mathcal{D}$ in this case. Next, consider $G_{\mathrm{b}}$, which contains one maximal clique of order three: $\mathcal{C}_1 = \{1, 2, 3\}$. As shown in Fig. 7.1d, the zero set $\hat{v}^{-1}(0)$ is equivalent to $\mathcal{D}$ in this case. Accordingly, graph $G_{\mathrm{b}}$ can achieve




**(a)** Graph $G_\mathrm{a}$ **(b)** Graph $G_\mathrm{b}$

**(c)** Zero set $\hat{v}^{-1}(0)$ for $G_\mathrm{a}$ **(d)** Zero set $\hat{v}^{-1}(0)$ for $G_\mathrm{b}$

**Figure 7.1:** Comparison of graphs on control performance in terms of the zero sets $\hat{v}^{-1}(0)$: (a), (b) graphs $G_\mathrm{a}, G_\mathrm{b}$; (c), (d) the corresponding zero sets $\hat{v}^{-1}(0)$.

better performance than $G_\mathrm{a}$. This illustrates the importance of "cliques" rather than "edges".

## 7.3 Controller design

To obtain the approximate indicator according to Theorem 7.2, we need to design indicators $\hat{v}_k(x_{\mathcal{C}_k})$ of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$. A typical indicator is given as follows.

**Lemma 7.3.** For a non-empty set $\mathcal{D} \subset (\mathbb{R}^d)^n$ and a node subset $\mathcal{C}_k \subset \mathcal{N}$, the squared distance function of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$ given by

$$v_k(x_{\mathcal{C}_k}) = \frac{\gamma_k}{2}(\mathrm{dist}(x_{\mathcal{C}_k}, \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})))^2 \tag{7.17}$$

is an indicator of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$ for $\gamma_k > 0$.

*Proof.* This lemma follows from the definition of the distance function. □

From this lemma, an example of the functions satisfying the conditions in Theorem 7.2 is given as follows.

**Theorem 7.4.** For a graph $G$ and a non-empty set $\mathcal{D} \subset (\mathbb{R}^d)^n$,

$$\hat{v}(x_{\mathcal{N}}) = \sum_{k \in \mathrm{clq}(G)} \frac{\gamma_k}{2} (\mathrm{dist}(x_{\mathcal{C}_k}, \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})))^2 \tag{7.18}$$

for $\gamma_k > 0$ is the best approximate indicator of $\mathcal{D}$ having a distributed gradient for $G$, i.e., $\hat{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{app}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G)$ satisfies (7.11).

*Proof.* This theorem follows from Theorem 7.2 and Lemma 7.3. □

The meaning of the function (7.18) is explained from the viewpoint of optimization as follows. The target problem, namely, the generalized coordination (7.1), corresponds to solving the optimization problem

$$\mathrm{dist}(x_{\mathcal{N}}, \mathcal{D}) = \inf_{D \in \mathcal{D}} \|x_{\mathcal{N}} - D\|. \tag{7.19}$$

Because (7.19) depends on the states $x_1, x_2, \ldots, x_n$ of all the robots, it is solvable in a centralized way. Instead, by projecting (7.19) onto the $x_{\mathcal{C}_k}$-space, the optimization problem is reduced to

$$\mathrm{dist}(x_{\mathcal{C}_k}, \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})) = \inf_{D_k \in \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})} \|x_{\mathcal{C}_k} - D_k\|, \tag{7.20}$$

which is solvable in a distributed manner when $\mathcal{C}_k$ is a maximal clique. Combination of the solutions to (7.20) for all $\mathcal{C}_k$ ($k \in \mathrm{clq}(G)$) yields the best approximate solution to the target problem (7.19). This procedure corresponds to (7.18).

A distributed controller is designed as the gradient of the best approximate indicator (7.18) as shown in the following theorem, where $\mathrm{cl}(\cdot)$ represents the closure of a set and $\mathrm{col}_m(\cdot)$ is the $m$th element of a tuple.

**Theorem 7.5.** For a graph $G$ and a non-empty set $\mathcal{D} \subset (\mathbb{R}^d)^n$, assume that the solution $D_k$ to (7.20) exists as a function $\hat{D}_k : (\mathbb{R}^d)^{|\mathcal{C}_k|} \to \mathrm{cl}(\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D}))$ of $x_{\mathcal{C}_k}$ for each $k \in \mathrm{clq}(G)$. Then, the gradient-based controller (7.7) for $v(x_{\mathcal{N}}) = \hat{v}(x_{\mathcal{N}})$ in (7.18) is reduced to the distributed controller

$$c_i(x_{\mathcal{N}_i}) = -\kappa_i \sum_{k \in \mathrm{clq}_i(G)} \gamma_k (x_i - \mathrm{col}_{n_{ki}}(\hat{D}_k(x_{\mathcal{C}_k}))) \tag{7.21}$$

for $\kappa_i, \gamma_k > 0$, where $n_{ki} \in \{1, 2, \ldots, |\mathcal{C}_k|\}$ represents the order of $i \in \mathcal{N}$ in the maximal clique $\mathcal{C}_k$, i.e., $x_{\mathcal{C}_k} = (\ldots, \overset{n_{ki}}{x_i}, \ldots)$.

*Proof.* This theorem follows from Lemma C.1. □

The distributed controller (7.21) can be systematically designed according to $G$ and $\mathcal{D}$. To implement this controller, each robot has to know the maximal cliques $\mathcal{C}_k, k \in \mathrm{clq}_i(G)$ that it belongs to. How to find them is discussed in Section 7.6. Furthermore, each robot needs to solve the optimization problem (7.20), which is considered in the following chapters in some concrete cases.

## 7.4 Stability analysis

The stability of the system with the designed controller (7.21) is analyzed under some assumptions on the desired configuration set $\mathcal{D}$.

First, the Lagrange stability is ensured if $\mathcal{D}$ is compact.

**Theorem 7.6.** For a graph $G$ and a non-empty, compact set $\mathcal{D} \subset (\mathbb{R}^d)^n$, assume that the solution to (7.20) exists for each $k \in \mathrm{clq}(G)$. Then, the system (7.2) under the control input (7.3) with the distributed controller (7.21) is Lagrange stable and $(\partial \hat{v}/\partial x_{\mathcal{N}})^{-1}(0)$ is globally attractive for $\hat{v}(x_{\mathcal{N}})$ in (7.18).

*Proof.* From Theorem 7.5, the distributed controller (7.21) is derived from the gradient of $\hat{v}(x_{\mathcal{N}})$ in (7.18). Hence, the system is reduced to the gradient-flow system (7.4) of $v(x_{\mathcal{N}}) = \hat{v}(x_{\mathcal{N}})$. From Theorem 5.3, for the Lagrange stability and the global attractiveness of $(\partial \hat{v}/\partial x_{\mathcal{N}})^{-1}(0)$, it is sufficient to prove that $\hat{v}(x_{\mathcal{N}})$ is radially unbounded. Because $\mathcal{D}$ is non-empty and compact, $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$ is non-empty and compact, and thus $\mathrm{dist}(x_{\mathcal{C}_k}, \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D}))$ is radially unbounded for any $k \in \mathrm{clq}(G)$. When $\|x_{\mathcal{N}}\| \to \infty$, at least one vector satisfies $\|x_i\| \to \infty$, which leads to $\mathrm{dist}(x_{\mathcal{C}_k}, \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})) \to \infty$ for some $k \in \mathrm{clq}_i(G)$ because each node belongs to at least one maximal clique. Then, from (7.18), $\hat{v}(x_{\mathcal{N}}) \to \infty$ is satisfied, which implies that $\hat{v}(x_{\mathcal{N}})$ is radially unbounded. □

In the next chapter, the Lagrange stability will be ensured for non-compact sets $\mathcal{D}$, differently from Theorem 7.6.

Next, the asymptotic stability is ensured under some assumptions including the Lagrange stability of the system. See Appendix B for real analytic submanifolds.

**Theorem 7.7.** For a graph $G$ and a non-empty set $\mathcal{D} \subset (\mathbb{R}^d)^n$, assume that (i) the solution to (7.20) exists for each $k \in \mathrm{clq}(G)$, that (ii) the system (7.2) under the control input (7.3) with the distributed controller (7.21) is Lagrange stable, and that (iii) $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$ is a real analytic submanifold for each $k \in \mathrm{clq}(G)$. Then, $\hat{v}^{-1}(0)$ is asymptotically stable for $\hat{v}(x_{\mathcal{N}})$ in (7.18).

*Proof.* For $\hat{v}(x_{\mathcal{N}})$ in (7.18), the zero set $\hat{v}^{-1}(0)$ is non-empty because $\mathcal{D}$ is non-empty and $\hat{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{app}}(\mathcal{D})$. Because $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$ is a real analytic submanifold from the assumption, Lemma B.1 guarantees that $(\mathrm{dist}(x_{\mathcal{C}_k}, \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})))^2$ is a real analytic function in an open set containing $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$. Thus, $\hat{v}(x_{\mathcal{N}})$ in (7.18) is real analytic in an open set containing $\bigcap_{k\in\mathrm{clq}(G)}\{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{C}_k} \in \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})\} = \hat{v}^{-1}(0)$. Additionally because the system is assumed to be Lagrange stable, Theorem 5.6 guarantees that $\hat{v}^{-1}(0)$ is asymptotically stable. □

## 7.5 Existence of indicators

To solve Problem 7.2, a condition of $(\mathcal{D}, G)$ will be derived such that there exists an indicator of $\mathcal{D}$ having a distributed gradient for $G$, i.e., $\mathcal{V}_{\mathrm{ind}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G)$ is non-empty. According to Theorem 7.2, this condition is fulfilled if and only if the best approximate indicator $\hat{v}(x_{\mathcal{N}})$ in (7.13) satisfies (7.5) to be an indicator. Hence, it is sufficient to verify whether $\hat{v}(x_{\mathcal{N}})$ satisfies (7.5) or not. From this viewpoint, the following theorem is derived.

**Theorem 7.8.** For a graph $G$ and a non-empty set $\mathcal{D} \subset (\mathbb{R}^d)^n$, there exists an indicator of $\mathcal{D}$ having a distributed gradient for $G$, i.e., $\mathcal{V}_{\mathrm{ind}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G)$ is non-empty, if and only if the set framework $(\mathcal{D}, G)$ is clique rigid. Moreover, under this condition, a function $v : (\mathbb{R}^d)^n \to \mathbb{R}$ belongs to $\mathcal{V}_{\mathrm{ind}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G)$ if and only if it can be of the form (7.13) for $\hat{v}(x_{\mathcal{N}}) = v(x_{\mathcal{N}})$ with indicators $\hat{v}_k(x_{\mathcal{C}_k})$ of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$ for the maximal cliques $\mathcal{C}_k$, $k \in \mathrm{clq}(G)$.

*Proof.* From Theorem 7.2 and the discussion just before this theorem, it is sufficient to verify that (7.5) holds for $v(x_{\mathcal{N}}) = \hat{v}(x_{\mathcal{N}})$ in (7.13), where $\hat{v}_k(x_{\mathcal{C}_k})$ are indicators of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$. From $\hat{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{app}}(\mathcal{D})$, $\hat{v}^{-1}(0) \supset \mathcal{D}$ always holds. Hence, (7.5) holds if and only if the converse inclusion

$$\hat{v}^{-1}(0) = \bigcap_{k \in \mathrm{clq}(G)} \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{C}_k} \in \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})\} \subset \mathcal{D}$$

holds, which is equivalent to the definition (4.10) of clique rigidity. □

## 7.6 Notes and references

In the conventional research, the gradient-flow approach is employed with an edge-based function $v(x_{\mathcal{N}}) = \sum_{\{i,j\} \in \mathcal{E}} \psi_{ij}(x_i, x_j)$, as mentioned in Chapter 6. The point of this chapter is the shift of objective functions from edge-based functions to clique-based ones $v(x_{\mathcal{N}}) = \sum_{k \in \mathrm{clq}(G)} v_k(x_{\mathcal{C}_k})$ in Theorem 7.1. Because each edge is contained by a maximal clique, the set of clique-based functions contains that of edge-based functions. Hence, clique-based functions always have potential to enhance the control performance. Actually, Theorem 7.2 guarantees that the best performance is obtained in terms of the generalized coordination by clique-based functions. This result was obtained in Sakurama *et al.*, 2012; Sakurama *et al.*, 2015. According to this shift, the graph conditions are generalized from conventional ones, e.g., connectivity and global rigidity, into clique rigidity. Actually, Theorem 7.8 shows that clique rigidity is a necessary and sufficient condition for achieving the generalized coordination, which was first pointed out by Sakurama, 2021b. See Section 4.4 for correspondence between clique rigidity and conventional graph conditions.

One drawback of employing clique-based functions is that each robot has to know the maximal cliques $\mathcal{C}_k, k \in \mathrm{clq}_i(G)$ that it belongs to from the subgraph $G|_{\mathcal{N}_i}$ of neighbors. To construct $G|_{\mathcal{N}_i}$, the information on the connections between neighbors is required. Hence, whether this information is available or not determines the applicability of this method. Once $G|_{\mathcal{N}_i}$ is obtained, it is not difficult to list the maximal cliques $\mathcal{C}_k, k \in \mathrm{clq}_i(G)$ if there are not many neighbors, as discussed in Section 4.6.

# 8

# Generalized Coordination with "Relative" Measurements

Consider the generalized coordination problem

$$\lim_{t\to\infty} \operatorname{dist}(x_{\mathcal{N}}(t), \mathcal{D}) = 0 \tag{8.1}$$

with respect to a desired configuration set $\mathcal{D} \subset (\mathbb{R}^d)^n$. The setting different from the previous chapter is that the global and local coordinate frames $\Sigma, \Sigma_i(t)$ differ in general. As shown in Subsection 2.3.1, a global coordinate $p(t) \in \mathbb{R}^d$ and the corresponding local coordinate $p^{[i]}(t) \in \mathbb{R}^d$ are transformed into each other according to

$$p(t) = M_i(t)p^{[i]}(t) + b_i(t) \tag{8.2}$$

for $(M_i(t), b_i(t)) \in \mathcal{M} \ltimes \mathcal{B}$. Here, we assume that the frame transformation set $\mathcal{M} \ltimes \mathcal{B}$ has the structure of a semidirect product and is a subgroup of $\mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$. This assumption is fulfilled with typical frame transformation sets including the examples in Subsection 2.3.1.

We expect to design a distributed controller with relative measurements over a graph $G$ such that the generalized coordination (8.1) is achieved. Whether such a controller exists depends on the triple $(\mathcal{D}, G, \mathcal{M} \ltimes \mathcal{B})$. In this chapter, we derive a necessary and sufficient condition of $(\mathcal{D}, G, \mathcal{M} \ltimes \mathcal{B})$ for the existence of such a controller.

## 8.1 Problem formulation

A frame transformation set $\mathcal{M} \ltimes \mathcal{B}$ is given as a subgroup of $\mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$. As discussed in Section 2.3, under the coordinate transformation (8.2) for $(M_i(t), b_i(t)) \in \mathcal{M} \ltimes \mathcal{B}$, the kinematic model is given as

$$\dot{x}_i(t) = M_i(t) u_i(t) \tag{8.3}$$

and the relative position $x_j^{[i]}(t)$ of neighbor $j \in \mathcal{N}_i$ is given as

$$x_j^{[i]}(t) = M_i^{-1}(t)(x_j(t) - b_i(t)) = (M_i(t), b_i(t))^{-1} \bullet x_j(t) \tag{8.4}$$

from (3.8). For a graph $G = (\mathcal{N}, \mathcal{E})$, the admissible controller is of the form

$$u_i(t) = c_i(x_{\mathcal{N}_i}^{[i]}(t)) \tag{8.5}$$

with a function $c_i : (\mathbb{R}^d)^{|\mathcal{N}_i|} \to \mathbb{R}^d$, where $\mathcal{N}_i \subset \mathcal{N}$ is the neighbor set of robot $i$, defined in (4.1). A controller of the form (8.5) is called a *distributed controller with relative measurements.*

### 8.1.1 Gradient-flow approach

To achieve the generalized coordination (8.1) with respect to a set $\mathcal{D} \subset (\mathbb{R}^d)^n$, we expect to asymptotically stabilize $\mathcal{D}$. To design a controller for this purpose, the gradient-flow approach is employed. The gradient-flow system is given as

$$\dot{x}_i(t) = -\kappa_i \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}(t)) \tag{8.6}$$

with a non-negative, continuously differentiable function $v : (\mathbb{R}^d)^n \to \mathbb{R}_+$ and a positive constant $\kappa_i > 0$. As discussed in Subsection 7.1.1, to asymptotically stabilize $\mathcal{D}$, the objective function $v(x_{\mathcal{N}})$ is expected to be an indicator of $\mathcal{D}$, that is, $v(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{ind}}(\mathcal{D})$. Furthermore, to obtain the gradient-flow system (8.6) from the system (8.3) with the control input (8.5), the controller is of the form

$$c_i(x_{\mathcal{N}_i}^{[i]}) = -\kappa_i M_i^{-1} \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}), \tag{8.7}$$

where $x_j^{[i]} = (M_i, b_i)^{-1} \bullet x_j$ represents the relative position of neighbor $j \in \mathcal{N}_i$ for $(M_i, b_i) \in \mathcal{M} \ltimes \mathcal{B}$ according to (8.4). Here, we consider the situation that the value of $(M_i, b_i)$ is unknown. Hence, (8.7) has to hold for arbitrary $(M_i, b_i) \in \mathcal{M} \ltimes \mathcal{B}$.

A function $v(x_\mathcal{N})$ satisfying (8.7) can be characterized with $\mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$ defined as follows:

$$\begin{aligned}\mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B}) = \{v(x_\mathcal{N}) \in \mathcal{V}_{\mathrm{c1}} : \forall i \in \mathcal{N},\ \exists \bar{c}_i : (\mathbb{R}^d)^n \to \mathbb{R}^d \\ \text{s.t.}\ \ M_i^{-1} \frac{\partial v}{\partial x_i}(x_\mathcal{N}) = \bar{c}_i((M_i, b_i)^{-1} \bullet x_\mathcal{N}) \\ \forall x_\mathcal{N} \in (\mathbb{R}^d)^n,\ (M_i, b_i) \in \mathcal{M} \ltimes \mathcal{B}\}. \quad (8.8)\end{aligned}$$

A function $v(x_\mathcal{N}) \in \mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$ is said to have a *relative gradient*. Note that $\mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$ in (8.8) focuses on the relativity of the gradients rather than distributedness. Actually, to satisfy (8.7), $v(x_\mathcal{N})$ needs to have a distributed, relative gradient, i.e., $v(x_\mathcal{N}) \in \mathcal{V}_{\mathrm{dis}}(G) \cap \mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$, as follows.

**Proposition 8.1.** For a graph $G$ and a subgroup $\mathcal{M} \ltimes \mathcal{B}$ of $\mathrm{GL}(d) \ltimes \mathbb{R}^d$, there exists a function $c_i : (\mathbb{R}^d)^{|\mathcal{N}_i|} \to \mathbb{R}^d$ satisfying (8.7) with $\kappa_i > 0$, $x_j^{[i]} = (M_i, b_i)^{-1} \bullet x_j$ for any $x_\mathcal{N} \in (\mathbb{R}^d)^n$ and $(M_i, b_i) \in \mathcal{M} \ltimes \mathcal{B}$ for each $i \in \mathcal{N}$, if and only if $v(x_\mathcal{N}) \in \mathcal{V}_{\mathrm{dis}}(G) \cap \mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$.

*Proof.* We show only the sufficiency because the necessity can be shown in the same way. Consider a function $v(x_\mathcal{N}) \in \mathcal{V}_{\mathrm{dis}}(G) \cap \mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$. From (7.8) and (8.8),

$$\frac{\partial v}{\partial x_i}(x_\mathcal{N}) = M_i \bar{c}_i((M_i, b_i)^{-1} \bullet x_\mathcal{N}) = \tilde{c}_i(x_{\mathcal{N}_i}) \quad (8.9)$$

holds for any $x_\mathcal{N} \in (\mathbb{R}^d)^n$ and $(M_i, b_i) \in \mathcal{M} \ltimes \mathcal{B}$. Let $(M_i, b_i) = (I_d, 0) \in \mathcal{M} \ltimes \mathcal{B}$, and from (8.9),

$$\bar{c}_i(x_\mathcal{N}) = \tilde{c}_i(x_{\mathcal{N}_i}) \quad (8.10)$$

holds for any $x_\mathcal{N} \in (\mathbb{R}^d)^n$. By replacing $x_\mathcal{N}$ with $(M_i, b_i)^{-1} \bullet x_\mathcal{N}$ in (8.10), and from (8.9),

$$\tilde{c}_i((M_i, b_i)^{-1} \bullet x_{\mathcal{N}_i}) = \bar{c}_i((M_i, b_i)^{-1} \bullet x_\mathcal{N}) = M_i^{-1} \frac{\partial v}{\partial x_i}(x_\mathcal{N}) \quad (8.11)$$

is obtained. From (8.11), (8.7) is achieved for $c_i(x_{\mathcal{N}_i}^{[i]}) = -\kappa_i \tilde{c}_i(x_{\mathcal{N}_i}^{[i]})$. $\square$

### 8.1.2 Target problem

Now, we expect to design an objective function $v(x_{\mathcal{N}}) \in \mathcal{V}_{\rm ind}(\mathcal{D}) \cap \mathcal{V}_{\rm dis}(G) \cap \mathcal{V}_{\rm rel}(\mathcal{M} \ltimes \mathcal{B})$. The existence of such a function depends on the triple $(\mathcal{D}, G, \mathcal{M} \ltimes \mathcal{B})$. Our goal in this chapter is to identify the triples with which such an objective function exists. Moreover, we design a distributed controller with relative measurements with such an objective function, and show that this controller asymptotically stabilizes $\mathcal{D}$.

**Problem 8.1.** For a graph $G$, a non-empty set $\mathcal{D} \subset (\mathbb{R}^d)^n$, and a subgroup $\mathcal{M} \ltimes \mathcal{B}$ of $\mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$, specify the triples $(\mathcal{D}, G, \mathcal{M} \ltimes \mathcal{B})$ such that the intersection $\mathcal{V}_{\rm ind}(\mathcal{D}) \cap \mathcal{V}_{\rm dis}(G) \cap \mathcal{V}_{\rm rel}(\mathcal{M} \ltimes \mathcal{B})$ is non-empty. Next, characterize the functions belonging to this intersection. Then, design a distributed controller with relative measurements from the gradient of such a function when it exists. Finally, analyze the asymptotic stability of $\mathcal{D}$ for the system with the designed controller.

## 8.2 Characterization of indicators

As the solution to the first part of Problem 8.1, the strict condition of the triple $(\mathcal{D}, G, \mathcal{M} \ltimes \mathcal{B})$ for the non-emptiness of $\mathcal{V}_{\rm ind}(\mathcal{D}) \cap \mathcal{V}_{\rm dis}(G) \cap \mathcal{V}_{\rm rel}(\mathcal{M} \ltimes \mathcal{B})$ is derived as follows.

**Theorem 8.1.** For a graph $G$, a non-empty set $\mathcal{D} \subset (\mathbb{R}^d)^n$, and a subgroup $\mathcal{M} \ltimes \mathcal{B}$ of $\mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$, the set $\mathcal{V}_{\rm ind}(\mathcal{D}) \cap \mathcal{V}_{\rm dis}(G) \cap \mathcal{V}_{\rm rel}(\mathcal{M} \ltimes \mathcal{B})$ is non-empty if and only if the following two conditions are satisfied:

(A) The set framework $(\mathcal{D}, G)$ is clique rigid.

(B) The set $\mathcal{D}$ is of the following form with some non-empty set $\mathcal{X}^* \subset (\mathbb{R}^d)^n$.

$$\mathcal{D} = \mathrm{orb}_{\mathcal{M} \ltimes \mathcal{B}}(\mathcal{X}^*) \tag{8.12}$$

*Proof.* (Necessity) Assume that $\mathcal{V}_{\mathrm{ind}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G) \cap \mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$ is non-empty. Then, from the non-emptiness of $\mathcal{V}_{\mathrm{ind}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G)$, Theorem 7.8 guarantees that condition (A) holds. The rest of the proof is to derive condition (B). From the assumption, there exists a function $v(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{ind}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$. Then, from (7.6), $v(x_{\mathcal{N}})$ is non-negative and $v^{-1}(0) = \mathcal{D}$ is non-empty. Hence, Lemma 8.3 given below guarantees that $v(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$ is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$. Let $\tilde{x}_{\mathcal{N}} \in \mathcal{D}$ and $(\tilde{M}, \tilde{b}) \in \mathcal{M} \ltimes \mathcal{B}$. Then, from (7.6), $v(\tilde{x}_{\mathcal{N}}) = 0$ holds. Furthermore, from the definition (3.19) of the relative invariance, $v((\tilde{M}, \tilde{b}) \bullet \tilde{x}_{\mathcal{N}}) = |\det(\tilde{M})|^{\frac{2}{d}} v(\tilde{x}_{\mathcal{N}}) = 0$ holds, and $(\tilde{M}, \tilde{b}) \bullet \tilde{x}_{\mathcal{N}} \in v^{-1}(0) = \mathcal{D}$ is obtained from (7.6). Hence, $\mathcal{D}$ is $(\mathcal{M} \ltimes \mathcal{B})$-invariant. Finally, Lemma 3.2 guarantees that such a non-empty set $\mathcal{D}$ is of the form (8.12) with some non-empty set $\mathcal{X}^*$.

