Elucidation of Chaotic Market Hypothesis Based on Ergodic Theory

Ken Umeno

Abstract We develop a method of directly testifying the efficient market hypothesis (EMH) proposed by Fama by using the ergodic theory to connect microscopic price fluctuations with macroscopic behavior. After testing the validity of the method by using exactly solvable chaos, we found a novel periodic structure in the 5-minute chart data of the Nikkei averages in 2019 with our developed new correlation function based on the characteristic function. This *directly* denies the EMH. Statistical data of the empirical studies have shown that a stable law well describes the price fluctuations of financial markets as predicted by the most generalized version of the central limit theorem called the universal super generalized central limit theorem (USGCLT) we discovered recently. The concept and proof of the extension of the super generalized central limit theorem (SGCLT) is also given to illustrate the mechanism of *universality* such that a sum of random numbers with *nonidentical* power law distributions converges to a stable law in distribution. With these theoretical facts together with the empirical fact of the data denying the EMH, we propose the chaotic market hypothesis (CMH) based on the ergodic theory to capture the essential characteristics of the financial markets.

1 Introduction

Ergodicity is the fundamental concept that connects macroscopic behavior with the microscopic behavior for complex systems such as the financial market. To characterize the financial market, Fama (1970) propose a clear vision that the the financial market is essentially characterized by the random walk by posing basic assumptions of *efficiency* that the equality of opportunities exist for every participants under the fair condition that there is no insider trading with privileged information.

Ken Umeno
Kyoto University, Yoshida Honmachi, Sakyo-ku, Kyoto
e-mail: umeno.ken.8z@kyoto-u.ac.jp

This assumption has been supported by many empirical studies such that financial time series data such as logarithmic returns of price fluctuations have *no correlation* as firstly observed by Fama (1970). Since then, the efficient market hypothesis (EMH) has prevailed with this view by stating that the financial market is just a *random walk* and no one can predict a future price from information up to the present. Actual financial time series are, however, known to have a short-term memory effect showing *correlation* by practitioners participating in markets. Thus, there is a non-negligible *gap* between EMH and actual financial markets. The purpose of the present study is to clearly characterize the gap by giving the reason on the tail risk behavior characterized by a certain generalized version of the central limit theorem and introducing a new hypothesis called the chaotic market hypothesis (CMH). In the CMH, a small amount of correlation is *acceptable* for financial time series such as a *mixing* behavior representing *chaos*, together with the fundamental concepts: super generalized central limit theorem showing a universality of fat tail behavior. In Section 2, we review the super generalized central limit theorem (SGCLT) as the basic ingredients of its further generalized central limit theorem showing a universal tail behavior of financial markets, which will be presented in Section 3. In Section 4, a novel correlation function based on a characteristic function is introduce to test the EMH by using the empirical data of the 5-minute chart data of logarithmic returns of the Nikkei average in 2019. In Section 5, CMH is proposed to capture the essential characteristics of the financial market behavior as described above. Discussion about the difference between the CMH and the fractal market hypothesis is presented in Section 6. Section 7 summarizes and concludes the chapter.

2 Super Generalized Central Limit Theorem and its Generalization

As the central limit theorem is the fundamental law to characterize a limiting distribution for sums of random numbers with finite variance, the generalized central limit theorem (Gnedenko and Kolmogorov 1954) has a key role in characterizing a limiting power law distributions for sums of random numbers obeying the power law with infinite variance such as Cauchy law. Such a limiting power law can be characterized as the stable law in a *universal* manner, which is the crux of the ubiquitous nature of macroscopic distributions due to the generalized central limit theorem.

Stable Law $S(x;\alpha,\beta,\gamma,\mu)$ is given by the Fourier transform of the characteristic function $\phi(t)$

$$S(x;\alpha,\beta,\gamma,\mu)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(t)e^{-\mathrm{I}xt}dx$$

where characteristic function $\phi(t)$ with the four parameters: α,β,γ and μ has the form:

$$\phi(t)=\exp\{\mathrm{I}\mu t-\gamma^{\alpha}|t|^{\alpha}(1-\mathrm{I}\beta\mathrm{sgn}(t)w(\alpha,t)\} \quad (1)$$

with

$$w(\alpha, t) = \begin{cases} \tan(\pi\alpha/2) & \text{if} \quad \alpha \neq 1 \\ -2/\pi \ln|t| & \text{if} \quad \alpha = 1. \end{cases}$$

Here the parameters α, β, γ and μ are the scaling exponent parameter $\alpha \in (0, 2]$ representing the fatness of the tail, the skewness parameter $\beta \in [-1, 1]$, the scaling parameter $\gamma > 0$, and the location parameter $\mu \in \mathbb{R}$, respectively. When $\alpha = 2$, it corresponds to the Gauss law with $\beta = 0$ and when $\alpha = 1, \beta = 0$, it corresponds to the Cauchy law.
Here, we introduce the the **condition for the super generalized central limit theorem** according to Shintani and Umeno (2018) :

1. The random variables $C_+ > 0$ and $C_- > 0$ obey respectively the distributions $\mathrm{P}_{c_+}(c)$ and $\mathrm{P}_{c_-}(c)$, and satisfy

$$\mathrm{E}[C_+] < \infty, \text{ and } \mathrm{E}[C_-] < \infty.$$

2. The probability distribution function $f_i(x)$ of the random variables X_i has a following limiting form when $0 < \alpha < 2$:

$$f_i(x) \simeq \begin{cases} c_{+i} x^{-(\alpha+1)} & \text{for} \quad x \to \infty \\ c_{-i} |x|^{-(\alpha+1)} & \text{for} \quad x \to -\infty, \end{cases} \tag{2}$$

where c_{+i} and c_{-i} are samples obtained by C_+ and C_-, respectively.

The following theorem holds (Shintani and Umeno 2018).

Theorem 1 (Super Generalized Central Limit Theorem)
Suppose that Condition 1 and Condition 2 are satisfied. Then the following superposition S_n of independent and nonidentical random variables with power laws converges in density to a unique stable distribution $S(x; \alpha, \beta^*, \gamma^*, 0)$ for $n \to \infty$, where

$$S_n = \frac{\sum_{i=1}^{n} X_i - A_n}{n^{\frac{1}{\alpha}}} \xrightarrow{d} S(x; \alpha, \beta^*, \gamma^*, 0) \quad \text{for} \quad n \to \infty,$$

$$A_n = \begin{cases} 0 & \text{if} \quad 0 < \alpha < 1 \\ n \sum_{i=1}^{n} \Im \ln(\varphi_i(1/n)) & \text{if} \quad \alpha = 1 \\ \sum_{i=1}^{n} \mathrm{E}[X_i] & \text{if} \quad 1 < \alpha < 2 \end{cases}$$

with $\varphi_i(t)$ being a characteristic function of X_i as the expected value of $\exp(\mathrm{I}tX_i)$ and parameters $\beta^*, \gamma^*, \beta_i, \gamma_i$ are expressed as:

$$\beta^* = \frac{\mathrm{E}_{C_+,C_-}[\beta_i \gamma_i^\alpha]}{\mathrm{E}_{C_+,C_-}[\gamma_i^\alpha]}, \gamma^* = \{\mathrm{E}_{C_+,C_-}[\gamma_i^\alpha]\}^{\frac{1}{\alpha}}, \tag{3}$$

$$\beta_i = \frac{c_{+i} - c_{-i}}{c_{+i} + c_{-i}}, \gamma_i = \left\{ \frac{\pi(c_{+i} + c_{-i})}{2\alpha \sin(\pi\alpha/2)\Gamma(\alpha)} \right\}^{\frac{1}{\alpha}},$$

where $\mathrm{E}_{C_+,C_-}[X]$ denotes the expectation value of X with respect to the random parameter distributions $\mathrm{P}_{c_+}(c)$ and $\mathrm{P}_{c_-}(c)$. Here, $\Im$ is an imaginary part of the argument.

Consider a further generalization of the super generalized central limit theorem. Here, a *generalization* means that we extend the super generalized central limit theorem with a unique power index α to a further generalization of the generalized central limit theorem for superposition of random variables with *different* power indices α_m where

$$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < 2$$

for a sum of independent random numbers X_i obeying the probability distribution function f_i such that

$$f_i(x; \alpha_k) \simeq \begin{cases} c_{+i} x^{-(\alpha_k+1)} & \text{for} \quad x \to \infty \\ c_{-i} |x|^{-(\alpha_k+1)} & \text{for} \quad x \to -\infty \end{cases} \tag{4}$$

When $m = 1$, it corresponds to the SGCLT with a unique α. Thus, we consider the case $m \geq 2$.
The following conditions are essential:

Condition for the Universal Super Generalized Central Limit Theorem (USG-CLT)

1. Each random number X_i satisfies the conditions of the super generalized central limit theorem (SGCLT).
2. The power indices of the probability distribution are in the order that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < 2$ and the equality $\sum_{k=1}^{m} q_{\alpha_k} = 1$ is satisfied where $q_{\alpha_k} (0 < q_{\alpha_k} < 1)$ is a probability that the power index $\alpha_k (1 \leq k \leq m)$ is randomly selected from the m power indices $(\alpha_1, \alpha_2, \cdots, \alpha_m)$. Furthermore, the probabilities q_{α_k} are assumed to be positive constants.

