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A Re-evaluation of the Classical Chinese Mathematical Text *Wucao Suanjing*

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Abstract

In this paper, we try to give a re-evaluation for the classical Chinese mathematical text *Wucao Suanjing*. The text, written in the sixth century CE, was originally a basic arithmetic text for the use of lower ranking officials in early imperial China. It was later included in the textbook collection, *Ten Books of Mathematical Canons*, compiled officially at the beginning of the Tang empire (618 - 907) for the imperial examinations of mathematics. After reviewing earlier studies and carefully reading the text, we have some conclusions with the perspective of mathematics education. The problems in the *Wucao suanjing* can be solved with only four basic operations, but the alignments of some problems suggest implicit ideas of algebraic correspondence or even function. Many problems in the text can indeed be integrated into the teaching of elementary school mathematics, and some of its problems show a new stage of development for the concept of decimals in China.

§1. Introduction

In recent years, many mathematics educators around the world have tried to explore possibilities of integrating history of mathematics into mathematics teaching, to help improve students' competency and interests in mathematics, with certain levels of success.[1] However, materials developed so far are mainly suitable for high school or undergraduate mathematics, with relatively little contents for advanced mathematics or elementary school arithmetic. Using history of mathematics to teach advanced mathematics is beyond the scope of this article, but the two authors of this paper tried to look for mathematical texts in pre-modern East Asia that might be applied to elementary school teaching. The text the authors found is the well-known Chinese text *Wucao*

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[1] For a general discussion on the application of history in mathematics teaching and learning, refer to, for example, Katz and Tzanakis [1]. In Japan and Taiwan, there is also a trend of using contents in *Wasan* 和算 (traditional Japanese mathematics) as materials in teaching high school mathematics, especially with problems taken from *Sangaku* 算額 (mathematical tablets) dedicated in shrines or temples. See, for instance, 深川英俊、トニー ロスマン [4] and 蘇意雯 [5].

suanjing 五曹算經 (Mathematical Canon of the Five Agencies, sixth century).[2] In this paper, related literature and a brief introduction of the text will be given, and then there will be a re-evaluation for the *Wucao sunajing* with the perspective of mathematics education. Hopefully some lights can be shed on the application of the text in teaching mathematics.

§2. The context and contents of the *Wucao suanjing*

The *Wucao sunagjing* has been well studied by historians of mathematics. Literally speaking, *cao* 曹 is a term for lower-level government agencies in early-imperial China.[3] So the title tells us that the content of the text is about mathematics for the use of local government agencies. The text is believed to have been written in the sixth century by Zhen Luan 甄鸞 of the Northern Zhou (北周) Kingdom in North-Western China. It was likely that Zhen Luan collected arithmetic problems encountered by governments in the local levels, such as *zhou* 州 ("province") and *xian* 縣 ("county"), to form this text.[4] There is a total of 67 problems in the *Wucao suanjing*, all of which are considered relatively elementary by historians of mathematics. It takes only the four arithmetic operations on integers to follow all the procedures for the 67 problems, without resorting to fractions or root extractions.[5]Since the problems are so basic relative to many earlier and contemporaneous mathematical texts, there was no need to have a commentary for those problems. Therefore, the contents of the *Wucao suanjing* that survive to this day contain only its main text without a commentary by any mathematician.[6] In fact, some historians of mathematics believe that, compared to the works of He Chengtian 何承天, Zu Chongzhi 祖沖之 and Zu Geng 祖暅 in the kingdoms in southern China, the *Wucao suanjing* was one of many "popular" mathematical texts that appeared in the kingdoms in northern China in the fifth to sixth centuries.[7]

The first term of the title *Wucao* (five agencies) suggests that the problems are divided into five groups for five different government agencies in the local levels. However, there is no record to show that a single empire or kingdom in China by the sixth century had all five *cao* in their local government systems.[8] So, we shall translate the five *cao* in the text as five "categories" of problems. The five categories in the text are *Tiancao* 田曹 (Arable Land Category), *Bingcao* 兵曹 (Military Category), *Jicao* 集曹 (Assembly Category), *Cangcao* 倉曹 (Warehouse Category), and *Jincao* 金曹 (Valuables Category).