(Sufficiency) This part follows from Theorem 8.4 given below. □

According to Theorem 8.1, the requirement to the triple $(\mathcal{D}, G, \mathcal{M} \ltimes \mathcal{B})$ is decomposed into condition (A) of $(\mathcal{D}, G)$ and condition (B) of $(\mathcal{D}, \mathcal{M} \ltimes \mathcal{B})$. Condition (A) requires the set framework $(\mathcal{D}, G)$ to be clique rigid in the same way as the absolute measurement case in Theorem 7.8 for the distributedness of controllers. Condition (B) is the additional condition for the relativity by making $\mathcal{D}$ have the DOF corresponding to the ambiguity in measurements, represented by $\mathcal{M} \ltimes \mathcal{B}$.

The following two lemmas are used to prove Theorem 8.1.

**Lemma 8.2.** For a subgroup $\mathcal{M} \ltimes \mathcal{B}$ of $\mathrm{GL}(d) \ltimes \mathbb{R}^d$, a function $v(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$ of $x_{\mathcal{N}} \in (\mathbb{R}^d)^n$ satisfies

$$M \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}) = \frac{\partial v}{\partial x_i}((M, b) \bullet x_{\mathcal{N}}) \tag{8.13}$$

for any $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$ for each $i \in \mathcal{N}$.

*Proof.* By replacing $(M_i, b_i) = (I_d, 0)$ in the equation of (8.8), $\bar{c}_i(x_{\mathcal{N}}) = \partial v / \partial x_i(x_{\mathcal{N}})$ is obtained. By applying $(M_i, b_i) = (M, b)^{-1}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$ to the equation of (8.8), we obtain (8.13) as follows:

$$M \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}) = \bar{c}_i((M, b) \bullet x_{\mathcal{N}}) = \frac{\partial v}{\partial x_i}((M, b) \bullet x_{\mathcal{N}}).$$

$\square$

**Lemma 8.3.** For a subgroup $\mathcal{M} \ltimes \mathcal{B}$ of $\mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$, a non-negative, continuously differentiable function $v : (\mathbb{R}^d)^n \to \mathbb{R}$ such that $v^{-1}(0)$ is non-empty belongs to $\mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$ if and only if it is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$.

*Proof.* Consider a non-negative, continuously differentiable function $v : (\mathbb{R}^d)^n \to \mathbb{R}$ such that $v^{-1}(0)$ is non-empty. From the chain rule, the action on multiple vectors (3.2), and the action of a semidirect product (3.6),

$$\begin{aligned} \frac{\partial v((M,b) \bullet x_{\mathcal{N}})}{\partial x_i} &= \left(\frac{\partial((M,b) \bullet x_i)}{\partial x_i}\right)^{\top} \left.\frac{\partial v(y_{\mathcal{N}})}{\partial y_i}\right|_{y_{\mathcal{N}}=(M,b)\bullet x_{\mathcal{N}}} \\ &= M^{\top} \frac{\partial v}{\partial x_i}((M,b) \bullet x_{\mathcal{N}}) \end{aligned} \tag{8.14}$$

holds for any $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$.

(Sufficiency) Assume that $v(x_{\mathcal{N}})$ is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$. Partially differentiate the definition (3.19) of the relative invariance with respect to $x_i$ by replacing $H = (M_i, b_i) \in \mathcal{M} \ltimes \mathcal{B}$ and $\mu(H) = |\det(M_i)|^{\frac{2}{d}}$, and from (3.1) and (8.14), we obtain

$$\begin{aligned} &|\det(M_i)|^{\frac{2}{d}} M_i \frac{\partial v(x_{\mathcal{N}})}{\partial x_i} \\ &\quad = M_i \frac{\partial v((M_i,b_i) \bullet x_{\mathcal{N}})}{\partial x_i} = M_i M_i^{\top} \frac{\partial v}{\partial x_i}((M_i,b_i) \bullet x_{\mathcal{N}}) \\ &\quad = |\det(M_i)|^{\frac{2}{d}} \frac{\partial v}{\partial x_i}((M_i,b_i) \bullet x_{\mathcal{N}}). \end{aligned} \tag{8.15}$$

Apply $(M_i, b_i)^{-1}$ instead of $(M_i, b_i)$ to (8.15), and we obtain

$$M_i^{-1} \frac{\partial v(x_{\mathcal{N}})}{\partial x_i} = \frac{\partial v}{\partial x_i}((M_i,b_i)^{-1} \bullet x_{\mathcal{N}}) = \bar{c}_i((M_i,b_i)^{-1} \bullet x_{\mathcal{N}})$$

for $\bar{c}_i(x_{\mathcal{N}}) = \partial v / \partial x_i(x_{\mathcal{N}})$. Hence, $v(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$ holds from (8.8).

(Necessity) Assume that $v(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$. From (3.1), (8.13), and (8.14),

$$\begin{aligned}\frac{\partial v((M,b) \bullet x_{\mathcal{N}})}{\partial x_i} &= M^\top \frac{\partial v}{\partial x_i}((M,b) \bullet x_{\mathcal{N}}) = M^\top M \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}) \\ &= |\det(M)|^{\frac{2}{d}} \frac{\partial v(x_{\mathcal{N}})}{\partial x_i}\end{aligned}$$

holds for any $(M,b) \in \mathcal{M} \ltimes \mathcal{B}$, which leads to

$$\frac{\partial (v((M,b) \bullet x_{\mathcal{N}}) - |\det(M)|^{\frac{2}{d}} v(x_{\mathcal{N}}))}{\partial x_{\mathcal{N}}} = 0. \tag{8.16}$$

Integrate (8.16) with respect to $x_{\mathcal{N}}$ according to the gradient theorem, we obtain

$$v((M,b) \bullet x_{\mathcal{N}}) = |\det(M)|^{\frac{2}{d}} v(x_{\mathcal{N}}) + \xi(M,b) \tag{8.17}$$

with a function $\xi : \mathcal{M} \ltimes \mathcal{B} \to \mathbb{R}$ independent of $x_{\mathcal{N}}$. Consider $\tilde{x}_{\mathcal{N}} \in v^{-1}(0)$. Then, from (8.17),

$$v((M,b) \bullet \tilde{x}_{\mathcal{N}}) = |\det(M)|^{\frac{2}{d}} v(\tilde{x}_{\mathcal{N}}) + \xi(M,b) = \xi(M,b), \tag{8.18}$$

$$\begin{aligned}0 = v(\tilde{x}_{\mathcal{N}}) &= v(((M,b) * (M,b)^{-1}) \bullet \tilde{x}_{\mathcal{N}}) \\ &= v((M,b) \bullet ((M,b)^{-1} \bullet \tilde{x}_{\mathcal{N}})) \\ &= |\det(M)|^{\frac{2}{d}} v((M,b)^{-1} \bullet \tilde{x}_{\mathcal{N}}) + \xi(M,b)\end{aligned} \tag{8.19}$$

are obtained, where the associativity of the group action is used. From (8.18) and (8.19),

$$\xi(M,b) = v((M,b) \bullet \tilde{x}_{\mathcal{N}}) = -|\det(M)|^{\frac{2}{d}} v((M,b)^{-1} \bullet \tilde{x}_{\mathcal{N}})$$

is obtained, which yields $\xi(M,b) = 0$ for any $(M,b) \in \mathcal{M} \ltimes \mathcal{B}$ because $v(x_{\mathcal{N}})$ is non-negative. Hence, from (8.17), $v(x_{\mathcal{N}})$ satisfies the definition of the relative invariance (3.19) with weight $|\det(M)|^{\frac{2}{d}}$ for $(M,b) \in \mathcal{M} \ltimes \mathcal{B}$. □

As a solution to the second part of Problem 8.1, the indicators having distributed relative gradients, i.e., $v(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{ind}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G) \cap \mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$, are characterized as follows.

**Theorem 8.4.** For a graph $G$, a non-empty set $\mathcal{D} \subset (\mathbb{R}^d)^n$, and a subgroup $\mathcal{M} \ltimes \mathcal{B}$ of scaled$(\mathrm{O}(d)) \ltimes \mathbb{R}^d$, assume that conditions (A) and (B) in Theorem 8.1 hold. Then, a continuously differentiable function $\hat{v} : (\mathbb{R}^d)^n \to \mathbb{R}$ belongs to $\mathcal{V}_{\mathrm{ind}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G) \cap \mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$ if and only if it can be of the form

$$\hat{v}(x_{\mathcal{N}}) = \sum_{k \in \mathrm{clq}(G)} \hat{v}_k(x_{\mathcal{C}_k}) \tag{8.20}$$

with indicators $\hat{v}_k(x_{\mathcal{C}_k})$ of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$, relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$ for the maximal cliques $\mathcal{C}_k$, $k \in \mathrm{clq}(G)$.

*Proof.* (Sufficiency) Consider a continuously differentiable $\hat{v}(x_{\mathcal{N}})$ of the form (8.20) with indicators $\hat{v}_k(x_{\mathcal{C}_k})$ of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$, relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$. Theorem 7.8 guarantees that $\hat{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{ind}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G)$. From the relative $(\mathcal{M} \ltimes \mathcal{B})$-invariance of $\hat{v}_k(x_{\mathcal{C}_k})$, $\hat{v}(x_{\mathcal{N}})$ is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$ as follows:

$$\begin{aligned}\hat{v}((M, b) \bullet x_{\mathcal{N}}) &= \sum_{k \in \mathrm{clq}(G)} \hat{v}_k((M, b) \bullet x_{\mathcal{C}_k}) = \sum_{k \in \mathrm{clq}(G)} |\det(M)|^{\frac{2}{d}} \hat{v}_k(x_{\mathcal{C}_k}) \\ &= |\det(M)|^{\frac{2}{d}} \hat{v}(x_{\mathcal{N}}).\end{aligned}$$

Moreover, $\hat{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{ind}}(\mathcal{D})$ is non-negative and $\hat{v}^{-1}(0) = \mathcal{D}$ is non-empty, and thus Lemma 8.3 guarantees $\hat{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$.

(Necessity) Consider a continuously differentiable function $\hat{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{ind}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G) \cap \mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$. From Theorem 7.8, $\hat{v}(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{ind}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G)$ can be of the form (8.20) with indicators $\hat{v}_k(x_{\mathcal{C}_k})$ of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$ for $k \in \mathrm{clq}(G)$. Additionally, each of these $\hat{v}_k(x_{\mathcal{C}_k})$ can be chosen as a relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant function of weight $|\det(M)|^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$ from Lemma D.7 (b). $\square$

Compared to Theorem 7.8, which characterizes the functions belonging to $\mathcal{V}_{\mathrm{ind}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G)$, Theorem 8.4 imposes the additional condition of $\mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$. Accordingly, $\hat{v}_k(x_{\mathcal{C}_k})$ is required to be relatively invariant.

## 8.3 Controller design

A typical relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant indicator $v_k(x_{\mathcal{C}_k})$, required in Theorem 8.4, is given as follows.

**Lemma 8.5.** For a subgroup $\mathcal{M} \ltimes \mathcal{B}$ of $\mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$, assume that a set $\mathcal{D} \subset (\mathbb{R}^d)^n$ satisfies condition (B) in Theorem 8.1. Then,

$$\hat{v}_k(x_{\mathcal{C}_k}) = \frac{\gamma_k}{2}(\mathrm{dist}(x_{\mathcal{C}_k}, \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})))^2 \tag{8.21}$$

with $\gamma_k > 0$ is an indicator of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$, relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$.

*Proof.* The part of the indicator is obvious. From condition (B), $\mathcal{D}$ in (8.12) is an $(\mathcal{M} \ltimes \mathcal{B})$-orbit. Then, Lemmas 3.2 and 3.3 guarantee that $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$ is an $(\mathcal{M} \ltimes \mathcal{B})$-invariant subset of $(\mathbb{R}^d)^{|\mathcal{C}_k|}$. Hence, Lemma 3.4 guarantees that the distance function (8.21) is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{2}{d}}$. □

From this lemma, an example of the functions characterized in Theorem 8.4 is given as follows.

**Theorem 8.6.** For a graph $G$, a non-empty set $\mathcal{D} \subset (\mathbb{R}^d)^n$, and a subgroup $\mathcal{M} \ltimes \mathcal{B}$ of $\mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$, assume that conditions (A) and (B) in Theorem 8.1 hold. Then,

$$\hat{v}(x_{\mathcal{N}}) = \sum_{k \in \mathrm{clq}(G)} \frac{\gamma_k}{2}(\mathrm{dist}(x_{\mathcal{C}_k}, \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})))^2 \tag{8.22}$$

belongs to $\mathcal{V}_{\mathrm{ind}}(\mathcal{D}) \cap \mathcal{V}_{\mathrm{dis}}(G) \cap \mathcal{V}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$.

*Proof.* This theorem follows from Theorem 8.4 and Lemma 8.5. □

To employ the function $\hat{v}(x_{\mathcal{N}})$ in (8.22), the following optimization problem has to be solved:

$$
\begin{aligned}
&\operatorname{dist}(x_{\mathcal{C}_k}, \operatorname{proj}_{\mathcal{C}_k}(\mathcal{D})) \\
&\qquad = \inf_{D_k \in \operatorname{proj}_{\mathcal{C}_k}(\mathcal{D})} \|x_{\mathcal{C}_k} - D_k\| \\
&\qquad = \inf_{((M_k, b_k), \Xi_k) \in (\mathcal{M} \ltimes \mathcal{B}) \times \operatorname{proj}_{\mathcal{C}_k}(\mathcal{X}^*)} \|x_{\mathcal{C}_k} - (M_k, b_k) \bullet \Xi_k\|. \qquad (8.23)
\end{aligned}
$$

The last equation follows from

$$
\begin{aligned}
&\operatorname{proj}_{\mathcal{C}_k}(\mathcal{D}) \\
&\quad = \operatorname{proj}_{\mathcal{C}_k}(\operatorname{orb}_{\mathcal{M} \ltimes \mathcal{B}}(\mathcal{X}^*)) = \operatorname{orb}_{\mathcal{M} \ltimes \mathcal{B}}(\operatorname{proj}_{\mathcal{C}_k}(\mathcal{X}^*)) \\
&\quad = \{(M_k, b_k) \bullet \Xi_k \in (\mathbb{R}^d)^{|\mathcal{C}_k|} : (M_k, b_k) \in (\mathcal{M} \ltimes \mathcal{B}), \Xi_k \in \operatorname{proj}_{\mathcal{C}_k}(\mathcal{X}^*)\}
\end{aligned}
$$

for $\mathcal{D}$ in (8.12), where Lemma 3.1 is used. The optimization problem (8.23) can be analytically solved for typical $\mathcal{M}$, $\mathcal{B}$, and $\mathcal{X}^*$, as detailed in Appendix E.

A distributed controller with relative measurements is derived from $\hat{v}(x_{\mathcal{N}})$ in (8.22) as follows.

**Theorem 8.7.** For a graph $G$, a set $\mathcal{D} \subset (\mathbb{R}^d)^n$, and a subgroup $\mathcal{M} \ltimes \mathcal{B}$ of $\operatorname{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$, assume that condition (B) in Theorem 8.1 holds, and that the solution to (8.23) exists as a function $((\hat{M}_k, \hat{b}_k), \hat{\Xi}_k) : (\mathbb{R}^d)^{|\mathcal{C}_k|} \to \operatorname{cl}((\mathcal{M} \ltimes \mathcal{B}) \times \operatorname{proj}_{\mathcal{C}_k}(\mathcal{X}^*))$ of $x_{\mathcal{C}_k}$ for each $k \in \operatorname{clq}(G)$. Let

$$
\hat{D}_k(x_{\mathcal{C}_k}) = (\hat{M}_k(x_{\mathcal{C}_k}), \hat{b}_k(x_{\mathcal{C}_k})) \bullet \hat{\Xi}_k(x_{\mathcal{C}_k}),
$$

and the gradient-based controller (8.7) for $v(x_{\mathcal{N}}) = \hat{v}(x_{\mathcal{N}})$ in (8.22) is reduced to the distributed controller with relative measurements as

$$
c_i(x_{\mathcal{N}_i}^{[i]}) = -\kappa_i \sum_{k \in \operatorname{clq}_i(G)} \gamma_k (x_i^{[i]} - \operatorname{col}_{n_{ki}}(\hat{D}_k(x_{\mathcal{C}_k}^{[i]}))) \qquad (8.24)
$$

for $\kappa_i, \gamma_k > 0$, where $n_{ki} \in \{1, 2, \ldots, |\mathcal{C}_k|\}$ represents the order of $i \in \mathcal{N}$ in the maximal clique $\mathcal{C}_k$, i.e., $x_{\mathcal{C}_k} = (\ldots, \overset{n_{ki}}{x_i}, \ldots)$.

*Proof.* Under condition (B) in Theorem 8.1, Lemma 8.5 guarantees that $\hat{v}_k(x_{\mathcal{C}_k})$ in (8.21) is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$. This function is non-negative and $\hat{v}_k^{-1}(0)$ is non-empty. Thus, from Lemma 8.3, $\hat{v}_k(x_{\mathcal{C}_k}) \in \mathcal{F}_{\mathrm{rel}}(\mathcal{M} \ltimes \mathcal{B})$ holds. From the inverse (3.5) of a semidirect product, (8.13) in Lemma 8.2, and (C.3) in Lemma C.1,

$$\begin{aligned} M_i^{-1} \frac{\partial \hat{v}_k}{\partial x_i}(x_{\mathcal{C}_k}) &= \frac{\partial \hat{v}_k}{\partial x_i}((M_i, b_i)^{-1} \bullet x_{\mathcal{C}_k}) = \frac{\partial \hat{v}_k}{\partial x_i}(x_{\mathcal{C}_k}^{[i]}) \\ &= x_i^{[i]} - \mathrm{col}_{n_{ki}}(\hat{D}_k(x_{\mathcal{C}_k}^{[i]})) \end{aligned} \tag{8.25}$$

is obtained, where $x_j^{[i]} = (M_i, b_i)^{-1} \bullet x_j$. From (8.7), (8.22), and (8.25), (8.24) is obtained. $\square$

Theorem 8.7 does not require condition (A) in Theorem 8.1, that is, the clique rigidity of the set framework $(\mathcal{D}, G)$. Without this condition, the designed controller (8.24) is distributed with relative measurements, and provides the best performance in the sense that the objective function (8.22) is the best approximate indicator from Theorem 7.4.

## 8.4 Stability analysis

The stability of the system with the designed controller (8.24) is analyzed under some assumptions on $\mathcal{M}$, $\mathcal{B}$, and $\mathcal{X}^*$. The assumptions are fulfilled with typical semidirect products including the examples in Subsection 2.3.1.

First, Lagrange stability is ensured.

**Theorem 8.8.** For a graph $G$, a set $\mathcal{D} \subset (\mathbb{R}^d)^n$, and a subgroup $\mathcal{M} \ltimes \mathcal{B}$ of $\mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$, consider the system (8.3) under the control input (8.5) of the distributed controller (8.24) with relative measurements. Assume the assumptions in Theorem 8.7, and assume that $\mathcal{M}$, $\mathcal{B}$, and $\mathcal{X}^*$ satisfy either of the following conditions:

(a) $\mathcal{M}$, $\mathcal{B}$, and $\mathcal{X}^*$ are all non-empty and compact;

(b) $\mathcal{B} = \mathbb{R}^d$, and $\mathcal{M}$ and $\mathcal{X}^*$ are non-empty and compact;

(c) scaled($\{I_d\}$) is a subgroup of $\mathcal{M}$, and $\mathcal{B}$ and $\mathcal{X}^*$ are non-empty.

Then, the system is Lagrange stable, and $(\partial \hat{v}/\partial x_{\mathcal{N}})^{-1}(0)$ is globally attractive for $\hat{v}(x_{\mathcal{N}})$ in (8.22).

*Proof.* From condition (B) in Theorem 8.1, $\mathcal{D}$ is of the form (8.12). From Theorem 8.7, the system (8.3) with the control input (8.5) and (8.24) is equivalent to the gradient-flow system (8.6) for $\hat{v}(x_{\mathcal{N}})$ in (8.22).

First, under condition (a), $\mathcal{D}$ in (8.12) is non-empty and compact. Then, Theorem 7.6 guarantees the Lagrange stability and global attractiveness.

Next, assume condition (b). Without loss of generality, we assume that $G$ is connected. Otherwise, we just have to discuss the following for each connected component of $G$. We show that $\hat{v}(x_{\mathcal{N}})$ in (8.22) satisfies conditions (i) and (ii) in Theorem 5.4. As for condition (i), from $\mathcal{B} = \mathbb{R}^d$, $\{I_d\} \ltimes \mathbb{R}^d$ is a subgroup of $\mathcal{M} \ltimes \mathcal{B}$. Hence, $\hat{v}_k(x_{\mathcal{C}_k})$ in (8.21) can be guaranteed to be $(\{I_d\} \ltimes \mathbb{R}^d)$-invariant in the same way as Lemma 8.5. Hence, $\hat{v}(x_{\mathcal{N}})$ in (8.22) is $(\{I_d\} \ltimes \mathbb{R}^d)$-invariant. To verify condition (ii), consider a pair $i_1, i_2 \in \mathcal{C}_k$ in a maximal clique $\mathcal{C}_k$. According to (8.23), the expressions

$$\begin{aligned}
&\mathrm{dist}(x_{\mathcal{C}_k}, \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D}))^2 \\
&\qquad = \|x_{\mathcal{C}_k} - (\hat{M}_k, \hat{b}_k) \bullet \hat{\Xi}_k\|^2 = \sum_{i \in \mathcal{C}_k} \|x_i - \hat{M}_k \mathrm{col}_{n_{ki}}(\hat{\Xi}_k) - \hat{b}_k\|^2 \\
&\qquad \geq \|x_{i_1} - \hat{M}_k \mathrm{col}_{n_{ki_1}}(\hat{\Xi}_k) - \hat{b}_k\|^2 + \|x_{i_2} - \hat{M}_k \mathrm{col}_{n_{ki_2}}(\hat{\Xi}_k) - \hat{b}_k\|^2 \\
&\qquad \geq \frac{1}{2}\|x_{i_1} - \hat{M}_k \mathrm{col}_{n_{ki_1}}(\hat{\Xi}_k) - \hat{b}_k - (x_{i_2} - \hat{M}_k \mathrm{col}_{n_{ki_2}}(\hat{\Xi}_k) - \hat{b}_k)\|^2 \\
&\qquad = \frac{1}{2}\|x_{i_1} - x_{i_2} - \hat{M}_k(\mathrm{col}_{n_{ki_1}}(\hat{\Xi}_k) - \mathrm{col}_{n_{ki_2}}(\hat{\Xi}_k))\|^2 \\
&\qquad \geq \frac{1}{2}(\|x_{i_1} - x_{i_2}\| - |\det(\hat{M}_k)|^{\frac{1}{d}}\|\mathrm{col}_{n_{ki_1}}(\hat{\Xi}_k) - \mathrm{col}_{n_{ki_2}}(\hat{\Xi}_k)\|)^2
\end{aligned} \tag{8.26}$$

hold from the parallelogram law and (3.1). From (8.22) and (8.26),

$$
\begin{aligned}
&\|x_{i_1} - x_{i_2}\| \\
&\quad \le \sqrt{2}\mathrm{dist}(x_{\mathcal{C}_k}, \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})) + |\det(\hat{M}_k)|^{\frac{1}{d}} \|\mathrm{col}_{n_{ki_1}}(\hat{\Xi}_k) - \mathrm{col}_{n_{ki_2}}(\hat{\Xi}_k)\| \\
&\quad \le 2\sqrt{\frac{\hat{v}(x_{\mathcal{N}})}{\gamma_k}} + \sqrt{2}|\det(\hat{M}_k)|^{\frac{1}{d}} \|\hat{\Xi}_k\| \le \lambda_1 \sqrt{\hat{v}(x_{\mathcal{N}})} + \lambda_2 \qquad (8.27)
\end{aligned}
$$

is obtained, where

$$
\begin{aligned}
\lambda_1 &= \frac{2}{\sqrt{\min_{k \in \mathrm{clq}(G)} \gamma_k}} > 0 \\
\lambda_2 &= \sqrt{2} \max_{M \in \mathcal{M}} |\det(M)|^{\frac{1}{d}} \max_{k \in \mathrm{clq}(G)} \max_{\Xi \in \mathrm{proj}_{\mathcal{C}_k}(\mathcal{X}^*)} \|\Xi\| \ge 0.
\end{aligned}
$$

The constant $\lambda_2$ exists under the assumption that $\mathcal{M}$ and $\mathcal{X}^*$ are non-empty and compact. Note that each edge belongs to a maximal clique, and from the assumption of the connectivity of $G$, there is an $n$-tuple $(i_1, i_2, \dots, i_n)$ of the distinct elements in $\mathcal{N}$ such that for any $\ell \in \{2, 3 \dots, n\}$, there exists $\hat{\ell} < \ell$ satisfying $\{i_{\hat{\ell}}, i_\ell\} \in \mathcal{E}$. Moreover, for any $\ell \in \{2, 3 \dots, n\}$, (8.27) holds for $i_{\hat{\ell}}$ and $i_\ell$ instead of $i_1$ and $i_2$. Hence, condition (ii) in Theorem 5.4 is satisfied, and thus the system is Lagrange stable and $(\partial \hat{v}/\partial x_{\mathcal{N}})^{-1}(0)$ is globally attractive.

