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On Zhu Shijie's Elimination Theory

By

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Abstract

Zhu Shijie's siyuan shu(the four elements method) and his elimination of unknowns made a significant contribution to the history of mathematics in the world in the 14th century. The final equations given by Zhu Shijie in his book *Si Yuan Yu Jian* are sometimes simpler than the resultants obtained from the simultaneous equations. This shows that Zhu Shijie took additional special considerations in eliminating unknowns.

In §4.1 of the present note we will discuss the second problem in Section 6 of the last chapter of Book III of *Si Yuan Yu Jian* , and give two possible arguments which Zhu Shijie would have used to arrive at his final equation. We will also show that it is possible to simplify his final equation for the fifth problem in the same section by using additional considerations as in the second problem. However, he did not use such arguments. This shows that Zhu Shijie did not use the additional considerations systematically.

In §4.3 we will give the list of of the given conditions, the simultaneous equations, the resultant I calculated and Zhu Shijie's final equations of all problems of the section 8 of the last chapter of Book III of *Si Yuan Yu Jian* , which Zhu Shijie used four unknowns to solve the problems. The list shows that only the last problem of Section 8 requires to use of four unknowns, while three unknowns are sufficient for other problems.

Introduction

In northern China in the 12th and 13th centuries, the theory of equations was highly developed, and equations of any order of one unknown could be solved numerically using a kind of synthetic division. Also invented was the tianyuan shu (天元術, in 13th century it was called ruji shu (如積術)(see [Liz, p.32])). The tianyuan shu is a method of writing equations of one unknown. This was followed by the invention of the eruyuna shu (二元術, the two elements method) and the sanyaun shu(三元術, the three elements method). These are methods to write equations of two or three unknowns.

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Finally, Zhu Shijie invented the siyuan shu(四元術, the four elements method). The siyuan shu is a method of writing equations of four unknowns and then eliminating the unknowns to obtain a final equation for one unknown. Zhu Shijie wrote a text book *Si Yuan Yu Jian* (四元玉鑑, Jade Mirror of the Four Unknowns, 1301) and he used these methods to solve the problems in the last chapter of Book III of *Si Yuan Yu Jian* .

In siyuan shu, the coefficients of the equation are arranged as in Figure 1. Not all monomials can be represented in this way. In some cases, ad hoc arrangements are necessary (see, for example, Figure 3 below). On the other hand, the elimination of unknowns was one of the most important results of the time. However, *Si Yuan Yu Jian* only contains a concrete description of how to eliminate the unknown y in the following simultaneous equations.

$$A(x) + B(x)y = 0$$
$$C(x) + D(x)y = 0$$

It is discussed in "Liang Yi Hua Yuan" (両儀化元) of the Section "Si Xiang Xi Cao Xi Ling Zhi Tu"(四象細草仮令之図). Here we see that Zhu Shijie uses a simple factorization to reduce the degree of final equation (resultant) (see §3 below). The following natural question arises here. Did Zhu Shijie use any other factorization? In §2.2 we show that it is possible that he was aware of the factorization. However, since there was no way to represent factorization diagrammatically in the siyuan shu, it was very difficult for him to use factorization systematically.

In his *Si Yuan Yu Jian* , Zhu Shijie only gave the final equation, but did not explain how he derived the final equation from the simultaneous equations of two, or three, or four unknowns. We can only deduce his method from his final equations. In some cases, however, his final equation does not match the resultant obtained by eliminating unknowns. The degree of the final equation is too law compared to the resultant of the original simultaneous equations. Such is the case in the second problem of Section 6 "Zuo You Feng Yuan(左右逢元). In this case, it is likely that Zhu Shijie made additional consideration in order to obtain the final equation of lower degree. In §4.1, we will consider conjectured methods to obtain the final equation. Neither mathematicians nor historians have discussed such a possibility on this issue (see, for example [Lu, Vol.2, pp.906–910] and [Li, p.226]. They discussed only the resultant to obtain the final equation of degree 4.). Also in §4.1 we will take up the fifth problem in the same Section and see that Zhu Shijie used neither factorization nor the fact that the solutions are integers. Thus, Zhu Shijie's method of finding the final equation seems to have been an ad hoc application of the elimination method to individual problems, rather than a systematic one.

Zhu Shijie's siyuan shu did not develop further in Chinese mathematics. The reason for this was that there was no environment that required the kind of advanced mathematics that required the siyuan shu, and there were few mathematicians who could understand the siyuan shu. Another reason was that the notation of the siyuan shu was not suitable for constructing a general theory. Of course, the fact that in Zhu Shijie's time equations were a tool for solving problems and not an object of mathematical research was also a major factor. It took more than 380 years for Takakazu Seki([Se]) and later Bézout([Be]) to establish the general theory of elimination.

Let us explain briefly the content of the present paper. In §1 we briefly explain the siyuan shu. In §2 we will explain multiplication of polynomials in the siyuan shu and discuss the possibility of factorization in the framework of the siyuan shu. In §3 the elimination process described in "Liang Yi Hua Yuan" (両儀化元) of Section "Si Xiang Xi Cao Xi Ling Zhi Tu"(四象細草仮令之図) is discussed. Here, the simultaneous equations are finally reduced to the following form

$$A(x) + B(x)y = 0$$
$$C(x) + D(x)y = 0.$$

Then,when A and C or B and D have a common divisor, Zhu Shijie uses the factorization to reduce the degree of the final equation. In §4.1 we will discuss the second and fifth problems of Section 6 "Zuo You Feng Yuan(左右逢元) of the last chapter of Book III of *Si Yuan Yu Jian* . For the second problem we present the expected methods for obtaining the final equation. In §4.2 and §4.3, we will give the list of of the given conditions, the simultaneous equations, the resultant I calculated and Zhu Shijie's final equations of all problems of Section 7 and Section 8 of the last chapter of Book III of *Si Yuan Yu Jian* , respectively. Zhu Shijie used three unknowns and four unknowns to solve the problems in Section 7 and Section 8, respectively. The list shows that only the last problem of Section 8 requires to use of four unknowns, while three unknowns are sufficient for other problems.