The *Wucao suanjing* was later included in the textbook collection, *Suanjing shishu* 算經十書 (Ten Books of Mathematical Canons), compiled officially at the beginning of the Tang empire (618 - 907) for the imperial education and examinations of mathematics.[9] So, the text was indeed

[2] The version of the text we use to investigate is from 郭書春 [11], pp.297-308.
[3] Refer to, for example, 嚴耕望 [7].
[4] 錢寶琮 [8].
[5] 郭書春 [6].
[6] 錢寶琮 [8].
[7] For example, 郭書春 [6].
[8] 嚴耕望 [7] and 郭書春 [6].
[9] 郭書春 [6].

used in mathematics teaching in pre-modern times, and there could also be some values in modern mathematics education. In the following we shall briefly discuss the contents of the five categories before we move on to the re-evaluations with the perspective of mathematics education.

***Tiancao* 田曹 (Arable Land Category)**: This category gives procedures for various land shapes in 19 problems, all of which seem to be practical problems in real-world arable land shapes. There are basic shapes of square (*fangtian* 方田), oblong (*zhitain* 直田), triangle (*guitian* 圭田), right trapezoid (*xiaotian* 簫田), isosceles trapezoid, (*jitian* 箕田) and circle (*yuantian* 圓田), with the same mathematically correct procedures that can also be found in several other earlier and contemporaneous Chinese mathematical texts. There are also several problems for more complicated rectilinear or curvilinear shapes, with procedures that give approximate values of their area. Some of the procedures are relatively accurate, while some are wide off the mark.[10] Other than these two groups of problems, one problem stands out. Problem 11 of this category gives one half of the length of the diagonal of a square, and asks the reader to find the side of the square. The answer could be found easily with an application of the Pythagorean theorem, but the solution in the text does not do so. Instead of the square root of two, it uses *fangwu xieqi* 方五斜七 ("square five and oblique seven"), an approximate ratio of 5 to 7 between the side and the diagonal of a square, to find the answer. In this way, one needs only basic arithmetic but not root extraction to solve the problem. This clearly shows the text's tendency towards real-world application, and its avoidance of any method that requires procedures beyond simple arithmetic.

***Bingcao* 兵曹 (Military Category)**: This category uses mainly simple multiplication and division to allocate military-related commodities or soldiers in 12 problems. It is worth noting that the *Wucao suanjing* was the earliest Chinese mathematical text that contains a separate category of problems related to military commodities. It might imply that the author, living in the time when regimes changed quickly and different kingdoms in northern and southern China wage wars against one another, felt the need for local government officials to be familiarised with fundamental problems of military resource allocations. The context of the problems in this category are military, but the arithmetic needed to solve the problems are still very simple. We shall discuss one particular problem later in this article in our re-evaluation.

***Jicao* 集曹 (Assembly Category)**: There is a total of 14 problems in this category. Most of these problems involve exchanges between different kinds of grains according to pre-determined ratios to ensure the barter is fair. The method used in pre-modern China to handle those exchange problems is called *jingyou* procedure (今有術).[11] Mathematically speaking, the procedure can be understood in this way: given a, b, and c, find x if $a: b = c: x$. Again, finding x takes no more than multiplication and division. There is also another group of problems which we call the "guest seat" problems. One also needs only multiplication and division to solve them. But, there are again some interesting features about these problems that we believe worth noting, and we shall discuss later

[10] 陳巍、鄒大海 [9].

[11] For a discussion of *jinyou* procedure, refer to, for instance, 傅海倫、石玉華、賈冠軍 [10].

in the paper.

***Cangcao* 倉曹 (Warehouse Category)**: This category contains 12 problems related to tax collection, stacked grain volumes and warehouse capacities. Tax collection problems are also about simple multiplication and division. Grain volumes and warehouse capacities are basically about volumes of solids, including the cuboid, the cylinder, and the whole or a part of the cone. The procedures are all correct. These problems are again obviously meant for real-world applications for local officials.

***Jincao* 金曹 (Valuables Category)**: There is a total of 10 problems in this category. Similar to the Assembly Category, these problems are also about exchanges. The difference is that the commodities being exchanged in this category are relatively valuable, such as different kinds of silk, currency, or precious metal. Like the Assembly Category, the main method in this category is also *jinyou* procedure.

After a quick glance into the individual problems of the *Wucao suanjing*, it is obvious that many of the problems are related to basic arithmetic, and can be used in elementary school teaching, with some modification and translation into modern language and symbols. However, there is more to the text than most educators originally thought about the text, if we put some problems into groups, or put them in the greater context of traditional East-Asian mathematics. In the following sections we shall try to elaborate some of the interesting points in the text with a perspective of mathematics education.

§3. Algebraic correspondence and the concept of function in the *Wucao suanjing*

Algebraic relations and the concept of function are usually taught at the end of elementary school or the beginning of high school. Mathematics educators often try to teach the concept of function with a realistic approach with different representations and models.[12] Although all problems in the *Wucao suangjing* can be solved with arithmetic methods, some problems can be put together and seen as realistic situations suggesting algebraic correspondence or even the concept of function in a verbal representation.

