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# Unitary conjugacy for type III subfactors and W*-superrigidity

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**Abstract.** Let $A, B \subset M$ be inclusions of $\sigma$-finite von Neumann algebras such that $A$ and $B$ are images of faithful normal conditional expectations. In this article, we investigate Popa's intertwining condition $A \preceq_M B$ using modular actions on $A$, $B$, and $M$. In the main theorem, we prove that if $A \preceq_M B$, then an intertwining element for $A \preceq_M B$ also intertwines some modular flows of $A$ and $B$. As a result, we deduce a new characterization of $A \preceq_M B$ in terms of the continuous cores of $A$, $B$, and $M$. Using this new characterization, we prove the first W$^*$-superrigidity type result for group actions on amenable factors. As another application, we characterize stable strong solidity for free product factors in terms of their free product components.

**Keywords.** W$^*$-superrigidity, Popa's intertwining theory, Tomita–Takesaki theory, strong solidity

## Contents

## 1. Introduction

In [35], Sorin Popa obtained the first uniqueness result for certain Cartan subalgebras in non-amenable type II$_1$ factors up to unitary conjugacy. He used this result to compute some invariants of von Neumann algebras and succeeded in giving the first examples of type II$_1$ factors which have trivial fundamental groups, solving a long-standing open problem in von Neumann algebra theory. This breakthrough work led to great progress

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in the classification of non-amenable von Neumann algebras over the last years, which is now called Popa's *deformation/rigidity theory* (see the surveys [26, 40, 50]).

An important technical ingredient in his theory is the *intertwining-by-bimodules* technique [35, 37]. Let $M$ be a finite von Neumann algebra and $A, B \subset M$ von Neumann subalgebras. The *intertwining condition*, which will be written as $A \preceq_M B$, is defined as a weaker version of unitary conjugacy from $A$ into $B$ (see Definition 2.4). Popa proved that this condition is equivalent to an analytic condition: non-existence of a net of unitaries in $A$ with a certain convergence condition. This equivalence provides a very powerful tool to obtain unitary conjugacy between certain subalgebras, and it is now regarded as a fundamental tool to study relations between *general* subalgebras in a von Neumann algebra.

The proof of this analytic characterization relies on the bimodule structure via GNS representations of *traces*. The finiteness assumption of $M$ is hence crucial in this context. However, since there are many natural questions for non-tracial von Neumann algebras (more specifically, for type III factors) which should be studied in deformation/rigidity theory, there have been many attempts to generalize the intertwining machinery to type III von Neumann algebras. In a joint work with C. Houdayer [15], we succeeded in proving the aforementioned analytic characterization in the case when $A$ is finite (and $B \subset M$ can be general), but the general case is still open. See also [2, 7, 18, 22, 28, 48, 49] for other partial generalizations of this technique.

In the present article, we focus on this problem. We will investigate Popa's intertwining condition $A \preceq_M B$ for general inclusions of von Neumann algebras. Before proceeding, we prepare some terminology. For a (possibly non-unital) inclusion of von Neumann algebras $A \subset M$, we say that $A \subset M$ is *with expectation* if there is a faithful normal conditional expectation $E_A \colon 1_A M 1_A \to A$, where $1_A$ is the unit of $A$. For any such expectation $E_A$, we say that a faithful normal positive functional $\varphi \in M_*$ is *preserved by* $E_A$ if it satisfies $\varphi = \varphi(1_A \cdot 1_A) + \varphi(1_A^\perp \cdot 1_A^\perp)$ and $\varphi \circ E_A = \varphi$ on $1_A M 1_A$, where $1_A^\perp := 1_M - 1_A$.

Now we introduce the main theorem in this article. The theorem shows that the intertwining condition $A \preceq_M B$ is equivalent to the same condition but together with additional conditions on *Tomita–Takesaki's modular actions*. More precisely, an intertwining element, which implements a weak unitary conjugacy for $A \preceq_M B$, also intertwines some modular flows for $A$ and $B$. As a result, the condition $A \preceq_M B$ is equivalent to a condition on the continuous cores of $A$, $B$, and $M$ (see item (3) below). This provides a new perspective on the intertwining machinery in type III von Neumann algebra theory. In the theorem below, $\sigma^\varphi$ is the modular action and $C_\varphi(M)$ is the continuous core of $M$ (with respect to $\varphi \in M_*^+$); see Section 2. Recall that a factor $N$ is a *type* $\mathrm{III}_1$ *factor* if its continuous core is a factor. See Definitions 3.4 and 3.7 for intertwining conditions with modular actions and with conditional expectations.

**Theorem A.** *Let $M$ be a $\sigma$-finite von Neumann algebra and $A, B \subset M$ (possibly non-unital) von Neumann subalgebras with expectations. Fix any faithful normal conditional expectation $E_B \colon 1_B M 1_B \to B$ and any faithful state $\varphi \in M_*$ which is preserved by $E_B$. Then the following two conditions are equivalent:*

- *$A \preceq_M B$.*
- *$(A, \sigma^\psi) \preceq_M (B, \sigma^\varphi)$ for some faithful state $\psi \in M_*$ such that $\sigma_t^\psi(A) = A$ for all $t \in \mathbb{R}$ (or equivalently such that $\psi$ is preserved by some conditional expectation onto $A$).*

*Moreover, for any fixed faithful normal conditional expectation $E_A\colon 1_A M 1_A \to A$, any faithful state $\psi \in M_*$ which is preserved by $E_A$, and any $\sigma$-finite type $\mathrm{III}_1$ factor $N$ equipped with a faithful state $\omega \in N_*$, the following conditions are equivalent:*