§ 1. On Siyuan Shu

The siyuan shu(四元術, four elements method) is a method to write down algebraic equations in four unknowns on a piece of paper by using symbols of counting-rods and and eliminate unknowns to get the final equation of one unknown.

$$\begin{array}{cccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & y^4w^3 & y^3w^3 & y^2w^3 & yw^3 & w^3 & zw^3 & z^2w^3 & z^3w^3 & \cdots \\
\cdots & y^4w^2 & y^3w^2 & y^2w^2 & yw^2 & w^2 & zw^2 & z^2w^2 & z^3w^2 & \cdots \\
\cdots & y^4w & y^3w & y^2w & yw & w & zw & z^2w & z^3w & \cdots \\
\cdots & y^4 & y^3 & y^2 & y & \text{太} & z & z^2 & z^3 & \cdots \\
\cdots & y^4x & y^3x & y^2x & yx & x & zx & z^2x & z^3x & \cdots \\
\cdots & y^4x^2 & y^3x^2 & y^2x^2 & yx^2 & x^2 & zx^2 & z^2x^2 & z^3x^2 & \cdots \\
\cdots & y^4x^3 & y^3x^3 & y^2x^3 & yx^3 & x^3 & zx^3 & z^2x^3 & z^3x^3 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Figure 1. For example, if we put the 4 at the place y^2x, it represents $4y^2x$. The above diagram shows that there are no places for monomials of three unknowns or four unknowns like xyz, xyzw.

In the following, we will use Arabic numerals instead of the symbol for the counting-rods.

There were also eryuan shu (二元術, two elements method), sanyuan shu(三元術, three elements method). Four unknowns are named as tianyuan(天元), diyuan(地元), renyuan(人元), wuyuan(物元), respectively.

In siyuan shu, the symbols represented by the counting-rods represent the coefficients of the monomials, and each coefficient is placed in the place indicated by the monomial, as was shown in Figure 1. The central Tai (太, abbreviation for 太極 (Taiji))) corresponds to the constant term, and the coefficients of the terms $x^m y^n$, $x^m z^n$, $y^m w^n$ and $z^n w^m$ are well placed. For example, Figure 2 represents $x^2 + y^2 - z^2 = 0$ and $x^2 - y - z = 0$, respectively.

y^2	y		z	z^2	
1		太		-1	
					x
		1			x^2

y^2	y		z	z^2	
	-1	太	-1		
					x
		1			x^2

Figure 2. The left configuration represents $x^2+y^2-z^2 = 0$. The right configuration represents $x^2 - y - z = 0$.

However, for example, there is not place for the term yz. In Figure 3, $4yz$ is placed at the top right corner of the constant term 太. This is ad hoc, and there is no general rule about how to arrange the coefficients of the terms like y^2z^2, xy^2z^2.

y^2	y	$\boxed{4}$	z	z^2	
		太		-3	
	-7		12		x
		-12			x^2

Figure 3. $4yz - 3z^2 + (-7y + 12z)x - 12x^2 = 0$

The eryuan (two elements method) shu is free to express all polynomials and equations in two unknowns, while sanyuan shu (three elements method) and siyuan shu (four elements method) are restricted in the expression. The vector notation (k, l, m, n) for a monomial $x^k y^l z^m w^n$ compensates for this shortcoming and can be generalized to more than four unknowns, but the theory has not developed in this directions, mainly because the unknowns were not explicitly described.

§ 2. Multiplication and Factorization in Siyuan Shu

In this section we will describe the multiplication of polynomials, which is an important process when setting upequations in siyuan shu. We will also explain the factorization of polynomials in siyuan shu.

§ 2.1. Multiplication

Multiplication by the m-th power of an unknown is explained by shifting the diagram by m-lines segments.

For example, multiplying $(x - y - xz^2)$ by x and x^2 is represented as in Figure 4.

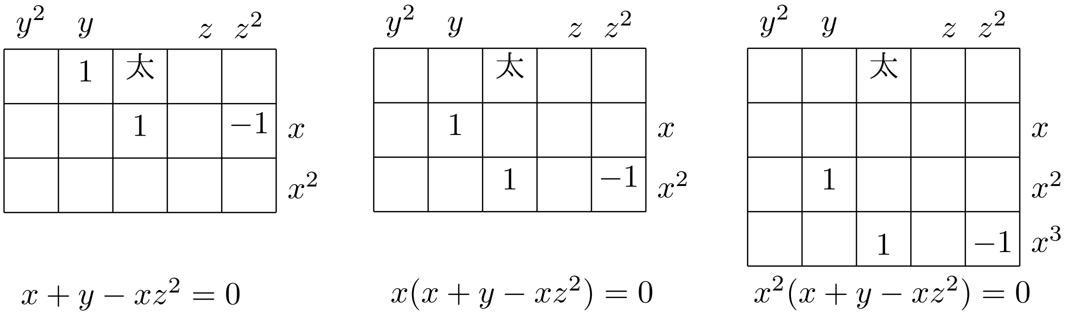

Figure 4. In the left configuration if we shift the configuration by 1 line below, we get the configuration which represents $x(x + y + xz^2)$. If we shift the first configuration by 2 lines below we get we get the one representing $x^2(x + y - xz^2)$.

However, if $x + y + xz^2$ is multiplied by y^2, then special care is needed. If we shift the configuration by 2 lines to the direction of y, we get $xy^2 + y^3 + x$, not $y^2(x + y + xz^2)$. It is difficult to find a suitable place to represent the term xy^2z^2. The best place seems to be the bottom right corner of 太, but this place is also suitable for the term xyz.

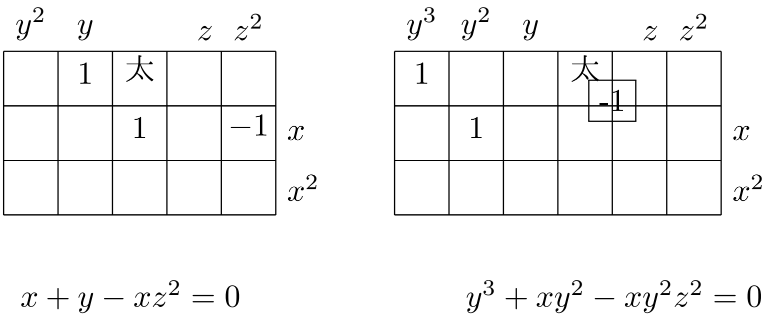

Figure 5. It is difficult to express an equation $y^3 + xy^2 - xy^2z^2 = 0$.

§ 2.2. Factorization

Factorization can be thought of as the inverse process of the multiplication. Although it is difficult to express the factorization in the frame work of siyuan shu, it is not surprising that Zhu Shijie actually knew how to factorize in certain cases. However it is difficult to find any evidence that this is true.