Problems 9 to 12 of the Assembly Category are as follows:[13]

A9. 今有席一領，坐客一十二人。有席一千五百三十八領，問客幾何？
There is a piece of seat mat on which 12 guests can sit. Now there are 1538 pieces of such seat mats. How many guests can sit on them?

A10. 又有席一領，坐客二十三人。有席一千五百領，問客幾何？
Again there is a piece of seat mat on which 23 guests can sit. Now there are 1500 pieces of

[12] Refer to, for example, Makonye [2].
[13] 郭書春 [11], p.303. The original problems and their English translation are listed, and the problem numbers are attached by the authors.

such seat mats. How many guests can sit on them?

A11. 今有席一領，坐客二十三人。有客五十三萬三千六百八十人，問席幾何？
There is a piece of seat mat on which 23 guests can sit. Now there are 533,680 guests. How many such pieces of seat mats are necessary?

A12. 又有席一領，坐客一十五人。有客四萬四千六百二十五人，問席幾何？
Again there is a piece of seat mat on which 15 guests can sit on. Now there are 44,625 guests. How many such pieces of seat mats are necessary?

All four problems are solved in the text by a simple step of multiplication or division, and if one looks at just an individual problem among the four, a simple step of arithmetic might be all that can be seen. However, Zhen Luan put the four problems together, and that might give the reader a feeling of correspondence between the number of pieces of seat mats and that of guests. In particular, the correspondences between the two quantities in these four problems are obviously the following.

number of seat mats × guests per mat = number of guests
number of guests / guests per mat = number of seat mats

For most other kinds of problems in the *Wucao suanjing*, only one question is posed. The combination of the four similar problems here suggests not only that Zhen Luan wanted the reader to do more practice, but he might also implicitly stress the algebraic correspondence or relations between two quantities.

The group of four problems in the Assembly Category is not the only case in the *Wucao suanjing* that might be interpreted as suggesting algebraic relations or the concept of function. Problems 3 to 6 of the Military Category is another group of problems with similar ideas, which are listed as follows:[14]

M3. 今有兵九百七十人，人給米七升。問計幾何？
There are 970 soldiers. Each is given 7 *sheng* of rice. How much rice is given?

M4. 今有兵三千八百三十七人，人給錢五百五十六文。問計幾何？
There are 3,837 soldiers. Each is given 556 *wen* of money. How much money is given?

M5. 今有兵三千一百四十八人，人給布一丈二尺三寸。問計幾何？
There are 3,148 soldiers. Each is given 1 *zhang* 2 *chi* 3 *cun* of cloth. How much cloth is given?

[14] 郭書春 [11], p.301. Again, the original problems and their English translation are listed, and the problem numbers are attached by the authors.

M6. 今有兵一千三百六十二人，人給絹二丈八尺五寸。問計幾何？

There are 1,362 soldiers. Each is given 2 *zhang* 8 *chi* 5 *cun* of silk. How much silk is given?

Mathematically speaking, all four problems have an identical solution with a simple step of multiplication. However, when the four problems are put together, they give the reader a sense of correspondence that is close to the concept of function, as Figure 1 shows.

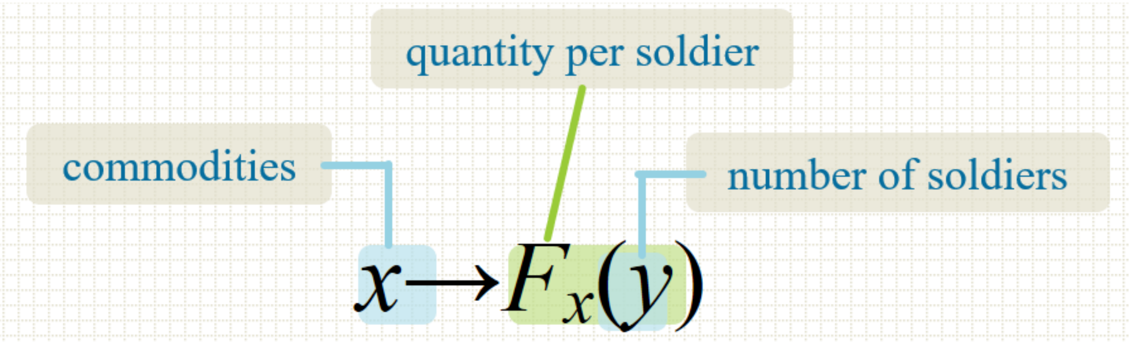

Figure 1. Diagram showing the concept of function in Problems 3 to 6 of the Military Category

In Figure 1, x represents commodities (rice, money, cloth, silk) in the four problems. For each commodity, there is exactly one corresponding procedure (or function F_x) that calculates the total amount of commodity needed to give certain number of soldiers. When one knows the actual number of soldiers y, then one can find the total amount of commodity in question, $F_x(y)$.