Finally, consider condition (c). Then, $\mathrm{scaled}(\{I_d\}) \ltimes \{0\}$ is a subgroup of $\mathcal{M} \ltimes \mathcal{B}$. In the same way as Lemma 8.5, we can show that $\hat{v}_k(x_{\mathcal{C}_k})$ in (8.21) is relatively $(\mathrm{scaled}(\{I_d\}) \ltimes \{0\})$-invariant of weight $s^2$ for $(sI_d, 0) \in (\mathrm{scaled}(\{I_d\}) \ltimes \{0\})$ with $s > 0$, and so is $\hat{v}(x_{\mathcal{N}})$ in (8.22). Hence, Theorem 5.5 guarantees that the system is Lagrange stable and $(\partial \hat{v}/\partial x_{\mathcal{N}})^{-1}(0)$ is globally attractive. □

Next, the asymptotic stability of $\hat{v}^{-1}(0)$ is guaranteed under the assumption that $\mathcal{M}$, $\mathcal{B}$, and $\mathcal{X}^*$ are real analytic submanifolds. (See Appendix B for the real analyticity of submanifolds.)

**Theorem 8.9.** For a graph $G$, a set $\mathcal{D} \subset (\mathbb{R}^d)^n$, and a subgroup $\mathcal{M} \ltimes \mathcal{B}$ of $\mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$, consider the system (8.3) under the control input (8.5) of the distributed controller (8.24) with relative measurements. Assume that the assumptions in Theorem

8.7 are satisfied, that the system is Lagrange stable, and that $\mathcal{M}$, $\mathcal{B}$, and $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{X}^*)$ are real analytic submanifolds for any $k \in \mathrm{clq}(G)$. Then, $\hat{v}^{-1}(0)$ is asymptotically stable for $\hat{v}(x_{\mathcal{N}})$ in (8.22). Moreover, if the set framework $(\mathcal{D}, G)$ is clique rigid, $\mathcal{D}$ is asymptotically stable.

*Proof.* The part of the asymptotic stability of $\hat{v}^{-1}(0)$ follows from Theorem 7.7. The asymptotic stability of $\mathcal{D}$ is achieved because if $(\mathcal{D}, G)$ is clique rigid, $\hat{v}^{-1}(0) = \mathcal{D}$ holds from Theorem 7.8. □

## 8.5 Relations between coordination, measurement, and networks

According to Theorem 8.1, this section provides the relations of the triple $(\mathcal{D}, G, \mathcal{M} \ltimes \mathcal{B})$ to achieve the generalized coordination by a distributed controller with relative measurements.

### 8.5.1 Relations between desired configuration and measurement information

First, we focus on condition (B) in Theorem 8.1. Let $\mathfrak{D}(\mathcal{M} \ltimes \mathcal{B}) \subset \mathrm{pow}((\mathbb{R}^d)^n)$ be defined as the family of the desired configuration sets $\mathcal{D}$ satisfying condition (B), that is,

$$\mathfrak{D}(\mathcal{M} \ltimes \mathcal{B}) := \{\mathrm{orb}_{\mathcal{M}\ltimes\mathcal{B}}(\mathcal{X}^*) \subset (\mathbb{R}^d)^n : \mathcal{X}^* \subset (\mathbb{R}^d)^n\}. \tag{8.28}$$

From Theorem 8.1, the generalized coordination (8.1) with respect to $\mathcal{D}$ is achievable by a distributed controller with relative measurements for $\mathcal{M} \ltimes \mathcal{B}$ over some graph $G$, if and only if $\mathcal{D} \in \mathfrak{D}(\mathcal{M} \ltimes \mathcal{B})$. The following theorem shows a relation between the achievable configuration set $\mathcal{D}$ and the frame transformation set $\mathcal{M} \ltimes \mathcal{B}$ through $\mathfrak{D}(\mathcal{M} \ltimes \mathcal{B})$.

**Theorem 8.10.** For subgroups $\mathcal{M} \ltimes \mathcal{B}$ and $\tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}$ of $\mathrm{GL}(d) \ltimes \mathbb{R}^d$, the following relation holds:

$$\mathcal{M} \ltimes \mathcal{B} \subset \tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}} \;\Leftrightarrow\; \mathfrak{D}(\mathcal{M} \ltimes \mathcal{B}) \supset \mathfrak{D}(\tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}). \tag{8.29}$$

*Proof.* ($\Rightarrow$) Assume the left part of (8.29). Consider a set $\tilde{\mathcal{D}} \in \mathfrak{D}(\tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}})$, and we prove $\tilde{\mathcal{D}} \in \mathfrak{D}(\mathcal{M} \ltimes \mathcal{B})$ to show the right part of (8.29). From

(8.28), there exists $\tilde{\mathcal{X}}^* \subset (\mathbb{R}^d)^n$ such that $\tilde{\mathcal{D}} = \mathrm{orb}_{\tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}}(\tilde{\mathcal{X}}^*)$. As shown below,

$$\mathrm{orb}_{\tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}}(\tilde{\mathcal{X}}^*) = \mathrm{orb}_{\mathcal{M} \ltimes \mathcal{B}}(\mathrm{orb}_{\tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}}(\tilde{\mathcal{X}}^*)) \tag{8.30}$$

holds, and $\tilde{\mathcal{D}} = \mathrm{orb}_{\tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}}(\tilde{\mathcal{X}}^*) \in \mathfrak{D}(\mathcal{M} \ltimes \mathcal{B})$ is obtained because the set in the right-hand side of (8.30) belongs to $\mathfrak{D}(\mathcal{M} \ltimes \mathcal{B})$ from (8.28).

We show (8.30). From the definition (3.11) of the group orbit and the property of the group action,

$$\begin{aligned}
&\mathrm{orb}_{\mathcal{M} \ltimes \mathcal{B}}(\mathrm{orb}_{\tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}}(\tilde{\mathcal{X}}^*)) \\
&\quad = \bigcup_{(M,b) \in \mathcal{M} \ltimes \mathcal{B}} (M, b) \bullet \Big( \bigcup_{(\tilde{M},\tilde{b}) \in \tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}} (\tilde{M}, \tilde{b}) \bullet \tilde{\mathcal{X}}^* \Big) \\
&\quad = \bigcup_{(M,b) \in \mathcal{M} \ltimes \mathcal{B}} \ \bigcup_{(\tilde{M},\tilde{b}) \in \tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}} ((M, b) * (\tilde{M}, \tilde{b})) \bullet \tilde{\mathcal{X}}^* \\
&\quad = \bigcup_{(\hat{M},\hat{b}) \in \tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}} (\hat{M}, \hat{b}) \bullet \tilde{\mathcal{X}}^* = \mathrm{orb}_{\tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}}(\tilde{\mathcal{X}}^*)
\end{aligned}$$

is obtained, where $(\hat{M}, \hat{b}) = (M, b) * (\tilde{M}, \tilde{b}) \in \tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}$ holds for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$ and $(\tilde{M}, \tilde{b}) \in \tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}$ from the left part of (8.29). Hence, (8.30) holds.

($\Leftarrow$) Assume that the right part of (8.29) holds but the left one does not, and we show contradiction. From the second assumption, there exists $(\bar{M}, \bar{b}) \in \mathcal{M} \ltimes \mathcal{B}$ such that $(\bar{M}, \bar{b}) \notin \tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}$. From the first assumption and (8.28), if $\mathcal{D} \in \mathfrak{D}(\tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}})$, i.e., there exists $\tilde{\mathcal{X}}^* \subset (\mathbb{R}^d)^n$ such that $\mathcal{D} = \mathrm{orb}_{\tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}}(\tilde{\mathcal{X}}^*)$, then $\mathcal{D} \in \mathfrak{D}(\mathcal{M} \ltimes \mathcal{B})$ holds, i.e., there exists $\mathcal{X}^*(\tilde{\mathcal{X}}^*) \subset (\mathbb{R}^d)^n$ such that $\mathcal{D} = \mathrm{orb}_{\mathcal{M} \ltimes \mathcal{B}}(\mathcal{X}^*(\tilde{\mathcal{X}}^*))$. Accordingly, for any $\tilde{\mathcal{X}}^* \subset (\mathbb{R}^d)^n$, there exists $\mathcal{X}^*(\tilde{\mathcal{X}}^*) \subset (\mathbb{R}^d)^n$ such that

$$\mathrm{orb}_{\tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}}(\tilde{\mathcal{X}}^*) = \mathrm{orb}_{\mathcal{M} \ltimes \mathcal{B}}(\mathcal{X}^*(\tilde{\mathcal{X}}^*)). \tag{8.31}$$

By operating $(\bar{M}, \bar{b})$ to (8.31), we obtain

$$\begin{aligned}
(\bar{M}, \bar{b}) \bullet \mathrm{orb}_{\tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}}(\tilde{\mathcal{X}}^*) &= (\bar{M}, \bar{b}) \bullet \mathrm{orb}_{\mathcal{M} \ltimes \mathcal{B}}(\mathcal{X}^*(\tilde{\mathcal{X}}^*)) \\
&= \bigcup_{(M,b) \in \mathcal{M} \ltimes \mathcal{B}} ((\bar{M}, \bar{b}) * (M, b)) \bullet \mathcal{X}^*(\tilde{\mathcal{X}}^*) \\
&= \bigcup_{(\hat{M},\hat{b}) \in \mathcal{M} \ltimes \mathcal{B}} (\hat{M}, \hat{b}) \bullet \mathcal{X}^*(\tilde{\mathcal{X}}^*) \\
&= \mathrm{orb}_{\mathcal{M} \ltimes \mathcal{B}}(\mathcal{X}^*(\tilde{\mathcal{X}}^*)) = \mathrm{orb}_{\tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}}(\tilde{\mathcal{X}}^*),
\end{aligned} \tag{8.32}$$

where $(\hat{M}, \hat{b}) = (\bar{M}, \bar{b}) * (M, b) \in \mathcal{M} \ltimes \mathcal{B}$ holds from $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$ and $(\bar{M}, \bar{b}) \in \mathcal{M} \ltimes \mathcal{B}$. Let $\tilde{\mathcal{X}}^* = \{x^*_{\mathcal{N}}\}$ for some $x^*_{\mathcal{N}} \in (\mathbb{R}^d)^n$, and from (8.32), for any $(M_1, b_1) \in \tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}$, there exists $(M_2, b_2) \in \tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}$ such that

$$(\bar{M}, \bar{b}) \bullet ((M_1, b_1) \bullet x^*_{\mathcal{N}}) = ((\bar{M}, \bar{b}) * (M_1, b_1)) \bullet x^*_{\mathcal{N}} = (M_2, b_2) \bullet x^*_{\mathcal{N}}.$$

If $n \geq \mathrm{fanum}(\mathcal{M} \ltimes \mathcal{B})$, from (3.31), $(\bar{M}, \bar{b}) * (M_1, b_1) = (M_2, b_2)$ is obtained for almost every $x^*_{\mathcal{N}}$. Then, $(\bar{M}, \bar{b}) = (M_2, b_2) * (M_1, b_1)^{-1} \in \tilde{\mathcal{M}} \ltimes \tilde{\mathcal{B}}$ holds, which contradicts the assumption. If $n < \mathrm{fanum}(\mathcal{M} \ltimes \mathcal{B})$, we can consider multiple $x^*_{\mathcal{N}}$ and the same discussion holds with the sufficient number of $x^*_{\mathcal{N}}$. □

Theorem 8.10 indicates that as $\mathcal{M} \ltimes \mathcal{B}$ is larger (measurement information is more ambiguous), $\mathfrak{D}(\mathcal{M} \ltimes \mathcal{B})$ is smaller (the range of achievable configuration sets $\mathcal{D}$ is narrower).

Let us consider the desired configuration set $\mathcal{D}$ in (2.16), that is,

$$\mathcal{D} = \mathrm{orb}_{\mathcal{S} \ltimes \mathcal{T}}(x^*_{\mathcal{N}}) \tag{8.33}$$

with a coordination freedom set $\mathcal{S} \ltimes \mathcal{T}$ and a desired configuration $x^*_{\mathcal{N}} \in (\mathbb{R}^d)^n$. Then, the following corollary follows from Theorem 8.10.

**Corollary 8.11.** For subgroups $\mathcal{S} \ltimes \mathcal{T}$ and $\mathcal{M} \ltimes \mathcal{B}$ of $\mathrm{GL}(d) \ltimes \mathbb{R}^d$, the following relation holds:

$$\mathcal{M} \ltimes \mathcal{B} \subset \mathcal{S} \ltimes \mathcal{T} \;\Leftrightarrow\; \mathrm{orb}_{\mathcal{S} \ltimes \mathcal{T}}(x^*_{\mathcal{N}}) \in \mathfrak{D}(\mathcal{M} \ltimes \mathcal{B}) \;\; \forall x^*_{\mathcal{N}} \in (\mathbb{R}^d)^n. \tag{8.34}$$

Corollary 8.11 indicates that if we wish to achieve the generalized coordination with respect to the desired configuration set $\mathcal{D}$ in (8.33), more precise measurement is required than the DOF of $\mathcal{D}$ in the sense that $\mathcal{M} \ltimes \mathcal{B} \subset \mathcal{S} \ltimes \mathcal{T}$.

### 8.5.2 Relation between desired configuration and network topology

Next, we consider condition (A) in Theorem 8.1, namely, clique rigidity. The condition (4.10) of clique rigidity is sometimes difficult to check.

Here, we give an intuitive condition of the clique rigidity for the set $\mathcal{D}$ in (8.33) by using the free action number and the intersection graph as follows.

**Theorem 8.12.** Assume that a set $\mathcal{D} \subset (\mathbb{R}^d)^n$ is of the form (8.33) with a subgroup $\mathcal{S} \ltimes \mathcal{T}$ of $\mathrm{GL}(d) \ltimes \mathbb{R}^d$ and some $x^*_{\mathcal{N}} \in (\mathbb{R}^d)^n$. For a graph $G$, if the fanum$(\mathcal{S} \ltimes \mathcal{T})$-intersection graph of the maximal cliques in $G$, say $\Gamma_{\mathrm{fanum}(\mathcal{S}\ltimes\mathcal{T})}(G)$, is connected, the set framework $(\mathcal{D}, G)$ is clique rigid for almost every $x^*_{\mathcal{N}} \in (\mathbb{R}^d)^n$.

*Proof.* Assume that $\Gamma_{\mathrm{fanum}(\mathcal{S}\ltimes\mathcal{T})}(G)$ is connected. Let $x_{\mathcal{N}} \in (\mathbb{R}^d)^n$ satisfy the assumption part of the definition (4.10) of clique rigidity. Consider distinct $k, \ell \in \mathrm{clq}(G)$ such that $\{k, \ell\}$ is an edge of $\Gamma_{\mathrm{fanum}(\mathcal{S}\ltimes\mathcal{T})}(G)$. Then,

$$|\mathcal{C}_k \cap \mathcal{C}_\ell| \geq \mathrm{fanum}(\mathcal{S} \ltimes \mathcal{T}) \tag{8.35}$$

holds from (4.16). From the assumption part of (4.10), $x_{\mathcal{C}_k} \in \mathrm{proj}_{\mathcal{C}_k}(\mathcal{D})$ and $x_{\mathcal{C}_\ell} \in \mathrm{proj}_{\mathcal{C}_\ell}(\mathcal{D})$ hold, equivalent to

$$x_{\mathcal{C}_k} = (S_k, \tau_k) \bullet x^*_{\mathcal{C}_k},\ x_{\mathcal{C}_\ell} = (S_\ell, \tau_\ell) \bullet x^*_{\mathcal{C}_\ell} \tag{8.36}$$

with some $(S_k, \tau_k), (S_\ell, \tau_\ell) \in \mathcal{S} \ltimes \mathcal{T}$ from (3.15). Take the elements of tuples corresponding to the intersection $\mathcal{C}_k \cap \mathcal{C}_\ell$ from (8.36), and we obtain

$$x_{\mathcal{C}_k\cap\mathcal{C}_\ell} = (S_k, \tau_k) \bullet x^*_{\mathcal{C}_k\cap\mathcal{C}_\ell} = (S_\ell, \tau_\ell) \bullet x^*_{\mathcal{C}_k\cap\mathcal{C}_\ell}. \tag{8.37}$$

From (8.35), $\mathcal{S} \ltimes \mathcal{T}$ is free to $(\mathbb{R}^d)^{|\mathcal{C}_k\cap\mathcal{C}_\ell|} \backslash \mathcal{Z}_{k\ell}$ with a set $\mathcal{Z}_{k\ell}$ of measure zero, and (8.37) yields $(S_k, \tau_k) = (S_\ell, \tau_\ell)$ for $x^*_{\mathcal{C}_k\cap\mathcal{C}_\ell} \in (\mathbb{R}^d)^{|\mathcal{C}_k\cap\mathcal{C}_\ell|} \backslash \mathcal{Z}_{k\ell}$ from (3.31). From the connectivity of $\Gamma_{\mathrm{fanum}(\mathcal{S}\ltimes\mathcal{T})}(G)$, $(S_k, \tau_k)$ coincides with some $(S, \tau) \in \mathcal{S} \ltimes \mathcal{T}$ for every $k \in \mathrm{clq}(G)$ for $x^*_{\mathcal{N}} \in (\mathbb{R}^d)^n$ that satisfies $x^*_{\mathcal{C}_k\cap\mathcal{C}_\ell} \in (\mathbb{R}^d)^{|\mathcal{C}_k\cap\mathcal{C}_\ell|} \backslash \mathcal{Z}_{k\ell}$ for any edge $\{k, \ell\}$ of $\Gamma_{\mathrm{fanum}(\mathcal{S}\ltimes\mathcal{T})}(G)$. Hence, $x_{\mathcal{N}} = (S, \tau) \bullet x^*_{\mathcal{N}}$ holds from (8.36) because each node belongs to a maximal clique. Then, $x_{\mathcal{N}} \in \mathcal{D}$ holds for $\mathcal{D}$ in (8.33), and thus the conclusion part of (4.10) is satisfied. As a result, the set framework $(\mathcal{D}, G)$ is clique rigid. □

Theorem 8.12 indicates that as the coordination freedom set $\mathcal{S} \ltimes \mathcal{T}$ in (8.33) is larger (coordination is more flexible), the required number fanum($\mathcal{S} \ltimes \mathcal{T}$) of connections between maximal cliques in $G$ is larger (required network topology is denser). The number fanum($\mathcal{S} \ltimes \mathcal{T}$) of connections between maximal cliques is required in order to make the parameters $(S_k, \tau_k)$ coincide with each other in (8.37).

**Example 8.1.** For graphs $G_{\mathrm{a}}$, $G_{\mathrm{b}}$ in Figs. 4.4a, 4.4b, and the space of dimension $d = 2$, consider $\mathcal{S} = \mathrm{scaled}(\{I_2\})$, SO(2), or scaled(SO(2)) with $\mathcal{T} = \mathbb{R}^2$. Then, from Table 3.1, fanum($\mathcal{S} \ltimes \mathbb{R}^2$) = 2 holds. Figs. 4.4c and 4.4d depict $\Gamma_2(G_{\mathrm{a}})$ and $\Gamma_2(G_{\mathrm{b}})$, that is, the 2-intersection graphs of the maximal cliques in $G_{\mathrm{a}}$ and $G_{\mathrm{b}}$. Only the intersection graph $\Gamma_2(G_{\mathrm{a}})$ is connected, and thus Theorem 8.12 guarantees that the set framework $(\mathcal{D}, G_{\mathrm{a}})$ is clique rigid for $\mathcal{D}$ in (8.33) for almost every $x^*_{\mathcal{N}} \in (\mathbb{R}^2)^n$.

### 8.5.3 Comparison with conventional formation control

The conditions obtained here are compared with the conventional results. From Corollary 8.11, the generalized coordination with respect to $\mathcal{D}$ in (8.33) is achievable for the frame transformation set $\mathcal{M} \ltimes \mathcal{B}$ if and only if $\mathcal{S} \ltimes \mathcal{T} \supset \mathcal{M} \ltimes \mathcal{B}$. Let $\mathcal{S} \ltimes \mathcal{T} = \mathcal{M} \ltimes \mathcal{B}$, with which the most precise coordination is achieved among $\mathcal{D}$ of the form (8.33). Then, the following results are obtained for concrete $\mathcal{S} \ltimes \mathcal{T}$.

- Case of $\mathcal{S} \ltimes \mathcal{T} = \mathcal{M} \ltimes \mathcal{B} = \{I_d\} \ltimes \mathbb{R}^d$: From Example 2.5, $\mathcal{D}$ in (8.33) corresponds to displacement-based formation in Example 2.1. As for graph topology, Proposition 4.2 guarantees that the set framework $(\mathcal{D}, G)$ is clique rigid if and only if $G$ is connected. This result corresponds to the conventional results on displacement-based formation control (Olfati-Saber *et al.*, 2007; Fax and Murray, 2004).

- Case of $\mathcal{S} \ltimes \mathcal{T} = \mathcal{M} \ltimes \mathcal{B} = \mathrm{O}(d) \ltimes \mathbb{R}^d$: From Example 2.5, $\mathcal{D}$ in (8.33) corresponds to distance-based formation of Example 2.2. In this case, Proposition 4.3 guarantees that the set framework $(\mathcal{D}, G)$ is clique rigid if and only if $(x^*_{\mathcal{N}}, G)$ is globally rigid. This result is associated with the conventional results on distance-based formation control, e.g., Anderson *et al.*, 2008; Krick *et al.*, 2009; Queiroz *et al.*, 2019.

- Case of $\mathcal{S} \ltimes \mathcal{T} = \mathcal{M} \ltimes \mathcal{B} = \mathrm{SO}(d) \ltimes \mathbb{R}^d$: From Example 2.7, $\mathcal{D}$ in (8.33) corresponds to reflection-free formation. In this case, the set framework $(\mathcal{D}, G)$ is clique rigid only if $(x^*_{\mathcal{N}}, G)$ is rigid, as discussed just after Proposition 4.3.

## 8.6 Notes and references

It has been known that relative measurements can be expressed with groups such as $\mathrm{SO}(d)$ as shown in Tron *et al.*, 2016. The point of this chapter is to specify the achievable desired configuration sets by those groups through the orbit in Theorem 8.1. Then, the objective functions to achieve the generalized coordination is characterized by clique-based functions consisting of relatively invariant functions in Theorem 8.4. Accordingly, the required graph condition is shown to be clique rigidity. Furthermore, the clique rigidity is characterized with the free action number and the intersection graph in Theorem 8.12. In this way, group- and graph-theoretic concepts are deeply associated in the multi-robot coordination problems. A part of the results were obtained in Sakurama, 2016; Sakurama *et al.*, 2019 in the case $\mathcal{M} \ltimes \mathcal{B} = \mathrm{SO}(d) \times \mathbb{R}^d$, and the results were completed in Sakurama, 2018; Sakurama, 2021b.

# 9

# Application Examples

In this chapter, four coordination problems are considered to demonstrate how to design distributed controllers according to the results in Chapters 7 and 8. The first two problems, "formation selection" and "scaling reflection-free formation", can be solved straightforwardly. The latter two problems, "position assignment with local indices" and "formation control of non-holonomic robots", are advanced applications.

## 9.1 Formation selection

Consider the formation selection problem in Example 2.12 under the assumption that the absolute positions of neighbors are available and the graph $G$ is connected. Let us design a distributed controller according to the results in Chapter 7.

From the assumption, each robot is governed by the single-integrator system (7.2) with the control input (7.3). This task is described by the generalized coordination (7.1) with respect to the desired configuration set

$$\mathcal{D} = \bigcup_{q \in \mathcal{Q}} \{x_{\mathcal{N}}^{*q}\}. \tag{9.1}$$

Here, $x_{\mathcal{N}}^{*q} \in (\mathbb{R}^d)^n$ for $q \in \mathcal{Q} = \{1, 2, \ldots, p\}$ are the prescribed configuration patterns, one of which is expected to form. Assume that $x_i^{*q} \neq x_i^{*\tilde{q}}$ holds for any $i \in \mathcal{N}$ and $q, \tilde{q} \in \mathcal{Q}$, $q \neq \tilde{q}$.