Now let us consider whether the following factorization is possible in siyuan shu.

$$-x^2 - 8xy + 6xz - 8yz + 7z^2 = (x + z)(-x - 8y + 7z)$$

In sanyuan shu the polynomial $-x^2 - 8xy + 6xz - 8yz + 7z^2$ or the equation $-x^2 - 8xy + 6xz - 8yz + 7z^2 = 0$ is expressed in the form

y^2	y	$\boxed{-8}$	z	z^2	
		太		7	
	-8		6		x
		-1			x^2

Looking at the configuration we may find that the polynomial is the sum of two polynomials of the similar type.

y^2	y		z	z^2	
		太			
	-8		7		x
		-1			x^2

$-x^2 - 8xy + 7xz$

y^2	y	$\boxed{-8}$	z	z^2	
		太		7	
			-1		x
					x^2

$-zx - 8zy + 7z^2$

The left-hand configuration is obtained by shifting one line below the following configuration (multiplication by x)and the right-hand configuration is obtained by shifting the following configuration one line to the right of the configuration (multiplication by z).

y^2	y		z	z^2	
	-8	太	7		
		-1			x
					x^2

$-x - 8y + 7z$

y^2	y		z	z^2	
	-8	太	7		
		-1			x
					x^2

$-x - 8y + 7z$

y^2	y		z	z^2	
		太			
	-8		7		x
		-1			x^2

$x(-x - 8y + 7z)$
$= -x^2 - 8xy + 7xz$

y^2	y	$\boxed{-8}$	z	z^2	
		太		7	
			-1		x
					x^2

$z(-x - 8y + 7z)$
$= -xz - 8yz + 7z^2$

The original configuration is the sum of these two configurations, that is $-x^2 - 8xy + 6xz - 8yz + 7z^2$ is obtained by multiplying $-x - 8y + 7z$ by $x + z$. Thus, finding such a factorization is quite possible. On the other hand it is very difficult to express the factorization within the frame work of the sanyuan shu (three elements method) or siyuan shu (four elements method). It is possible to explain it in words, though.

y^2	y	$\boxed{-8}$	z	z^2	
		太		7	
	-8		6		x
		-1			x^2

$$(x+z)(-x-8y+7z) \\ = -x^2 - 8xy + 6xz - 8yz + 7z^2$$

§ 3. Elimination theory

The most important part of siyuan shu (the four elements method) is the elimination of unknowns.

In Section "Si Xiang Xi Cao Xi Ling Zhi Tu"(四象細草仮令之図) Zhu Shijie outlined a method for eliminating unknowns. Three problems were given and solved. The problem of "Liang Yi Hua Yuan" (両儀化元) deals with two unknowns and the basic tools of the elimination. In the problem, Zhu Shijie dealt with a right triangle.

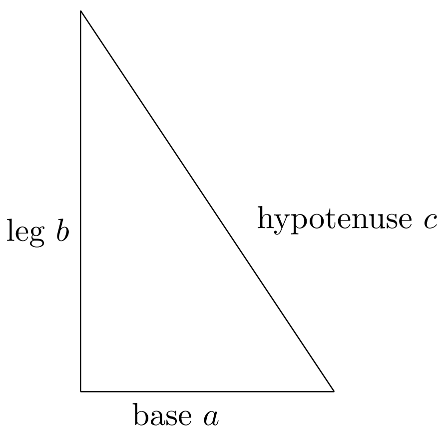

Figure 6. A right triangle.

Problem 3.1. *Let us consider a right triangle The difference between the square of the leg (b^2) and the difference between the hypotenuse and the difference between the leg and base ($c-(b-a)$) is equal to the product of the base and the leg (ab). It is also said that the sum of the square of the base (a^2), the sum of the hypotenuse and the difference of the leg and the base ($c+(b-a)$) is equal to the product of the base and the hypotenuse (ac). Find the leg.*[1]

Let the unknown tiyuan (x) be the leg (b) and the unknown diyuan (y) be the sum of the base and the hypotenuse ($a+c$).

Zhu Shijie gave two equations

$$x^2 + 2xy + 2x^2y - 2y^2 - xy^2 = 0$$
$$x^2 + 2xy + 2y^2 - xy^2 = 0$$

[1] Original text: 今有股冪減弦較較與股乘句等只云句冪加弦較和與句乘弦同問股幾何.

but did not describe the procedure how to obtain these equations.

The problem gives three conditions given be the following equations.

$$
\begin{aligned}
b^2 - \{c - (b-a)\} &= ab \\
a^2 + \{c + (b-a)\} &= ac \\
a^2 + b^2 &= c^2
\end{aligned}
$$

Therefore, strictly speaking, the problem is an example of sanyaun shu. We need to change from the unknowns a, b and c to unknowns a, $x = b$ and $y = a + c$, and eliminate first a to obtain the above simultaneous equations in x and y. Since Zhu Shijie only gave the results by the elimination of a, let us try to eliminate a. The above equations can be rewritten as follows.

$$x^2 - (y - x) = ax \tag{3.1}$$

$$a^2 + (x + y - 2a) = a(y - a) \tag{3.2}$$

$$x^2 - y^2 + 2ya = 0 \tag{3.3}$$

By (3.3) we have

$$2ay = y^2 - x^2.$$

Hence, multiplying $2y$ to the both sides of (3.1) and replacing $2ay$ by $y^2 - x^2$, we obatin

$$2x^2y + 2xy - 2y^2 = xy^2 - x^3.$$

The left hand side and the right hand side are expressed as follows.

$$2x^2y + 2xy - 2y^2 \quad \begin{array}{ccc} -2 & 0 & \text{太} \\ 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}, \qquad xy^2 - x^3 \quad \begin{array}{ccc} 0 & 0 & \text{太} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array}$$

Subtracting the right hand side from the left hand side we obtain the first equation

$$x^3 + 2(x + x^2)y - (2 + x)y^2 = 0 \quad \begin{array}{ccc} -2 & 0 & \text{太} \\ -1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{array} \tag{3.4}$$

To obtain the second equation, first add a^2 to the both side of (3.2) to obtain the equation

$$2a^2 + x + y^2a = ay$$

Then, multiply $2y^2$ to the both sides of the equation to obtain

$$4a^2y^2 + 2xy^2 + 2y^3 - 4ay^2 = 2ay^3$$

Then, replace $2ay$ by $y^2 - x^2$ to obtain

$$(y^2 - x^2)^2 + 2xy^2 + 2y^3 - 2y(y^2 - x^2) = 2y^2(y^2 - x^2)$$

Thus we obtain the equation

$$x^4 + 2x^2y + 2xy^2 - x^2y^2 = 0$$

Dividing the both side by x we obtain

$$x^3 + 2xy + (2-x)y^2 = 0 \qquad \begin{array}{ccc} 1 & 0 & \text{太} \\ -1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \tag{3.5}$$

In this way we obtain two equations given by Zhu Shijie.