We have the following theorem:

Theorem 2 (Universal Super Generalized Central Limit Theorem (USGCLT)) *If independent random variables $X_1, X_2, \ldots$ satisfy the* **condition for the universal super generalized central limit theorem (USGCLT)**, *then a superposition*

$$S_n \equiv \frac{\sum_{i=1}^{n} X_i - A_n}{n^{\frac{1}{\alpha_1}}}$$

converges in density to a stable law as

$$S_n \xrightarrow{d} S(x; \alpha_1, \beta^*[\alpha_1], \gamma^*[\alpha_1]\{q_{\alpha_1}\}^{\frac{1}{\alpha_1}}, 0),$$

where

$$A_n = \sum_{k=1}^{m} A_{nq_{\alpha_k}}[\alpha_k] \tag{5}$$

with $A_{nq_{\alpha_k}}[\alpha_k]$'s sum A_n being a sample mean when α_k is selected and the parameters $\beta^*[\alpha_1]$ and $\gamma^*[\alpha_1]$ are a skewness parameter and a scaling parameter, respectively which corresponds to the parameters of a limiting stable distribution with **the minimum characteristic exponent** α_1 *via the Super Generalized Central Limit Theorem when $nq_{\alpha_1} \to \infty$.*

Proof We first note that the following decomposition always holds for $\sum_{i=1}^{n} X_i$:

$$\mathrm{E}\left[\mathrm{Exp}\left(\mathrm{I}\left(\sum_{i=1}^{n} X_i\right)\right)\right] = \prod_{k=1}^{m} \mathrm{E}\left[\mathrm{Exp}\left(\mathrm{I}\left(\sum_{j=1}^{n(k)} X_j[\alpha_k]\right)\right)\right]$$

where $\lim_{n\to\infty} \dfrac{n(k)}{n} = q_{\alpha_k}$ is satisfied according to the condition for the USGLT. Since each stable law is an *infinitely divisible distribution* and its characteristic function $\phi_{\alpha,\beta}(t)$ always satisfies

$$|\phi_{\alpha,\beta}(t)| \le 1,$$

then the decomposition is interchangeable even for $n \to \infty$ as:

$$\mathrm{E}\left[\mathrm{Exp}\left(\mathrm{I}\left(\sum_{i=1}^{n} X_i\right)\right)\right] = \prod_{k=1}^{m} \mathrm{E}\left[\mathrm{Exp}\left(\mathrm{I}\left(\sum_{j=1}^{n(k)} X_j[\alpha_k]\right)\right)\right] \to \prod_{k=1}^{m} \mathrm{E}\left[\mathrm{Exp}\left(\mathrm{I}\left(\sum_{j=1}^{nq_k} X_j[\alpha_k]\right)\right)\right] \tag{6}$$

as $n \to \infty$. Here we can thus apply the super generalized central limit theorem to a superposition $\sum_{j=1}^{nq_k} X_j[\alpha_k]$ with the power index α_k, then for each $\alpha_k (1 \le k \le m)$, the relation

$$\frac{\sum_{j=1}^{nq_{\alpha_k}} X_j[\alpha_k] - A_{nq_{\alpha_k}}}{\{nq_{\alpha_k}\}^{\frac{1}{\alpha_k}}} \xrightarrow{d} S(x; \alpha_k, \beta^*[\alpha_k], \gamma^*[\alpha_k], 0) \tag{7}$$

as $n \to \infty$ holds. In particular, for the case when $k = 1$, the relation

$$\frac{\sum_{j=1}^{nq_{\alpha_1}} X_j[\alpha_1] - A_{nq_{\alpha_1}}}{\{nq_{\alpha_1}\}^{\frac{1}{\alpha_1}}} \xrightarrow{d} S(x; \alpha_1, \beta^*[\alpha_1], \gamma^*[\alpha_1], 0) \tag{8}$$

as $n \to \infty$ holds. By seeing the obvious relation $\alpha_2 (> \alpha_1)$ for example, we have the relations:

$$\frac{\sum_{j=1}^{nq_{\alpha_2}} X_j[\alpha_2] - A_{nq_{\alpha_2}}}{\{nq_{\alpha_1}\}^{\frac{1}{\alpha_1}}} = \frac{\sum_{j=1}^{nq_{\alpha_2}} X_j[\alpha_2] - A_{nq_{\alpha_2}}}{\{nq_{\alpha_2}\}^{\frac{1}{\alpha_2}}} \frac{\{nq_{\alpha_2}\}^{\frac{1}{\alpha_2}}}{\{nq_{\alpha_1}\}^{\frac{1}{\alpha_1}}}$$

$$= \frac{\sum_{j=1}^{nq_{\alpha_2}} X_j[\alpha_2] - A_{nq_{\alpha_2}}}{\{nq_{\alpha_2}\}^{\frac{1}{\alpha_2}}} \cdot \frac{1}{n^{\frac{1}{\alpha_1} - \frac{1}{\alpha_2}}} \cdot \frac{q_{\alpha_2}^{\frac{1}{\alpha_2}}}{q_{\alpha_1}^{\frac{1}{\alpha_1}}} \xrightarrow{d} \delta(x)$$

as $n \to \infty$. Here, we use the SGCLT for α_2 and the obvious fact that $\dfrac{1}{\alpha_1} - \dfrac{1}{\alpha_2} > 0$ and then as $n \to \infty$

$$\frac{1}{n^{\frac{1}{\alpha_1} - \frac{1}{\alpha_2}}} \to 0.$$

This means that the density function of the random variable $\frac{\sum_{j=1}^{nq_{\alpha_2}} X_j[\alpha_2] - A_{nq_{\alpha_2}}}{\{nq_{\alpha_1}\}^{\frac{1}{\alpha_1}}}$ approaches to Dirac's delta function $\delta(x)$ as $n \to \infty$ whose characteristic function is just unity.

In the same way, for more general $\alpha_k (k \geq 2)$ by using the obvious relation $\frac{1}{\alpha_1} - \frac{1}{\alpha_k} > 0$, the relation

$$\frac{\sum_{j=1}^{nq_{\alpha_k}} X_j[\alpha_k] - A_{nq_{\alpha_k}}}{\{nq_{\alpha_1}\}^{\frac{1}{\alpha_1}}} \xrightarrow{d} \delta(x) \quad \text{for} \quad n \to \infty$$

holds. Thus, in the limit $n \to \infty$

$$\frac{\sum_{i=1}^{n} X_i - A_n}{\{nq_{\alpha_1}\}^{\frac{1}{\alpha_1}}} \xrightarrow{d} S(x; \alpha_1, \beta^*[\alpha_1], \gamma^*[\alpha_1], 0).$$

Therefore, we conclude:

$$\frac{\sum_{i=1}^{n} X_i - A_n}{n^{\frac{1}{\alpha_1}}} \xrightarrow{d} S(x; \alpha_1, \beta^*[\alpha_1], \gamma^*[\alpha_1]\{q_{\alpha_1}\}^{\frac{1}{\alpha_1}}, 0).$$

This universal super generalized central limit theorem (USGCLT) proposed and proven here says that the *minimum power index* α_1 corresponding to the *biggest tail-risk* component is dominant factor to characterize a limit theorem for a mixed superposition of power laws. In other words, we can say that the biggest risk represented by the minimum tail power index α_1 is *dominant* in a limiting behavior of macroscopic risk, which is conceptually similar to Gresham's Law that states "bad money drives out good" because "goodness" of money can be measured by tail risk indicator α. In addition to this feature, this USGCLT gives a theoretical explanation about the reason why stable laws appear universally in empirical distributions of price fluctuations.

3 Exactly Solvable Chaos and Stable Law to Test Universal Super Generalized Central Limit Theorem

In this section, we introduce the connection between a certain ideal class of chaotic mappings and stable laws by ergodic theory. Umeno (1998) showed that the following chaotic mapping

$$Y = \frac{1}{2}\left(X - \frac{1}{X}\right) \tag{9}$$

has the Cauchy distribution with $\alpha = 1$ as an ergodic invariant measure while more generalized map

$$Y = \frac{1}{2}\left||X|^{\alpha} - \frac{1}{|X|^{\alpha}}\right|^{\frac{1}{\alpha}} \cdot \mathrm{SGN}\left[X - \frac{1}{X}\right], \quad (\alpha > 0) \tag{10}$$

has the power law with the power index α whose superposition converges to a stable law by the generalized central limit theorem (GCLT) for $0 < \alpha < 2$. All the chaotic mappings can be categorized as *exactly solvable chaos* which has an exact solution and the explicit ergodic invariant measure (Umeno 1997). Figure 1 shows the return maps of the above exactly solvable chaos. Empirical density obtained by 100, 000 iterations of the chaotic mapping (9) is shown in Fig 2 and it remarkably matches the exact Cauchy probability density. According to Fig. 3, empirical density obtained by 100, 000 iterations of the chaotic mapping (10) at $\alpha = 3/2$ also matches an analytical density

$$\rho(x;\alpha) = \frac{\alpha|x|^{\alpha-1}}{\pi(1+|x|^{2\alpha})} \tag{11}$$

at $\alpha = 3/2$ very well (Umeno 1998). Thus, we can say that *ergodic theory works well* for characterizing a logical connection between statistics and deterministic chaos, especially for the case of exactly solvable chaos. Note that there are other more chaotic mappings with ergodic Cauchy distribution such as the generalized Boole transformation in Eq. (12) with an analytical Lyapunov exponent λ in Eq. (13) (Umeno and Okubo 2016) and the super generalized Boole transformations (Okubo and Umeno 2018, 2021) that have Cauchy distribution as an ergodic invariant measure.