From Theorem 7.4, $\hat{v}(x_{\mathcal{N}})$ in (7.18) is the best approximate indicator. Theorem 7.5 guarantees that a distributed controller is derived from its gradient as (7.21), which is reduced to

$$c_i(x_{\mathcal{N}_i}) = -\kappa_i \sum_{k \in \mathrm{clq}_i(G)} \gamma_k (x_i - x_i^{*\hat{q}_k}) \tag{9.2}$$

for

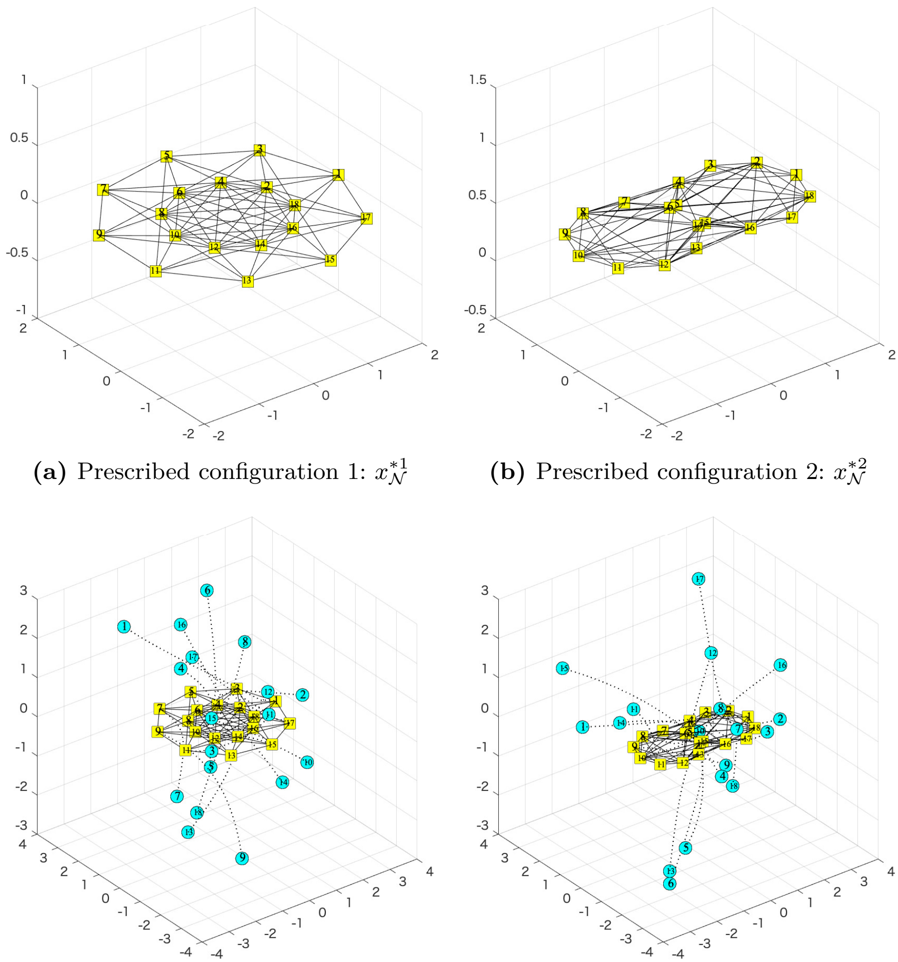
$$\hat{q}_k \in \operatorname*{argmin}_{q \in \mathcal{Q}} \| x_{\mathcal{C}_k} - x_{\mathcal{C}_k}^{*q} \|. \tag{9.3}$$

Note that (9.3) is obtained from (7.20) because

$$\mathrm{proj}_{\mathcal{C}_k}(\mathcal{D}) = \bigcup_{q \in \mathcal{Q}} \{ x_{\mathcal{C}_k}^{*q} \} \tag{9.4}$$

holds for $\mathcal{D}$ in (9.1). By using the distributed controller (9.2), $\mathcal{D}$ in (9.1) is asymptotically stable, which is shown as follows. From the assumptions of $G$ and $x_{\mathcal{N}}^{*q}$, Proposition 4.1 guarantees that the set framework $(\mathcal{D}, G)$ is clique rigid, and Theorem 7.8 guarantees that $\hat{v}(x_{\mathcal{N}})$ is an indicator of $\mathcal{D}$, that is, $\hat{v}^{-1}(0) = \mathcal{D}$ holds. Because $\mathcal{D}$ is compact, the system is Lagrange stable from Theorem 7.6. From this result and the real analyticity of (9.4), Theorem 7.7 guarantees that $\hat{v}^{-1}(0) = \mathcal{D}$ is asymptotically stable.

We conduct simulations of $n = 18$ robots in $d = 3$-dimensional space with the control gains $\kappa_i = 2$ and $\gamma_k = 1$ for all $i \in \mathcal{N}$ and $k \in \mathrm{clq}(G)$ in (9.2). Let $p = 2$ be the number of the prescribed configuration patterns, which are given by $x_{\mathcal{N}}^{*1}, x_{\mathcal{N}}^{*2} \in (\mathbb{R}^d)^n$ depicted in Figs. 9.1a and 9.1b, respectively. The edges of the graph $G$ are given is these figures. Figs. 9.1c and 9.1d show the simulation results from different initial positions, where the circles and squares with numbers represent the positions of the robots at $t = 0$ and 20, respectively, and the dotted lines represent the trajectories. We can see that either of the patterns $x_{\mathcal{N}}^{*1}$ or $x_{\mathcal{N}}^{*2}$ is successfully achieved at $t = 20$ in each of Figs. 9.1c and 9.1d. Notably, both the patterns are attained with the same controller.



**(a)** Prescribed configuration 1: $x_{\mathcal{N}}^{*1}$

**(b)** Prescribed configuration 2: $x_{\mathcal{N}}^{*2}$

**(c)** Simulation result 1

**(d)** Simulation result 2

**Figure 9.1:** Simulation results of formation selection: (a), (b) prescribed configuration patterns (numbered squares) and edges; (c), (d) trajectories from initial positions (numbered circles) to terminal positions (numbered squares).

## 9.2 Scaling reflection-free formation

We consider the situation of Example 2.18, that is, the frame transformation set is given as $\mathcal{M} \ltimes \mathcal{B} = \mathrm{scaled}(\mathrm{SO}(d)) \ltimes \mathbb{R}^d$, which involves transformation in rotation, translation, and scale, caused by sensors.

A distributed controller with relative measurements over a graph $G$ is designed according to the results in Chapter 8.

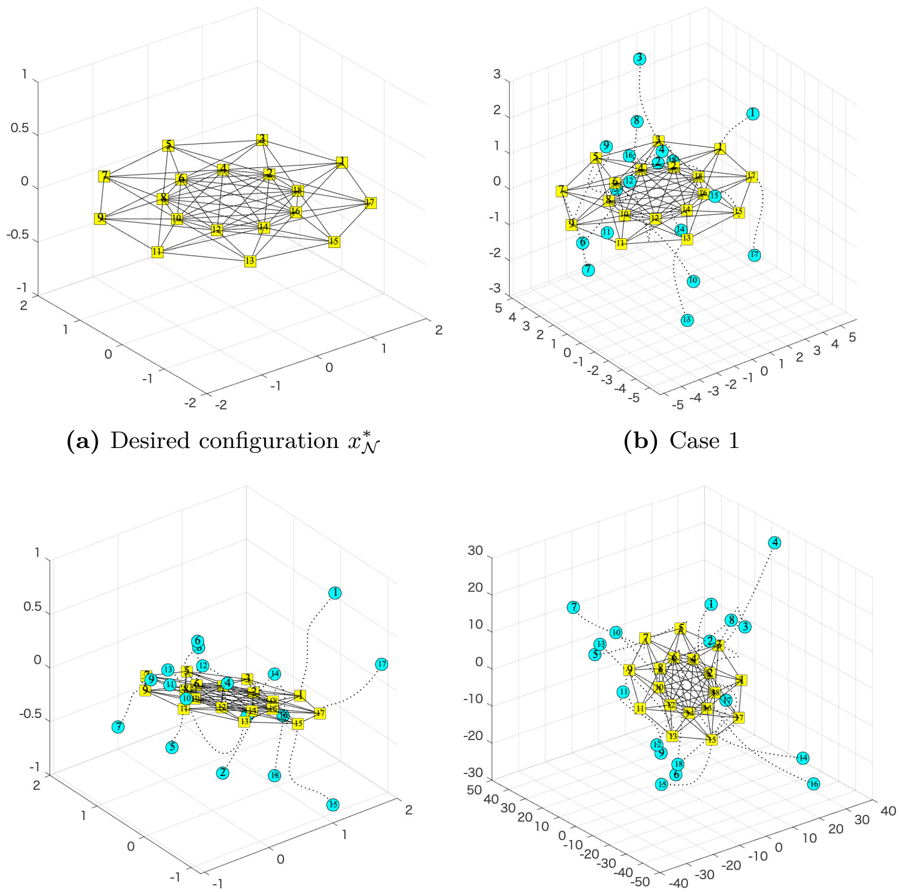
The kinematic model is given as (8.3) with the relative positions (8.4) and the control input (8.5) for $(M_i(t), b_i(t)) \in \mathcal{M} \ltimes \mathcal{B}$. We expect to achieve the generalized coordination (8.1) with respect to the desired configuration set $\mathcal{D}$ in (8.33) for a subgroup $\mathcal{S} \ltimes \mathcal{T}$ and a desired configuration $x^*_{\mathcal{N}} \in (\mathbb{R}^d)^n$. According to Corollary 8.11, $\mathcal{M} \ltimes \mathcal{B} \subset \mathcal{S} \ltimes \mathcal{T}$ has to be satisfied to achieve the coordination. Here, $\mathcal{S} \ltimes \mathcal{T} = \mathcal{M} \ltimes \mathcal{B}$ is chosen. Notably, this coordination is the scaling reflection-free formation in Example 2.11. In this way, from $\mathcal{M} \ltimes \mathcal{B}$ determined by available measurement information, the achievable coordination is naturally determined.

According to Theorem 8.7, a distributed controller with relative measurements is derived from the gradient of $\hat{v}(x_{\mathcal{N}})$ in (8.22) as (8.24), which involves the optimization problem (8.23). This problem is reduced to

$$\inf_{(M_k, b_k) \in \mathrm{scaled}(\mathrm{SO}(d)) \ltimes \mathbb{R}^d} \|x_{\mathcal{C}_k} - (M_k, b_k) \bullet x^*_{\mathcal{C}_k}\|,$$

which is analytically solvable as shown in Proposition E.4. Assume that the set framework $(\mathcal{D}, G)$ is clique rigid. Then, by using the controller (8.24), $\mathcal{D}$ is asymptotically stable, which is shown as follows. From Theorem 8.6, $\hat{v}(x_{\mathcal{N}})$ is an indicator of $\mathcal{D}$, that is, $\hat{v}^{-1}(0) = \mathcal{D}$ holds. Theorem 8.8 guarantees that the system is Lagrange stable because $\mathrm{scaled}(\{I_d\})$ is a subgroup of $\mathcal{M} = \mathrm{scaled}(\mathrm{SO}(d))$. Moreover, from Theorem 8.9, $\hat{v}^{-1}(0) = \mathcal{D}$ is asymptotically stable because $\mathcal{M} = \mathrm{scaled}(\mathrm{SO}(d))$, $\mathcal{B} = \mathbb{R}^d$, and $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{X}^*) = \{x^*_{\mathcal{C}_k}\}$ are all real analytic submanifolds.

We conduct simulations of $n = 18$ robots in $d = 3$-dimensional space, where the transformation matrices $M_i(t) \in \mathrm{scaled}(\mathrm{SO}(d))$ are randomly chosen, and the origins of the local coordinate frames are assigned to the robot positions as $b_i(t) = x_i(t)$. The control gains in (8.24) are chosen as $\kappa_i = 2$ and $\gamma_k = 1$ for all $i \in \mathcal{N}$ and $k \in \mathrm{clq}(G)$. The desired configuration $x^*_{\mathcal{N}}$ of $\mathcal{D}$ in (8.33) is depicted in Fig. 9.2a with the edges of the graph $G$. Because the 3-intersection graph of $G$ is connected, the set framework $(\mathcal{D}, G)$ is clique rigid from Theorem 8.12. Figs. 9.2b, 9.2c, and 9.2d show the trajectories of the robots from different initial positions at $t = 0$ (numbered circles). Note that the scales of the figures



**(a)** Desired configuration $x^*_{\mathcal{N}}$ **(b)** Case 1

**(c)** Case 2 **(d)** Case 3

**Figure 9.2:** Simulation results of scaling reflection-free formation: (a) desired configuration (numbered squares) and edges; (b)–(d) trajectories from initial positions (numbered circles) and terminal positions (numbered squares).

are all different, and we can see that the terminal positions (numbered squared) at $t = 20$ form the desired configuration $x^*_{\mathcal{N}}$ in any cases with different translations, rotations, and scales according to the DOF in $\mathcal{D}$ for $\mathcal{S} \ltimes \mathcal{T} = \mathrm{scaled}(\mathrm{SO}(d)) \ltimes \mathbb{R}^d$.

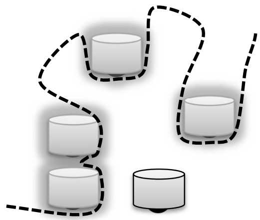


**Figure 9.3:** Outlines of other robots obtained by measuring distances of surroundings.

## 9.3 Position assignment with local indices

In this section, each robot is assumed to recognize its neighbors via *local indices*. The local indices are made by each robot under its own rule, which may be different from those of the others. Local indices enable us to relax requirements of sensors because they can be obtained from only the outlines of other robots, e.g., with LiDARs, as illustrated in Fig. 9.3. In contrast, the conventional distributed controllers use the *global indices*, which are common among all the robots. Obtaining global indices requires (i) extra sensors detecting additional features of robots, e.g., shapes, colors, and QR codes, to distinguish robots from each other and (ii) extra effort to make the correspondence between these features and the global indices.

Here, we consider the state-dependent graph $G(x_{\mathcal{N}}) = (\mathcal{N}, \mathcal{E}(x_{\mathcal{N}}))$ such that a pair of robots are connected when they are within distance $\delta > 0$. The edge set $\mathcal{E}(x_{\mathcal{N}})$ is given as

$$\mathcal{E}(x_{\mathcal{N}}) = \{\{i,j\} : \|x_i - x_j\| < \delta, i, j \in \mathcal{N}, i \neq j\}. \tag{9.5}$$

This graph is called a *proximity graph*. Let $\mathcal{N}_i(x_{\mathcal{N}}) \subset \mathcal{N}$ be the neighbor set of robot $i$, and let $n_i(x_{\mathcal{N}}) = |\mathcal{N}_i(x_{\mathcal{N}})|$ be the number of the neighbors of robot $i$. The set of local indices of robot $i$ is given as

$$\mathcal{N}_i^*(x_{\mathcal{N}}) = \{1, 2, \ldots, n_i(x_{\mathcal{N}})\}.$$

Each local index $j^* \in \mathcal{N}_i^*(x_{\mathcal{N}})$ corresponds to a global one $j \in \mathcal{N}_i(x_{\mathcal{N}})$ via a bijective function $\phi_i(x_{\mathcal{N}}) \in \mathcal{P}(\mathcal{N}_i^*(x_{\mathcal{N}}), \mathcal{N}_i(x_{\mathcal{N}}))$. The relation between the global and local indices is represented as $\phi_i(x_{\mathcal{N}})[j^*] = j$.

Note that the correspondence $\phi_i(x_\mathcal{N})$ between the global and local indices is unknown to anyone including robot $i$.

Assume that the absolute positions of the neighbors are available, and that each robot is governed by the single-integrator system (7.2). Note that although robot $i$ can use the values of the absolute positions $x_{\phi_i(x_\mathcal{N})[j^*]}(= x_j)$ of the neighbors $j^* \in \mathcal{N}_i^*(x_\mathcal{N})$, it cannot specify whose state $x_{\phi_i(x_\mathcal{N})[j^*]}$ is. Accordingly, the control input has to be generated as

$$u_i(t) = c_i(x_{\phi_i(x_\mathcal{N}(t))[1]}(t), \ldots, x_{\phi_i(x_\mathcal{N}(t))[n_i(x_\mathcal{N}(t))]}(t)) \tag{9.6}$$

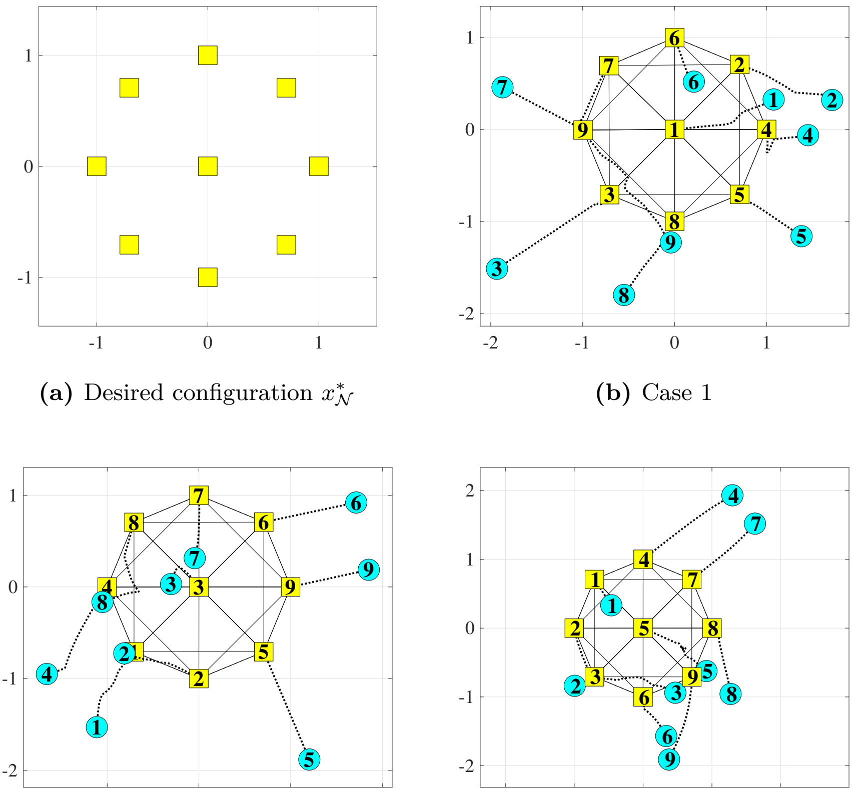
with a function $c_i : (\mathbb{R}^d)^{n_i(x_\mathcal{N}(t))} \to \mathbb{R}^d$. The controller of the form (9.6) is said to be *distributed over local indices*. Compared with the distributed controller over the global indices in (7.3), (9.6) contains the unknown bijective function $\phi_i(x_\mathcal{N}) \in \mathcal{P}(\mathcal{N}_i^*(X_\mathcal{N}), \mathcal{N}_i(X_\mathcal{N}))$. To properly define the control input by (9.6), the value of the control input has to be invariant for any $\phi_i(x_\mathcal{N}) \in \mathcal{P}(\mathcal{N}_i^*(X_\mathcal{N}), \mathcal{N}_i(X_\mathcal{N}))$, which makes designing this type of controller difficult.

To overcome this issue, we employ position assignment in Example 2.13 as a coordination task. This task can be expressed by the generalized coordination (7.1) with respect to the desired configuration set

$$\mathcal{D} = \bigcup_{\alpha \in \mathcal{P}_n} \{(x^*_{\alpha(1)}, x^*_{\alpha(2)}, \ldots, x^*_{\alpha(n)})\}, \tag{9.7}$$

where $\mathcal{P}_n$ represents the set of permutations of $n$ elements. Then, the controller is designed according to (7.21) with $\mathcal{D}$ in (9.7), which is distributed over local indices from Theorem 7.5 and the fact that the permutation $\alpha$ in $\mathcal{D}$ cancels the effect of the unknown bijective function $\phi_i(x_\mathcal{N})$. Furthermore, the asymptotic stability of $\mathcal{D}$ is shown in the same way as Section 9.1.

A collision avoidance term is added to the controller, and simulations are conducted for the proximity graph (9.5) with $\delta = 1.5$. Fig. 9.4a shows the desired configuration $x^*_\mathcal{N}$ of $\mathcal{D}$ in (9.7), and Figs. 9.4b, 9.4c, and 9.4d depict the trajectories of the robots from different initial positions (numbered circles). These result show that the terminal positions (numbered squared) at $t = 15$ form a shape of the desired configuration $x^*_\mathcal{N}$ in each case, while the positions of the robots are differently assigned. Hence, the position assignment is successfully achieved. The snapshots of the simulation in Case 1 in Fig. 9.4b are shown in Fig. 9.5, indicating that the graph topology varies according to the distances between the robots according to the proximity graph in (9.5).



**(a)** Desired configuration $x^*_{\mathcal{N}}$ **(b)** Case 1

**(c)** Case 2 **(d)** Case 3

**Figure 9.4:** Simulation results of position assignment: (a) desired configuration (numbered squares) and edges; (b)–(d) trajectories from initial positions (numbered circles) and terminal positions (numbered squares).

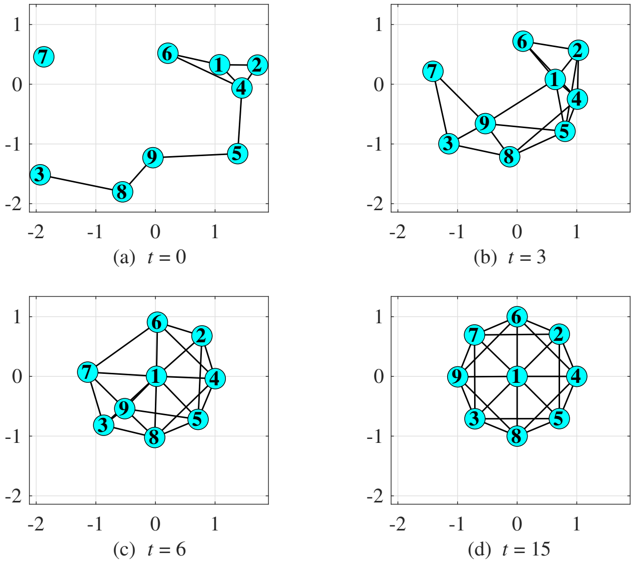



**Figure 9.5:** Snapshots of the simulation result in Case 1 in Fig. 9.4.

## 9.4 Formation control of non-holonomic robots

Many practical robots including unmanned ground vehicles (UGVs) and unmanned aerial vehicles (UAVs) have some non-holonomic constraints under which robots cannot slide laterally. In this section, formation control of such non-holonomic robots is treated.

We consider the frame transformation set $\mathcal{M} \ltimes \mathcal{B} = \mathrm{SO}(d) \ltimes \mathbb{R}^d$, which involves transformation in rotation and translation as shown in Example 2.17. The difference from the holonomic case in Subsection 2.3.2 is that (i) the orientation $M_i(t) \in \mathcal{M}$ can be controlled, and that (ii) the direction of the velocity $\dot{x}_i(t)$ is fixed in the local coordinate frame $\Sigma_i(t)$. According to (i) and (ii), the kinematic model of this system is given as

$$\dot{M}_i(t) = M_i(t) S_i(t) \tag{9.8}$$

$$\dot{x}_i(t) = M_i(t) h_i \nu_i(t) \tag{9.9}$$

for the state $(M_i(t), x_i(t)) \in \mathrm{SO}(d) \ltimes \mathbb{R}^d$ and the input $(S_i(t), \nu_i(t)) \in$

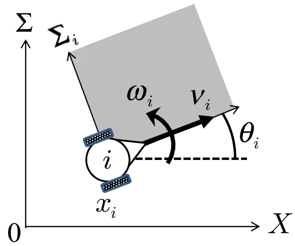



**Figure 9.6:** Robot under the non-holonomic constraint.

$\mathrm{Skew}(d) \times \mathbb{R}$ with a unit vector $h_i \in \mathbb{R}^d$. Here, $S_i(t) \in \mathrm{Skew}(d)$ and $\nu_i(t) \in \mathbb{R}$ correspond to the angular velocity and speed of robot $i$, respectively, where $\mathrm{Skew}(d)$ denotes the set of skew-symmetric matrices of dimension $d$. Equation (9.9) is derived from the holonomic model (8.3) by limiting the velocity $u_i(t)$ to a direction $h_i \in \mathbb{R}^d$ fixed in $\Sigma_i(t)$. Equation (9.8) represents the rotational motion, with which $M_i(t)$ always belongs to $\mathrm{SO}(d)$. Actually, the constraint of the orthogonal matrix, namely, $M_i^\top(t)M_i(t) = I_d$, is always satisfied because

$$\frac{\mathrm{d}M_i^\top(t)M_i(t)}{\mathrm{d}t} = \dot{M}_i^\top(t)M_i(t) + M_i^\top(t)\dot{M}_i(t) = S_i^\top(t) + S_i(t) = 0$$

holds from $S_i(t) \in \mathrm{Skew}(d)$.

For example, consider a robot in the $d = 2$-dimensional space as Fig. 9.6. Let $\theta_i(t) \in [0, 2\pi)$ be the orientation of the robot from the $X$-axis of the global coordinate frame $\Sigma$, and let $\omega_i(t) \in \mathbb{R}$ be the angular velocity. Then, for $M_i(t) = \mathrm{Rot}(\theta_i(t)) \in \mathrm{SO}(2)$, $h_i = [1\ 0]^\top$, and

$$S_i(t) = \begin{bmatrix} 0 & -\omega_i(t) \\ \omega_i(t) & 0 \end{bmatrix} \in \mathrm{Skew}(2),$$

the system consisting of (9.8) and (9.9) is reduced to the rolling coin model given as

$$\begin{aligned} \dot{\theta}_i(t) &= \omega_i(t) \\ \dot{x}_i(t) &= \begin{bmatrix} \cos(\theta_i(t)) \\ \sin(\theta_i(t)) \end{bmatrix} \nu_i(t). \end{aligned}$$

Assume that the relative positions (8.4) of the neighbors are available over a graph $G$. Then, distributed controllers with relative measurements are of the form

$$(S_i(t), \nu_i(t)) = (C_i(x_{\mathcal{N}_i}^{[i]}(t)), c_i(x_{\mathcal{N}_i}^{[i]}(t))) \tag{9.10}$$

with functions $C_i : (\mathbb{R}^d)^{|\mathcal{N}_i|} \to \mathrm{Skew}(d)$ and $c_i : (\mathbb{R}^d)^{|\mathcal{N}_i|} \to \mathbb{R}$. Following the discussion in Section 9.2, we expect to achieve the generalized coordination (8.1) with respect to the desired configuration set $\mathcal{D}$ in (8.33) for $\mathcal{S} \ltimes \mathcal{T} = \mathcal{M} \ltimes \mathcal{B} = \mathrm{SO}(d) \ltimes \mathbb{R}^d$.