Next we need to eliminate y from the two equations (3.4) and (3.5). In *Si Yuan Yu Jian* , Zhu Shie only wrote by "hu yin tong fen" (互隠通分) one obtains the interior polynomial

$$8 + 4x$$

and exterior polynomial

$$2x + x^2$$

Since these polynomials are equal we obtain the final equation

$$-8 - 2x + x^2 = 0$$

Zhu Shijie did not give the procedure how to get the interior and exterior polynomials. So, let us deduce the above result. By subtracting (3.5) from (3.4) we obtain

$$2x^2y - 4y^2 = 0 \qquad \begin{array}{ccc} -4 & 0 & \text{太} \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{array}$$

Dividing the equation by $2y$ we obtain

$$x^2 - 2y = 0 \qquad \begin{array}{ccc} 0 & -2 & \text{太} \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \tag{3.6}$$

To obtain the second equation, multiplying (3.6) by x and subtracting (3.4) from the equation thus obtained we get the equation

$$(-4x - 2x^2)y + (2+x)y^2 = 0 \qquad \begin{array}{ccc} 2 & 0 & \text{太} \\ 1 & -4 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{array}$$

Dividing the both side by y we obtain the second equation

$$(-4x - 2x^2) + (2+x)y = 0 \qquad \begin{array}{ccc} 0 & 2 & \text{太} \\ 0 & 1 & -4 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{array} \tag{3.7}$$

The product of the coefficient of y of (3.6) and the the coefficient of the degree 0 term of (3.7) is $8x + 4x^2$. Dividing it by x we obtain the interior polynomial. The product of the coefficient of y of (3.7) and the the coefficient of the degree 0 term of (3.6) is $2x^2 + x^3$. Dividing it by x we obtain the exterior polynomial. By cancellation of these polynomials, we obtain the final equation

$$-8 - 2x - x^2 = 0$$

In this way we can reconstruct the whole process of the elimination of unknowns.

Also for other two problems in this section he did not give the elimination process. Thus, Zhu Shijie did not describe how to eliminate unknowns except equations of the basic type

$$A + By = 0$$
$$C + Dy = 0$$

In this case the elimination of y is given by

$$AD - BC = 0.$$

Moreover, the above consideration shows that if A and C or B and D have a common divisor one can factor out the common divisor to obtain the final equation.

For the higher degree equations in y one need to find methods to deduce to this simple case. For the simultaneous equations of degree two in y we can use the following process to reduce to the basic situation.

$$a_0 + a_1 y + a_2 y^2 = 0 \tag{3.8}$$
$$b_0 + b_1 y + b_2 y^2 = 0 \tag{3.9}$$

we can proceed as follows. To eliminate y^2 from the equations

$$(a_0 + a_1 y) + a_2 y^2 = 0$$
$$(b_0 + b_1 y) + b_2 y^2 = 0$$

we get the equation

$$(a_0 b_2 - a_2 b_0) + (a_1 b_2 - a_2 b_1) y = 0 \tag{3.10}$$

Of course this equation can be obtained by

$$b_2 \times (3.8) - a_2 \times (3.9) = 0.$$

Also eliminating y by the equations

$$a_0 + (a_1 + a_2 y) y = 0$$
$$b_0 + (b_1 + b_2 y) y = 0$$

we get the equation

$$a_0(b_1 + b_2 y) - b_0(a_1 + a_2 y) = (a_0 b_1 - a_1 b_0) - (a_0 b_2 - a_2 b_0) y = 0 \tag{3.11}$$

The equation can be also obtained by

$$\{a_0 \times (3.9) - b_0 \times (3.8)\}/y = 0$$

The eliminating y from the equations (3.10), (3.11), we get

$$\begin{aligned}&(a_0 b_2 - a_2 b_0) - a_0 b_2 (a_0 b_2 - a_2 b_0) - (a_1 b_2 - a_2 b_1)(a_0 b_1 - a_1 b_0)\\ &\qquad = a_0^2 b_2^2 + a_1^2 b_0 b_2 + a_0 a_2 b_1^2 + a_2^2 b_0^2 - a_0 a_1 b_1 b_2 - 2 a_0 a_2 b_0 b_2 - a_1 a_2 b_0 b_1 = 0\end{aligned}$$

On the other hand Sylvester's resultant is given by

$$\begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & 0 \\ 0 & b_2 & b_1 & b_0 \end{vmatrix} = a_2 \begin{vmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 & 0 \\ b_2 & b_1 & b_0 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_0 & 0 \\ a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 \end{vmatrix}$$
$$= a_2(a_2b_0^2 + a_0b_1^2 - a_0b_0b_2 - a_1b_0b_1) + b_2(a_1^2b_0 + a_0^2b_2 - a_0a_1b_1 - a_0a_2b_0)$$
$$= a_0^2b_2^2 + a_1^2b_0b_2 + a_0a_2b_1^2 + a_2^2b_0^2 - a_0a_1b_1b_2 - 2a_0a_2b_0b_2 - a_1a_2b_0b_1 = 0.$$

Hence the both resultants coincide.

Other cases Zhu Shijie used ad hoc method to reduce the simultaneous equations to the. basic ones.

§ 4. The last chapter of Book III of *Si Yuan Yu Jian*

In the last chapter of *Si Yuan Yu Jian* , Zhu Shijie treated the problems solvable by using two or many unknowns. Zhu Shijie only gave the final equations obtained by eliminating unknowns but did not give his elimination processes. Therefore, we only guess his method by looking at his final equations. Sometimes the degree of the final equation is less than the one of the resultant, we may guess that Zhu Shijie used not only the elimination of the unknowns but also additional consideration. Here we shall discuss the second problem of Section 6"Zuo You Feng Yuan (左右逢元)" . Zhu Shijie's final equation is linear but we cannot get a linear resultant by eliminating unknowns. We shall give two possible arguments how Zhu Shijie would get the linear equation.

In the rest of this section we shall give all the simultaneous equations, the resultants and Zhu Shijie's final equations of all the problems of Sections 7 and 8 of Chpter III, in which simultaneous equations of three and four unknowns are treated.

The list of simultaneous equations shows that almost all problems in these sections are artificial and can be solved with fewer unknowns in practice. This is because the siyuan shu gives a strong restriction on the polynomials that can be represented.