$$Y = aX - \frac{b}{X}, \quad (0 < a < 1, b > 0) \tag{12}$$

$$\lambda = \log\left(1 + 2\sqrt{a(1-a)}\right), \quad (0 \le a \le 1). \tag{13}$$

Next, we use these exactly solvable chaotic mappings to generate a mixed type of superposition to test the universal super generalized central limit theorem (USGCLT). Let us consider a simple mixed type of superposition composed of chaotic dynamics with Cauchy distribution ($\alpha = 1$) and chaotic dynamics with the power law ($\alpha = 3/2$). Thus, this example corresponds to the case that $\alpha_1 = 1$ and $\alpha_2 = 3/2$. In Fig. 4, a mixed superposition of Cauchy chaos mapping in Eq. (9) with $\alpha_1 = 1$ and chaos mapping in Eq. (10) with $\alpha_2 = 3/2$ where $q_{\alpha_1} = q_{\alpha_2} = 1/2$ is shown to converge to the Cauchy distribution which is a confirmation of the validity of the USGCLT. Here, Cauchy chaos at $\alpha = 1$ is *dominant* as predicted by the USGCLT.

4 Testing Efficient Market Hypothesis and Discovery of Novel Periodic Structure

Now we consider the test of the efficient market hypothesis (EMH) by empirical data. If EMH holds, then there must be *no correlation* in price fluctuations. Thus we investigate a *correlation structure* of empirical fluctuations of financial markets. We use the 5-minute (high-frequency) chart of logarithmic returns of the Nikkei Averages in 2019 to test the EMH. Many empirical studies have shown that such high-frequency charts of logarithmic returns show no correlation or random fluctuations as predicted by EMH. Is it true? We solve the very question on EMH. In Figs. 5 and 6, a normalized standard correlation $C(l) = \text{Const.} \sum_{i=1}^{N} X_i X_{i+l}$ of the 5-minute chart of logarithmic returns of the Nikkei averages in 2019 is depicted, which shows that no nontrivial correlation exists and EMH *seems to hold*. The reason why the correlation looks likely to be zero can be explained by the fact that the variance of price fluctuations corresponding to the normalized constant of the correlation is generally quite large due to the fat tail distribution and then the normalized correlation $C(l)$ computed by the division of the variance becomes near zero even if there exist a non-trivial correlation in data. Thus, with this normal correlation, we cannot conclude that EMH holds for the 5-minute chart of logarithmic returns of the Nikkei averages in 2019 at this stage because there might be a *non-trivial correlation structure* in data. To investigate a new possibility, we consider an essence of the financial time series data by ergodic theory.

Consider a sequence of data $X_1, X_2, \ldots \in M$ satisfying ergodicity (Arnold and Avez 1968) in a sense that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(X_j) = \int_M f(x)\mu(dx) \quad \text{a.e.} \tag{14}$$

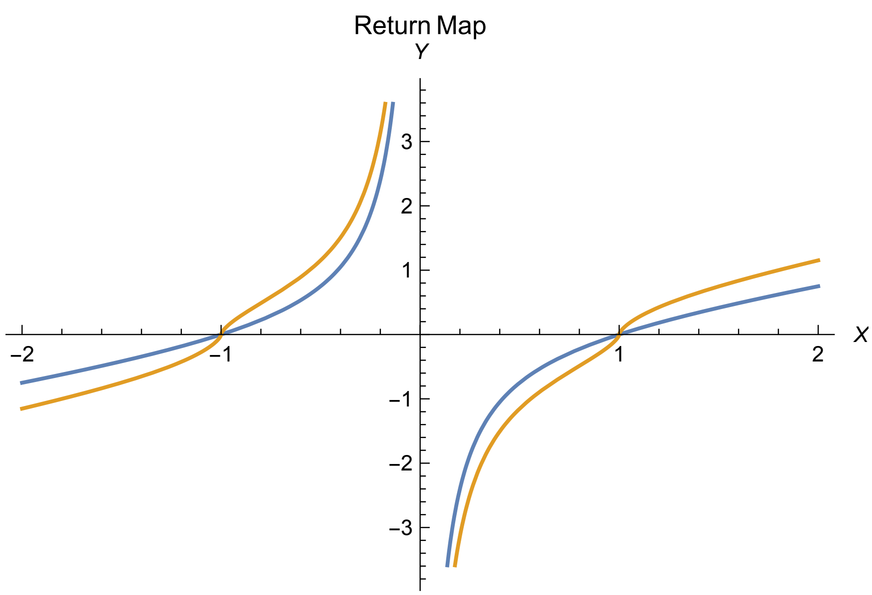

Fig. 1 Chaotic maps generating Cauchy distribution $\alpha = 1$ (blue) and Levy's stable law with $\alpha = 3/2$ (orange).

By ergodicity, the characteristic function of the probability measure $\mu(dx) = \rho(x)dx$ can thus be computed by

$$\phi(k) = \int_{-\infty}^{\infty} \exp(\mathrm{I}kx)\rho(x)dx = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \exp(\mathrm{I}kX_n) \quad \text{a.e.}$$

In this case, an empirical characteristic function $\phi(k; N) = \frac{1}{N}\sum_{n=1}^{N} \exp(\mathrm{I}X_n k)$ computed by finite N points converges to the characteristic function. That is the theoretical foundation of analysis called Chaos Fourier Transform (Umeno 2016). Furthermore, we can compute the mean square deviation of the characteristic function by $V_k(N)$ as

$$V_k(N) \equiv E[|\phi(k; N) - \phi(k)^2] = \frac{D_k}{N} + \frac{E_k}{N^2}$$

where D_k and E_k are:

$$D_k = 1 - \phi(k)^2 + 2\sum_{l=1}^{N} \{\langle e^{\mathrm{I}k(X_0 - X_l)}\rangle - |\phi(k)|^2\}$$

$$E_k = -2\sum_{l=1}^{N} l\{\langle e^{\mathrm{I}k(X_0 - X_l)}\rangle - |\phi(k)|^2\}.$$

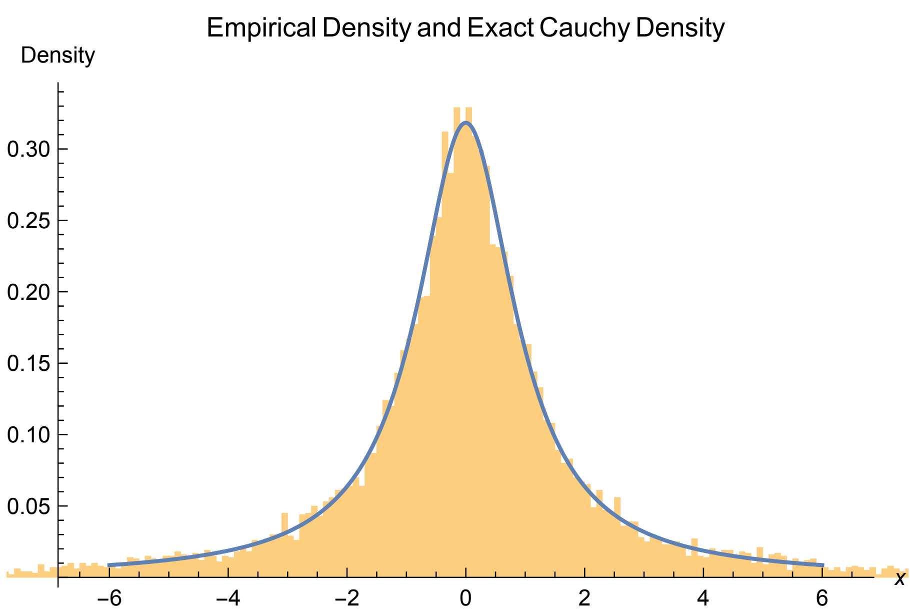

Fig. 2 Empirical Density $N = 100,000$ and Cauchy Probability Density

In this representation, a term $2\sum_{l=1}^{N}\{\langle e^{\mathrm{I}k(X_0-X_l)}\rangle - |\phi(k)|^2\}$ corresponds to a *correlation term* that plays a key role in estimating fluctuations of deviations for the ergodic sum (Umeno 2000). Because an investigation of characteristic function is more essential to capture a feature of fat-tail distributions such as stable law rather than an investigation of simple distribution function [see Fukunaga and Umeno (2017) and Kakinaka and Umeno (2020a, 2020b) for detailed explanation on this matter], we are now motivated to define a new correlation function $C_k(l)$ as

$$C_k(l) \equiv \langle e^{\mathrm{I}k(X_0-X_l)}\rangle - |\phi(k)|^2. \tag{15}$$

This correlation function can be computed by the following formulae:

$$\hat{C}_k(l;N) \equiv \frac{1}{N}\sum_{n=1}^{N} e^{\mathrm{I}k(X_n-X_{n+l})} - \left|\frac{1}{N}\sum_{n=1}^{N} e^{\mathrm{I}kX_n}\right|^2$$

$$\lim_{N\to\infty}\hat{C}_k(l;N) = C_k(l) \quad \text{a.e.}$$

Thus, we call this novel correlation function $C_k(l)$ characteristic function-based correlation function or CF-based correlation function.