To design a controller, the gradient-flow approach is adapted for the non-holonomic system. Concretely, the gradient-based controller (8.7) with an objective function $v(x_{\mathcal{N}})$ is modified into

$$\begin{aligned} C_i(x_{\mathcal{N}_i}^{[i]}) = &-\kappa_i (I_d - h_i h_i^\top) M_i^{-1} \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}) h_i^\top \\ &+ \kappa_i ((I_d - h_i h_i^\top) M_i^{-1} \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}) h_i^\top)^\top, \end{aligned} \tag{9.11}$$

$$c_i(x_{\mathcal{N}_i}^{[i]}) = -\kappa_i h_i^\top M_i^{-1} \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}) \tag{9.12}$$

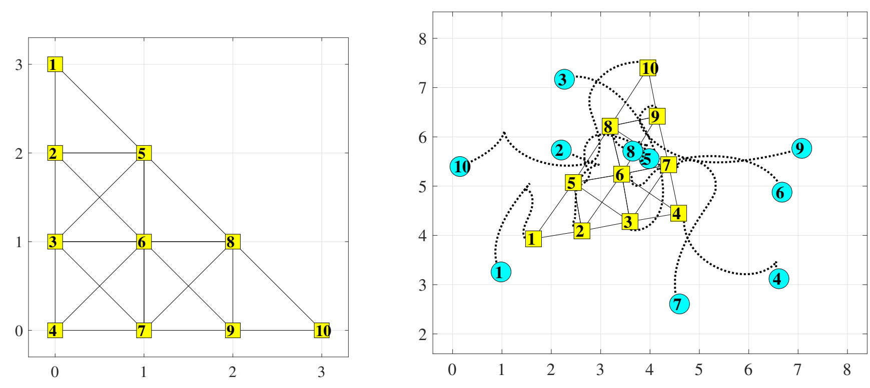
by projections onto the subspaces associated with $h_i$, where $\kappa_i > 0$ is a gain. Now, we assign the objective function $v(x_{\mathcal{N}})$ as $\hat{v}(x_{\mathcal{N}})$ in (8.22). Then, from Theorem 8.7, there exist functions $C_i(x_{\mathcal{N}_i}^{[i]})$ and $c_i(x_{\mathcal{N}_i}^{[i]})$ satisfying (9.11) and (9.12). Furthermore, it can be shown that $\hat{v}^{-1}(0)$ is asymptotically stable under some assumptions. If the set framework $(\mathcal{D}, G)$ is clique rigid, $\hat{v}(x_{\mathcal{N}})$ is an indicator of $\mathcal{D}$, that is, $\hat{v}^{-1}(0) = \mathcal{D}$ holds from Theorem 8.6, and thus $\mathcal{D}$ is asymptotically stable.

A simulation is conducted for the $d = 2$-dimensional space with $h_i = [1\ 0]^\top$. The control gains are chosen as $\kappa_i = 2$, $\gamma_k = 1$ in (9.11) and (9.12) for $v(x_{\mathcal{N}}) = \hat{v}(x_{\mathcal{N}})$ in (8.22). Fig. 9.7a shows the desired configuration $x_{\mathcal{N}}^* \in (\mathbb{R}^d)^n$ of $\mathcal{D}$ in (8.33), and Fig. 9.7b shows the trajectories of the robots from initial positions (numbered circles). Motions proper to the non-holonomic system are observed along the trajectories, and the desired configuration is finally achieved with some rotation and translation at the terminal positions (numbered squares) in Fig. 9.7b.

## 9.5 Notes and references

The results given in this chapter, formation selection, scaling reflection-free formation, position assignment with local indices, and formation control of non-holonomic robots, have been obtained in Sakurama *et al.*, 2015, Sakurama, 2021b, Sakurama and Ahn, 2020, and Sakurama, 2020, respectively. These results show that various complex coordination problems can be treated in the same manner.

These problems have been individually addressed in the existing papers. A part of the literature is listed as follows. Formation selection has been considered in Yu and Barca, 2015. Scaling formation has been investigated by Han *et al.*, 2016; Sakurama *et al.*, 2018, and is realizable by bearing-based, angle-based formations (Zhao and Zelazo, 2016; Zhao and Zelazo, 2019; Chen *et al.*, 2020), and affine formation (Lin *et al.*, 2016; Zhao, 2018). Position assignment is a sort of task assignment (allocation), which has been investigated in decades (Kuhn, 1955), and is recently considered for multi-agent systems (Ji *et al.*, 2006; Michael *et al.*, 2008; Smith and Bullo, 2009; Zavlanos *et al.*, 2008; Kanjanawanishkul and Zell, 2010; Kingston and Egerstedt, 2010; Liu



(a) Desired configuration $x^*_{\mathcal{N}}$ (b) Trajectories

**Figure 9.7:** Simulation result of formation control of non-holonomic robots: (a) desired configuration (numbered squares) and edges; (b) trajectories from initial positions (numbered circles) and terminal positions (numbered squares).

and Shell, 2011; Bürger *et al.*, 2012; Mosteo *et al.*, 2017). Note that the methods developed in the existing papers usually rely on global indices. Formation control of non-holonomic robot systems has been addressed in several papers (Lin *et al.*, 2005; Dimarogonas and Kyriakopoulos, 2008; Liu and Jiang, 2013; Montijano *et al.*, 2016; Zhao, 2018).

# 10

## Concluding Remarks

This monograph provided a systematic design theory for multi-robot coordination problems, based on group, graph, and gradient-flow theories. Especially, the generalized coordination problem has been formulated with a desired configuration set $\mathcal{D}$ to describe various coordination tasks in a unified way. Moreover, relative measurements are described by a semidirect product $\mathcal{M} \ltimes \mathcal{B}$ from the viewpoint of the transformation of the global and local coordinate frames, which provide the unified treatment of various types of measurement information. The network topology of robots is modeled by a graph $G$. Then, a necessary and sufficient condition of the triple $(\mathcal{D}, G, \mathcal{M} \ltimes \mathcal{B})$ has been derived under which the generalized coordination is achievable by distributed control with relative measurements. This condition reveals the following two points. (i) The class of the achievable configuration sets $\mathcal{D}$ is given by the $(\mathcal{M} \ltimes \mathcal{B})$-orbits. (ii) The required network topology is the clique rigidity of the set framework $(\mathcal{D}, G)$. Furthermore, a distributed controller with relative measurements has been designed with the best approximate indicator, which is decomposable into relatively invariant, clique-based functions. Finally, through the application to "formation selection", "scaling reflection-free formation", "position assignment with local indices", and "formation control of non-holonomic robots", it has been shown that various complex coordination problems can be solved through the approach of this monograph.

# Acknowledgements

The authors are grateful to Professor Shun-ichi Azuma and Professor Hyo-sung Ahn for collaboration on this research.

A part of this work was supported by JSPS KAKENHI Grant Numbers 17H03281, 19K04439, and 21H01352.

# Appendices

# A

## Examples of Frame Transformation Sets

In Subsection 2.3.1, various types of local coordinate frame $\Sigma_i$ were introduced according to frame transformation sets $\mathcal{M} \times \mathcal{B}$, which is reviewed as follows. Let $p \in \mathbb{R}^d$ be a global position of an object in the global coordinate frame $\Sigma$, and let $p^{[i]} \in \mathbb{R}^d$ be the corresponding local position in $\Sigma_i$. Then, the following relation between the positions holds for $(M_i, b_i) \in \mathcal{M} \times \mathcal{B}$:

$$p = M_i p^{[i]} + b_i. \tag{A.1}$$

Now, from the results in Sakurama, 2021b, we show how to derive transformation sets $\mathcal{M} \times \mathcal{B}$ shown in Subsection 2.3.1 from the measurement values of sensors in the case of $d = 2$-dimensional space.

**Example A.1.** Consider the situation that each robot is equipped with a camera or laser-range-finder to measure the distance and relative bearing of the object as shown in Fig. A.1. Let $x_i \in \mathbb{R}^2$ be the position of robot $i$, which faces to the direction of an angle $\theta_i \in [-\pi, \pi)$ from the $X$-axis of $\Sigma$. Assume that the distance $d_{pi} \geq 0$ and relative bearing $\phi_{pi} \in [-\pi, \pi)$ of the object from robot $i$ are

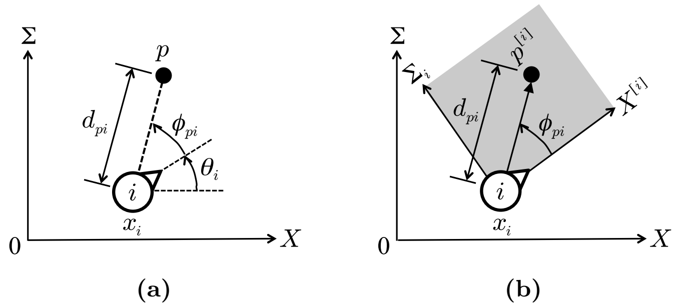



**Figure A.1:** Relation between the global and local coordinate frames, and their coordinates for $M_i \in \mathrm{SO}(2)$ and $b_i = x_i$: (a) the global coordinate $p$; (b) the corresponding local coordinate $p^{[i]}$.

measurable. From Fig. A.1a, the global position is expressed as

$$p = x_i + d_{pi} \begin{bmatrix} \cos(\theta_i + \phi_{pi}) \\ \sin(\theta_i + \phi_{pi}) \end{bmatrix}. \tag{A.2}$$

Assume that the $X^{[i]}$-axis of $\Sigma_i$ is set toward the face of robot $i$ as Fig. A.1b. Then, the local position is expressed as

$$p^{[i]} = d_{pi} \begin{bmatrix} \cos\phi_{pi} \\ \sin\phi_{pi} \end{bmatrix}, \tag{A.3}$$

which can be obtained from the measurable information. From (A.2) and (A.3), the relation between global and local coordinates (A.1) holds with $M_i = \mathrm{Rot}(\theta_i) \in \mathrm{SO}(2)$ and $b_i = x_i \in \mathbb{R}^2$. Hence, $\mathcal{M} \times \mathcal{B} = \mathrm{SO}(2) \times \mathbb{R}^2$ is obtained.

**Example A.2.** When each robot carries a compass in addition to the equipment in Example A.1, the absolute bearing is available. Then, the angle $\theta_i$ from the $X$-axis of $\Sigma$ in Fig. A.1a is available. By using $\theta_i$ in addition to $d_{pi}$ and $\phi_{pi}$, the local position $p^{[i]}$ can be defined as

$$p^{[i]} = d_{pi} \begin{bmatrix} \cos(\theta_i + \phi_{pi}) \\ \sin(\theta_i + \phi_{pi}) \end{bmatrix}. \tag{A.4}$$

This means that the $X^{[i]}$-axis of $\Sigma_i$ and the $X$-axis of $\Sigma$ are aligned. According to (A.2) and (A.4), (A.1) is achieved with $M_i = I_2$ and $b_i = x_i$. Hence, $\mathcal{M} \times \mathcal{B} = \{I_2\} \times \mathbb{R}^2$ is obtained.

**Example A.3.** Suppose that the measured distance from the object includes an unknown scale in Example A.1. Let $\hat{d}_{pi} \geq 0$ be the measured value, which satisfies $d_{pi} = s_i \hat{d}_{pi}$ for a scale factor $s_i > 0$ according to the actual distance $d_{pi} \geq 0$. Then, the local coordinate $p^{[i]}$ of the object is described as

$$p^{[i]} = \hat{d}_{pi} \begin{bmatrix} \cos \phi_{pi} \\ \sin \phi_{pi} \end{bmatrix} \tag{A.5}$$

instead of (A.3). From (A.2) and (A.5), (A.1) holds with $M_i = s_i \mathrm{Rot}(\theta_i) \in \mathrm{scaled}(\mathrm{SO}(2))$ and $b_i = x_i$. Thus, $\mathcal{M} \times \mathcal{B} = \mathrm{scaled}(\mathrm{SO}(2)) \times \mathbb{R}^2$ is achieved. The resultant local coordinate frame is illustrated in Fig. 2.10b.

**Example A.4.** Suppose that each robot detects a beacon on the object by two receivers as shown in Fig. A.2a. Without loss of generality, let $e_{21}$ and $-e_{21}$ be the local coordinates of the receivers, where $e_{21} = [1\ 0]^\top$. The distances $d_{pi1}, d_{pi2} \geq 0$ of the beacon from

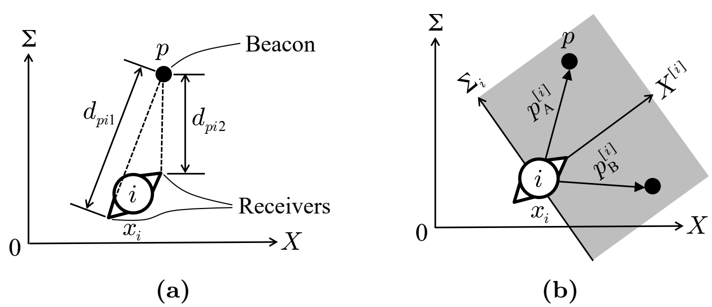



**Figure A.2:** Relation between the global and local coordinate frames, and their coordinates for $M_i \in \mathrm{O}(2)$ and $b_i = x_i$: (a) a global coordinate $p$; (b) the corresponding local coordinate $p^{[i]} = p_{\mathrm{A}}^{[i]}$ or $p_{\mathrm{B}}^{[i]}$.

the receivers are measured. Then, the local coordinate $p^{[i]}$ of the object satisfies

$$\|p^{[i]} - e_{21}^\top\| = d_{pi1}, \ \|p^{[i]} + e_{21}^\top\| = d_{pi2}. \tag{A.6}$$

Fig. A.2b indicates that from (A.6), $p_{\mathrm{A}}^{[i]}, p_{\mathrm{B}}^{[i]} \in \mathbb{R}^2$ can be the two possible coordinates for $p^{[i]}$, where

$$p_{\mathrm{A}}^{[i]} = d_{pi} \begin{bmatrix} \cos \hat{\phi}_{pi} \\ \sin \hat{\phi}_{pi} \end{bmatrix}, \ p_{\mathrm{B}}^{[i]} = d_{pi} \begin{bmatrix} \cos \hat{\phi}_{pi} \\ -\sin \hat{\phi}_{pi} \end{bmatrix} \tag{A.7}$$

with

$$d_{pi} = \sqrt{\frac{d_{pi1}^2 + d_{pi2}^2}{2} - 1}, \ \hat{\phi}_{pi} = \cos^{-1} \frac{d_{pi2}^2 - d_{pi1}^2}{4d_{pi}} \in [0, \pi].$$

Note that the relative bearing $\phi_{pi} \in (-\pi, \pi]$ of the object is equal to either $\hat{\phi}_{pi}$ or $-\hat{\phi}_{pi}$, and from (A.7), we obtain

$$p^{[i]} \in \{p_{\mathrm{A}}^{[i]}, p_{\mathrm{B}}^{[i]}\} = \left\{ d_{pi} \begin{bmatrix} \cos \phi_{pi} \\ \sin \phi_{pi} \end{bmatrix}, d_{pi} \mathrm{Refl}(w) \begin{bmatrix} \cos \phi_{pi} \\ \sin \phi_{pi} \end{bmatrix} \right\} \tag{A.8}$$

with the reflection matrix $\mathrm{Refl}(w) \in \mathbb{R}^{2\times 2}$ for $w = [0\ 1]^\top$. From (A.2) and (A.8), (A.1) is satisfied with $M_i = \mathrm{Rot}(\theta_i)$ or $\mathrm{Rot}(\theta_i)\mathrm{Refl}(w)$, and $b_i = x_i$. The duality of the possible positions is called a flip ambiguity, which can be matched to either $M_i = \mathrm{Rot}(\theta_i)$ or $\mathrm{Rot}(\theta_i)\mathrm{Refl}(w)$ for all the neighbors by using a landmark beacon. Then,

$$\begin{aligned} \mathcal{M} &= \{\mathrm{Rot}(\theta) : \theta \in [-\pi, \pi)\} \cup \{\mathrm{Rot}(\theta)\mathrm{Refl}(w) : \theta \in [-\pi, \pi)\} \\ &= \mathrm{O}(2) \end{aligned}$$

is obtained with $\mathcal{B} = \mathbb{R}^2$.

# B

## Real Analytic Functions

Properties of real analytic functions are used in Section 5.3 for proving the asymptotic stability of the gradient-flow system. Hence, some basics of real analytic functions are summarized here. The contents here are based on Krantz and Parks, 2002.

For an open subset $\Omega \subset \mathbb{R}^n$, a function $f : \Omega \to \mathbb{R}$ is said to be *real analytic* at $\bar{x} \in \Omega$ if there exists a power series of $x$ converging to $f(x)$ around $\bar{x}$, that is, there exist sequences $(c(k))_{k \in \mathbb{Z}_+}$ and $(p_i(k))_{k \in \mathbb{Z}_+}$ for $i \in \{1, 2, \ldots, n\}$ of elements $c(k) \in \mathbb{R}$ and $p_i(k) \in \mathbb{Z}_+$ in an open set containing $\bar{x}$ such that

$$f(x) = \sum_{k=0}^{\infty} c(k) \prod_{i=1}^{n} (x_i - \bar{x}_i)^{p_i(k)},$$

where $x = (x_1, x_2, \ldots, x_n)$ and $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$. A vector-valued function $f : \Omega \to \mathbb{R}^m$ is said to be *real analytic* at $\bar{x} \in \Omega$ if each component of $f(x)$ is real analytic at $\bar{x} \in \Omega$. A function $f : \Omega \to \mathbb{R}^m$ is said to be *real analytic* in a set $\bar{\Omega} \subset \Omega$ if $f(x)$ is real analytic at each $\bar{x} \in \bar{\Omega}$.

A set $\mathcal{D} \subset \mathbb{R}^m$ is called an *$n$-dimensional real analytic submanifold* if for each $y \in \mathcal{D}$, there exists an open set $\Omega \subset \mathbb{R}^n$ and a real analytic function $f : \Omega \to \mathbb{R}^m$ such that open subsets of $\Omega$ are mapped to relatively open subsets of $\mathcal{D}$, $y \in f(\Omega)$, and

$$\operatorname{rank}\frac{\partial f}{\partial x}(x) = n \quad \forall x \in \Omega.$$

This definition means that a real analytic submanifold is locally parameterized through a real analytic function as the following example.

**Example B.1.** The $(\mathrm{scaled}(\mathrm{SO}(d)) \ltimes \mathbb{R}^d)$-orbit of $x_{\mathcal{N}}^* \in (\mathbb{R}^d)^n$, i.e., $\mathrm{orb}_{\mathrm{scaled}(\mathrm{SO}(d))\ltimes\mathbb{R}^d}(x_{\mathcal{N}}^*)$, is a real analytic submanifold. Actually, if $x_{\mathcal{N}} \in (\mathbb{R}^d)^n$ belongs to the orbit, $x_{\mathcal{N}} = (M, b)\bullet x_{\mathcal{N}}^*$ holds for $(M, b) \in \mathrm{scaled}(\mathrm{SO}(d)) \ltimes \mathbb{R}^d$, where $M \in \mathrm{scaled}(\mathrm{SO}(d))$ is parametrized as $M = s\exp(N)$ with $s > 0$ and $N \in \mathrm{Skew}(d)$. Here, $\mathrm{Skew}(d)$ denotes the set of skew-symmetric matrices of dimension $d$, which is parameterized by the upper (or lower) off-diagonal entries, and $\exp(\cdot)$ represents the exponential matrix.

The squared distance function from a real analytic submanifold $\mathcal{D}$ is a real analytic function as follows.

**Lemma B.1.** [Theorem 6.5.23 in Krantz and Parks, 2002] Let $\mathcal{D}$ be a closed subset of $\mathbb{R}^n$. Then, the squared distance function $(\mathrm{dist}(x, \mathcal{D}))^2$ is a real analytic function of $x \in \mathbb{R}^n$ in an open set containing $\bar{x} \in \mathcal{D}$ if and only if $\mathcal{D}$ is a real analytic submanifold in an open set containing $\bar{x}$.

The following lemma provides important properties of the real analytic functions, called Łojasiewicz's inequalities.

**Lemma B.2.** [Theorem 6.3.4 in Krantz and Parks, 2002, Kurdyka, 1998] Let $f : \Omega \to \mathbb{R}$ be a real analytic function in an open set $\Omega \subset \mathbb{R}^n$. Assume that the zero set $f^{-1}(0)$ is non-empty in $\Omega$. Then, for a compact subset $\Omega_1$ of $\Omega$, there exist $\beta_1(\Omega_1) > 0$, $\theta_1(\Omega_1) > 0$ such that

$$|f(x)| \geq \beta_1(\Omega_1)(\mathrm{dist}(x, f^{-1}(0)))^{\theta_1(\Omega_1)} \quad \forall x \in \Omega_1. \tag{B.1}$$

For a bounded open set $\Omega_2 \subset \Omega$, there exist $\beta_2(\Omega_2) > 0$, $\theta_2(\Omega_2) > 0$,

$\rho_2(\Omega_2) > 0$ such that

$$|f(x)| \leq \beta_2(\Omega_2) \left\| \frac{\partial f}{\partial x}(x) \right\|^{\theta_2(\Omega_2)} \quad \forall x \in \mathrm{int}(\mathcal{L}^-_{\rho_2(\Omega_2)}(|f|)) \cap \Omega_2. \quad \text{(B.2)}$$

Here, $\mathcal{L}^-_\rho(|f|) = \{x \in \mathbb{R}^n : |f(x)| \leq \rho\}$ is the sublevel set of $|f(x)|$ for $\rho \geq 0$ and $\mathrm{int}(\cdot)$ represents the interior of a set.

# C

# Gradients of Squared Distance Functions

Consider the squared distance function

$$v(x_{\mathcal{N}}) = \frac{1}{2}(\mathrm{dist}(x_{\mathcal{N}}, \mathcal{D}))^2, \tag{C.1}$$

where $\mathcal{N} = \{1, 2, \ldots, n\}$ and the distance function of $x_{\mathcal{N}} \in (\mathbb{R}^d)^n$ from a set $\mathcal{D} \subset (\mathbb{R}^d)^n$ is defined as

$$\mathrm{dist}(x_{\mathcal{N}}, \mathcal{D}) = \inf_{D \in \mathcal{D}} \|x_{\mathcal{N}} - D\|. \tag{C.2}$$

The gradient of the function (C.1) is used in Theorems 7.5 and 8.7, which is derived as follows.

**Lemma C.1.** For a set $\mathcal{D} \subset (\mathbb{R}^d)^n$, if $v(x_{\mathcal{N}})$ in (C.1) is partially differentiable with respect to $x_i$ at $x_{\mathcal{N}} \in (\mathbb{R}^d)^n$, the following holds:

$$\frac{\partial v}{\partial x_i}(x_{\mathcal{N}}) = x_i - \mathrm{col}_i(\hat{D}(x_{\mathcal{N}})), \tag{C.3}$$

where $\hat{D} : (\mathbb{R}^d)^n \to \mathrm{cl}(\mathcal{D})$ is the solution $D$ to (C.2).

Here, $\mathrm{col}_i(\cdot)$ represents the $i$th element of a tuple and $\mathrm{cl}(\cdot)$ is the closure of a set.

In order to prove Lemma C.1, the directional derivative of a function $F : (\mathbb{R}^d)^n \to (\mathbb{R}^d)^n$ at $x_\mathcal{N} \in (\mathbb{R}^d)^n$ in the direction of $y_\mathcal{N} \in (\mathbb{R}^d)^n$ is defined as

$$\nabla_{y_\mathcal{N}} F(x_\mathcal{N}) = \lim_{h\to 0} \frac{1}{h}(F(x_\mathcal{N} + hy_\mathcal{N}) - F(x_\mathcal{N})). \tag{C.4}$$

Then, the following lemma is obtained.