§ 4.1. The second and fifth problems of Section 6 "Zuo You Feng Yuan (左右逢元)" of Book III

There are several problems in which Zhu Shijie's final equations are different from the one obtained by usual elimination of unknowns. Let us discuss one of such problems, the second problem of §6 "Zuo You Feng Yuan (左右逢元)" of the Book III.

Problem 4.1. *For a right triangle the product of the base* (a) *and hypotenuse* (c) *exceeds the product of the base* (a) *and leg* (b) *by* 3*. Also it is said that the product of the leg* (b) *and the hypotenuse* (c) *is less than the square of the hypotenuse* (c) *by* 5 *bu (步). Find the leg.*[2]

Zhu Shijie chooses the tianyuan (x) as the leg (b) and the diyuan (y) as the difference between the hypotenuse and the leg ($c - b$). The final equation Zhu Shijie gave is

$$12 - 3x = 0.$$

[2]Original text: 今有句弦相乘比直積多三步只云股弦相乘比弦冪少五步問句股幾何.

So far nobody has succeeded to obtain the final equation by eliminating the unknowns. The conditions given in the problem are

$$ac - ab = 3 \tag{4.1}$$

$$c^2 - bc = 5 \tag{4.2}$$

where a, b, c are the lengths of the base, the leg and the hypotenuse of a right triangle, respectively. By (4.1) and (4.2) we obtain

$$a(c - b) = 3 \tag{4.3}$$

$$c(c - b) = 5 \tag{4.4}$$

Since Zhu Shijie considered only integral solutions, we may assume a, b, c are natural numbers. Hence, the first equation (4.3) implies $a = 1$ and $c - b = 3$ or $a = 3$ and $c - b = 1$. Similarly the second equation (4.4) implies that $c = 5$ and $c - b = 1$ or $c = 1$ and $c - b = 5$. Therefore, we conclude that $a = 3$, $c = 5$ and $c - b = 1$. Then form (4.1) we infer $12 - 3x = 0$. Thus, Zhu Shiujie may use the condition of integral solutions. However, this solution does not use the elimination.

Therefore, the following argument seems to be more plausible as the one used by Zhu Shijie. The equations (4.3) and (4.4) can be rewritten in the form

$$ay - 3 = 0 \tag{4.5}$$

$$cy - 5 = 0 \tag{4.6}$$

Eliminating y from the both equations we obtain

$$5a - 3c = 0$$

Hence $c = 5a/3$. By $a^2 + b^2 = c^2$ we get $b = 4a/3$, since b is positive. Hence $y = c - b = a/3$. Then by (4.5) $a^2 = 9$, hence $a = 3$. Then, again by (4.1) or by $b = 4a/3$ we have $3x - 12 = 0$.

If we do not use positivity of a, b, c, we need to argue as follows, which give the correct resultant. From (4.5) and (4.6) we infer

$$(xy)^2 = (cy)^2 - (ay)^2 = 5^2 - 4^2 = 4^2. \tag{4.7}$$

The equations (4.6) and (4.7) imply

$$c^2 = \frac{25}{16}x^2.$$

As $y = c - x$ we have

$$5 = cy = c^2 - cx = \frac{25}{16}x^2 - cx.$$

Hence

$$cx = \frac{25}{16}x^2 - 5.$$

and

$$y^2 = (c - x)^2 = c^2 - 2cx + x^2 = \frac{25}{16}x^2 - 2(\frac{25}{16}x^2 - 5) + x^2 = 10 - \frac{9}{16}x^2$$

Therefore, we deduce the equation

$$16 = (xy)^2 = x^2(10 - \frac{9}{16}x^2).$$

Thus

$$256 - 160x^2 + 9x^4 = 0 \tag{4.8}$$

is the resultant of the simultaneous equations

$$\begin{aligned} ax &= 3 \\ (x+y)y &= 5 \\ a^2 + x^2 &= (x+y)^2 \end{aligned}$$

The roots of the equation (4.8) are $x = \pm 4$ and $\pm 4/3$. For $x = 4$ we have$y = 1$, $a = 3$, for $x = -4$ we have $y = -1$, $a = -3$, for $x = 4/3$ we have $y = -3$, $a = -1$ and for $x = -4/3$ we have$y = 1$, $a = 1$. They are the solutions of the simultaneous equations. This means that the degree of the resultant of the simultaneous equations is at least four. This also supports the guess that Zhu Shijie did not use the elimination procedure fully to obtain his linear equation.

However, Zhu Shijie did not systematically use the integral approach. This can be seen, for example, in the fifth problem of Section 6 of Book III.

Problem 4.2. *There is a rectangle such that by adding the short side* (a) *to the area and by subtracting three times of the long side* (b) *from the result we get* 3 *bu* (歩). *It is said that the square of the long side is equal to the difference of the long side and the short side. Find the area.* [3]

The conditions of the problem are given by the following equations.

$$ab + a - 3b = 3 \tag{4.9}$$

$$a^2 = b - a \tag{4.10}$$

By (4.10) we get $b = a^2 + a$. Then, substituting it to (4.9) we obtain the equation

$$ab + a - 3(a^2 + a) = a(b - 2 - 3a) = 3$$

If we assume that a is a natural number, from the equation we infer that $a = 3$ and $b-2-3a = 1$, or $a = 1$ and $b - 2 - 3a = 3$. The first case is $a = 3$, $b = 12$ and the second case $a = 1$, $b = 6$. The second case contradicts (4.10). Hence $a = 3$, $b = 12$ and the area is 36. However, in this solution we do not use the elimination of unknowns.

Also if Zhu Shijie knew the factorization the first equation (4.9) gives

$$(a-3)(b+1) = 0$$

so that we get immediately $a = 3$ without assuming integrability of the solutions.

Of course the problem asks the area, so it is expected to find the equation satisfying the area. This is the reason why Zhu Shijie did not take such integral approach.

Let x be the area ab and y be the long side b. Hence, $a = x/y$. Then, (4.9) is rewritten in the form

$$x + \frac{x}{y} - 3y - 3 = 0$$

[3] Original text: 今有直積加一平減三長餘有三步只云平冪與較等問積幾何.

Hence we have

$$x + (-3 + x)y - 3y^2 = (x - 3y)(1 + y) = 0 \tag{4.11}$$

Similarly by (4.10) we get

$$x^2 + xy - y^3 = 0 \tag{4.12}$$

If Zhu Shijie knew the factorization (4.11), then he would get $y = x/3$, since $y > 0$. Then, (4.12) implies $36x^2 - x^3 = 0$. Hence the final equation is $36 - x = 0$. However, his final equation is of degree 3.