(1) *$(A, \sigma^\psi) \preceq_M (B, \sigma^\varphi)$.*

(2) *$(A, E_A) \preceq_M (B, E_B)$.*

(3) *$\Pi(C_{\psi\otimes\omega}(A \mathbin{\overline{\otimes}} N)) \preceq_{C_{\varphi\otimes\omega}(M \overline{\otimes} N)} C_{\varphi\otimes\omega}(B \mathbin{\overline{\otimes}} N)$, where $\Pi\colon C_{\psi\otimes\omega}(M \mathbin{\overline{\otimes}} N) \to C_{\varphi\otimes\omega}(M \mathbin{\overline{\otimes}} N)$ is the canonical $*$-isomorphism given by the Connes cocycle.*

The following immediate corollary gives a new characterization of $A \preceq_M B$ in terms of the continuous cores of $A$, $B$, and $M$. Since all continuous cores are semifinite, up to cutting down by a finite projection, one can use the analytic characterization of the intertwining condition at the level of continuous cores.

**Corollary B.** *Keep the setting of Theorem A and fix a type $\mathrm{III}_1$ factor $N$ and a faithful state $\omega \in N_*$. Then $A \preceq_M B$ if and only if item (3) in Theorem A holds for some $E_A$ and $\psi$.*

We emphasize that this corollary *fails* if we do not take tensor products with a type $\mathrm{III}_1$ factor. In fact, there is an inclusion $B \subset M = A$ such that $M \not\preceq_M B$ but $C_\varphi(M) \preceq_{C_\varphi(M)} C_\varphi(B)$ (see [16, Theorem 4.9]). Hence the type $\mathrm{III}_1$ factor $N$ is necessary.

Here we explain the idea behind Theorem A. In [38, 39], Popa proved his celebrated cocycle superrigidity theorem. He developed a way of using his intertwining machinery to study cocycles of actions. If two discrete group actions $\Gamma \curvearrowright^\alpha M$ and $\Gamma \curvearrowright^\beta M$ on a finite von Neumann algebra $M$ are cocycle conjugate (so that $M \rtimes_\beta \Gamma = M \rtimes_\alpha \Gamma$), then the intertwining condition $\mathbb{C}1_M \rtimes_\beta \Gamma \preceq_{M \rtimes_\alpha \Gamma} \mathbb{C}1_M \rtimes_\alpha \Gamma$ is equivalent to a weak conjugacy condition for $\alpha$ and $\beta$ (see Definition 3.1). In [19], by assuming the subalgebra $A$ is trivial (but $B \subset M$ can be general), Houdayer, Shlyakhtenko, and Vaes applied this idea to the case of modular actions. They combined it with *Connes cocycles* and deduced a new characterization of intertwining conditions, in terms of the states of $A$, $B$, and $M$. This new characterization enabled them to identify specific states on von Neumann algebras, and they applied it to the classification of free Araki–Woods factors.

Our Theorem A is strongly motivated by these works. In fact, when the subalgebra $A$ is finite, Theorem A can be proved (without tensoring by a type $\mathrm{III}_1$ factor) by developing ideas in these works. Hence the main interest of Theorem A is the case that $A$ is of type III. It is technically more challenging, since the proofs of [38, 39] and [19] can no longer be adapted. We will use another characterization of $A \preceq_M B$ which holds without the finiteness assumption (see Theorem 2.5(2)). By taking tensor products with a type $\mathrm{III}_1$ factor $N$ and by analyzing operator valued weights on basic constructions, we will connect this condition on $M$ to the one of $C_\varphi(M \mathbin{\overline{\otimes}} N)$. See Lemmas 2.3 and 3.12 for the use of type $\mathrm{III}_1$ factors.

### *Application: $W^*$-superrigidity for actions on amenable factors*

Our first application of Theorem A is to *$W^*$-superrigidity* of group actions on amenable factors. For a group action $\Gamma \curvearrowright^{\beta} B$ on a von Neumann algebra $B$, W$^*$-superrigidity of $\beta$ means that the isomorphism class of the action $\beta$ can be *recovered* from the one of the von Neumann algebra (or the W$^*$-algebra) $B \rtimes_\beta \Gamma$. More precisely, for any action $\Lambda \curvearrowright^{\alpha} A$, if $B \rtimes_\beta \Gamma \simeq A \rtimes_\alpha \Lambda$ as von Neumann algebras, then $\beta \simeq \alpha$ as actions. Here for the action $\alpha$, we only assume natural conditions in the framework (e.g. free and ergodic action) and do not impose any technical assumptions. W$^*$-superrigidity is one of the highlights of deformation/rigidity theory.

The first example of W$^*$-superrigid actions was discovered by Popa and Vaes [42]. They proved that for a large class of amalgamated free groups, any free ergodic probability measure preserving action is W$^*$-superrigid. After this breakthrough work, many examples have been obtained [1,8,17,24,25,34,43,44,51]. All these works are on actions on probability spaces, namely, actions on commutative von Neumann algebras.

In the present article, we investigate actions on *amenable factors*. Recall that a von Neumann algebra $M$ (with separable predual) is *amenable* if it is generated by an increasing union of (countably many) finite-dimensional von Neumann algebras. The amenable von Neumann algebras are the easiest class of von Neumann algebras, which contains all commutative von Neumann algebras. Hence it is natural to ask if a W$^*$-superrigidity phenomenon occurs for actions on non-commutative amenable von Neumann algebras. However, because of the technical difficulties coming from non-commutativity, none of W$^*$-superrigidity type results for such actions is known so far (even for type II$_1$ factors).