**Lemma C.2.** For a set $\mathcal{D} \subset (\mathbb{R}^d)^n$, let $\hat{D} : (\mathbb{R}^d)^n \to \mathrm{cl}(\mathcal{D})$ be a solution to (C.2). If $\nabla_{y_\mathcal{N}} \hat{D}(x_\mathcal{N})$ exists for $x_\mathcal{N}, y_\mathcal{N} \in (\mathbb{R}^d)^n$, the following holds:

$$\langle \nabla_{y_\mathcal{N}} \hat{D}(x_\mathcal{N}), x_\mathcal{N} - \hat{D}(x_\mathcal{N}) \rangle = 0. \tag{C.5}$$

*Proof.* From (C.2),

$$\|x_\mathcal{N} - \hat{D}(x_\mathcal{N})\| \le \|x_\mathcal{N} - \tilde{D}\| \quad \forall \tilde{D} \in \mathrm{cl}(\mathcal{D})$$

holds, and by applying $\tilde{D} = \hat{D}(x_\mathcal{N} + hy_\mathcal{N})$, we obtain

$$\begin{aligned}
&\langle \hat{D}(x_\mathcal{N} + hy_\mathcal{N}) - \hat{D}(x_\mathcal{N}), x_\mathcal{N} - \hat{D}(x_\mathcal{N}) \rangle \\
&\quad = \langle \hat{D}(x_\mathcal{N} + hy_\mathcal{N}) - x_\mathcal{N}, x_\mathcal{N} - \hat{D}(x_\mathcal{N}) \rangle + \|x_\mathcal{N} - \hat{D}(x_\mathcal{N})\|^2 \\
&\quad \le \langle \hat{D}(x_\mathcal{N} + hy_\mathcal{N}) - x_\mathcal{N}, x_\mathcal{N} - \hat{D}(x_\mathcal{N}) \rangle \\
&\qquad + \frac{1}{2}\|x_\mathcal{N} - \hat{D}(x_\mathcal{N})\|^2 + \frac{1}{2}\|x_\mathcal{N} - \hat{D}(x_\mathcal{N} + hy_\mathcal{N})\|^2 \\
&\quad = \frac{1}{2}\|(x_\mathcal{N} - \hat{D}(x_\mathcal{N})) - (x_\mathcal{N} - \hat{D}(x_\mathcal{N} + hy_\mathcal{N}))\|^2 \\
&\quad = \frac{1}{2}\|\hat{D}(x_\mathcal{N} + hy_\mathcal{N}) - \hat{D}(x_\mathcal{N})\|^2.
\end{aligned} \tag{C.6}$$

From (C.6), for $h > 0$, we obtain

$$\begin{aligned}
&\frac{1}{h}\langle \hat{D}(x_\mathcal{N} + hy_\mathcal{N}) - \hat{D}(x_\mathcal{N}), x_\mathcal{N} - \hat{D}(x_\mathcal{N}) \rangle \\
&\qquad\qquad \le \frac{h}{2}\left\| \frac{1}{h}(\hat{D}(x_\mathcal{N} + hy_\mathcal{N}) - \hat{D}(x_\mathcal{N})) \right\|^2.
\end{aligned} \tag{C.7}$$

From (C.4), as $h \to 0+$, the left hand side of (C.7) converges as

$$\begin{aligned}
&\lim_{h\to 0+} \langle \frac{1}{h}(\hat{D}(x_\mathcal{N} + hy_\mathcal{N}) - \hat{D}(x_\mathcal{N})), x_\mathcal{N} - \hat{D}(x_\mathcal{N}) \rangle \\
&\qquad\qquad = \langle \nabla_{y_\mathcal{N}} \hat{D}(x_\mathcal{N}), x_\mathcal{N} - \hat{D}(x_\mathcal{N}) \rangle
\end{aligned}$$

while the right hand side of (C.7) converges to 0. Hence,

$$\langle \nabla_{y_{\mathcal{N}}} \hat{D}(x_{\mathcal{N}}), x_{\mathcal{N}} - \hat{D}(x_{\mathcal{N}}) \rangle \leq 0$$

holds. By considering $h \to 0-$, the converse inequality is obtained. Thus, (C.5) is achieved. □

*Proof of Lemma C.1.* $(\mathrm{dist}(x_{\mathcal{N}}, \mathcal{D}))^2 = \|x_{\mathcal{N}} - \hat{D}(x_{\mathcal{N}})\|^2$ holds because $\hat{D}(x_{\mathcal{N}})$ is the solution to (C.2). Partially differentiate this equation with respect to $x_{ji}$, $j$th component of $x_i$, and we obtain

$$\begin{aligned}
\frac{\partial(\mathrm{dist}(x_{\mathcal{N}}, \mathcal{D}))^2}{\partial x_{ji}} &= \frac{\partial}{\partial x_{ji}} \langle x_{\mathcal{N}} - \hat{D}(x_{\mathcal{N}}), x_{\mathcal{N}} - \hat{D}(x_{\mathcal{N}}) \rangle \\
&= 2 \left\langle e_{dj} e_{ni}^\top - \frac{\partial \hat{D}}{\partial x_{ji}}(x_{\mathcal{N}}), x_{\mathcal{N}} - \hat{D}(x_{\mathcal{N}}) \right\rangle \\
&= 2 \left\langle e_{dj} e_{ni}^\top - \nabla_{e_{dj} e_{ni}^\top} \hat{D}(x_{\mathcal{N}}), x_{\mathcal{N}} - \hat{D}(x_{\mathcal{N}}) \right\rangle \\
&= 2 \langle e_{dj} e_{ni}^\top, x_{\mathcal{N}} - \hat{D}(x_{\mathcal{N}}) \rangle = 2\mathrm{tr}(e_{ni} e_{dj}^\top (x_{\mathcal{N}} - \hat{D}(x_{\mathcal{N}}))) \\
&= 2 e_{dj}^\top (x_{\mathcal{N}} - \hat{D}(x_{\mathcal{N}})) e_{ni} = 2 e_{dj}^\top (x_i - \mathrm{col}_i(\hat{D}(x_{\mathcal{N}})))
\end{aligned} \tag{C.8}$$

from (C.4) and Lemma C.2 for $y_{\mathcal{N}} = e_{dj} e_{ni}^\top$, where $e_{ni} \in \mathbb{R}^n$ is the $i$th unit vector and $n$-tuples in $(\mathbb{R}^d)^n$ are regarded as matrices in $\mathbb{R}^{d \times n}$. By collecting (C.8) for all $j \in \{1, 2, \ldots, d\}$, the equation (C.3) is achieved. □

# D

# Partial Difference

A mathematical operation, called partial difference, is introduced and its properties will be summarized. This operation plays an essential role in proving Theorem 7.1. Some contents here are based on Sakurama *et al.*, 2015.

## D.1 Partial difference and high-order partial difference

Consider a function $w : (\mathbb{R}^d)^n \to \mathbb{R}^m$ of $x_\mathcal{N}$, where $\mathcal{N} = \{1, 2, \ldots, n\}$. The *partial difference* of $w(x_\mathcal{N})$ with respect to $x_i$ for a vector $a \in \mathbb{R}^d$ is defined with the operator $\triangle_a^{x_i}$ as follows:

$$\triangle_a^{x_i} w(x_\mathcal{N}) := w(x_\mathcal{N}) - w(x_\mathcal{N})|_{x_i=a}, \tag{D.1}$$

where the notation $\cdot|_{x_i=a}$ represents the replacement of $x_i$ by $a$, i.e., $w(x_\mathcal{N})|_{x_i=a} = w(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n)$.

The partial difference is commutative as follows.

**Lemma D.1.** The following holds.

$$\triangle_{a_i}^{x_i} \triangle_{a_j}^{x_j} w(x_\mathcal{N}) = \triangle_{a_j}^{x_j} \triangle_{a_i}^{x_i} w(x_\mathcal{N}) \quad \forall x_\mathcal{N} \in (\mathbb{R}^d)^n, a_i, a_j \in \mathbb{R}^d \tag{D.2}$$

*Proof.* From (D.1), the following equations hold, and (D.2) is derived.

$$\begin{aligned}
&\triangle_{a_i}^{x_i}\triangle_{a_j}^{x_j} w(x_{\mathcal{N}}) \\
&\quad = \triangle_{a_i}^{x_i}(w(x_{\mathcal{N}}) - w(x_{\mathcal{N}})|_{x_j=a_j}) \\
&\quad = (w(x_{\mathcal{N}}) - w(x_{\mathcal{N}})|_{x_j=a_j}) - (w(x_{\mathcal{N}})|_{x_i=a_i} - w(x_{\mathcal{N}})|_{x_j=a_j,x_i=a_i}) \\
&\quad = (w(x_{\mathcal{N}}) - w(x_{\mathcal{N}})|_{x_i=a_i}) - (w(x_{\mathcal{N}})|_{x_j=a_j} - w(x_{\mathcal{N}})|_{x_i=a_i,x_j=a_j}) \\
&\quad = \triangle_{a_j}^{x_j}(w(x_{\mathcal{N}}) - w(x_{\mathcal{N}})|_{x_i=a_i}) = \triangle_{a_j}^{x_j}\triangle_{a_i}^{x_i} w(x_{\mathcal{N}})
\end{aligned}$$

□

From the commutativity, we can properly define the high-order partial difference. For $\mathcal{I} = \{i_1, i_2, \ldots, i_{|\mathcal{I}|}\} \subset \mathcal{N}$ and $a_{\mathcal{I}} \in (\mathbb{R}^d)^{|\mathcal{I}|}$, the partial difference of $w(x_{\mathcal{N}})$ with respect to $x_{\mathcal{I}}$ is defined as

$$\triangle_{a_{\mathcal{I}}}^{x_{\mathcal{I}}} w(x_{\mathcal{N}}) := \triangle_{a_{i_1}}^{x_{i_1}}\triangle_{a_{i_2}}^{x_{i_2}} \cdots \triangle_{a_{i_{|\mathcal{I}|}}}^{x_{i_{|\mathcal{I}|}}} w(x_{\mathcal{N}}). \tag{D.3}$$

## D.2 Verification of dependency of functions

The dependency of a function from a specific variable can be verified through the partial difference with respect to the variable. A function $w : (\mathbb{R}^d)^n \to \mathbb{R}^m$ of $x_{\mathcal{N}}$ is said to be *independent from* $x_i$ if there exists a function $\hat{w} : (\mathbb{R}^d)^{n-1} \to \mathbb{R}^m$ such that

$$w(x_{\mathcal{N}}) = \hat{w}(x_{\mathcal{N}\setminus\{i\}}) \;\; \forall x_{\mathcal{N}} \in (\mathbb{R}^d)^n. \tag{D.4}$$

The following lemma is obtained.

**Lemma D.2.** A function $w : (\mathbb{R}^d)^n \to \mathbb{R}^m$ of $x_{\mathcal{N}}$ satisfies

$$\triangle_a^{x_i} w(x_{\mathcal{N}}) = 0 \;\; \forall x_{\mathcal{N}} \in (\mathbb{R}^d)^n, a \in \mathbb{R}^d \tag{D.5}$$

if and only if it is independent from $x_i$.

*Proof.* Assume that (D.5) holds. Then, $w(x_{\mathcal{N}}) = w(x_{\mathcal{N}})|_{x_i=a}$ is obtained from (D.1), and (D.4) holds with $\hat{w}(x_{\mathcal{N}\setminus\{i\}}) = w(x_{\mathcal{N}})|_{x_i=a}$ for any $a \in \mathbb{R}^d$. Hence, $w(x_{\mathcal{N}})$ is independent from $x_i$.

Conversely, assume that $w(x_{\mathcal{N}})$ is independent from $x_i$. Then, from (D.4),

$$w(x_{\mathcal{N}})|_{x_i=a} = \hat{w}(x_{\mathcal{N}\setminus\{i\}})|_{x_i=a} = \hat{w}(x_{\mathcal{N}\setminus\{i\}}) \tag{D.6}$$

is obtained for any $x_{\mathcal{N}} \in (\mathbb{R}^d)^n$ and $a \in \mathbb{R}^d$. From (D.1), (D.4), and (D.6),

$$\triangle_a^{x_i} w(x_{\mathcal{N}}) = w(x_{\mathcal{N}}) - w(x_{\mathcal{N}})|_{x_i=a} = \hat{w}(x_{\mathcal{N}\setminus\{i\}}) - \hat{w}(x_{\mathcal{N}\setminus\{i\}}) = 0$$

holds, and (D.5) is derived. □

The high-order version of Lemma D.2 is given as follows.

**Lemma D.3.** For a sets $\mathcal{I} \subset \mathcal{N}$, a function $w : (\mathbb{R}^d)^n \to \mathbb{R}^m$ of $x_{\mathcal{N}}$ satisfies

$$\triangle_{a_{\mathcal{I}}}^{x_{\mathcal{I}}} w(x_{\mathcal{N}}) = 0 \quad \forall x_{\mathcal{N}} \in (\mathbb{R}^d)^n, a_{\mathcal{I}} \in (\mathbb{R}^d)^{|\mathcal{I}|} \tag{D.7}$$

if it is independent from $x_i$ for some $i \in \mathcal{I}$.

*Proof.* Under the assumption, from Lemma D.2, (D.2), and (D.3),

$$\triangle_{a_{\mathcal{I}}}^{x_{\mathcal{I}}} w(x_{\mathcal{N}}) = \triangle_{a_{\mathcal{I}\setminus\{i\}}}^{x_{\mathcal{I}\setminus\{i\}}} \triangle_{a_i}^{x_i} w(x_{\mathcal{I}}) = 0$$

is obtained for any $x_{\mathcal{N}} \in (\mathbb{R}^d)^n, a_{\mathcal{I}} \in (\mathbb{R}^d)^{|\mathcal{I}|}$, and (D.7) is derived. □

Only a sufficient condition of (D.7) is provided by Lemma D.3. The necessary and sufficient condition is given as follows.

**Lemma D.4.** For a set $\mathcal{I} = \{i_1, i_2, \ldots, i_{|\mathcal{I}|}\} \subset \mathcal{N}$, a function $w : (\mathbb{R}^d)^n \to \mathbb{R}^m$ of $x_{\mathcal{N}}$ satisfies (D.7) if and only if it can be of the form

$$w(x_{\mathcal{N}}) = \sum_{k=1}^{|\mathcal{I}|} w_k(x_{\mathcal{N}\setminus\{i_k\}}) \tag{D.8}$$

with functions $w_k : (\mathbb{R}^d)^{n-1} \to \mathbb{R}^m$ for $k \in \{1, 2, \ldots, |\mathcal{I}|\}$. If $w(x_{\mathcal{N}})$ is scalar and non-negative, each $w_k(x_{\mathcal{N}\setminus\{i_k\}})$ can be chosen as a scalar, non-negative function. One of such $w_k(x_{\mathcal{N}\setminus\{i_k\}})$ is given as

$$w_k(x_{\mathcal{N}\setminus\{i_k\}}) = \inf_{x_{i_k} \in \mathbb{R}^d} w(x_{\mathcal{N}}). \tag{D.9}$$

*Proof.* (Sufficiency) Assume that $w(x_{\mathcal{N}})$ is of the form (D.8). Each $w_k(x_{\mathcal{N}\setminus\{i_k\}})$ in (D.8) is independent from $x_{i_k}$ for $i_k \in \mathcal{I}$, and from Lemma D.3, $\triangle_{a_{\mathcal{I}}}^{x_{\mathcal{I}}} w_k(x_{\mathcal{N}\setminus\{i_k\}}) = 0$ holds. Thus, (D.7) holds.

(Necessity) We use the mathematical induction with respect to $|\mathcal{I}|$. The case of $|\mathcal{I}| = 1$ follows from Lemma D.2. Next, assume that this lemma holds for $|\mathcal{I}| = \ell - 1$, and we consider the case of $|\mathcal{I}| = \ell$. Assume that (D.7) is satisfied. From (D.2), (D.3), and (D.7),

$$\triangle_{a_{\mathcal{I}\setminus\{i_\ell\}}}^{x_{\mathcal{I}\setminus\{i_\ell\}}} \triangle_{a_{i_\ell}}^{x_{i_\ell}} w(x_{\mathcal{N}}) = \triangle_{a_{\mathcal{I}}}^{x_{\mathcal{I}}} w(x_{\mathcal{N}}) = 0 \tag{D.10}$$

is obtained. Note that $|\mathcal{I}\setminus\{i_\ell\}| = \ell - 1$, and under the assumption of the mathematical induction, from (D.10),

$$\triangle_{a_{i_\ell}}^{x_{i_\ell}} w(x_{\mathcal{N}}) = \sum_{k=1}^{\ell-1} w_k(x_{\mathcal{N}\setminus\{i_k\}}) \tag{D.11}$$

holds with some functions $w_k : (\mathbb{R}^d)^{n-1} \to \mathbb{R}$ for $k \in \{1, 2, \ldots, \ell - 1\}$. Then, from (D.1) and (D.11),

$$w(x_{\mathcal{N}}) = w(x_{\mathcal{N}})|_{x_{i_\ell} = a_{i_\ell}} + \sum_{k=1}^{\ell-1} w_k(x_{\mathcal{N}\setminus\{i_k\}}) \tag{D.12}$$

holds, and (D.8) is obtained for $|\mathcal{I}| = \ell$ with

$$w_\ell(x_{\mathcal{N}\setminus\{i_\ell\}}) = w(x_{\mathcal{N}})|_{x_{i_\ell} = a_{i_\ell}}. \tag{D.13}$$

To show the latter part, we assume that $w(x_{\mathcal{N}})$ is scalar and non-negative. We continue the mathematical induction. First, we consider the case that $w(x_{\mathcal{N}})$ has a minimum point with respect to $x_{i_\ell}$, i.e., $a_{i_\ell} \in \operatorname{argmin}_{x_{i_\ell} \in \mathbb{R}^d} w(x_{\mathcal{N}})$ exists, which is independent from $x_{i_\ell}$. Then, $w_\ell(x_{\mathcal{N}\setminus\{i_\ell\}})$ in (D.13) is reduced to (D.9) for $k = \ell$. Note that this $w_\ell(x_{\mathcal{N}\setminus\{i_\ell\}})$ is non-negative from the non-negativeness of $w(x_{\mathcal{N}})$. Moreover,

$$\triangle_{a_{i_\ell}}^{x_{i_\ell}} w(x_{\mathcal{N}}) = w(x_{\mathcal{N}}) - w(x_{\mathcal{N}})|_{x_{i_\ell} = a_{i_\ell}} = w(x_{\mathcal{N}}) - \min_{x_{i_\ell} \in \mathbb{R}^d} w(x_{\mathcal{N}}) \geq 0$$

holds. From the assumption of the mathematical induction, the non-negative function $\triangle_{a_{i_\ell}}^{x_{i_\ell}} w(x_{\mathcal{N}})$ can be of the form (D.11) with non-negative functions $w_k(x_{\mathcal{N}\setminus\{i_k\}})$ for any $k \in \{1, \ldots, \ell - 1\}$.

In the case that $w(x_{\mathcal{N}})$ does not have a minimum point with respect to $x_{i_\ell}$, the infimum exists because of the non-negativeness of $w(x_{\mathcal{N}})$. Then, there exists a sequence $(a^\iota_{i_\ell})_{\iota\in\{1,2,\ldots\}}, a^\iota_{i_\ell} \in \mathbb{R}^d$ such that

$$\inf_{x_{i_\ell}\in\mathbb{R}^d} w(x_{\mathcal{N}}) = \lim_{\iota\to\infty} w(x_{\mathcal{N}})|_{x_{i_\ell}=a^\iota_{i_\ell}}.$$

The above discussion holds for $a^\iota_{i_\ell}$, which yields the non-negativeness of $w_k(x_{\mathcal{N}\setminus\{i_k\}})$ in (D.9). □

Some properties (the zero set and relative invariance) of $w(x_{\mathcal{N}})$ are inherited by $w_k(x_{\mathcal{N}\setminus\{i_k\}})$ in (D.9) as follows.

**Lemma D.5.** For a scalar and non-negative function $w : (\mathbb{R}^d)^n \to \mathbb{R}_+$ of $x_{\mathcal{N}}$, let $w_k(x_{\mathcal{N}\setminus\{i_k\}})$ be the function in (D.9). (a) If $w^{-1}(0)$ is non-empty, $w_k(x_{\mathcal{N}\setminus\{i_k\}})$ is an indicator of $\mathrm{proj}_{\mathcal{N}\setminus\{i_k\}}(w^{-1}(0))$. (b) For a subgroup $\mathcal{M} \ltimes \mathcal{B}$ of $\mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$, if $w(x_{\mathcal{N}})$ is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{d}{2}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$, $w_k(x_{\mathcal{N}\setminus\{i_k\}})$ is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{d}{2}}$.

*Proof.* (a) Assume that $w^{-1}(0)$ is non-empty. From the definition of the projection, for $\hat{x}_{\mathcal{N}\setminus\{i_k\}} \in (\mathbb{R}^d)^{n-1}$, there exists $\hat{x}_{i_k} \in \mathbb{R}^d$ such that $w(\hat{x}_{\mathcal{N}}) = 0$ if and only if $\hat{x}_{\mathcal{N}\setminus\{i_k\}} \in \mathrm{proj}_{\mathcal{N}\setminus\{i_k\}}(w^{-1}(0))$. Hence, if $\hat{x}_{\mathcal{N}\setminus\{i_k\}} \in \mathrm{proj}_{\mathcal{N}\setminus\{i_k\}}(w^{-1}(0))$, from (D.9) and the non-negativeness of $w(x_{\mathcal{N}})$,

$$w_k(\hat{x}_{\mathcal{N}\setminus\{i_k\}}) = \inf_{x_{i_k}\in\mathbb{R}^d} w(x_{\mathcal{N}})|_{x_{\mathcal{N}\setminus\{i_k\}}=\hat{x}_{\mathcal{N}\setminus\{i_k\}}} = w(\hat{x}_{\mathcal{N}}) = 0$$

holds. Otherwise, $w_k(\hat{x}_{\mathcal{N}\setminus\{i_k\}}) > 0$ holds. Therefore, $w_k(x_{\mathcal{N}\setminus\{i_k\}})$ is an indicator of $\mathrm{proj}_{\mathcal{N}\setminus\{i_k\}}(w^{-1}(0))$.

(b) Assume that $w(x_{\mathcal{N}})$ is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{d}{2}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$. From the assumption and the nonsingularity of $M$, $w_k(x_{\mathcal{N}\setminus\{i_k\}})$ in (D.9) satisfies

$$\begin{aligned} w_k((M,b)\bullet x_{\mathcal{N}\setminus\{i_k\}}) &= \inf_{y_{i_k}\in\mathbb{R}^d} w(y_{\mathcal{N}})|_{y_{\mathcal{N}\setminus\{i_k\}}=(M,b)\bullet x_{\mathcal{N}\setminus\{i_k\}}} \\ &= \inf_{y_{i_k}\in\mathbb{R}^d} w((M,b)\bullet x_{\mathcal{N}})|_{x_{i_k}=(M,b)^{-1}\bullet y_{i_k}} \\ &= |\det(M)|^{\frac{2}{d}} \inf_{y_{i_k}\in\mathbb{R}^d} w(x_{\mathcal{N}})|_{x_{i_k}=(M,b)^{-1}\bullet y_{i_k}} \\ &= |\det(M)|^{\frac{2}{d}} \inf_{\hat{y}_{i_k}\in\mathbb{R}^d} w(x_{\mathcal{N}})|_{x_{i_k}=\hat{y}_{i_k}} \\ &= |\det(M)|^{\frac{2}{d}} w_k(x_{\mathcal{N}\setminus\{i_k\}}), \end{aligned}$$

where $\hat{y}_{i_k} = (M,b)^{-1}\bullet y_{i_k}$. □

## D.3 Relations to integrals and partial derivatives

Consider a scalar, continuously differentiable function $w(x_{\mathcal{N}})$. Its partial difference in (D.1) can be described in the integral form as

$$\triangle_a^{x_i} w(x_{\mathcal{N}}) = \int_a^{x_i} \frac{\partial w}{\partial x_i}(x_{\mathcal{N}})\mathrm{d}x_i \tag{D.14}$$

from the gradient theorem. Moreover, the partial derivative is described by the partial difference in the following way:

$$\left[\frac{\partial w}{\partial x_i}(x_{\mathcal{N}})\right]_p = \lim_{h\to 0} \frac{\triangle_{x_i-he_{dp}}^{x_i} w(x_{\mathcal{N}})}{h}, \tag{D.15}$$

where $[\cdot]_p$ represents the $p$th component of a vector and $e_{dp}\in\mathbb{R}^d$ is the $p$th unit vector.

Partial derivative and partial difference are commutative as follows.

**Lemma D.6.** For a continuously differentiable function $w:(\mathbb{R}^d)^n\to\mathbb{R}$ of $x_{\mathcal{N}}$, the following holds:

$$\triangle_a^{x_j}\frac{\partial w}{\partial x_i}(x_{\mathcal{N}}) = \frac{\partial}{\partial x_i}\triangle_a^{x_j} w(x_{\mathcal{N}}) \quad \forall x_{\mathcal{N}}\in(\mathbb{R}^d)^n, a\in\mathbb{R}^d. \tag{D.16}$$

*Proof.* From (D.2) and (D.15), the $p$th component of the left-hand side of (D.16) is calculated as

$$\begin{aligned}
\left[\triangle_a^{x_j}\frac{\partial w}{\partial x_i}(x_{\mathcal{N}})\right]_p &= \triangle_a^{x_j}\left[\frac{\partial w}{\partial x_i}(x_{\mathcal{N}})\right]_p \\
&= \triangle_a^{x_j}\lim_{h\to 0}\frac{\triangle_{x_i-he_{dp}}^{x_i}w(x_{\mathcal{N}})}{h} \\
&= \lim_{h\to 0}\triangle_a^{x_j}\triangle_{x_i-he_{dp}}^{x_i}\frac{w(x_{\mathcal{N}})}{h} \\
&= \lim_{h\to 0}\triangle_{x_i-he_{dp}}^{x_i}\triangle_a^{x_j}\frac{w(x_{\mathcal{N}})}{h} \\
&= \lim_{h\to 0}\frac{\triangle_{x_i-he_{dp}}^{x_i}(\triangle_a^{x_j}w)(x_{\mathcal{N}})}{h} \\
&= \left[\frac{\partial(\triangle_a^{x_j}w)}{\partial x_i}(x_{\mathcal{N}})\right]_p,
\end{aligned}$$

and the right-hand side of (D.16) is obtained. □

## D.4 Decomposition of functions

Lemma D.4 shows that any function can be decomposed as (D.8) as long as it satisfies (D.7). Repeating this process, we can obtain a decomposition form consisting of functions which do not satisfy (D.7). This decomposition form is used to derive the form (7.12) of functions belonging to $\mathcal{V}_{\mathrm{dis}}(G)$ in Theorem 7.1.