If you look at only one of those problems, then it is just one simple formula. When you put the four problems together, you can see that Zhen Luan seemed to implicitly stress that for each commodity, there is exactly one way (function) to distribute it to the soldiers.

From the discussions above, we can see that the two groups of problems may give readers a sense of algebraic correspondence or even function.[15] We want to stress that this does not mean we should believe that the author of the *Wucao suanjing* already possessed the concept of the function *per se* in the sixth century. But it might mean that when compiling the text, Zhen Luan developed some ideas of algebraic correspondence without knowing he did it, and these two groups of problems, in our opinion, could be modified into materials to teach algebraic relations and the concept function in late elementary-school level or early high-school level.

In the next section, we will discuss another aspect of the *Wucao suanjing* with the perspective of mathematics education, which might shed light on not just teaching but also history.

§4. The use of decimals in the *Wucao suanjing*

[15] The idea that the *Wucao suanjing* may contain procedures that can be seen as functions was originally suggested to the authors by Professor Alexei Volkov of National Tsing Hua University. We would like to express our thanks to him.

In Qian Baocong's studies of the *Wucao suanjing*, he mentioned that "In the ninth problem of the Military Category …, the concept of decimals has a new development, and it deserves our attention."[16] However, Qian did not elaborate what he meant by "new development" in any of his works. If we use some well-established perspective of mathematics education, we may understand what Qian meant.

According to educator and philosopher Anna Sfard, when a student learns a mathematical concept, the mental process goes through three stages: interiorisation, condensation, and reification.[17] At the stage of interiorisation, a learner gets acquainted with the processes which will eventually give rise to a new concept (like counting which leads to natural numbers, subtracting which yields negatives, or algebraic manipulations which turn into functions). These processes are operations performed on lower-level mathematical objects. For instance, when learning the concept of rational numbers, the learner first gets more and more familiar with long division. At the next stage, when a student's concept learning goes into condensation, she gets into a period of "squeezing" lengthy sequences of operations into more manageable units. At this stage a person becomes more and more capable of thinking about a given process as a whole, without feeling an urge to go into details. From now on the learner would refer to the process in terms of input-output relations rather than by indicating any operations. For the same example, when learning the concept of rational numbers, at this stage the learner knows that when she has a dividend and a divisor, then she can get a quotient. Finally, only when a person becomes capable of conceiving the notion as a fully-fledged object, we could say that the concept has been "reified", that is, it becomes an object. Various representations of the concept become semantically unified by this abstract, purely imaginary construct. The new entity is soon detached from the process which produced it and begins to draw its meaning from the fact of its being a member of a certain category. For the case of learning the concept of rational numbers, at this stage the learner could see a rational number not just as the result of division but as a "thing", or an object with which one can use to calculate in other procedures. Rational numbers have really become members of "numbers".

How can we use the framework introduced in the previous paragraph to understand the decimals in the *Wucao suanjing*? We need to put them into a greater context. The famous *Jiuzhang suanshu* 九章算術 (Nine Chapters of Mathematical Art) compiled around the beginning of the Common Era and Liu Hui's commentary on it in the third century were considered the paradigm of mathematics in early imperial China.[18] In the main text of the *Jiuzhang suanshu*, all (positive) numbers less than one are represented with fractions. Only in Liu Hui's commentary two centuries later can we see the use of decimals. In Liu Hui's commentary, decimals can be found multiple times, but they always appear as the results of division or root extraction, never as objects to be calculated with. In other words, the stage of development about the concept of decimals could still be at "interiorisation" or "condensation" for Chinese practitioners of mathematics in the third century.

16 錢寶琮 [12].

17 All explanation of Anna Sfard's theory of mathematics concept learning in this section is quoted from her work Sfard [3].