We prepare some terminology. We say that a countable discrete group $\Gamma$ is in the *class $\mathcal{C}$* [52] if it is non-amenable and for any trace preserving cocycle action $\Gamma \curvearrowright B$ on a finite von Neumann algebra $B$, the following condition holds:

- for any projection $p \in B \rtimes \Gamma =: M$ and any amenable von Neumann subalgebra $A \subset pMp$, if $A' \cap pMp \subset A$ and if $\mathcal{N}_{pMp}(A)'' \subset pMp$ has essentially finite index, then $A \preceq_M B$.

Here an inclusion $P \subset N$ of finite von Neumann algebras has *essentially finite index* if there is a projection $p \in P' \cap N$ which is arbitrary close to 1 such that $Pp \subset pNp$ has finite Jones index. The class $\mathcal{C}$ contains all weakly amenable groups $\Gamma$ with $\alpha_1^{(2)}(\Gamma) > 0$ [43], all non-amenable hyperbolic groups [44] and all non-amenable free products [25, 51]. As explained in [52], groups in this class do not contain any infinite amenable subgroups. Recall that a faithful normal state $\varphi$ on a von Neumann algebra $M$ is *weakly mixing* if the fixed point algebra of the modular action of $\varphi$ is trivial. In this case $M$ must be a type III$_1$ factor, and the unique amenable type III$_1$ factor admits such a state.

The following theorem is the main application of Theorem A. This is the first W$^*$-superrigidity type result for actions on amenable factors. As we will explain below, the proof of this theorem uses modular theory in a crucial way, and hence cannot be adapted to type II$_1$ factors.

**Theorem C.** *Let $\Gamma$ be an ICC countable discrete group in the class $\mathcal{C}$, $B_0$ a type* III$_1$ *amenable factor with separable predual, and $\varphi_0$ a faithful normal state on $B_0$ which* *is weakly mixing. Then the Bernoulli shift action* $\Gamma \curvearrowright^{\beta} \bigotimes_{\Gamma}(B_0, \varphi_0)$ $(=: (B, \varphi))$ *is* W*-superrigid *in the following sense.*

*Let* $\Lambda \curvearrowright^{\alpha} (A, \psi)$ *be any state preserving outer action of a discrete group* $\Lambda$ *on an amenable factor* $A$ *with a faithful normal state* $\psi$*. If* $B \rtimes_{\beta} \Gamma \simeq A \rtimes_{\alpha} \Lambda$*, then there exist*

- *a finite normal subgroup* $\Lambda_0 \leq \Lambda$ *and a cocycle action* $\Lambda/\Lambda_0 \curvearrowright^{\alpha^{\Lambda/\Lambda_0}} (A \rtimes_{\alpha} \Lambda_0, \psi')$ *by a fixed section* $s\colon \Lambda/\Lambda_0 \to \Lambda$*, where* $\psi'$ *is the canonical extension of* $\psi$ *on* $A \rtimes_{\alpha} \Lambda_0$*;*
- *a state preserving cocycle action* $(\mathrm{Ad}(u_g))_{g\in\Gamma}$ *of* $\Gamma$ *on a type* I *factor* $(\mathbb{B}, \omega)$ *equipped with a faithful normal state,*

*such that the actions* $\Lambda/\Lambda_0 \curvearrowright^{\alpha^{\Lambda/\Lambda_0}} (A \rtimes_{\alpha} \Lambda_0, \psi')$ *and* $\Gamma \curvearrowright^{\beta\otimes\mathrm{Ad}(u)} (B \mathbin{\overline{\otimes}} \mathbb{B}, \varphi \otimes \omega)$ *are conjugate via a state preserving isomorphism.*

The Bernoulli action in this theorem was intensively studied in [52, 53] where similar conclusions were obtained if the action $\Lambda \curvearrowright^{\alpha} (A, \psi)$ is also a Bernoulli action of a group in the class $\mathcal{C}$. Now thanks to our Theorem C, we can take *arbitrary* actions as $\Lambda \curvearrowright^{\alpha} (A, \psi)$.

The conclusion of Theorem C is optimal. Indeed, subgroups and type I factors in the theorem can appear always, since the amenable type III$_1$ factor $B$ has decompositions such as $B = A \rtimes \Lambda_0$ and $B = B \mathbin{\overline{\otimes}} \mathbb{B}$. Note also that the cocycle action $\Lambda/\Lambda_0 \curvearrowright^{\alpha^{\Lambda/\Lambda_0}}$ $(A \rtimes_{\alpha} \Lambda_0, \psi')$ above depends on the choice of the section $s$, but this dependence affects the cocycle action $\mathrm{Ad}(u)$ on a type I factor only.

The proof of Theorem C splits into two steps. Firstly, we prove a unique crossed product decomposition theorem: we identify the base algebra $B$ from the von Neumann algebra $B \rtimes_{\beta} \Gamma$, so that the associated groups are isomorphic and the two actions are cocycle conjugate. Secondly, we prove a cocycle superrigidity type theorem: the corresponding cocycle is cohomologous to a coboundary, so that the two actions are conjugate.

The next theorem treats the first step. Such a unique crossed product decomposition theorem has been intensively studied during the last decade for actions on finite von Neumann algebras [10, 22, 33, 44] (and see aforementioned works for W*-superrigidity). Thanks to our Theorem A, we can take type III factors as base algebras $B$.