Efficient Market Hypothesis

Now we test the EMH (Fama 1970) by investigating whether there exist a nontrivial correlation structure. If the efficient market hypothesis holds, then

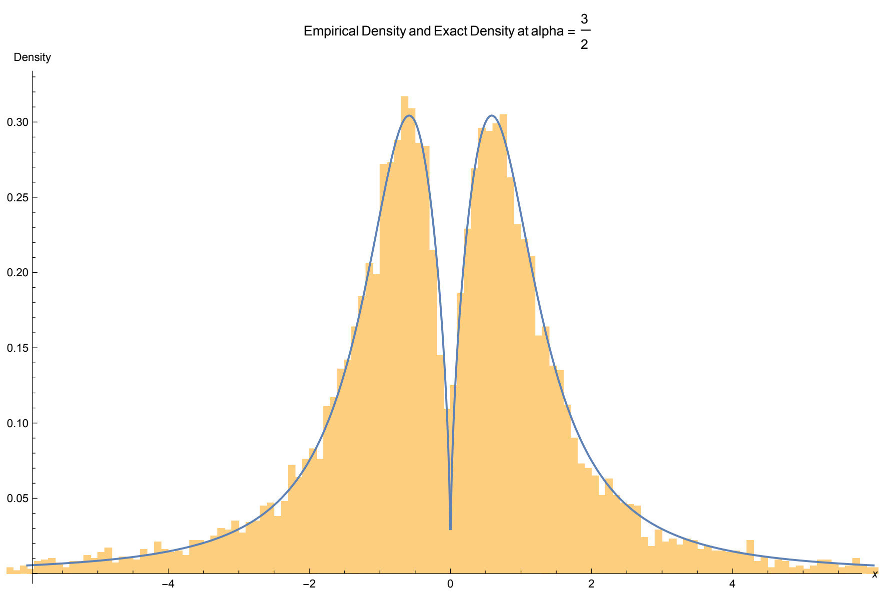

Fig. 3 Empirical Density $N = 100,000$ and Exact Probability Density at $\alpha = \frac{3}{2}$

$$C_k(l) = 0 \quad \text{for} \quad \text{any} \quad l(\neq 0).$$

Thus, to deny the EMH, it is sufficient to detect some correlation such that

$$C_k(l) \neq 0 \quad \text{for} \quad \text{some} \quad l(\neq 0) \text{ and } k(\neq 0).$$

Note that $C_0(l) = 0$ for any l. In Fig. 7 the absolute value of characteristic function-based correlation at $k = 10$ defined by Eq. (15) is depicted for the 5-minute chart of logarithmic returns of Nikkei averages in 2019, which shows no clear correlation supporting EMH. However, in Fig. 8, a *nontrivial periodic structure* (small correlation) is shown to exist for the absolute value of characteristic function-based correlation at $k = 3$ with the 5-minute chart of logarithmic returns of Nikkei averages in 2019, which clearly *deny* the EMH in a direct manner. A clear periodic structure with 5-hour periodicity is shown in Fig. 9, which is an enlarged version of Fig. 8. This 5-hour periodicity is exactly the same as the trading duration per day in the Nikkei stock market in 2019 (the daily trading time is: 9:00−11:30 and 12:30−15:00 in Japan standard time). In Fig. 10, the real part of the characteristic function-based correlation at $k = 3$ with the 5-minute chart of logarithmic returns of Nikkei averages in 2019 is depicted, which shows clear periodic structure in correlation with the *5-hour periodicity* which corresponds to a 60-lag periodicity satisfying the simple relation

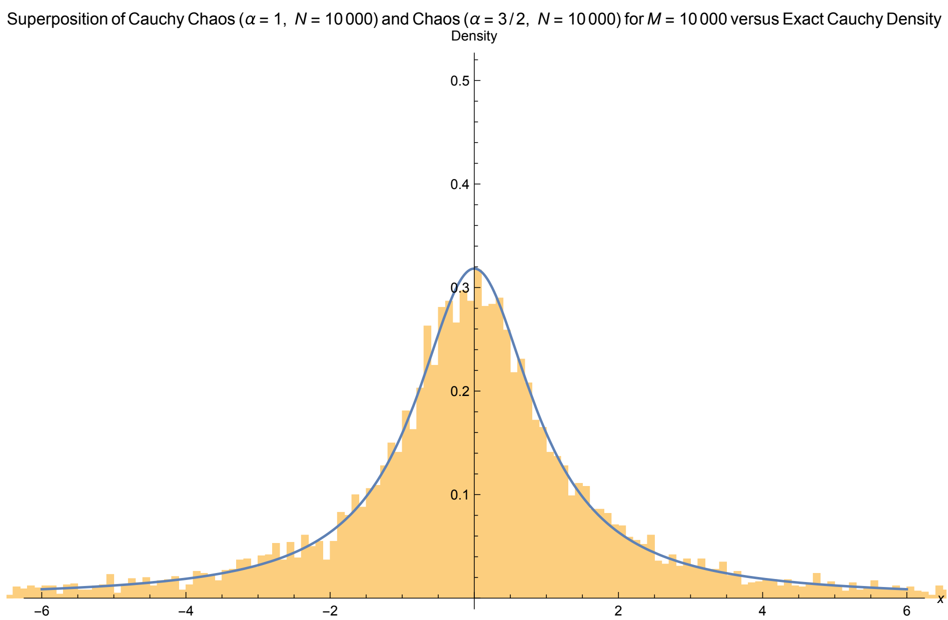

Fig. 4 Mixed superposition of Cauchy Chaos ($\alpha = 1$, $N = 10,000$) and Chaos ($\alpha = 3/2$, $N = 10,000$) for $M = 10,000$.

$$\text{Period} = 60 \text{ (lags)} \times 5 \text{ minutes} = 5 \text{ hours (trading time duration).} \quad (16)$$

On the contrary, no correlation structure is detected in the imaginary part of the characteristic function-based correlation at $k = 3$ as depicted in Fig. 11. This novel periodicity according to the trading time duration also appears in the absolute value of characteristic function-based correlation at $k = 1$ for the 5-minute chart of logarithmic returns of Nikkei averages in 2019 as depicted in Fig. 12. Thus, this periodicity detected with CF-based correlation is NOT a peculiar feature at special $k(\neq 0)$ but rather a *universal* phenomenon capturing the market periodicity corresponding to the daily periodicity with the market. Such an exploration of novel periodic structure of financial markets such as the commodity market is now extensively investigated in our group (Shiihashi 2020).

5 Elucidation of Chaotic Market Hypothesis

We are now approaching the construction of new model of financial markets based on our findings about the universal super generalized central limit theorem (USGCLT) and nontrivial periodic structure in CF-based correlation functions as discussed in the previous sections. While the efficient market hypothesis (EMH) well captures the random structure of financial markets, we are now focused on the detailed distribution structure and its mechanism to show the universality of stable law and nontrivial correlation structure, both of which cannot be captured by EMH. In Fig. 13, the whole time series of the rescaled logarithmic returns fo the Nikkei Averages in 2019 is depicted, which clearly shows randomness. Thus, a new model should also possess a capability of explaining randomness of the financial markets in addition to the capabilities of explaining the universality of stable law and nontrivial correlation structure. In Fig. 14, an empirical probability density of rescaled logarithmic returns of the Nikkei Averages in 2019 is compared to the stable law with $\alpha = 1.4$, which shows that the USGCLT seems to hold such that stable law at $\alpha = 1.4$ matches the empirical probability density for the Nikkei Averages in 2019. After seeing the evidence to support USGCLT and nonrandom time correlation structure in randomness look-like time series of price fluctuaions, we are now motivated to propose the following chaotic market hypothesis capturing these properties as follows.

Chaotic Market Hypothesis:

For financial market we say that *chaotic market hypothesis* (CMH) holds (1) if stable law can *universally* capture the averaged data (index data) of price fluctuations by the universal super generalized central limit theorem (USGCLT) -*random fat-tailed* behavior- and (2) if the characteristic function based correlation functions show non-zero (sometimes periodic) correlation structure as

$$C_k(l) = \langle e^{\mathrm{I}k(X_0 - X_l)} \rangle - |\phi(k)|^2 \neq 0 \text{ for } l(\neq 0) \text{ and } k(\neq 0)$$

-*nonrandom (sometimes periodic)* time structure-.