18 For a thorough study of the *Jiuzhang suanshu*, refer to, for instance, 郭書春 [13].

Between Liu Hui's commentary of the *Jiuzhang suanshu* in the third century and Zhen Luan's *Wucao suanjing* in the sixth century, there were several important mathematical treatises in China, such as the *Sunzi suanjing* 孫子算經 (Sunzi's Mathematical Canon) and the *Zhang Chiujian suanjing* 張丘建算經 (Zhang Chiujian's Mathematical Canon) in the fifth century.[19] In the *Zhang Chiujian* suanjing there are very few appearances of decimals, all of which also appear as results of calculation, while in the *Sunzi suanjing* there are more appearances of decimals, most of which are also results of calculation. Only in a few cases in the *Sunzi suanjing* the procedure can be explained as either using integers or decimals to calculate. A detailed discussion of the decimal use of the *Sunzi suanjing* is beyond the scope of this paper, but it is our opinion that it is difficult to conclude that the author of the *Sunzi suanjing* treated decimals as numbers that can be used in calculation.

Now we turn to the *Wucao suanjing* in the sixth century. In problem 9 of the Military Category, the question, answer and procedure are as follows:

> 今有軍糧米三千二百四十六斛八斗七升，每斛直錢四百八十二文。問計幾何？
> 荅曰：一千五百六十四貫九百九十一文三分四釐
> 術曰：列米三千二百四十六斛八斗七升，以四百八十二文乘之，即得。
> There are 3246 *hu* 8 *dou* 7 *sheng* of military provision. Each *hu* is worth 482 *wen* of money. How much is the total value?
> The answer says: 1564 strings 991 *wen* 3 *fen* 4 *li*.
> The procedure says: List 3246 *hu* 8 *dou* 7 *sheng*, and multiply it with 482 *wen*. Then the answer is obtained.

The question gives the unit price of the provision in *wen* 文 per *hu* 斛, so the units in the provision under *hu* (*dou* 斗 and *sheng* 升, with 1 *hu* = 10 *dou* = 100 *sheng*) can be seen as decimals. The amount of provision is then multiplied by the unit price, and the result is represented in *wen*, and the units under *wen* (*fen* 分 and *li* 釐, with 1 *wen* = 10 *fen* = 100 *li*) should again be considered as decimals.[20] Therefore, in Hindu-Arabic numerals, the process of calculation can be represented as

$$3246.87 \times 482 = 1564991.34.$$

So, in this problem we can see that decimals are used both as the *object* to be calculated with and as the *result*. In other words, the concept has reached the stage of "reification" for mathematical practitioners in sixth-century China. We believe this explains the "new development" in Qian Baocong's words, that decimals had been used as an object in calculation since the *Wucao suanjing*.

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[19] Both can be found in the collection 郭書春 [11]. We would like to thank Dr Zhou Xiaohan 周霄漢 of Chinese Academy of Sciences for suggesting us to investigate the two treatises.
[20] An in-depth discussion about the units the measurements in pre-modern and modern China can be found in 吳洛 [14].

§5. Concluding remarks

After re-examining the contents of the *Wucao suanjing*, we are now in a position to give a re-evaluation for the mathematical treatise, in a perspective of mathematics education. Earlier studies have told us that the text contains problems that require no more than basic arithmetic to solve, which is indeed the case if one takes a look into its contents. These problems can obviously be modified and then integrated into elementary school teaching. However, there is more in the treatise.

Individual problems may be solved without any mathematics other than simple arithmetic, but as we have shown in the paper, there are at least two groups of problems in the *Wucao suanjing* that, when considered in groups, suggest implicit algebraic relations or even the concept of function. Although there is still not enough evidence to say that Zhen Luan possessed the concept of function *per se*, the two groups of problems can be, with proper modification, used in teaching algebraic relations and the concept function in late elementary-school level or early high-school level. This is a new discovery that earlier historians or educators of mathematics might not have noticed.

Also, we can verify a claim in Qian Baocong's work that the *Wucao suanjing* had reached a new stage of development regarding the concept of decimals in Chinese mathematics. According to a well-established model for mathematics concept learning, the stage of development about the concept of decimals was still at the stage of "interiorisation" or "condensation" for Chinese practitioners of mathematics by the fifth century. And in the *Wucao suanjing*, we can clearly see that decimals are used both as an object and the result of calculation, so the concept has reached the stage of "reification" for mathematical practitioners in sixth-century China. This is another value that was not clarified by earlier studies.

Therefore, beside showing basic arithmetic method necessary for lower-level government officials in early-imperial China, we believe that the contents of the *Wucao suanjing* can be transformed into materials to help in teaching elementary school arithmetic, and also in teaching algebraic relations and the concept of function in a verbal representation in elementary or high school. And the text does give evidence to show that in the sixth century the use of decimals in Chinese mathematics reached a new level. These are the re-evaluations we give to the classical Chinese mathematical treatise.

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