**Theorem D.** *Let* $\Gamma$ *be an ICC countable discrete group in the class* $\mathcal{C}$*,* $B$ *a* $\sigma$*-finite, amenable, diffuse factor, and* $\Gamma \curvearrowright^{\beta} B$ *an outer action.*

*Assume that* $B \rtimes_{\beta} \Gamma \simeq A \rtimes_{\alpha} \Lambda$ *for some outer action* $\Lambda \curvearrowright^{\alpha} A$ *of a countable discrete group* $\Lambda$ *on a* $\sigma$*-finite, amenable, diffuse factor* $A$*. Then there is an amenable normal subgroup* $\Lambda_0 \leq \Lambda$ *such that the induced cocycle action* $\Lambda/\Lambda_0 \curvearrowright^{\alpha^{\Lambda/\Lambda_0}} A \rtimes_{\alpha} \Lambda_0$ *is cocycle conjugate to* $\beta$*. In particular if* $\Lambda$ *has no amenable normal subgroups, then* $\beta$ *and* $\alpha$ *are cocycle conjugate.*

If $\Lambda$ is an ICC group in the class $\mathcal{C}$, then it has no amenable subgroups, hence we get the following corollary, which generalizes [43, Theorem 1.10].

**Corollary E.** *Let* $\Gamma \curvearrowright^{\beta} B$ *and* $\Lambda \curvearrowright^{\alpha} A$ *be outer actions of countable discrete ICC groups on* $\sigma$*-finite, amenable, diffuse factors such that* $B \rtimes_{\beta} \Gamma \simeq A \rtimes_{\alpha} \Lambda$*. If* $\Gamma$ *and* $\Lambda$ *are in the class* $\mathcal{C}$*, then* $\beta$ *and* $\alpha$ *are cocycle conjugate.*

We next need a cocycle superrigidity type theorem for the second step. Appropriate adaptations of techniques in [36, 39] (see also [32, 52]) to our setting easily provide the following proposition. This proposition is however *not* useful in our study. As we explain shortly, we will use Popa's argument in the proof of this proposition.

**Proposition F.** *Let* $\Gamma$ *be a non-amenable countable discrete group,* $(B_0, \varphi_0)$ *an amenable factor with separable predual and with a faithful normal state, and* $\Gamma \curvearrowright^{\beta} \bigotimes_{\Gamma}(B_0, \varphi_0) =: (B, \varphi)$ *the Bernoulli shift action. Assume that either* $\Gamma$ *is a direct product of two infinite groups or it has a normal subgroup with relative property* (T).

*Assume that* $\beta$ *is cocycle conjugate to some state preserving outer action* $\Lambda \curvearrowright^{\alpha} (A, \psi)$ *of a countable discrete group* $\Lambda$ *on an amenable factor* $A$ *with a faithful normal state* $\psi$*. Then there exists an inner action* $(\mathrm{Ad}(u_g))_{g\in\Gamma}$ *of* $\Gamma$ *on a type* I *factor* $\mathbb{B}$ *such that the actions* $\alpha$ *and* $\beta \otimes \mathrm{Ad}(u)$ *are conjugate.*

### *Idea of the proof of Theorem C*

The proof uses modular theory in a crucial way. Consider two actions $\beta$ and $\alpha$ as in Theorem C.

Since the group $\Gamma$ is in the class $\mathcal{C}$, we can first apply Theorem D. Then an induced *cocycle action* $\alpha^{\Lambda/\Lambda_0}$ is cocycle conjugate to $\beta$. If this cocycle action is a genuine action, by assuming that $\Gamma$ is a direct product or has property (T), one can apply Proposition F and obtain a conjugacy result. However, it is not clear when the cocycle action, which comes from a section $s\colon \Gamma \simeq \Lambda/\Lambda_0 \to \Lambda$, is a genuine action. In other words, we do not know when the exact sequence $1 \to \Lambda_0 \to \Lambda \to \Gamma \to 1$ splits, where $\Lambda_0$ is amenable and $\Gamma$ is in the class $\mathcal{C}$ satisfying the assumption of Proposition F. This is the main technical issue in proving the W*-superrigidity theorem in our setting, and this is why such a result is not known even for type $\mathrm{II}_1$ factors.

In the present article, to avoid this problem, we use modular actions. Since we have assumed that $\beta$ and $\alpha$ are state preserving, there is an isomorphism

$$B \rtimes_{\beta\times\sigma^{\varphi}} (\Gamma \times \mathbb{R}) \simeq A \rtimes_{\alpha\times\sigma^{\psi}} (\Lambda \times \mathbb{R})$$

such that the corresponding (possibly cocycle) actions are cocycle conjugate. By assuming that $\varphi_0$ is weakly mixing (which means $\sigma^{\varphi}$ is weakly mixing), and combining with some rigidity property of Bernoulli actions, one can apply an argument similar to the proof of Proposition F to the direct product group $\Gamma \times \mathbb{R}$. Here we emphasize that $\mathbb{R}$-actions are always *genuine actions*, so we can avoid the above problem in this context. Thus the cocycle is cohomologous to a coboundary as $\mathbb{R}$-actions. Since $\mathbb{R} \leq \Gamma \times \mathbb{R}$ is normal and $\sigma^{\varphi}$ is weakly mixing, the same conclusion actually holds for $\Gamma \times \mathbb{R}$-actions and we can finish the proof. This is the main idea of the proof of Theorem C.