The following lemma provides the decomposition form.

**Lemma D.7.** A function $w : (\mathbb{R}^d)^n \to \mathbb{R}^m$ of $x_{\mathcal{N}}$ can be of the form

$$w(x_{\mathcal{N}}) = \sum_{k=1}^{h} w_k(x_{\mathcal{I}_k}) \tag{D.17}$$

with some functions $w_k : (\mathbb{R}^d)^{|\mathcal{I}_k|} \to \mathbb{R}^m$ and sets $\mathcal{I}_k \subset \mathcal{N}$ for $k \in \{1, 2, \ldots, h\}$ satisfying

$$\triangle_{a_{\mathcal{I}_k}}^{x_{\mathcal{I}_k}} w_k(x_{\mathcal{I}_k}) \neq 0 \;\; \exists x_{\mathcal{I}_k}, a_{\mathcal{I}_k} \in (\mathbb{R}^d)^{|\mathcal{I}_k|} \tag{D.18}$$

$$\mathcal{I}_k \not\subset \mathcal{I}_\ell \;\; \forall k, \ell \in \{1, 2, \ldots, h\}, k \neq \ell. \tag{D.19}$$

(a) If $w(x_\mathcal{N})$ is scalar, non-negative, and $w^{-1}(0)$ is non-empty, then $w_k(x_{\mathcal{I}_k})$ can be chosen as an indicator of $\mathrm{proj}_{\mathcal{I}_k}(w^{-1}(0))$. (b) For a subgroup $\mathcal{M} \ltimes \mathcal{B}$ of $\mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$, if $w(x_\mathcal{N})$ is scalar, non-negative, and relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{d}{2}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$, then $w_k(x_{\mathcal{I}_k})$ can be chosen as a relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant function of weight $|\det(M)|^{\frac{d}{2}}$.

*Proof.* Repeat the decomposition operation to each term in $w(x_\mathcal{N})$ according to Lemma D.4, and we obtain the form

$$w(x_\mathcal{N}) = \sum_{k=1}^{\hat{h}} \hat{w}_k(x_{\mathcal{I}_k}) \tag{D.20}$$

with some functions $\hat{w}_k : (\mathbb{R}^d)^{|\mathcal{I}_k|} \to \mathbb{R}^m$ and distinct sets $\mathcal{I}_k \subset \mathcal{N}$ for $k \in \{1, 2, \ldots, \hat{h}\}$ satisfying

$$\triangle_{a_{\mathcal{I}_k}}^{x_{\mathcal{I}_k}} \hat{w}_k(x_{\mathcal{I}_k}) \neq 0 \quad \exists x_{\mathcal{I}_k}, a_{\mathcal{I}_k} \in (\mathbb{R}^d)^{|\mathcal{I}_k|}. \tag{D.21}$$

This form is necessarily achieved with finite operations from Lemma D.4. Let $h \leq \hat{h}$ be the maximum number of the sets $\mathcal{I}_k$ which do not contain each other. Then, without loss of generality, we can assume that the sets $\mathcal{I}_\ell$ for $\ell \in \{1, 2, \ldots, h\}$ satisfy (D.19), and there exist $\varphi_\ell \in \{h+1, \ldots, \hat{h}\}$ such that $\varphi_1 = h+1$, $\varphi_{h+1} = \hat{h}+1$, $\varphi_\ell \leq \varphi_{\ell+1}$, and

$$\mathcal{I}_{\varphi_\ell}, \mathcal{I}_{\varphi_\ell+1}, \ldots, \mathcal{I}_{\varphi_{\ell+1}-1} \subsetneq \mathcal{I}_\ell. \tag{D.22}$$

We define

$$w_\ell(x_{\mathcal{I}_\ell}) = \hat{w}_\ell(x_{\mathcal{I}_\ell}) + \sum_{k=\varphi_\ell}^{\varphi_{\ell+1}-1} \hat{w}_k(x_{\mathcal{I}_k}), \tag{D.23}$$

with which (D.20) is reduced to (D.17). Furthermore, (D.19) holds from the choice of the sets $\mathcal{I}_\ell$ for $\ell \in \{1, 2, \ldots, h\}$, and (D.18) holds as follows:

$$\begin{aligned}\triangle_{a_{\mathcal{I}_\ell}}^{x_{\mathcal{I}_\ell}} w_\ell(x_{\mathcal{I}_\ell}) &= \triangle_{a_{\mathcal{I}_\ell}}^{x_{\mathcal{I}_\ell}} \hat{w}_\ell(x_{\mathcal{I}_\ell}) + \sum_{k=\varphi_\ell}^{\varphi_{\ell+1}-1} \triangle_{a_{\mathcal{I}_\ell}}^{x_{\mathcal{I}_\ell}} \hat{w}_k(x_{\mathcal{I}_k}) \\ &= \triangle_{a_{\mathcal{I}_\ell}}^{x_{\mathcal{I}_\ell}} \hat{w}_\ell(x_{\mathcal{I}_\ell}) \neq 0 \quad \exists x_{\mathcal{I}_\ell}, a_{\mathcal{I}_\ell} \in (\mathbb{R}^d)^{|\mathcal{I}_\ell|}\end{aligned}$$

from (D.23), Lemma D.3 and (D.22), and (D.21) in order.

(a) Assume that $w(x_{\mathcal{N}})$ is scalar, non-negative, and $w^{-1}(0)$ is non-empty. Then, from Lemma D.4 and Lemma D.5 (a), each $\hat{w}_k(x_{\mathcal{I}_k})$ in (D.20) can be chosen as an indicator of $\mathrm{proj}_{\mathcal{I}_k}(w^{-1}(0))$. From (D.22),

$$\{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{I}_\ell} \in \mathrm{proj}_{\mathcal{I}_\ell}(w^{-1}(0))\} \\ \subset \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{I}_k} \in \mathrm{proj}_{\mathcal{I}_k}(w^{-1}(0))\} \tag{D.24}$$

holds for any $k \in \{\varphi_\ell, \varphi_\ell + 1, \ldots, \varphi_{\ell+1} - 1\}$. From the non-negativeness of $\hat{w}_k(x_{\mathcal{I}_k})$, (D.23), and (D.24),

$$\begin{aligned}
&\{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{I}_\ell} \in w_\ell^{-1}(0)\} \\
&\quad = \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{I}_\ell} \in \hat{w}_\ell^{-1}(0)\} \\
&\qquad \cap \bigcap_{k=\varphi_\ell}^{\varphi_{\ell+1}-1} \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{I}_k} \in \hat{w}_k^{-1}(0)\} \\
&\quad = \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{I}_\ell} \in \mathrm{proj}_{\mathcal{I}_\ell}(w^{-1}(0))\} \\
&\qquad \cap \bigcap_{k=\varphi_\ell}^{\varphi_{\ell+1}-1} \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{I}_k} \in \mathrm{proj}_{\mathcal{I}_k}(w^{-1}(0))\} \\
&\quad = \{x_{\mathcal{N}} \in (\mathbb{R}^d)^n : x_{\mathcal{I}_\ell} \in \mathrm{proj}_{\mathcal{I}_\ell}(w^{-1}(0))\}
\end{aligned}$$

holds, which shows that $w_\ell(x_{\mathcal{I}_\ell})$ is an indicator of $\mathrm{proj}_{\mathcal{I}_\ell}(w^{-1}(0))$.

(b) For a subgroup $\mathcal{M} \ltimes \mathcal{B}$ of $\mathrm{scaled}(\mathrm{O}(d)) \ltimes \mathbb{R}^d$, assume that $w(x_{\mathcal{N}})$ is scalar, non-negative, and relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{d}{2}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$. Then, from Lemma D.4 and Lemma D.5 (b), each $\hat{w}_k(x_{\mathcal{I}_k})$ in (D.20) can be chosen as a relatively invariant function of weight $|\det(M)|^{\frac{d}{2}}$. Then, because $w_\ell(x_{\mathcal{I}_\ell})$ in (D.23) is the sum of these functions, it is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{d}{2}}$. □

We characterize the functions belonging to $\mathcal{V}_{\mathrm{dis}}(G)$ by using partial difference in the following two lemmas.

**Lemma D.8.** For a graph $G = (\mathcal{N}, \mathcal{E})$, a continuously differntiable function $v : (\mathbb{R}^d)^n \to \mathbb{R}$ belongs to $\mathcal{V}_{\mathrm{dis}}(G)$ if and only if

$$\triangle_{a_i}^{x_i} \triangle_{a_j}^{x_j} v(x_{\mathcal{N}}) = 0 \quad \forall x_{\mathcal{N}} \in (\mathbb{R}^d)^n, a_i, a_j \in \mathbb{R}^d \tag{D.25}$$

holds for any $i, j \in \mathcal{N}$ such that $\{i, j\} \notin \mathcal{E}$, $i \neq j$.

*Proof.* From (D.14) and (D.16), we obtain

$$\begin{aligned}\int_{a_i}^{x_i} \triangle_{a_j}^{x_j} \frac{\partial v}{\partial x_i}(x_\mathcal{N}) \mathrm{d}x_i &= \int_{a_i}^{x_i} \frac{\partial}{\partial x_i} \triangle_{a_j}^{x_j} v(x_\mathcal{N}) \mathrm{d}x_i \\ &= \triangle_{a_i}^{x_i} \triangle_{a_j}^{x_j} v(x_\mathcal{N}).\end{aligned} \tag{D.26}$$

(Necessity) Assume that $v(x_\mathcal{N}) \in \mathcal{V}_{\mathrm{dis}}(G)$. Consider a pair $i, j \in \mathcal{N}$ such that $\{i, j\} \notin \mathcal{E}$, $i \neq j$. Then, because $j \notin \mathcal{N}_i$, from (7.8) and Lemma D.2,

$$\triangle_{a_j}^{x_j} \frac{\partial v}{\partial x_i}(x_\mathcal{N}) = \triangle_{a_j}^{x_j} \tilde{c}_i(x_{\mathcal{N}_i}) = 0 \quad \forall x_\mathcal{N} \in (\mathbb{R}^d)^n, a_j \in \mathbb{R}^d \tag{D.27}$$

is obtained. From (D.26) and (D.27), we obtain (D.25).

(Sufficiency) Assume that (D.25) holds for any $i, j \in \mathcal{N}$ such that $\{i, j\} \notin \mathcal{E}$, $i \neq j$. Then, from (D.26),

$$\triangle_{a_j}^{x_j} \frac{\partial v}{\partial x_i}(x_\mathcal{N}) = 0 \quad \forall x_\mathcal{N} \in (\mathbb{R}^d)^n, a_j \in \mathbb{R}^d$$

holds, and thus from Lemma D.2, $\partial v / \partial x_i(x_\mathcal{N})$ is independent from $x_j$ for any $j \in \mathcal{N} \backslash \mathcal{N}_i$. Hence, according to (D.4), there exists a function $\tilde{c}_i(x_{\mathcal{N} \backslash (\mathcal{N} \backslash \mathcal{N}_i)}) = \tilde{c}_i(x_{\mathcal{N}_i})$ satisfying

$$\frac{\partial v}{\partial x_i}(x_\mathcal{N}) = \tilde{c}_i(x_{\mathcal{N}_i}) \quad \forall x_\mathcal{N} \in (\mathbb{R}^d)^n.$$

Now, (7.8) is achieved, and $v(x_\mathcal{N}) \in \mathcal{V}_{\mathrm{dis}}(G)$ holds. □

Theorem 7.1 follows from the following lemma.

**Lemma D.9.** Consider a continuously differentiable function $v : (\mathbb{R}^d)^n \to \mathbb{R}$ of $x_\mathcal{N}$ and a graph $G = (\mathcal{N}, \mathcal{E})$. Let $\mathcal{I}_k \subset \mathcal{N}$ for $k \in \{1, 2, \ldots, h\}$ be the sets given in Lemma D.7 for $w(x_\mathcal{N}) = v(x_\mathcal{N})$, and let $\mathcal{C}_\psi \subset \mathcal{N}$ for $\psi \in \mathrm{clq}(G)$ be the maximal cliques in $G$. Then, $v(x_\mathcal{N})$ belongs to $\mathcal{V}_{\mathrm{dis}}(G)$ if and only if there exists $\psi(k) \in \mathrm{clq}(G)$ such that $\mathcal{I}_k \subset \mathcal{C}_{\psi(k)}$ holds for each $k \in \{1, 2, \ldots, h\}$.

*Proof.* The sufficiency part is the same as Theorem 7.1, which is proved there. The necessity is shown. To show this lemma by contradiction, we assume that a function $v(x_{\mathcal{N}}) \in \mathcal{V}_{\mathrm{dis}}(G)$, but there exists $\ell \in \{1, 2, \ldots, h\}$ such that $\mathcal{I}_\ell \not\subset \mathcal{C}_\psi$ holds for any maximal cliques $\mathcal{C}_\psi$, $\psi \in \mathrm{clq}(G)$. Then, there exist $i, j \in \mathcal{I}_\ell$ such that $\{i, j\} \notin \mathcal{E}$. From (D.25) in Lemma D.8, (D.2), and (D.3),

$$\begin{aligned} \triangle_{a_{\mathcal{I}_\ell}}^{x_{\mathcal{I}_\ell}} v(x_{\mathcal{N}}) &= \triangle_{a_{\mathcal{I}_\ell \setminus \{i,j\}}}^{x_{\mathcal{I}_\ell \setminus \{i,j\}}} \triangle_{a_i}^{x_i} \triangle_{a_j}^{x_j} v(x_{\mathcal{N}}) \\ &= 0 \ \ \forall x_{\mathcal{N}} \in (\mathbb{R}^d)^n, a_{\mathcal{I}_\ell} \in (\mathbb{R}^d)^{|\mathcal{I}_\ell|} \end{aligned} \tag{D.28}$$

is obtained. On the other hand, from (D.19), $\mathcal{I}_\ell \backslash \mathcal{I}_k$ is non-empty for any $k \neq \ell$. Hence, from Lemma D.3,

$$\triangle_{a_{\mathcal{I}_\ell}}^{x_{\mathcal{I}_\ell}} w_k(x_{\mathcal{I}_k}) = 0 \ \ \forall x_{\mathcal{I}_k}, a_{\mathcal{I}_k} \in (\mathbb{R}^d)^{|\mathcal{I}_k|}$$

holds. From this equation, (D.17), and (D.18),

$$\begin{aligned} \triangle_{a_{\mathcal{I}_\ell}}^{x_{\mathcal{I}_\ell}} v(x_{\mathcal{N}}) &= \sum_{k=1}^{h} \triangle_{a_{\mathcal{I}_\ell}}^{x_{\mathcal{I}_\ell}} w_k(x_{\mathcal{I}_k}) = \triangle_{a_{\mathcal{I}_\ell}}^{x_{\mathcal{I}_\ell}} w_\ell(x_{\mathcal{I}_\ell}) \\ &\neq 0 \ \ \forall x_{\mathcal{N}} \in (\mathbb{R}^d)^n, a_{\mathcal{I}_\ell} \in (\mathbb{R}^d)^{|\mathcal{I}_\ell|} \end{aligned} \tag{D.29}$$

is derived. Consequently, (D.28) and (D.29) contradict. □

# E

## Procrustes Problems

For a Cartesian product $\mathcal{M} \times \mathcal{B}$ of a subset of $\mathbb{R}^{d\times d} \times \mathbb{R}^d$ and vectors $x_i, x_i^* \in \mathbb{R}^d$ for $i \in \mathcal{N} = \{1, 2, \ldots, n\}$, consider the optimization problem

$$\inf_{(M,b)\in\mathcal{M}\times\mathcal{B}} \sum_{i=1}^{n} \|x_i - (Mx_i^* + b)\|^2, \tag{E.1}$$

which is called a *Procrustes problem* (Gower and Dijksterhuis, 2004). This problem corresponds to (8.23) with a singleton $\mathcal{X}^* = \{x_{\mathcal{N}}^*\}$, which needs to be solved to use the controller designed in Theorem 8.7.

The Procrustes problem is associated with rotation fitting (Kanatani, 1994), which determines the orientation of a rigid body from feature points in pictures, as illustrated in Fig. E.1. Let $x_i, x_i^* \in \mathbb{R}^d$ be feature points in different pictures corresponding to each other in a rigid body, and their relation is expressed as

$$x_i = Mx_i^* + b \tag{E.2}$$

with some $(M, b) \in \mathcal{M} \times \mathcal{B}$. If there is no scale difference between the pictures, $\mathcal{M} \times \mathcal{B} = \mathrm{SO}(d) \times \mathbb{R}^d$ is employed; otherwise $\mathcal{M} \times \mathcal{B} = \mathrm{scaled}(\mathrm{SO}(d)) \times \mathbb{R}^d$ is employed. Note that the correspondence (E.2) between $x_i$ and $x_i^*$ does not accurately hold in general due to noise. In such a case, it is expected to find the element $(M, b) \in \mathcal{M} \times \mathcal{B}$ that

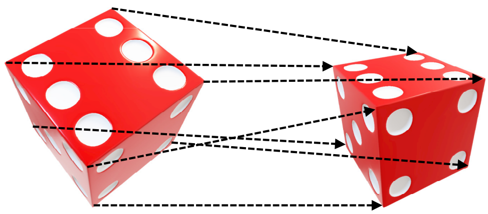


**Figure E.1:** Rotation fitting of feature points in different pictures.

minimizes the errors in terms of (E.2) for any feature points $i \in \mathcal{N}$. This problem is reduced to the Procrustes problem (E.1).

For typical $\mathcal{M} \times \mathcal{B}$ in Subsection 2.3.1, analytic solutions are obtained as follows. Here, the $n$-tuples $x_{\mathcal{N}}, x_{\mathcal{N}}^* \in (\mathbb{R}^d)^n$ are regarded as the corresponding matrices in $\mathbb{R}^{d\times n}$, $\mathrm{ave}(x_{\mathcal{N}}) := \sum_{i\in\mathcal{N}} x_i/n$ is the average of the elements of $x_{\mathcal{N}}$, and $\mathrm{cen}(x_{\mathcal{N}}) := x_{\mathcal{N}} - (\mathrm{ave}(x_{\mathcal{N}}), \ldots, \mathrm{ave}(x_{\mathcal{N}}))$ is the center of $x_{\mathcal{N}}$.

**Proposition E.1.** For $\mathcal{M} \times \mathcal{B} = \{I_d\} \times \mathbb{R}^d$, the solution to the Procrustes problem (E.1) with $x_{\mathcal{N}}, x_{\mathcal{N}}^* \in (\mathbb{R}^d)^n$ is given as

$$(M, b) = (I_d, \mathrm{ave}(x_{\mathcal{N}} - x_{\mathcal{N}}^*)). \tag{E.3}$$

*Proof.* For $(I_d, b) \in \{I_d\} \times \mathbb{R}^d$, the partial derivative of the function in (E.1) with respect to $b$ is calculated as

$$\frac{\partial}{\partial b} \sum_{i=1}^{n} \|x_i - (x_i^* + b)\|^2 = 2\sum_{i=1}^{n}(x_i^* + b - x_i) = 2(nb - \sum_{i=1}^{n}(x_i - x_i^*)),$$

which is zero when $b = \sum_{i=1}^{n}(x_i - x_i^*)/n$, and (E.3) is obtained. □

**Proposition E.2.** For $\mathcal{M} \times \mathcal{B} = \mathrm{scaled}(\{I_d\}) \times \{0\}$, the solution to the Procrustes problem (E.1) with $x_{\mathcal{N}}, x_{\mathcal{N}}^* \in (\mathbb{R}^d)^n$ is given as

$$(M, b) = (\chi(x_{\mathcal{N}}, x_{\mathcal{N}}^*) I_d, 0), \tag{E.4}$$

where the function $\chi : (\mathbb{R}^d)^n \times (\mathbb{R}^d)^n \to \mathbb{R}_+$ is defined as

$$\chi(x_{\mathcal{N}}, x^*_{\mathcal{N}}) := \max\{\frac{\langle x_{\mathcal{N}}, x^*_{\mathcal{N}}\rangle}{\|x^*_{\mathcal{N}}\|^2}, 0\}. \tag{E.5}$$

For $\mathcal{M} \times \mathcal{B} = \mathrm{scaled}(\{I_d\}) \times \mathbb{R}^d$, the solution is given as

$$(M, b) = (\chi(\mathrm{cen}(x_{\mathcal{N}}), \mathrm{cen}(x^*_{\mathcal{N}}))I_d, \mathrm{ave}(x_{\mathcal{N}} - Mx^*_{\mathcal{N}})). \tag{E.6}$$

*Proof.* For $(sI_d, 0) \in \mathrm{scaled}(\{I_d\}) \times \{0\}$ with $s > 0$, the partial derivative of the function in (E.1) with respect to $s$ is reduced to

$$\frac{\partial}{\partial s}\|x_{\mathcal{N}} - sx^*_{\mathcal{N}}\|^2 = 2\langle x^*_{\mathcal{N}}, sx^*_{\mathcal{N}} - x_{\mathcal{N}}\rangle = 2(s\|x^*_{\mathcal{N}}\|^2 - \langle x^*_{\mathcal{N}}, x_{\mathcal{N}}\rangle),$$

which is zero for $s = \langle x_{\mathcal{N}}, x^*_{\mathcal{N}}\rangle / \|x^*_{\mathcal{N}}\|^2$ if this $s$ is positive. Otherwise, $s = 0$ is the solution. Hence, (E.4) is obtained with (E.5).

Consider $(sI_d, b) \in \mathrm{scaled}(\{I_d\}) \times \mathbb{R}^d$. From Proposition E.1, the solution is given as $b = \mathrm{ave}(x_{\mathcal{N}} - sx^*_{\mathcal{N}})$. Then, from the first part of this proposition, (E.6) is derived. □

**Proposition E.3.** (Kanatani, 1994) For $\mathcal{M} \times \mathcal{B} = \mathrm{SO}(d) \times \{0\}$, the solution to the Procrustes problem (E.1) with $x_{\mathcal{N}}, x^*_{\mathcal{N}} \in (\mathbb{R}^d)^n$ is given as

$$(M, b) = (\Phi(x_{\mathcal{N}}, x^*_{\mathcal{N}}), 0). \tag{E.7}$$

Here, the matrix-valued function $\Phi : (\mathbb{R}^d)^n \times (\mathbb{R}^d)^n \to \mathrm{SO}(d)$ is defined as

$$\Phi(x_{\mathcal{N}}, x^*_{\mathcal{N}}) := VDU^\top, \tag{E.8}$$

where $U, V \in \mathrm{O}(d)$ are matrices satisfying

$$x^*_{\mathcal{N}} x^\top_{\mathcal{N}} = U\Sigma V^\top \tag{E.9}$$

with a diagonal matrix $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_d)$ whose entries $\sigma_1, \sigma_2, \ldots, \sigma_d$ satisfy $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_d \geq 0$, and

$$D = \mathrm{diag}(1, 1, \ldots, 1, \det(U)\det(V)) \in \mathbb{R}^{d\times d}. \tag{E.10}$$

For $\mathcal{M} \times \mathcal{B} = \mathrm{SO}(d) \times \mathbb{R}^d$, the solution is given as

$$(M, b) = (\Phi(\mathrm{cen}(x_\mathcal{N}), \mathrm{cen}(x^*_\mathcal{N})), \mathrm{ave}(x_\mathcal{N} - Mx^*_\mathcal{N})).$$

Note that the singular value decomposition (SVD) is performed in (E.9) to compare the $n$-tuples $(x_1, \dots, x_n)$ and $(x^*_1, \dots, x^*_n)$.

As a preliminary to proving Proposition E.3, the following lemma is given.

**Lemma E.1.** For $x_\mathcal{N}, x^*_\mathcal{N} \in (\mathbb{R}^d)^n$, let $\mathcal{W} \subset \mathbb{R}^{d\times d}$ be defined as

$$\mathcal{W} = \{M \in \mathbb{R}^{d\times d} : x^*_\mathcal{N} x^\top_\mathcal{N} M \text{ is symmetry}\}. \tag{E.11}$$

Then, $M = \Phi(x_\mathcal{N}, x^*_\mathcal{N})$ is a solution to

$$\min_{M \in \mathcal{W} \cap \mathrm{SO}(d)} \|x_\mathcal{N} - Mx^*_\mathcal{N}\|^2, \tag{E.12}$$

where $\Phi(x_\mathcal{N}, x^*_\mathcal{N})$ is given in (E.8).