Here we use the term "chaotic" in the chaotic market hypothesis coined here because (1) random universal fat-tailed behavior via USGCLT and (2) nonrandom (sometimes periodic) time correlation structure are *compatible* in chaos such as the model of exactly solvable chaos in Eqs. (9) and (10). Empirical data in Fig. 14 and the discovered periodic (nonrandom) correlation structure supports this chaotic market hypothesis (CMH) while the periodic (nonrandom) correlation structure does not support the efficient market hypothesis (EMH). On the contrary, we now see how artificial market price fluctuations generated by superposition of chaotic dynamics mimic the empirical market fluctuations and can support that CMH holds. In Fig. 15, Levy walk (deterministic diffusion process) generated by superposition of 1000 deterministic chaos mappings in Eq. (10) at $\alpha = \frac{3}{2}$ is depicted, which clearly shows that this artificial randomness (Levy walk) matches the empirical randomness generated by an accumulation of logarithmic rate of returns of the Nikkei Average in 2019 (Fig. 16). Thus, this constructive approach of the chaotic market hypothesis (CMH) via the use of chaotic mapping can serve us a "good model" to simulate fat-tailed randomness of price fluctuations to compute tail-risk in the financial market. Because chaotic dynamics generally has a mixing property such that correlation has an *exponential decay* structure whose decay rate is characterized by the Lyapunov exponent $\lambda > 0$, it is highly expected that such an artificial market constructed by superposition of chaotic dynamics also has an exponential decay correlation. In Fig. 17, the real part of characteristic function-based correlation for the CMH-based artificial market by the superposition of 10,000 chaotic dynamics in Eq. (10) at $\alpha = \frac{3}{2}$ is depicted, and clearly shows *exponential decay* structure in correlation as expected. The enlarged figure of Fig. 17 is depicted in Fig. 18, which shows *short-term memory effect* represented by exponential decay in correlation where the short-term memory effect is a universal characteristic of financial markets. Thus, CMH not only characterizes the price fluctuation of financial markets but can also work as an effective method of simulating tail-risk in financial markets, which is an important topic in finance. Thus, CMH can well match the real market and conversely the real market can be well simulated by the model based on CMH. For those reasons, we see CMH as a valid working hypothesis for financial markets.

6 Discussions

The proposed chaotic market hypothesis (CMH) seems similar to the fractal market hypothesis (FMH) proposed by Peter (1991) which is also based on chaos theory. It is known that FMH can also capture the market behavior, in particular persistent behavior with long memory ("trends"") *better* than the efficient market hypothesis (EMH). What is the difference between the CMH and FMH? While the FMH *assumes* self-similarity (fractal property) of a price time-series quantified by the Hurst exponent or the fractal dimension, CMH does NOT assume self-similar structure in the time-series but assume a power-law with an index α in a distribution of price fluctuations, which is theoretically founded by the universal super generalized central limit theorem (USGCLT). Although the power index α in the CMH is related to the Hurst exponent in the FMH, the α is but one of four parameters α, β, γ, and μ to characterize a stable law in CMH via the USGCLT. Furthermore, a skewness parameter β can also be important in characterizing the financial market (Fukunaga and Umeno 2017) as in CMH while there is no similar counterpart parameter such as a skewness parameter in the FMH. Furthermore, a discovered *periodic* structure in CF-based correlation in Section 4 cannot be explained by the FMH where a corresponding correlation structure must also have self-similar structure in FMH and CF-based correlation function is not provided in the theoretical framework of FMH. Thus, the self-similarity assumption in FMH does not match the observed periodicity in CF-based correlation in the empirical study. Because the self-similarity assumption is the crux of FMH, we can argue that the crux of FMH can miss important trading characteristics of financial markets such as daily trading time duration.

7 Conclusions

A generalization of the super generalized central limit theorem (SGCLT) is proposed to show a universality of stable laws in the price fluctuations in financial markets. The validity of the USGCLT is confirmed by numerical simulations using superposition

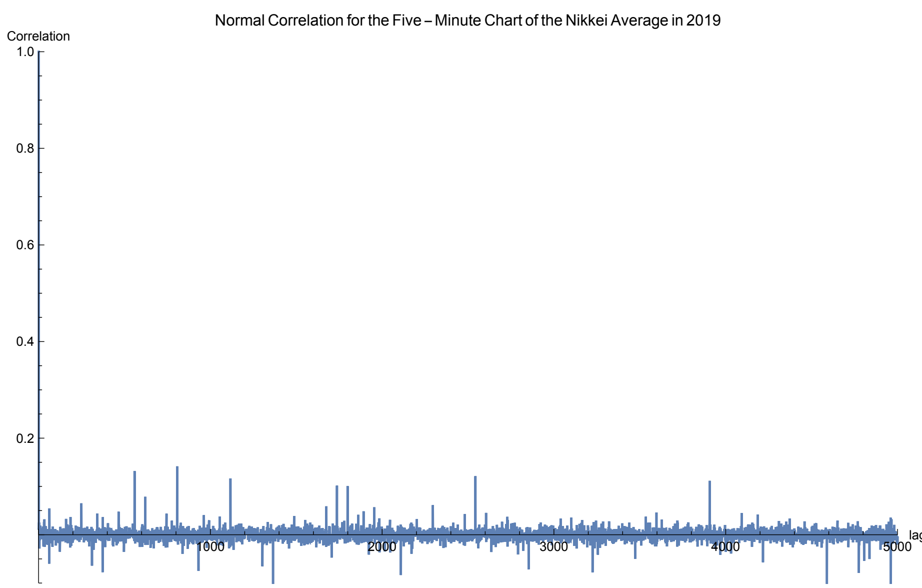

Fig. 5 Normal correlation for the 5-minute chart of log-returns of the Nikkei averages in 2019.

of different kinds of exactly solvable chaotic mappings with power laws with different power indices. Then by using ergodic theory, we propose CF-based correlation to test the efficient market hypothesis (EMH) and directly deny EMH by finding a novel periodic (non-random) structure in CF-based correlation whose periodicity estimated 5 hours is exactly the same as the trading time duration in the 5-minute chart Nikkei averages in 2019, while the normal correlation function cannot capture such kind of periodic structure because of the general fat tail distribution of the price fluctuations making the variance (a normalized constant) divergent to output no correlation. Thus, we conclude that the very CF-based correlation function method based on ergodic theory correctly captures a fine time structure in financial markets. Based on these findings and USGCLT, we propose chaotic market hypothesis (CMH) to capture the essential characteristics of financial market that EMH and other models like the fractal market hypothesis fail to capture. Furthermore, this CMH can be a good model or good simulation method to measure a tailed risk in finance because a constructive approach to modeling of financial markets is easy by direct applications of USGCLT. To conclude, the CMH is based on ergodic theory and is a valid working hypothesis for modeling, featuring and simulating financial markets with theoretical foundation of the universal super generalized central limit theorem (USGCLT).

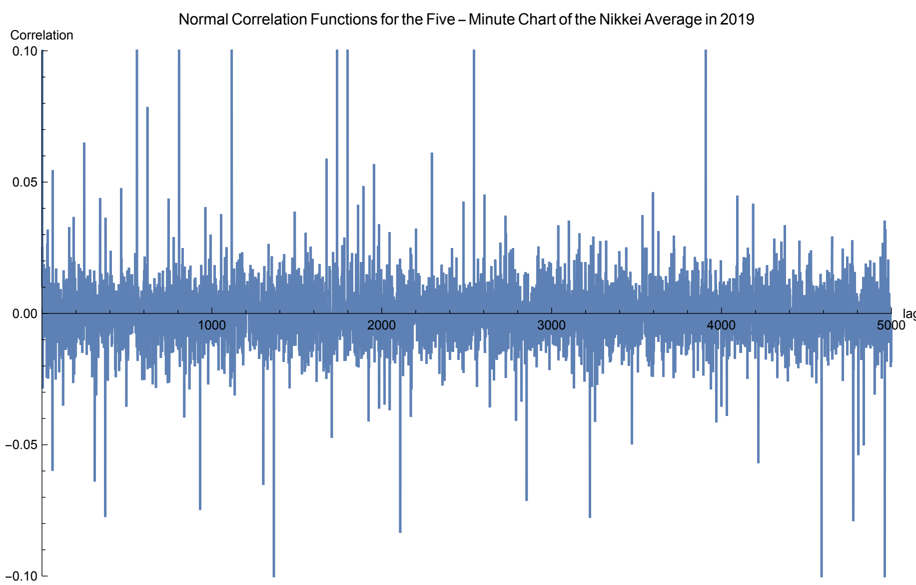

Fig. 6 Enlarged figure of normal correlation for the 5-minute chart of log-returns of the Nikkei averages in 2019.