### *Application: stable strong solidity of free product factors*

The next application is to the structure of amalgamated free product von Neumann algebras. We will generalize Ioana's work [25] to the type III setting.

Recall that for any (possibly non-unital) inclusions $A, B \subset M$ with expectations and with $1_B = 1_M$, we say that $A$ *is injective relative to* $B$ *in* $M$ [29,33] if there is a conditional expectation $E\colon 1_A\langle M, B\rangle 1_A \to A$ which is faithful and normal on $1_A M 1_A$. Recall that for any von Neumann algebra $M$ with a decomposition $M = M_a \oplus M_d$, where $M_a$ is atomic and $M_d$ is diffuse, we say that $M$ is *strongly solid* (resp. *stably strongly solid*) [3,33] if for any diffuse amenable von Neumann algebra $A \subset M_d$ with expectation, $\mathcal{N}_{M_d}(A)''$ (resp. $s\mathcal{N}_{M_d}(A)''$) remains amenable. Here $s\mathcal{N}_{M_d}(A)$ is the set of all elements $x \in M_d$ such that $xAx^* \subset A$ and $x^*Ax \subset A$, and such elements are called *stable normalizers*. Then $\mathcal{N}_{M_d}(A)$ is given by $s\mathcal{N}_{M_d}(A) \cap \mathcal{U}(M_d)$ and its elements are called *normalizers*. Note that these two notions of strong solidity coincide if $M$ is properly infinite. By definition, a strongly solid non-amenable factor $M$ does not admit any crossed product decomposition $M = A \rtimes \Gamma$ (for amenable $A$), so strong solidity should be understood as a strong *indecomposability* of $M$.

The following theorem is a generalization of Ioana's theorem [25, Theorem 1.6] (see also [3, 21, 51]). As a corollary, we characterize stable strong solidity of free product factors; see [25, Theorem 1.8] for the same characterization for type II$_1$ factors.

**Theorem G.** *Let* $B \subset M_i$ *be inclusions of* $\sigma$*-finite von Neumann algebras with expectations* $E_i$ *for* $i = 1, 2$. *Let* $M := (M_1, E_1) *_B (M_2, E_2)$ *be the amalgamated free product von Neumann algebra,* $p \in M$ *a projection, and* $A \subset pMp$ *a von Neumann subalgebra with expectation. Assume that* $A$ *is injective relative to* $B$ *in* $M$ *and assume that* $A' \cap pMp \subset A$. *Then at least one of the following conditions holds true:*

(i) $A \preceq_M B$;

(ii) $s\mathcal{N}_{pMp}(A)'' \preceq_M M_i$ *for some* $i \in \{1, 2\}$;

(iii) $s\mathcal{N}_{pMp}(A)''$ *is injective relative to* $B$.

**Corollary H.** *Let* $I$ *be a set and* $(M_i, \varphi_i)_{i \in I}$ *a family of nontrivial von Neumann algebras with faithful normal states. Put* $M := *_{i \in I}(M_i, \varphi_i)$. *Then* $M$ *is stably strongly solid if and only if so are all* $M_i$*'s.*

Factoriality of free product von Neumann algebras was studied in [47]. Examples of stably strongly solid factors have been obtained in several articles [3, 4, 6, 20]. Also all amenable von Neumann algebras are stably strongly solid. Using these algebras, Corollary H provides plenty of new examples of stably strongly solid factors.

## 2. Preliminaries

### *Tomita–Takesaki theory*

Let $M$ be a von Neumann algebra and $\varphi$ a faithful normal semifinite weight on $M$. Throughout the paper, for objects in Tomita–Takesaki's modular theory, we will use the following notation. The *modular operator, conjugation*, and *action* are denoted by $\Delta_\varphi$, $J_\varphi$, and $\sigma^\varphi$ respectively. The *continuous core*, which is the crossed product von Neumann algebra $M \rtimes_{\sigma^\varphi} \mathbb{R}$, is denoted by $C_\varphi(M)$, and $\mathrm{Tr}_\varphi$ and $L_\varphi\mathbb{R}$ mean the canonical trace on

$C_\varphi(M)$ and the canonical copy of $L\mathbb{R}$ in $C_\varphi(M)$ respectively. The *centralizer algebra* $M_\varphi$ is the fixed point algebra of the modular action. The norm $\|\cdot\|_\infty$ is the operator norm of $M$, while $\|\cdot\|_{2,\varphi}$ (or $\|\cdot\|_\varphi$) is the $L^2$-norm determined by $\varphi$. See [45] for definitions of all these objects.

For any continuous action $G \curvearrowright^\alpha M$ of a locally compact group $G$, in this article we will use the following canonical embeddings for crossed products: $\pi_\alpha\colon M \to M \rtimes_\alpha G$ by $(\pi_\alpha(x)\xi)(g) = \alpha_{g^{-1}}(x)\xi(g)$ for all $\xi \in L^2(G, L^2(M))$ and $g \in G$; and $G \to M \rtimes_\alpha G$ by $g \mapsto 1_M \otimes \lambda_g$ for all $g \in G$. Via these embeddings, we often regard $M$ and $LG$ as subalgebras of $M \rtimes_\alpha G$.

### *Connes cocycle*

Let $G$ be a locally compact group, $M$ a von Neumann algebra and $G \curvearrowright^\alpha M$ a continuous action (see [45, Definition X.1.1] for continuity). Let $p \in M$ be a non-zero projection. We say that a $\sigma$-strongly continuous map $u\colon G \to pM$ is a *generalized cocycle for* $\alpha$ (*with support projection* $p$) if

- $u_{gh} = u_g\alpha_g(u_h)$ for all $g, h \in G$;
- $u_g u_g^* = p$ and $u_g^* u_g = \alpha_g(p)$ for all $g \in G$.