Let us try to eliminate y of the simultaneous equations:

$$x + (-3 + x)y - 3y^2 = 0 \tag{4.13}$$

$$x^2 + xy - y^3 = 0 \tag{4.14}$$

(4.13) $\times\, y -$ (4.14) $\times\, 3$ gives the equation:

$$-3x^2 - 2xy + (-3 + x)y^2 = 0 \tag{4.15}$$

Then, (4.13) $\times\, (-3 + x) +$ (4.15) $\times\, 3$ gives the equation:

$$x(-3 - 8x) + (9 - 12x + x^2)y = 0 \tag{4.16}$$

Similarly, ((4.13) $\times\, 3x +$ (4.15))$/y$ gives the equation:

$$x(-11 + 3x) - (3 + 8x)y = 0 \tag{4.17}$$

Eliminating y from (4.16) and (4.17) we obtain the resultant:

$$x(-108 + 111x - 111x^2 - +3x^3) = 3x(-36 + 37x - 37x^2 + x^3) = 3x(-36 + x)(1 - x + x^2) = 0$$

Of course as we saw in §3, we may factor out x of the degree 0 terms of (4.16) and (4.17) to obtain the final equation. Indeed, Zhu Shijie's final equation is

$$-36 + 37x - 37x^2 + x^3 = 0$$

§ 4.2. Section 7 "San Cai Bian Tong"(三才變通) of Book III

In Section 7 "San Cai Bian Tong" of Book III eleven problems which can be solved by using three unknowns are treated. The following is the list of the conditions of the problems, the simultaneous equations, the resultant I calculated and Zhu Shijie's final equations.

The symbols a, b and c denote the unknowns to describe the conditions given in the problem. The three unknowns, tianyuan, diyuan, renyuna are denoted by x, y and z respectively, and they are represented by a, b and c as noted below.

First four problems the simultaneous equations contain linear equations so that the simultaneous equations are easily reduced to the ones of two unknowns. From Problem 5 to 10 the same conditions are given and one can easily see that these conditions implies $b = 4a/3$ and $c = 5a/4$. Hence, again the simultaneous equations are reduced to the one of two unknwons.

In the following list we use freely the following notations.

五和和 (sum of the five sums)

$$A = (a + b) + (b + c) + (c + a) + \{c + (a + b)\} + \{c + (b - a)\} \tag{4.18}$$

五較和 (sum of the five differences)

$$B = (b - a) + (c - a) + (c - b) + \{c - (b - a)\} + \{(a + b) - c\} \tag{4.19}$$

Prob. 1. Conditions:

$$ab - \{c + (b - a)\} + (c - b)^2 = \{c - (b - a)\}^2$$
$$a^2 - (a + b + c) = c + (b - c)$$
$$a^2 + b^2 = c^2$$

$x = a$, $y = b$, $z = c$. Find the value of z.
Simultaneous equations:

$$x - x^2 - (1 + 2x)c + (-1 + 3x)y = 0$$
$$x^2 - 2z - 2y = 0$$
$$x^2 - z^2 + y^2 = 0$$

Resultant:

$$-68 + 38z - 2z^2 = -2(34 - 19z + z^2) = (z - 17)(z - 2) = 0$$

Zhu Shiejie's final equation:

$$34 - 19z + z^2 = 0$$

Prob. 2. Conditions:

$$\sqrt{ab + (a + b - c)^2} = 2(c - a)$$
$$\sqrt{\{c - (b - a)\}^2 - (a + b)} = b - (c - b)$$
$$a^2 + b^2 = c^2$$

$x = a$, $y = b$, $z = a + b + c$. Find the value of z.
Simultaneous equations:

$$4yz - 3z^2 + (-7y + 12z)x - 12x^2 = 0$$
$$-y - 5y^2 + 2yz + (-1 - 6y + 2z)x - x^2 = 0$$
$$2yz - z^2 + 2(-y + z)x = 0$$

Resultant:

$$2z(-504 + 1974z - 161z^2) = -14z(-72 - 282z + 23z^2) = 0$$

Zhu Shiejie's final equation:

$$-864 - 4104z + 42228z^2 - 53998z^3 + 4209z^4 = (-72 - 282z + 23z^2)(-12 - 104z + 183z^2) = 0$$

Prob. 3. Conditions:

$$a(ab)/b - a - 3(b - a) = a$$
$$b(a + b) - a = 2ab + b - a$$
$$a^2 + b^2 = c^2$$

$x = a,\ y = b,\ z = (a+b)^2 + c^2 + (b-a)^2 + (ab + a + b) = 3(a^2 + b^2) + ab + a + b$. Find the value of z.
Simultaneous equations:

$$\begin{aligned} -3y + x + x^2 &= 0 \\ -1 + y - x &= 0 \\ -z + y + 3y^2 + (1+y)x + 3x^2 &= 0 \end{aligned}$$

Resultant:

$$-94 + z = 0$$

Zhu Shiejie's final equation:

$$188 - 96z + z^2 = (-2 + z)(-94 + z) = 0$$

Prob. 4. Conditions:

$$\begin{aligned} \frac{\dfrac{b(ab) - 2a}{a} + 2(b-a)}{b} - a &= b - a \\ \frac{ab - b(b-a)}{b} &= 2(b-a) \\ a^2 + b^2 &= c^2 \end{aligned}$$

$x = a,\ y = b,\ z = (ab)^2 + (a+b)^2 + b^2 + a^2 + (b-a)^2 + (a+b) + 2a + 3b + 4(b-a)$. Find the value of z.
Simultaneous equations:

$$\begin{aligned} -(1+x) + y &= 0 \\ 4x - 3y &= 0 \\ -z - x + 3x^2 + 8y + (3 + x^2)y^2 &= 0 \end{aligned}$$