*Proof.* Consider a matrix $M \in \mathcal{W} \cap \mathrm{SO}(d)$, and from (E.9) and (E.11), we obtain

$$\begin{aligned}(x^*_\mathcal{N} x^\top_\mathcal{N} M)^2 &= (x^*_\mathcal{N} x^\top_\mathcal{N} M)(x^*_\mathcal{N} x^\top_\mathcal{N} M)^\top = (x^*_\mathcal{N} x^\top_\mathcal{N}) M M^\top (x^*_\mathcal{N} x^\top_\mathcal{N})^\top \\ &= U\Sigma V^\top (U\Sigma V^\top)^\top = U\Sigma^2 U^\top. \end{aligned} \tag{E.13}$$

The square root of (E.13) is reduced to

$$x^*_\mathcal{N} x^\top_\mathcal{N} M = U\Sigma\bar{D}U^\top \tag{E.14}$$

for a diagonal matrix $\bar{D} \in \mathbb{R}^{d\times d}$ with each entry 1 or $-1$. Take the determinant of each side of (E.14), and from (E.9), we obtain

$$\begin{aligned}&\det(x^*_\mathcal{N} x^\top_\mathcal{N} M) = \det(U\Sigma V^\top M) = \det(U)\det(\Sigma)\det(V) \\ &\det(U\Sigma\bar{D}U^\top) = \det(U)^2 \det(\Sigma)\det(\bar{D}) = \det(\Sigma)\det(\bar{D})\end{aligned}$$

because $\det(M) = 1$ and $\det(U) \in \{1, -1\}$ from $M \in \mathrm{SO}(d)$ and $U \in \mathrm{O}(d)$. From (E.14), these determinants are equal, and

$$\det(\bar{D}) = \det(U)\det(V) \tag{E.15}$$

is obtained. From (E.14), the function in (E.12) is reduced to

$$\begin{aligned}
&\|x_{\mathcal{N}} - Mx_{\mathcal{N}}^*\|^2 = \|x_{\mathcal{N}}\|^2 + \|x_{\mathcal{N}}^*\|^2 - 2\langle x_{\mathcal{N}}, Mx_{\mathcal{N}}^*\rangle \\
&\ = \|x_{\mathcal{N}}\|^2 + \|x_{\mathcal{N}}^*\|^2 - 2\mathrm{tr}(x_{\mathcal{N}}^\top Mx_{\mathcal{N}}^*) = \|x_{\mathcal{N}}\|^2 + \|x_{\mathcal{N}}^*\|^2 - 2\mathrm{tr}(x_{\mathcal{N}}^* x_{\mathcal{N}}^\top M) \\
&\ = \|x_{\mathcal{N}}\|^2 + \|x_{\mathcal{N}}^*\|^2 - 2\mathrm{tr}(U\Sigma\bar{D}U^\top) = \|x_{\mathcal{N}}\|^2 + \|x_{\mathcal{N}}^*\|^2 - 2\mathrm{tr}(\Sigma\bar{D}).
\end{aligned} \tag{E.16}$$

The function (E.16) is minimized with respect to a diagonal matrix $\bar{D} \in \mathbb{R}^{d\times d}$ with each entry 1 or $-1$ satisfying (E.15), if $\bar{D} = D$ for $D$ in (E.10) because $\Sigma$ is the diagonal matrix with entries $\sigma_1, \dots, \sigma_d$ satisfying $\sigma_1 \geq \cdots \geq \sigma_d \geq 0$. Hence, if a matrix $M \in \mathcal{W} \cap \mathrm{SO}(d)$ satisfies (E.14) for $\bar{D} = D$, it is a solution to (E.12). Actually, $M = \Phi(x_{\mathcal{N}}, x_{\mathcal{N}}^*) \in \mathrm{SO}(d)$ is such a matrix because from (E.8) and (E.9),

$$x_{\mathcal{N}}^* x_{\mathcal{N}}^\top M = U\Sigma V^\top (VDU^\top) = U\Sigma DU^\top$$

holds, which indicates that $M \in \mathcal{W}$ and that (E.14) is satisfied for $\bar{D} = D$. $\square$

*Proof of Proposition E.3.* Let $M \in \mathrm{SO}(d)$. Then, $M^\top M = I_d$ and $\det(M) = 1$ hold. By considering the first equation as a constraint of the optimization problem (E.1), the Lagrangian $L(M, \Lambda) \in \mathbb{R}$ is given as

$$L(M,\Lambda) = \|x_{\mathcal{N}} - Mx_{\mathcal{N}}^*\|^2 + \langle \Lambda, M^\top M - I_d\rangle \tag{E.17}$$

for a symmetric matrix $\Lambda \in \mathbb{R}^{d\times d}$, the Lagrangian multiplier. Partially differentiate (E.17) with respect to $M$ and $\Lambda$, and we obtain

$$\begin{aligned}
\frac{\partial L}{\partial M} &= \frac{\partial L}{\partial M}\mathrm{tr}(x_{\mathcal{N}}^\top x_{\mathcal{N}} - 2x_{\mathcal{N}}^\top Mx_{\mathcal{N}}^* + (x_{\mathcal{N}}^*)^\top M^\top Mx_{\mathcal{N}}^* + \Lambda(M^\top M - I_d)) \\
&= \frac{\partial L}{\partial M}\mathrm{tr}(x_{\mathcal{N}}^\top x_{\mathcal{N}} - 2x_{\mathcal{N}}^* x_{\mathcal{N}}^\top M + x_{\mathcal{N}}^*(x_{\mathcal{N}}^*)^\top M^\top M + \Lambda M^\top M - \Lambda) \\
&= 2(-x_{\mathcal{N}}(x_{\mathcal{N}}^*)^\top + Mx_{\mathcal{N}}^*(x_{\mathcal{N}}^*)^\top + M\Lambda) &\text{(E.18)} \\
\frac{\partial L}{\partial \Lambda} &= M^\top M - I_d. &\text{(E.19)}
\end{aligned}$$

If $M \in \mathrm{SO}(d)$ is a solution to the optimization problem, (E.18) and (E.19) are equal to zero, that is,

$$x_{\mathcal{N}}(x_{\mathcal{N}}^*)^\top = M(x_{\mathcal{N}}^*(x_{\mathcal{N}}^*)^\top + \Lambda),\ \ M^\top M = I_d$$

hold. From these equations,

$$x_{\mathcal{N}}^* x_{\mathcal{N}}^\top M = (M(x_{\mathcal{N}}^*(x_{\mathcal{N}}^*)^\top + \Lambda))^\top M = x_{\mathcal{N}}^*(x_{\mathcal{N}}^*)^\top + \Lambda$$

is obtained. Hence, $x_{\mathcal{N}}^* x_{\mathcal{N}}^\top M$ is symmetry, and $M \in \mathcal{W}$ holds for $\mathcal{W}$ defined in (E.11). From this fact, this $M$ is also a solution to (E.12), and thus Lemma E.1 guarantees that $M = \Phi(x_{\mathcal{N}}, x_{\mathcal{N}}^*)$ holds. Now, (E.7) is achieved.

The case of $\mathcal{M} \times \mathcal{B} = \mathrm{SO}(d) \times \mathbb{R}^d$ can be solved from Proposition E.1 in the same way as Proposition E.2. □

**Proposition E.4.** For $\mathcal{M} \times \mathcal{B} = \mathrm{scaled}(\mathrm{SO}(d)) \times \{0\}$, the solution to the Procrustes problem (E.1) is given as

$$(M, b) = (\Psi(x_{\mathcal{N}}, x_{\mathcal{N}}^*), 0),$$

where the matrix-valued function $\Psi : (\mathbb{R}^d)^n \times (\mathbb{R}^d)^n \to \mathrm{scaled}(\mathrm{SO}(d))$ is defined as follows with $\Sigma, D, U, V$ in Proposition E.3:

$$\Psi(x_{\mathcal{N}}, x_{\mathcal{N}}^*) := \frac{\langle \Sigma, D \rangle}{\|x_{\mathcal{N}}^*\|^2} V D U^\top. \tag{E.20}$$

For $\mathcal{M} \times \mathcal{B} = \mathrm{scaled}(\mathrm{SO}(d)) \times \mathbb{R}^d$, the solution is given as

$$(M, b) = (\Psi(\mathrm{cen}(x_{\mathcal{N}}), \mathrm{cen}(x_{\mathcal{N}}^*)), \mathrm{ave}(x_{\mathcal{N}} - M x_{\mathcal{N}}^*)).$$

*Proof.* For the optimization problem (E.1) with $\mathcal{M} \times \mathcal{B} = \mathrm{scaled}(\mathrm{SO}(d)) \times \{0\}$, let $sR \in \mathrm{scaled}(\mathrm{SO}(d))$ be a solution for $s > 0$ and $R \in \mathrm{SO}(d)$. According to Proposition E.3, $R = \Phi(x_{\mathcal{N}}, x_{\mathcal{N}}^*)$ holds for $\Phi(\cdot, \cdot)$ in (E.8) because even if $x_{\mathcal{N}}^*$ is multiplied by a non-negative scalar, only $\Sigma$ is multiplied by the scalar while $U$ and $V$ are unchanged in (E.9). From Proposition E.2, $s$ is derived by replacing $x_{\mathcal{N}}^*$ with $Rx_{\mathcal{N}}^*$ in (E.5), and

$$\begin{aligned} s = \chi(x_{\mathcal{N}}, Rx_{\mathcal{N}}^*) &= \max\{\frac{\langle x_{\mathcal{N}}, Rx_{\mathcal{N}}^* \rangle}{\|Rx_{\mathcal{N}}^*\|^2}, 0\} = \max\{\frac{\mathrm{tr}(x_{\mathcal{N}}^\top R x_{\mathcal{N}}^*)}{\|x_{\mathcal{N}}^*\|^2}, 0\} \\ &= \max\{\frac{\mathrm{tr}((x_{\mathcal{N}}^* x_{\mathcal{N}}^\top) R)}{\|x_{\mathcal{N}}^*\|^2}, 0\} = \max\{\frac{\mathrm{tr}((U\Sigma V^\top)(VDU^\top))}{\|x_{\mathcal{N}}^*\|^2}, 0\} \\ &= \max\{\frac{\mathrm{tr}(\Sigma D)}{\|x_{\mathcal{N}}^*\|^2}, 0\} = \frac{\langle \Sigma, D \rangle}{\|x_{\mathcal{N}}^*\|^2} \end{aligned}$$

is obtained from (E.8) and (E.9), where the last equation holds because

$$\mathrm{tr}(\Sigma D) = \langle \Sigma, D \rangle = \sigma_1 + \sigma_2 + \cdots \pm \sigma_d \geq 0$$

holds from $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_d \geq 0$. The resultant $sR$ is reduced to (E.20).

The case of $\mathcal{M} \times \mathcal{B} = \mathrm{scaled}(\mathrm{SO}(d)) \times \mathbb{R}^d$ is shown from Proposition E.1. $\square$

# References

Ahn, H. (2020). *Formation Control*. Springer.

Anderson, B. D. O., Z. Sun, T. Sugie, S. Azuma, and K. Sakurama. (2017). “Formation shape control with distance and area constraints”. *IFAC Journal of Systems and Control*. 1: 2–12.

Anderson, B. D. O., C. Yu, B. Fidan, and J. M. Hendrickx. (2008). “Rigid graph control architectures for autonomous formations”. *IEEE Control Syst. Mag.* 28(6): 48–63.

Bai, H., M. Arcak, and J. Wen. (2011). *Cooperative Control Design*. Springer.

Bolloás, B. (1998). *Modern Graph Theory*. Springer Science+Business Media.

Boutin, M. and G. Kemper. (2004). “On reconstructing $n$-point configurations from the distribution of distances or areas”. *Adv. Appl. Math.* 32(4): 709–735.

Bullo, F., J. Cortes, and S. Martinez. (2009). *Distributed Control of Robotic Networks: A Mathematical Approach to Motion Coordination Algorithms*. Princeton University Press.

Bullo, F. and A. D. Lewis. (2004). *Geometric Control of Mechanical Systems: Modeling, Analysis, and Design for Simple Mechanical Control Systems*. Springer Science+Business Media.

Bürger, M., G. Notarstefano, F. Bullo, and F. Allgöwer. (2012). "A distributed simplex algorithm for degenerate linear programs and multi-agent assignments". *Automatica*. 48(9): 2298–2304.

Cao, Y., W. Yu, W. Ren, and G. Chen. (2012). "An overview of recent progress in the study of distributed multi-agent coordination". *IEEE Trans. Ind. Informat.* 9(1): 427–438.

Chen, L., M. Cao, and C. Li. (2020). "Angle rigidity and its usage to stabilize multi- agent formations in 2D". *IEEE Trans. Autom. Control.*

Clarke, F. (1983). *Optimization and Nonsmooth Analysis*. Wiley.

Cortes, J. and M. Egerstedt. (2017). "Coordinated cntrol of multi-robot systems: a survey". *SICE Journal of Control, Measurement, and System Integration*. 10(6): 495–503.

Cortés, J., S. Martinez, T. Karatas, and F. Bullo. (2004). "Coverage control for mobile sensing networks". *IEEE Trans. Robot. Autom.* 20(2): 243–255.

Darmanin, R. N. and M. K. Bugeja. (2017). "A review on multi-robot systems categorised by application domain". In: *IEEE the 25th Mediterranean Conf. on Control and Automation*.

Dimarogonas, D. V. and K. J. Kyriakopoulos. (2008). "A connection between formation infeasibility and velocity alignment in kinematic multi-agent systems". *Automatica*. 44(10): 2648–2654.

Dörfler, F. and B. Francis. (2010). "Geometric analysis of the formation problem for autonomous robots". *IEEE Trans. Autom. Control.* 55(10): 2379–2384.

Eren, T., W. Whiteley, A. S. Morse, B. D. O. Anderson, and P. N. Belhumeur. (2004). "Information structures to secure control of globally rigid formations". In: *Proc. of the American Control Conference*. 4945–4950.

Fax, J. A. and R. M. Murray. (2004). "Information flow and cooperative control of vehicle formations". *IEEE Trans. Autom. Control.* 49(9): 1465–1476.

Gower, J. C. and G. B. Dijksterhuis. (2004). *Procrustes Problems*. Oxford University Press.

Haddad, W. M. and V. Chellaboina. (2008). *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*. Princeton Univ Pr ess.

Han, Z., L. Wang, Z. Lin, and R. Zheng. (2016). "Formation control with size scaling via a complex Laplacian-based approach". *IEEE Trans. Cybernetics*. 46(10): 2348–2359.

Hatanaka, T., N. Chopra, M. Fujita, and M. Spong. (2015). *Passivity-Based Control and Estimation in Networked Robotics*. Springer.

Igarashi, Y., T. Hatanaka, M. Fujita, and M. W. Spong. (2009). "Passivity-based attitude synchronization in SE(3)". *IEEE Trans. Control Syst. Technol.* 17(5): 1119–1134.

Ji, M., S. Azuma, and M. Egerstedt. (2006). "Role assignment in multi-agent coordination". *International Journal of Assistive Robotics and Mechatronics*. 7(1): 32–40.

Kanatani, K. (1994). "Analysis of 3-D rotation fitting". *IEEE Trans. Pattern Anal. Mach. Intell.* 16(5): 543–549.

Kanjanawanishkul, K. and A. Zell. (2010). "Distributed role assignment in multi-robot formation". In: *Proc. of the 7th IFAC Symposium on Intelligent Autonomous Vehicles*.

Kannan, A. A., B. Fidan, G. Mao, and B. D. Anderson. (2007). "Analysis of flip ambiguities in distributed network localization". In: *Conf. Proc. 2007 Information, Decis. Control. IDC*.

Khalil, H. K. (2002). *Nonlinear Systems*. Third. Prentice Hall.

Kim, T.-H., S. Hara, and Y. Hori. (2010). "Cooperative control of multi-agent dynamical systems in target-enclosing operations using cyclic pursuit strategy". *International Journal Control*. 83(10): 2040–2052.

Kingston, P. and M. Egerstedt. (2010). "Index-free multi-agent systems: an Eulerian approach". In: *Proc. of the 2nd IFAC Workshop on Distributed Estimatimation and Control in Networked Systems*.

Krantz, S. G. and H. R. Parks. (2002). *A Primer of Real Analytic Functions: Second Edition*. Springer Science+Business Media.

Krick, L., M. E. Broucke, and B. A. Francis. (2009). "Stabilisation of infinitesimally rigid formations of multi-robot networks". *International Journal Control*. 82(3): 423–439.

Kuhn, H. W. (1955). "The Hungarian method for the assignment problem". *Naval Research Logistics Quarterly*. 2: 83–97.

Kurdyka, K. (1998). “On gradients of functions definable in o-minimal structures”. *Ann. Inst. Fourier*. 48(3): 769–783.

Lima, P. U. and L. M. Custodio. (2005). “Multi-robot systems”. In: *Innovations in Robot Mobility and Control*. Springer. 1–64.

Lin, P. and Y. Jia. (2010). “Distributed rotating formation control of multi-agent systems”. *Systems & Control Letters*. 59(10): 587–595.

Lin, Z., L. Wang, Z. Chen, M. Fu, and Z.Han. (2016). “Necessary and sufficient graphical conditions for affine formation control”. *IEEE Trans. Autom. Control*. 61(10): 2877–2891.

Lin, Z., B. Francis, and M. Maggiore. (2005). “Necessary and sufficient graphical conditions for formation control of unicycles”. *IEEE Trans. Autom. Control*. 50(1): 121–127.

Liu, L. and D. A. Shell. (2011). “Multi-level partitioning and distribution of the assignment problem for large-scale multi-robot task allocation”. In: *Proc. of Robotics: Science and Systems*. 185–192.

Liu, T. and Z. P. Jiang. (2013). “Distributed formation control of nonholonomic mobile robots without global position measurements”. *Automatica*. 49(2): 592–600.

Łojasiewicz, S. (1965). *Ensembles semi-analytiques*. Institut des Hautes Etudes Scientifiques.

Łojasiewicz, S. and M. Zurro. (1999). “On the gradient inequality”. *Bulletin of the Polish Academy of Sciences. Mathematics*. 47(2): 143–145.

Marshall, J. A., M. E. Broucke, and B. A. Francis. (2004). “Formations of vehicles in cyclic pursuit”. *IEEE Trans. Autom. Control*. 49(11): 1963–1974.

Martínez, S., J. Cortés, and F. Bullo. (2007). “Motion cordination with distributed information”. *IEEE Control Syst. Mag.* 27(4): 75–88.

McKee, T. A. and F. R. McMorris. (1999). *Topics in Intersection Graph Theory*. SIAM.

Mesbahi, M. and M. Egerstedt. (2010). *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press.

Michael, N., M. M. Zavlanos, V. Kumar, and G. J. Pappas. (2008). “Distributed multi-robot task assignment and formation control”. In: *Proc. of the IEEE International Conference on Robotics and Automation*.

Montijano, E., E. Cristofalo, M. Schwager, and C. Sagues. (2016). "Distributed formation control of non-holonomic robots without a global reference frame". *Proc. of the IEEE Int. Conf. Robot. Autom.* 2016-June: 5248–5254.

Mosteo, A. R., E. Montijano, and D. Tardioli. (2017). "Optimal role and position assignment in multi-robot freely reachable formations". *Automatica.* 81(July 2017): 305–313.

Oh, K., M. Park, and H. Ahn. (2015). "A survey of multi-agent formation control". *Automatica.* 53(3): 424–440.

Olfati-Saber, R. (2006). "Flocking for multi-agent dynamic systems: algorithms and theory". *IEEE Trans. Autom. Control.* 51(3): 401–420.

Olfati-Saber, R., J. A. Fax, and R. M. Murray. (2007). "Consensus and cooperation in networked multi-agent systems". *Proc. of the IEEE.* 95(1): 215–233.

Olfati-Saber, R. and R. M. Murray. (2004). "Consensus poblems in networks of agents with switching topology and time-delays". *IEEE Trans. Autom. Control.* 49(9): 1520–1533.

Olver, P. J. (1999). *Classical Invariant Theory.* Cambridge University Press.

Olver, P. J. (1995). *Equivalence, Invariants, and Symmetry.* Cambridge University Press.

Queiroz, M. de, X. Cai, and M. Feemster. (2019). *Formation control of multi-agent systems: a graph rigidity approach.* Wiley.

Reid, M. and B. Szendroi. (2005). *Geometry and Topology.* Cambridge University Press.

Ren, W. (2010). "Distributed cooperative attitude synchronization and tracking for multiple rigid bodies". *IEEE Trans. Control Syst. Technol.* 18(2): 383–392.

Ren, W. and R. Beard. (2008). *Distributed Consensus in Multi-vehicle Cooperative Control Theory and Applications.* Springer.

Ren, W. and Y. Cao. (2011). *Distributed Coordination of Multi-agent Networks.* Springer-Verlag.

Rozenheck, O., S. Zhao, and D. Zelazo. (2015). "A proportional-integral controller for distance-based formation tracking". In: *Proc. of 2015 European Control Conference.*

Sakurama, K. (2016). “Distributed control of networked multi-agent systems for formation with freedom of special Euclidean group”. In: *Proc. of the 55th IEEE Conference on Decision and Control*. 928–932.

Sakurama, K. (2018). “Formation control of multi-agent systems over generalized relative measurement”. In: *Proc. of the 57th IEEE Conference on Decision and Control*. 2878–2882.

Sakurama, K., S. Azuma, and T. Sugie. (2012). “Parameterization of all distributed controllers based on gradient-flow method for networked multi-agent systems”. In: *Proc. of the IEEE Conference on Control and Decision*.

Sakurama, K., S. Azuma, and T. Sugie. (2015). “Distributed controllers for multi-agent coordination via gradient-flow approach”. *IEEE Trans. Autom. Control*. 60(6): 1471–1485.

Sakurama, K., S. Azuma, and T. Sugie. (2018). “Multi-agent coordination to high-dimensional target subspaces”. *IEEE Trans. Control Netw. Syst.* 5(1): 345–357.

Sakurama, K., S. Azuma, and T. Sugie. (2019). “Multi-agent coordination via distributed pattern matching”. *IEEE Trans. Autom. Control*. 64(8): 3210–3225.

Sakurama, K. (2020). “Formation control of non-holonomic multi-agent systems under relative measurements”. In: *Proc. of the 21th IFAC World Congress*.

Sakurama, K. (2021a). “Clique-based distributed PI control for multiagent coordination with heterogeneous,uncertain, time-varying orientations”. *IEEE Trans. Control Netw. Syst.* 7(4): 1712–1722.

Sakurama, K. (2021b). “Unified formulation of multi-agent coordination with relative measurements”. *IEEE Trans. Autom. Control*. 66(9).

Sakurama, K. and H. Ahn. (2020). “Multi-agent coordination over local indexes via clique-based distributed assignment”. *Automatica*. 112(February 2020): 108670.

Shamma, J., ed. (2008). *Cooperative control of distributed multi-agent systems*. Wiley-Interscience.

Shevitz, D. and B. Paden. (1994). “Lyapunov stability theory of nonsmooth systems”. *IEEE Trans. Autom. Control*. 39(9): 1910–1914.

Sidman, J. and A. S. John. (2017). "The rigidity of frameworks: theory and applications". *Notices of the AMS*. Oct.: 973–978.

Smith, S. L. and F. Bullo. (2009). "Monotonic target assignment for robotic networks". *IEEE Trans. Autom. Control*. 54(9): 2042–2057.

Sun, Z. (2018). *Cooperative Coordination and Formation Control for Multi-agent Systems*. Springer.

Sun, Z., M.-C. Park, B. D. Anderson, and H. Ahn. (2017). "Distributed stabilization control of rigid formations with prescribed orientation". *Automatica*. 78: 250–257.

Tanner, H. G., A. Jadbabaie, and G. J. Pappas. (2007). "Flocking in fixed and switching networks". *IEEE Trans. Autom. Control*. 52(5): 863–868.

Tomita, E., A. Tanaka, and H. Takahashi. (2006). "The worst-case time complexity for generating all maximal cliques and computational experiments". *Theoretical Computer Science*. 363: 28–42.

Tron, R., J. Thomas, G. Loianno, K. Daniilidis, and V. Kumar. (2016). "A distributed optimization framework for localization and formation control". *IEEE Control Syst. Mag.* 36(4): 22–44.

Yu, S. and J. C. Barca. (2015). "Autonomous formation selection for ground moving multi-robot systems". In: *Proc. of the 2015 IEEE International Conference on Advanced Intelligent Mechatronics*. 54–932.

Zavlanos, M. M., L. Spesivtsev, and G. J. Pappas. (2008). "A distributed auction algorithm for the assignment problem". In: *Proc. of the 47th IEEE Conference on Decision and Control*.

Zhao, S. (2018). "Affine formation maneuver control of multiagent systems". *IEEE Trans. Autom. Control*. 63(12): 4140–4155.

Zhao, S. and D. Zelazo. (2016). "Bearing rigidity and almost global bearing-only formation stabilization". *IEEE Trans. Autom. Control*. 61(5): 1255–1268.

Zhao, S. and D. Zelazo. (2019). "Bearing rigidity theory and its applications for control and estimation of network systems". *IEEE Control Syst. Mag.* 39(2): 66–83.