In this case, by putting $\alpha_g^u(pxp) := u_g\alpha_g(pxp)u_g^*$ for all $x \in M$ and $g \in G$, one has a continuous $G$-action on $pMp$. We have $p(M \rtimes_\alpha G)p \simeq pMp \rtimes_{\alpha^u} G$. When $p = 1$, we simply say that $u$ is a *cocycle*.

Let $N$ be another von Neumann algebra and consider continuous actions $G \curvearrowright^\alpha M$ and $G \curvearrowright^\beta N$. We say that $\alpha$ *is cocycle conjugate to* $\beta$ *via a generalized cocycle* if there exist a projection $p \in M$, a $*$-isomorphism $\pi\colon pMp \to N$ and a generalized cocycle $u\colon G \to pM$ for $\alpha$ with support projection $p$ such that

$$\pi^{-1} \circ \beta_g \circ \pi(a) = u_g\alpha_g(a)u_g^* \quad \text{for all } a \in pMp,\ g \in G.$$

In this case, by identifying $pMp = N$ by means of $\pi$, we can define a partial isometry $U\colon L^2(G, L^2(M)) \to L^2(G, L^2(M))$ by $(U\xi)(g) = u_{g^{-1}}\xi(g) = pu_{g^{-1}}\alpha_{g^{-1}}(p)\xi(g)$ for $g \in G$. Note that $U^*U = \pi_\alpha(p)$ and $UU^* = p \otimes 1_{L^2(G)}$. One has a $*$-isomorphism

$$\Pi_{\beta,\alpha} := \mathrm{Ad}(U)\colon p(M \rtimes_\alpha G)p \to pMp \rtimes_\beta G$$

satisfying $\Pi_{\beta,\alpha}(x) = x$ for $x \in pMp$ and $\Pi_{\beta,\alpha}(p\lambda_g^\alpha p) = pu_g\lambda_g^\beta p = u_g\lambda_g^\beta$ for $g \in G$. If one can choose $p = 1$, so that $u$ is a cocycle, then we simply say that $\alpha$ and $\beta$ are *cocycle conjugate*.

Let $M$ be a von Neumann algebra and $\varphi, \psi$ normal semifinite weights on $M$. Assume that $\varphi$ is faithful and let $s(\psi)$ be the support projection of $\psi$. Consider the modular actions $\sigma^\varphi$ on $M$ and $\sigma^\psi$ on $s(\psi)Ms(\psi)$. The *Connes cocycle* $([D\psi, D\varphi]_t)_{t\in\mathbb{R}}$ [11] is a generalized cocycle for $\sigma^\varphi$ with support projection $s(\psi)$ such that $\sigma^\varphi$ is cocycle conjugate to $\sigma^\psi$ via $([D\psi, D\varphi]_t)_{t\in\mathbb{R}}$. In particular, there is a canonical $*$-isomorphism

$$\Pi_{\psi,\varphi}\colon pC_\varphi(M)p = p(M \rtimes_{\sigma^\varphi} G)p \to pMp \rtimes_{\sigma^\psi} G = C_\psi(pMp).$$

See [45, V.III.3.19-20] for this non-faithful version of the Connes cocycle. In this article, we need the following important theorem.

**Theorem 2.1** ([11, Théorème 1.2.4]). *Let $M$ be a von Neumann algebra and $\varphi$ a faithful normal semifinite weight on $M$. Let $p \in M$ be a projection and $(u_t)_{t\in\mathbb{R}}$ a generalized cocycle for $(\sigma_t^\varphi)_t$ with support projection $p$. Then there is a unique normal semifinite weight $\psi$ on $M$ such that $s(\psi) = p$ and $u_t = [D\psi, D\varphi]_t$ for all $t \in \mathbb{R}$.*

Below, we record an elementary lemma. We use the notation $x\varphi y = \varphi(y \cdot x)$.

**Lemma 2.2.** *Let $M$ be a von Neumann algebra and $\varphi, \psi \in M_*$ faithful positive functionals.*

(1) *For any projection $e \in M_\psi$, we have*

$$[De\psi e, D\psi]_t = e \quad \text{and} \quad e[D\psi, D\varphi]_t = [De\psi e, D\varphi]_t.$$

*In particular we have a chain rule:*

$$[De\psi e, D\psi]_t\, [D\psi, D\varphi]_t = [De\psi e, D\varphi]_t.$$

(2) *Let $v \in M$ be a partial isometry such that $e := vv^* \in M_\psi$ and $f := v^*v \in M_\varphi$. Assume that $v\varphi v^* = e\psi e$ on $M$ (equivalently $f\varphi f = v^*\psi v$). Then*

$$v\sigma_t^\varphi(v^*xv)v^* = \sigma_t^\psi(exe), \quad v^*[D\psi, D\varphi]_t = \sigma_t^\varphi(v^*), \quad x \in M,\ t \in \mathbb{R}.$$