Rrsultant:

$$-z + 248 = 0$$

Zhu Shijie's final equation:

$$79608 - 569z + z^2 = (z - 321)(z - 248) = 0$$

From Problem 5 to Problem10 the same right triangle with conditions

$$\begin{aligned} &\frac{\left[\dfrac{\{ab - (a+b)\}a + b - a - ab}{b} + (a+b)\right]b + b}{a} - (2b + b - a) = a \\ &\frac{\dfrac{ab^2}{b} + ab - a^2}{a} = c \\ &a^2 + b^2 = c^2 \end{aligned}$$

is treated. From these condtions we can easily show that

$$b = \frac{4}{3}a, \quad c = \frac{5}{3}a$$

Prob. 5. $x = a$, $y = b$, $z = A + B + ab + c$. Find the value of z.
Simultaneous equations:

$$15 - 41x + 12x^2 = 0$$
$$3z + 57x + 4x^2 = 0$$

Prob. 6. $x = a$, $y = b$, $z = \{c - (b - a)\}ab$, Find the value of z.
Simultaneous equations:

$$15 - 41x + 12x^2 = 0$$
$$-9z + 16x^3 = 0$$

Resultant:

$$41(-6000 + 46781z - 972z^2) = 0$$

Zhu Shijie's final equation:

$$6000 - 46781z + 972z^2 = 41(-48 + z)(-125 + 972z) = 0$$

Prob. 7. $x = a$, $y = b$, $z = \{c + (a + b)\}(a + b - c)$. Find the value of z.
Simultaneous equations:

$$15 - 41x + 12x^2 = 0$$
$$-3z + 8x^2 = 0$$

Resultant:

$$7200 - 15852z + 648z^2 = -12(-600 + 1321z - 54z^2) = 0$$

Zhu Shijie's final equation:

$$-600 + 1321z - 54z^2 = -(-24 + z)(-25 + 54z) = 0$$

Prob. 8. $x = a$, $y = b$, $z = AB$. Find the value of z.
Simultaneous equations:

$$15 - 41x + 12x^2 = 0$$
$$-3z + 140x^2 = 0$$

Resultant:

$$3307500 - 416115z + 972z^2 = 9(367500 - 46235z + 108z^2) = 0$$

Zhu Shijie's final equation:

$$367500 - 46235z + 108z^2 = (-420 + z)(-875 + 108z) = 0$$

Prob. 9. $x = a$, $y = b$, $z = \frac{c}{B}(a + b + c)$. Find the value of z.

Simultaneous equations:

$$\begin{aligned} 15 - 41x + 12x^2 &= 0 \\ -z + 2x &= 0 \end{aligned}$$

Resultant:

$$30 - 41z + 6z^2 = 0$$

Zhu Shijie's final equation:

$$30 - 41z + 6z^2 = (-6 + z)(-5 + 6z) = 0$$

Prob. 10. $x = a$, $y = b$, $z = \dfrac{b}{A + B + a + b + c} \cdot (ab)^2$. Find the value of z.
Simultaneous equations:

$$\begin{aligned} 15 - 41x + 12x^2 &= 0 \\ -9z + x^4 &= 0 \end{aligned}$$

Resultant:

$$\begin{aligned} &- 2430000 + 725864112z - 80621568z^2 \\ &\qquad = -432(5625 - 1680241z + 186624z^2) = 0 \end{aligned}$$

Zhu Shijie's final equation:

$$5625 - 1680241z + 186624z^2 = (-9 + z)(-625 + 186624z) = 0$$

Prob. 11. Conditions:

$$\begin{aligned} &\sqrt{(c-a)(b+c) + (a+b)} = b + 1 \\ &\sqrt{(c-b)(c+b-a) - c} = (a+b-c)^2 - 3 \\ &a^2 + b^2 = c^2 \end{aligned}$$

$x = a$, $y = b$, $z = a + b - c$. Find the value of z.
Simultaneous equations:

$$\begin{aligned} (y-z)(x+2y-z) + x + y &= (y+1)^2 \\ 2y(x-z) - (z+y-x) &= (z^2-3)^2 \\ 2xy - 2(x+y)z + z^2 &= 0 \end{aligned}$$

Resultant:

$$\begin{aligned} &- 2(3560 + 760z - 7698z^2 - 1134z^3 + 6522z^4 + 720z^5 - 2829z^6 - 233z^7 \\ &\qquad + 670z^8 + 36z^9 - 82z^{10} - 2z^{11} + 4z^{12}) = 0 \end{aligned}$$

Zhu Shijie's final equation:

$$\begin{aligned} &3560 + 760z - 7698z^2 - 1134z^3 + 6522z^4 + 720z^5 - 2829z^6 - 233z^7 \\ &\qquad + 670z^8 + 36z^9 - 82z^{10} - 2z^{11} + 4z^{12} \\ &= (-2+z)(1+z)(2+z)(-890 + 700z + 1002z^2 - 766z^3 \\ &\qquad - 439z^4 + 318z^5 + 88z^6 - 60z^7 - 6z^8 + 4z^9) \\ &= 0 \end{aligned}$$

§ 4.3. Section 8 "Si Xiang Chao Yuan" (四象朝元) of Book III

In Book III, Section 8 "Si Xiang Chao Yuan"(四象朝元), six problems are solved using four unknowns. The following is a list of the conditions of the problems, the simultaneous equations, the resultant I obtained and the final equations of *Si Yuan Yu Jian* . The symbols a, b and c denote the unknowns to describe the conditions given in the problem. The four unknowns, tianyuan, diyuan, renyuna and wuyuan are denoted by x, y, z and w, respectively, and they are represented by a, b and c as noted below. Also the symbol A and B are the same given in (4.18) and (4.19).

The list below shows that only the last problem is a real problem of the siyuan shu. Other problems are essentially the problems of sanyuan shu, since in the simultaneous equations the first three equations consist of three unknowns and only the last one contains the fourth unknown, which apperas linearly.

Prob. 1. The conditions:

$$\sqrt[3]{5a+3b} = b-1$$
$$\sqrt{3(a+b)+4(b-a)} = a+2$$
$$a^2+b^2=c^2$$

$x=a$, $y=b$, $z=c$, $w=c-(b-c)+a+b-c=2a$. Find the value of w.

The simultaneous equations:

$$1+5x+3y^2-y^3=0$$
$$4+5x+x^2-7y=0$$
$$x^2+y^2-z^2=0$$
$$-2x+w=0$$

The resultant and Zhu Shijie's final equation.

$$39360+74080w+5520w^2-280w^3-264w^4-30w^5-w^6=0$$

Prob. 2. The conditions:

$$c+(b-a)=\frac{3}{8}b^2$$
$$\{c-(b-a)\}^2=\frac{1}{4}(a+c)^2$$
$$a^2+b^2=c^2$$

$x=a$, $y=b$, $z=c$, $w=2c+4a+2b$. Find the value of w.