References

1. Fama, F.:Efficient Capital Markets: A Review of Theory of Empirical Work. *The Journal of Finance* **25**, pp. 383–417 (1970)
2. Gnedenko, B.V., Kolmogorov, A.N.: Limit Distributions for Sums of Independent Random Variables. Addison-Wesley, Cambridge (1954)
3. Arnold, V.I., Avez, A.: Ergodic Problems of Classical Mechanics. Benjamin, New York (1968)
4. Umeno, K.:Method of constructing exactly solvable chaos. *Physical Review E* **55**,pp.5280–5284 (1997)
5. Umeno, K.:Superposition of chaotic processes with convergence to Levy's stable law. *Physical Review E* **58**,pp.2644–2647 (1998)
6. Umeno, K.:Chaotic Monte Carlo Computation: A Dynamical Effect of Random Number Generations. *Japanese Journal of Applied Physics* **39**, 1442–1456 (2000)
7. Umeno, K.:Ergodic Transformations on R preserving Cauchy laws. *NOLTA, IEICE* **7**, pp. 14–20 (2016)
8. Umeno, K., Okubo, K.:Exact Lyapunov exponents of the generalized Boole transformations. *Progress of Theoretical and Experimental Physics*, **21A01** doi:10.1093/ptep/ptv195 (2016)
9. Okubo, K., Umeno, K.: Universality of the route to chaos: Exact analysis. *Progress of Theoretical and Experimental Physics*,**103A01** doi:10.1093/ptep/pty094 (2018)
10. Fukunaga, T., Umeno, K.:Universal Levy's stable law of stock market and its characterization, arXiv:1709.06279 (2017)
11. Shintani, M., Umeno, K.:Super Generalized Central Limit Theorem -Limit Distributions for Sums of Non-identical Random Variables with Power Laws-. *Journal of the Physical Society of Japan* **87**, 043003 (2018) doi:10.7566/JPSJ.87.043003
12. Kakinaka, S., Umeno, K.:Characterizing Cryptocurrency Market with Levy's Stable Distributions. *Journal of the Physical Society of Japan* (2020a) , 024802, doi:10.7566/JPSJ.89.024802
13. Kakinaka, S., Umeno, K.:Flexible two-point selection approach for characteristic function-based parameter estimation of stable laws. *Chaos* (2020b), 073128, doi:10.1063/5.0013148

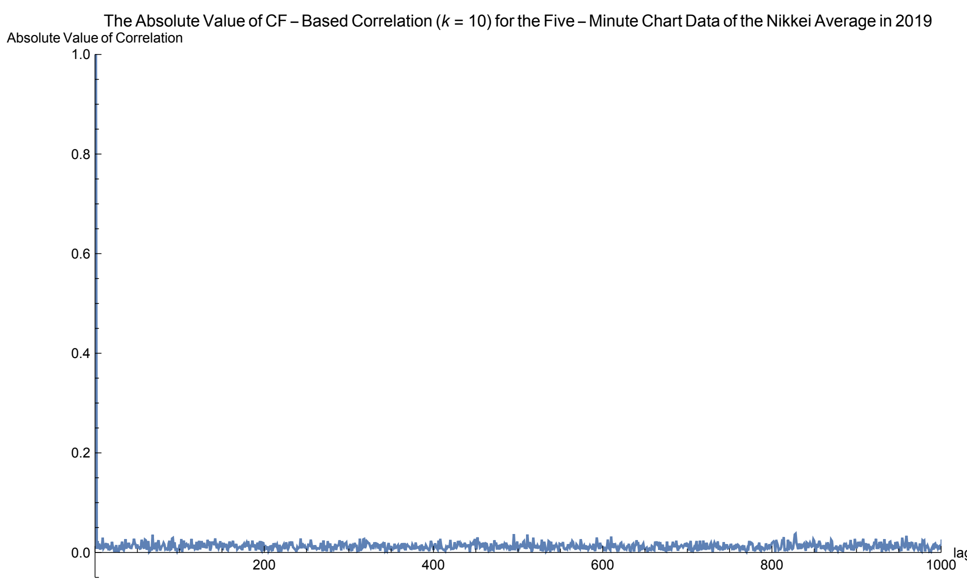

Fig. 7 The absolute value of characteristic function-based correlation ($k = 10$) for the 5-minute chart of log-returns of the Nikkei averages in 2019.

14. Okubo, K., Umeno, K.:Infinite ergodicity that preserves the Lebesgue measure. *Chaos* (2021), 033135, doi:10.1063/5.0029751
15. Shiihashi, K. Graduation thesis, Undergraduate School of Informatics and Mathematical Science, Kyoto University (2020) in Japanese
16. Peter, E.:Fractal Market Analysis: Applying Chaos Theory to Investment and Economics, p. 336. London: Wiley Finance. John Wiley Science (1991)

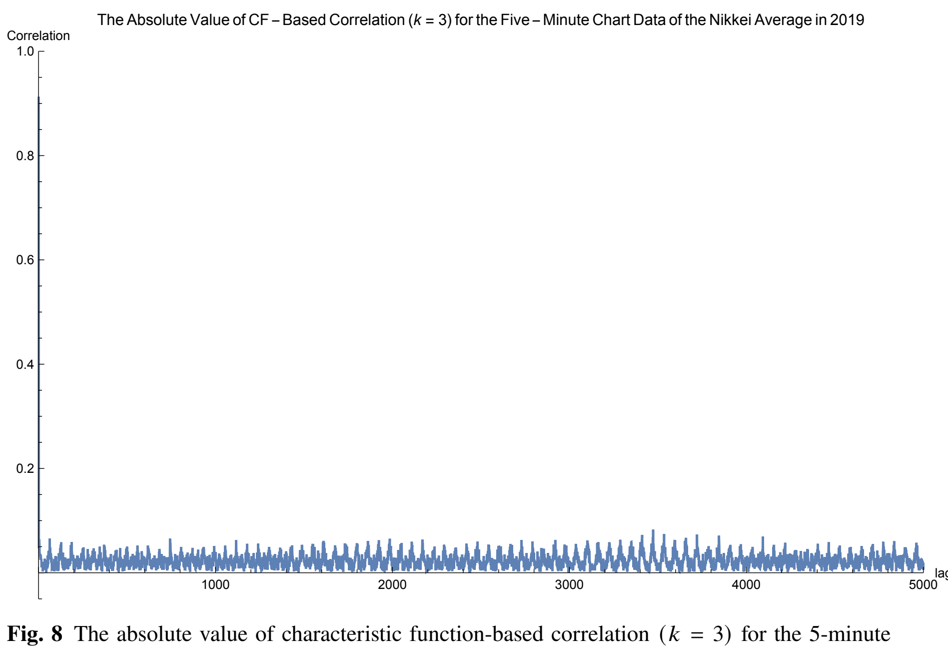

Fig. 8 The absolute value of characteristic function-based correlation ($k = 3$) for the 5-minute chart of log-returns of the Nikkei averages in 2019.

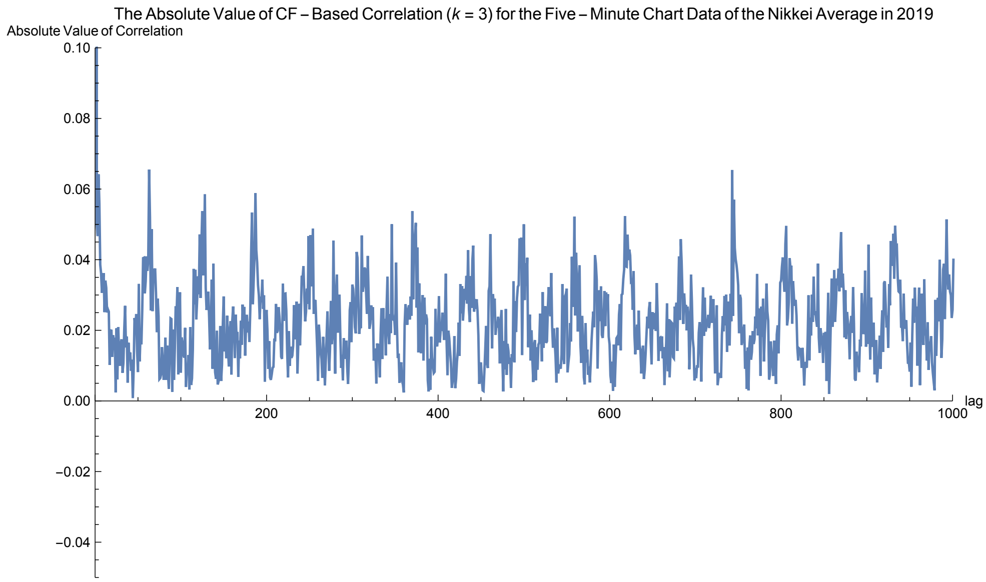

Fig. 9 Enlarged figure of the absolute value of characteristic function-based correlation ($k = 3$) for the 5-minute chart of log-returns of the Nikkei averages in 2019.

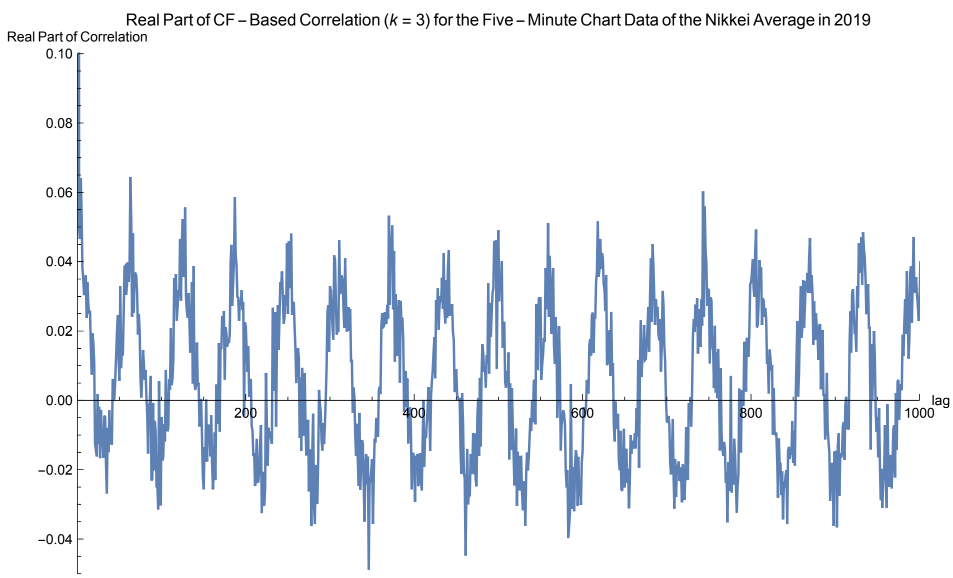

Fig. 10 Real part of characteristic function-based correlation ($k = 3$) for the 5-minute chart of log-returns of the Nikkei averages in 2019.