### *Cocycle actions*

A more general version of a group action is a cocycle action. We say that a locally compact group $G$ acts on a von Neumann algebra $M$ as a *cocycle action* if there exist continuous maps $\alpha: G \to \operatorname{Aut}(M)$ and $v: G \times G \to \mathcal{U}(M)$ such that

$$\alpha_e = \mathrm{id}, \quad \alpha_g \circ \alpha_h = \operatorname{Ad}(v(g,h)) \circ \alpha_{gh}, \quad v(g,h)v(gh,k) = \alpha_g(v(h,k))v(g,hk)$$

for all $g, h, k \in G$, where $e$ is the neutral element. The map $v$ is called a 2-*cocycle*. Two cocycle actions $G \curvearrowright^{(\alpha,v)} M$ and $G \curvearrowright^{(\beta,w)} N$ are said to be *cocycle conjugate* if there exist a $*$-isomorphism $\pi: M \to N$ and a continuous map $u: G \to \mathcal{U}(M)$ such that, for all $g, h \in G$,

$$\pi^{-1} \circ \beta_g \circ \pi = \operatorname{Ad}(u_g) \circ \alpha_g, \quad \pi^{-1}(w(g,h)) = u_g\alpha_g(u_h)v(g,h)u_{gh}^*.$$

In this article, cocycle actions appear in the following two contexts.

Let $\Gamma \curvearrowright^\alpha B$ be an action of a discrete group on a von Neumann algebra $B$. Let $p \in B$ be a projection and assume that $\alpha_g(p) \sim p$ in $B$ for all $g \in G$. Take any partial isometries $w_g \in B$ such that $w_g w_g^* = p$ and $w_g^* w_g = \alpha_g(p)$ for all $g \in \Gamma$. Define $\alpha_g^p(x) := w_g\alpha_g(x)w_g^*$ and $v^p(g,h) := w_g\alpha_g(w_h)w_{gh}^*$ for all $x \in pBp$, $g, h \in \Gamma$. Then $(\alpha^p, v^p)$ is a cocycle action on $pBp$ satisfying $p(B \rtimes_\alpha \Gamma)p \simeq pBp \rtimes_{(\alpha^p,v^p)} \Gamma$.

Let $\Gamma \curvearrowright^\alpha B$ be the same group action. Let $\Lambda \leq \Gamma$ be a normal subgroup and fix a section $s: \Gamma/\Lambda \to \Gamma$ such that $s(\Lambda)$ is the unit of $\Gamma$. Inside $B \rtimes_\alpha \Gamma$, for all $g, h \in \Gamma/\Lambda$, we define

$$\alpha_g^{\Gamma/\Lambda} := \operatorname{Ad}(\lambda_{s(g)}^\Gamma) \in \operatorname{Aut}(B \rtimes_\alpha \Lambda) \quad \text{and} \quad v(g,h) := \lambda_{s(g)s(h)s(gh)^{-1}}^\Gamma \in L\Lambda.$$

It is easy to verify that $\alpha^{\Gamma/\Lambda}$ and $v$ define a cocycle action of $\Gamma/\Lambda$ on $B \rtimes_\alpha \Lambda$ satisfying $B \rtimes_\alpha \Gamma \simeq (B \rtimes_\alpha \Lambda) \rtimes_{(\alpha^{\Gamma/\Lambda},v)} \Gamma/\Lambda$.

*Basic constructions and operator valued weights*

For operator valued weights, we refer the reader to [13, 14]. We will say that a unital inclusion $B \subset M$ of von Neumann algebras is *with operator valued weight* if there is an operator valued weight $E_B \colon M \to B$, which is always assumed to be normal, faithful and semifinite.

Let $B \subset M$ be a unital inclusion of $\sigma$-finite von Neumann algebras with expectation $E_B$. Fix a faithful normal state $\varphi$ on $M$ such that $\varphi = \varphi \circ E_B$. Put $L^2(M) := L^2(M, \varphi)$ and $J := J_\varphi$, and consider $B \subset M \subset \mathbb{B}(L^2(M))$. The von Neumann algebra $\langle M, B\rangle := (JBJ)'$ is called the *basic construction*, and is generated by $Me_BM$, where $e_B$ is the Jones projection for $E_B$. Using the inclusion $JBJ \subset JMJ$ with expectation $JE_BJ := \mathrm{Ad}(J) \circ E_B \circ \mathrm{Ad}(J)$, one can define a canonical operator valued weight $(JE_BJ)^{-1} \colon (JBJ)' \to (JMJ)' = M$. We will write as $\widehat{E}_B := (JE_BJ)^{-1}$. It satisfies $\widehat{E}_B(b^*e_Ba) = b^*a$ for all $a, b \in M$. See [30, 31] for the general theory of $\widehat{E}_B$.

Below we collect well known facts for basic constructions and operator valued weights, which we will need in this article.

- For any faithful $\psi \in M_*^+$, one can define a faithful normal semifinite weight $\widehat{\psi} := \psi \circ \widehat{E}_B$ on $\langle M, B\rangle$. We have
$$\sigma_t^{\widehat{\psi}}|_M = \sigma_t^{\psi} \quad \text{and} \quad [D\widehat{\psi}, D\widehat{\varphi}]_t = [D\psi, D\varphi]_t \quad \text{for all } t \in \mathbb{R}.$$
- Let $E_{C_\varphi(B)} \colon C_\varphi(M) \to C_\varphi(B)$ be the canonical conditional expectation such that $E_{C_\varphi(B)}|_M = E_B$ and $E_{C_\varphi(B)}|_{L_\varphi\mathbb{R}} = \mathrm{id}$. Using $\sigma_t^{\varphi} \circ \widehat{E}_B = \widehat{E}_B \circ \sigma_t^{\widehat{\varphi}}$ for all $t \in \mathbb{R}$, one can define an operator valued weight from $\langle M, B\rangle \rtimes_{\sigma^{\widehat{\varphi}}} \mathbb{R}$ to $M \rtimes_{\sigma^{\varphi}} \mathbb{R}$ whose restriction on $\langle M, B\rangle^+$ coincides with $\widehat{E}_B$. We will denote it by $\widehat{E}_B \rtimes \mathbb{R}$.
- We canonically have
$$\langle C_\varphi(M), C_\varphi(B)\rangle = C_{\widehat{\varphi}}(\langle M, B\rangle).$$
The left hand side has a canonical operator valued weight $\widehat{E}_{C_\varphi(B)}$ onto $C_\varphi(M)$, and the right hand side has $\widehat{E}_B \rtimes \mathbb{R}$. Since the constructions are canonical, these two operator valued weights coincide.