The simultaneous equations:

$$8(z+y-x)-3y^2=0$$
$$4\{z-(y-x)\}^2-(x+z)^2=0$$
$$x^2+y^2-z^2=0$$
$$-w+2z+4x+2y=0$$

Resultant:

$$6(1500 - 140w + 3w^2) = 6(-30 + w)(-50 + 3w) = 0$$

Zhu Shijie's final equation:

$$-60000 + 1100w + 300w^2 - 9w^3 = -(-30 + w)(-50 + 3w)(40 + 3w) = 0$$

Prob. 3. The conditions:

$$\begin{aligned} &b^2 - [\{(a+b) - c\}^2 + (c-a)^2] = c + a \\ &\{c + (a+b)\}^2 - (a+b)\{c + (a+b)\} + (c-a) - 2c^2 = a + b + c \end{aligned}$$

$x = a$, $y = b$, $z = c$, $w = a + b + c$. Find the value of w. Find the value of w.

The simultaneous equations:

$$\begin{aligned} &z + 2z^2 - 2zy + (1 - 4z + 2y)x + 2x^2 = 0 \\ &-z^2 + (-1 + z)y + (-2 + z)x = 0 \\ &-z^2 + y^2 + x^2 = 0 \\ &-w + z + y + x = 0 \end{aligned}$$

Resultant:

$$\begin{aligned} &20(48 - 136w + 35w^2 + 142w^3 - 108w^4 + 8w^5) \\ &\qquad = 20(-12 + w)(-1 + 2w)(4 - 3w - 4w^2 + 4w^3) = 0 \end{aligned}$$

Zhu Shijie's final equation:

$$\begin{aligned} &-48 + 424w - 851w^2 + 68w^3 + 960w^4 - 656w^5 + 48w^6 \\ &\qquad = (-12 + w)(-1 + 2w)(-1 + 6w)(4 - 3w - 4w^2 + 4w^3) = 0 \end{aligned}$$

Prob. 4. The conditions:

$$\begin{aligned} &A + (a + b + c) = 4ab + 2a \\ &(a + b + c) - B = [a + b + c + \{(a+b) - c\}] - b^2 \\ &a^2 + b^2 = c^2 \end{aligned}$$

$x = a$, $y = b$, $z = c$, $w = a + b + c + (a + b + c) + (a + b - c)$. Find the value of w.

The simultaneous equations:

$$\begin{aligned} &5z + 5y - 4xy + x = 0 \\ &z - 3y + y^2 - 3x = 0 \\ &-z^2 + y^2 + x^2 = 0 \\ &-w + z + 3y + x = 0 \end{aligned}$$

Resultant:

$$\begin{aligned}&-7680+60224w+2288w^2-1204w^3-33w^4+4w^5\\&\quad=(-20+w)(-8+w)(12+w)(-4+31w+4w^2)=0\end{aligned}$$

Zhu Shijie's final equation:

$$\begin{aligned}&7680-75584w+118160w^2+5780w^3-2375w^4-70w^5+8w^6\\&\quad=(-20+w)(-8+w)(12+w)(-1+2w)(-4+31w+4w^2)=0\end{aligned}$$

Prob. 5. The conditions:

$$\begin{aligned}&(a^2+b^2+c^2)-A=2(a+b-c)^2\\&A+2a^2+2b^2-8c=\frac{1}{2}B^2+(a+b-c)\\&a^2+b^2=c^2\end{aligned}$$

$x=a$, $y=b$, $z=c$, $w=\sqrt[3]{3a+5b+7c}$. Find the value of w.

The simultaneous equations:

$$\begin{aligned}&-x-2y-2xy-2z+2xz+2yz-z^2=0\\&\quad x+3y-3z=0\\&\quad x^2+y^2-z^2=0\\&-w^3+3x+5y+7z=0\end{aligned}$$

Resultant:

$$-21w^3(3072-112w^3+w^6)=-21w^3(-4+w)(16+4w+w^2)(-48+w^3)=0$$

or

$$w^4-64=0$$

The last equation is equal to the one obtained by Lu Shilin ([Lu, vol.3, p.1075) up to constant. Zhu Shijie's final equation:

$$3072-112w-41w^3=-(-4+w)(768+164w+41w^2)=0$$

Prob. 6.

$$\begin{aligned}Q&=(b-a)^2+(c-a)^2+(c-b)^2+\{c-(b-a)\}^2+\{(a+b)-c\}^2\\&=4a^2-2ab+4b^2-2ac-6bc+4c^2\\&=-2ab-2ac-6bc+8c^2\end{aligned}$$

The conditions:

$$\begin{aligned}&Q-3w+w^2+w^3-2w^4=0\\&(a+b)^2-2ab+(a+b+c)=(w^2)^2+c^2-b\\&\frac{1}{2}(a+b+c)+\{(a+b)-c\}=w^3\end{aligned}$$

$x = a$, $y = b$, $z = c$, w given above. Find the value of w.

The simultaneous equations:

$$\begin{aligned}
&-3w + w^2 + w^3 - 2w^4 - 2xy - 2xz - 6yz + 8z^2 = 0\\
&-w^4 + x + 2y + z = 0\\
&-2w^3 + 3x + 3y - z = 0\\
&\ x^2 + y^2 - z^2 = 0
\end{aligned}$$

The resultant and Zhu Shijie's final equation:

$$\begin{aligned}
1152 - 768w - 640w^2 + 1792w^3 - 384w^4 - 9008w^5 + 19112w^6 - 8799w^7 - 8795w^8\\
+ 12637w^9 + 2030w^{10} - 19168w^{11} + 22292w^{12} - 11112w^{13} + 2006w^{14} = 0
\end{aligned}$$

References

[Be] Bézout, Étienne : Recheres sur le degré des équations résultantes de l'évanouissement des inconnues, et sur les moyens qu'il convient d'employer pour trouver ces éqations. Hist. de l'Acad. Roy. des Sciences, Ann.1764, p.288–338.

[CG] Ch'en Tsai Hsin (English translation), Guo Shuchun (Modern Chinese translation) & Guo Jinhai (revised and supplemented) : *Jade Mirror of the Four Unknowns* I, II, Libray of Chinese classics, Liaoning Educations Press, 2006.

[Li] 李兆華 (Li Zhaohua) : 『四元玉鑑校証』(*Si Yuan Yu Jian Xiozheng*), 科学技術出版, 2007.

[Liz] 李治 (Li Zhi) : 『敬齋古今黈』, 中華書局出版，1995.

[Lu] 羅士琳 (Lu Shilin) : 『四元玉鑑細草』(*Si Yuan Yu Jian Xicao*), 1834, 商務印書館, 1937.

[Se] 關孝和 (Seki Takakzu) : 「解伏題之法」(*Kaifukudai no Ho*), 1683.