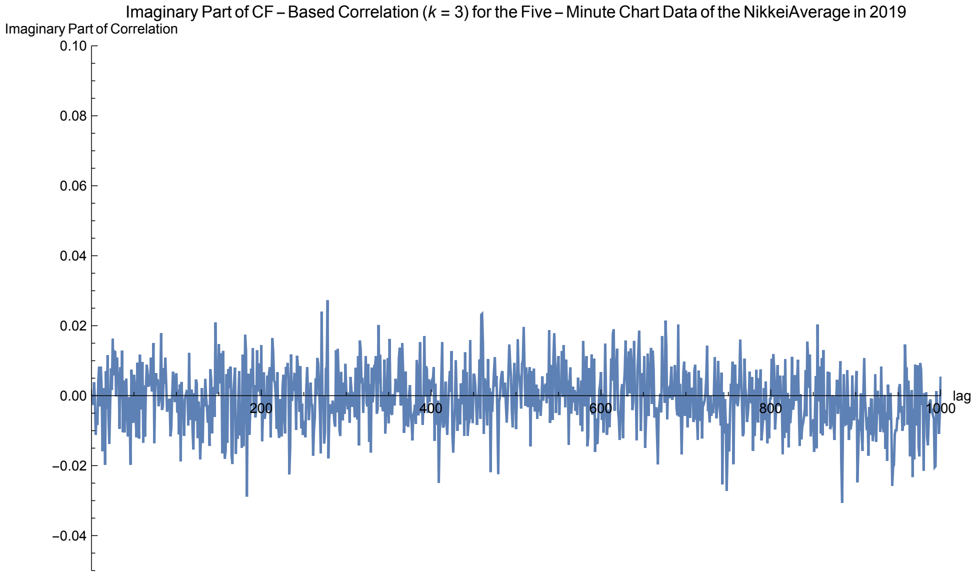

Fig. 11 Imaginary part of characteristic function-based correlation ($k = 3$) for the 5-minute chart of log-returns of the Nikkei averages in 2019.

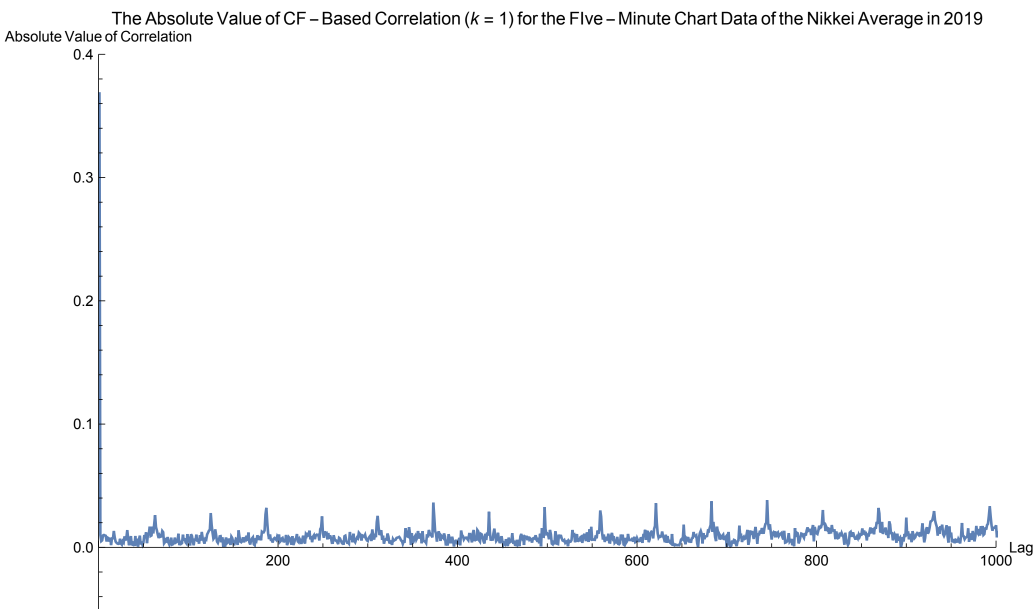

Fig. 12 The absolute value of characteristic function-based correlation ($k = 1$) for the 5-minute chart of log-returns of the Nikkei averages in 2019.

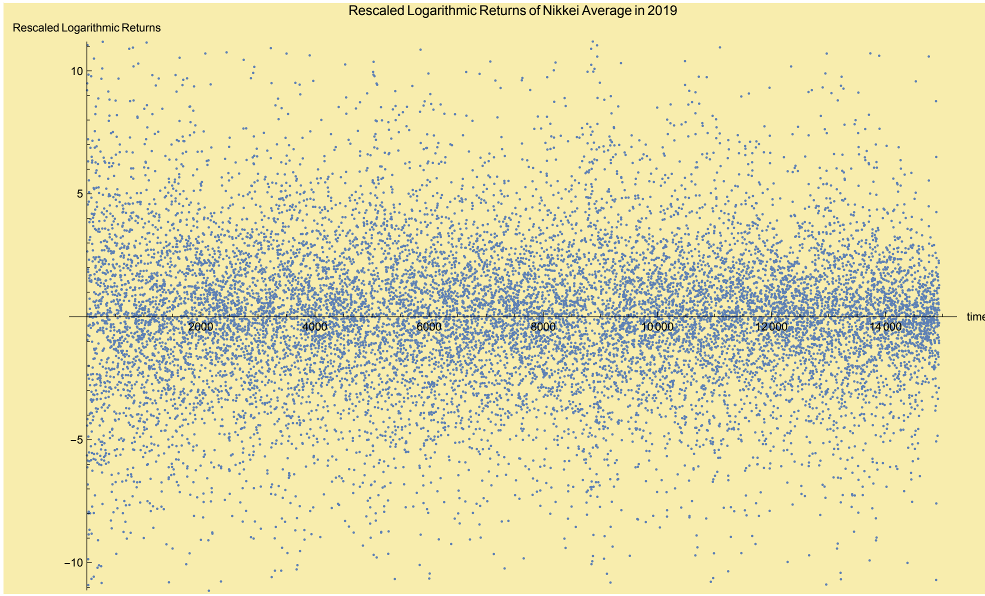

Fig. 13 The whole time series of the rescaled logarithmic returns of the Nikkei averages in 2019.

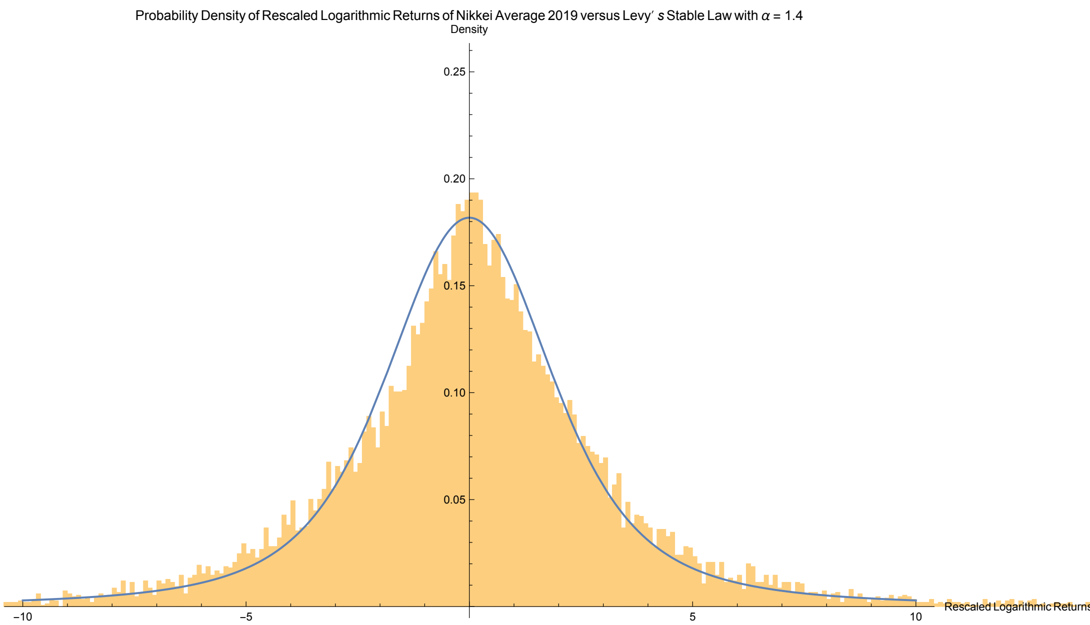

Fig. 14 Empirical probability fensity of rescaled logarithmic returns of the Nikkei averages in 2019 versus Levy's stable law with $\alpha = 1.4$

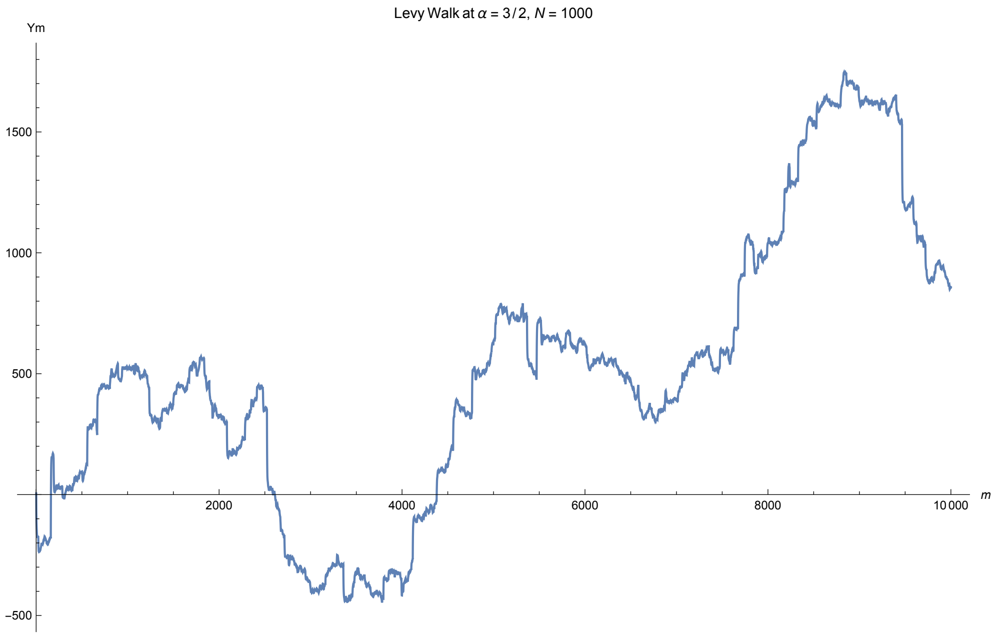

Fig. 15 Levy walk generated by superposition of chaos $\alpha = 3/2$, $N = 1000$.

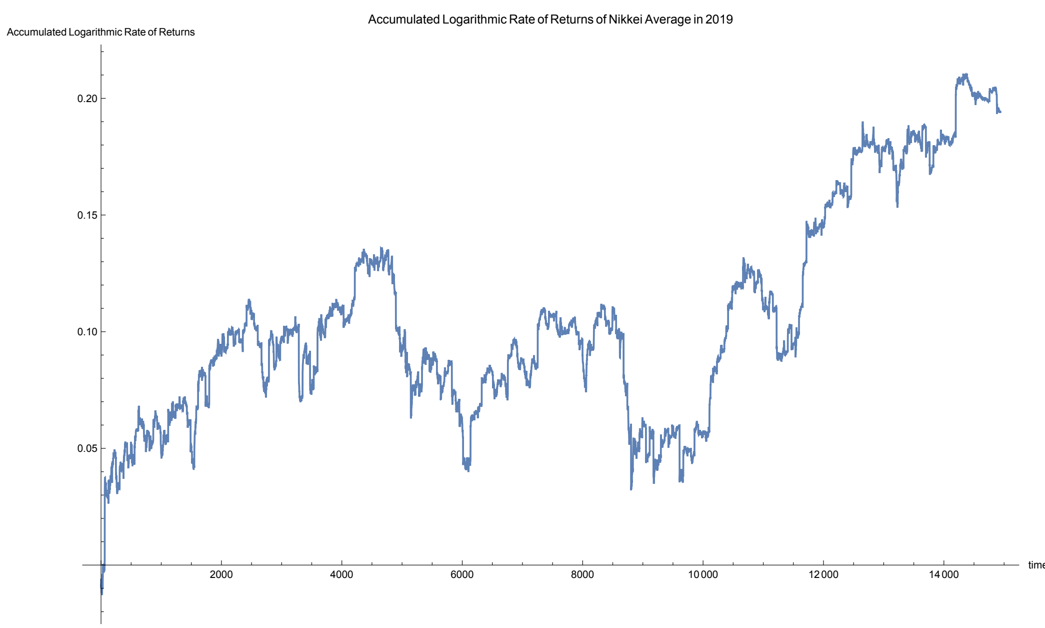

Fig. 16 Accumulated logarithmic rate of returns of the Nikkei averages in 2019.

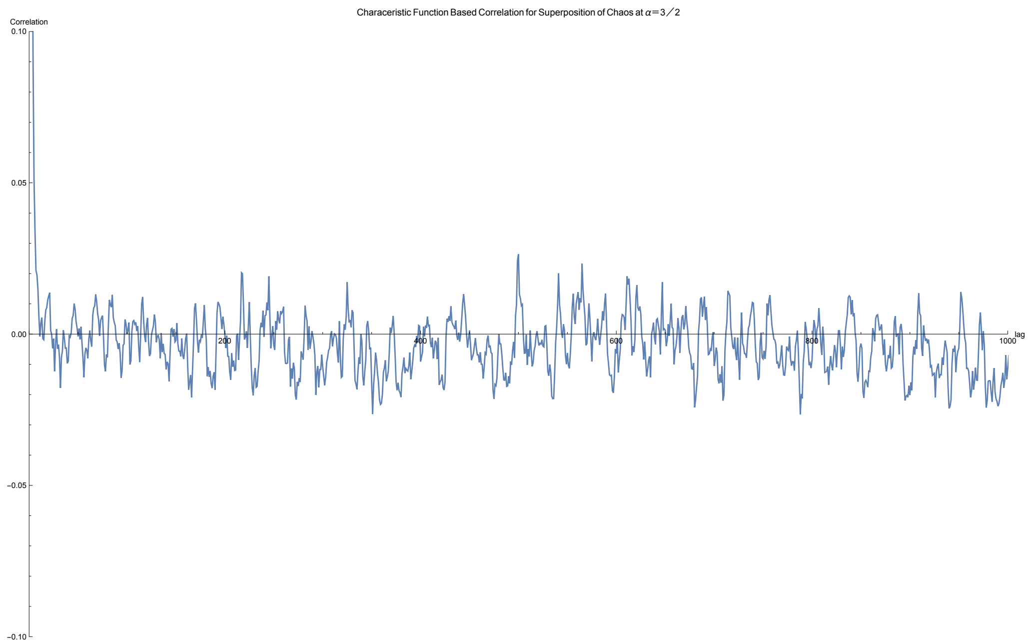

Fig. 17 Real part of characteristic function-based correlation for the superposition ($N = 10,000$) of chaos at $\alpha = 3/2$. Exponential decay of the correlation function can be observed.

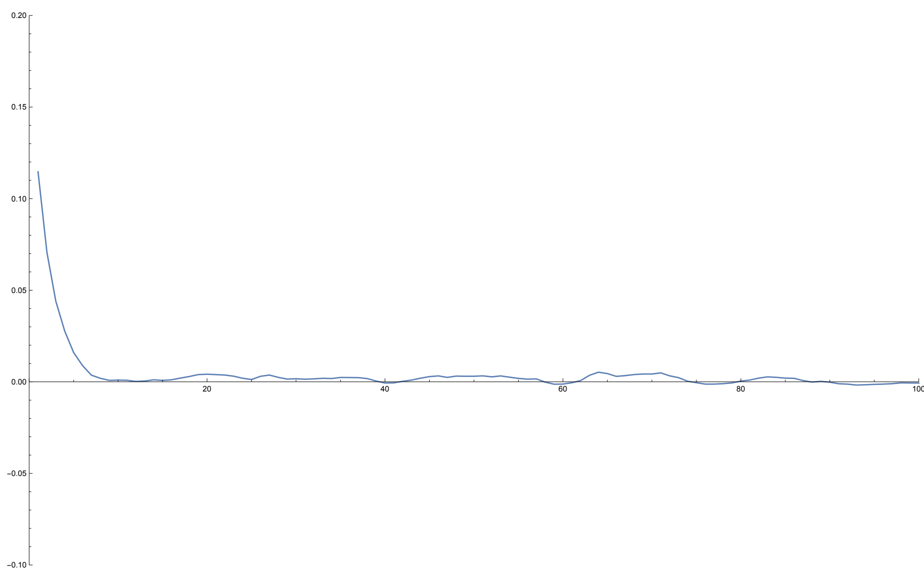

Fig. 18 Enlarged figure of the real part of characteristic function-based correlation for the superposition ($N = 10,000$) of chaos at $\alpha = 3/2$. *Exponential decay* of the correlation function can be observed, which shows the essential characteristic of chaotic nature. The horizontal axis corresponds to a lag while the vertical axis corresponds to the real part of characteristic function-based correlation.