Here we prove a lemma for type $\mathrm{III}_1$ factors.

**Lemma 2.3.** *Let $A \subset M$ be a unital inclusion of von Neumann algebras with an operator valued weight $E_A$. Fix a faithful $\psi_A \in A_*^+$, and put $\psi := \psi_A \circ E_A$. Let $N$ be a type* $\mathrm{III}_1$ *factor with a faithful normal semifinite weight $\omega$. Then*
$$C_{\psi\otimes\omega}(A \mathbin{\overline{\otimes}} N)' \cap C_{\psi\otimes\omega}(M \mathbin{\overline{\otimes}} N) = (A' \cap M_\psi) \otimes \mathbb{C}1_N \otimes \mathbb{C}1_{L^2(\mathbb{R})}.$$

*Proof.* The inclusion $\supset$ is clear, so we prove the converse.

Since $N$ is a type $\mathrm{III}_1$ factor, there is a faithful normal semifinite weight $\omega'$ such that $(N_{\omega'})' \cap N = \mathbb{C}$ (see [45, Theorem XII.1.7]). Thanks to the Connes cocycle, there is a canonical isomorphism from $C_{\psi\otimes\omega'}(M \mathbin{\overline{\otimes}} N)$ to $C_{\psi\otimes\omega}(M \mathbin{\overline{\otimes}} N)$ which sends

$C_{\psi\otimes\omega'}(A \mathbin{\overline{\otimes}} N)$ onto $C_{\psi\otimes\omega}(A \mathbin{\overline{\otimes}} N)$ and which is the identity on $M \mathbin{\overline{\otimes}} N$. Hence to prove this lemma, by exchanging $\omega'$ with $\omega$, we may assume that $N'_\omega \cap N = \mathbb{C}$.

For simplicity we write $L_{\psi\otimes\omega}\mathbb{R} = L\mathbb{R}$. Observe that (e.g. [18, Proposition 2.4])

$$C_{\psi\otimes\omega}(\mathbb{C}1_A \otimes \mathbb{C}1_N)' \cap C_{\psi\otimes\omega}(M \mathbin{\overline{\otimes}} N) \subset (M \mathbin{\overline{\otimes}} N)_{\psi\otimes\omega} \mathbin{\overline{\otimes}} L\mathbb{R}.$$

Since $(\mathbb{C}1_A \otimes N_\omega)' \cap (M \mathbin{\overline{\otimes}} N)_{\psi\otimes\omega} = M_\psi \otimes \mathbb{C}1_N$, we have

$$C_{\psi\otimes\omega}(\mathbb{C}1_A \otimes N_\omega)' \cap C_{\psi\otimes\omega}(M \mathbin{\overline{\otimes}} N) \subset M_\psi \mathbin{\overline{\otimes}} \mathbb{C}1_N \mathbin{\overline{\otimes}} L\mathbb{R}.$$

Since $C_\omega(N)$ is a factor (because $N$ is of type $\mathrm{III}_1$), we have $\pi_\omega(N)' \cap (\mathbb{C}1_N \otimes L_\omega\mathbb{R}) = \mathbb{C}1_N \otimes \mathbb{C}1_{L^2(\mathbb{R})}$, where $\pi_\omega(N)$ is the canonical image of $N$ in $C_\omega(N)$. This implies that

$$\begin{aligned} C_{\psi\otimes\omega}(\mathbb{C}1_A \otimes N)' \cap C_{\psi\otimes\omega}(M \mathbin{\overline{\otimes}} N) &\subset M_\psi \mathbin{\overline{\otimes}} [\pi_\omega(N)' \cap (\mathbb{C}1_N \otimes L\mathbb{R})] \\ &= M_\psi \mathbin{\overline{\otimes}} \mathbb{C}1_N \mathbin{\overline{\otimes}} \mathbb{C}1_{L^2(\mathbb{R})}. \end{aligned}$$

Using the canonical embedding $\pi_{\psi\otimes\omega}$, the last term coincides with $\pi_{\psi\otimes\omega}(M_\psi \otimes \mathbb{C}1_N)$, hence

$$\begin{aligned} C_{\psi\otimes\omega}(A \mathbin{\overline{\otimes}} N)' \cap C_{\psi\otimes\omega}(M \mathbin{\overline{\otimes}} N) &= \pi_{\psi\otimes\omega}(A \mathbin{\overline{\otimes}} \mathbb{C}1_N)' \cap \pi_{\psi\otimes\omega}(M_\psi \otimes \mathbb{C}1_N) \\ &= \pi_{\psi\otimes\omega}((A' \cap M_\psi) \otimes \mathbb{C}1_N) \\ &= (A' \cap M_\psi) \otimes \mathbb{C}1_N \otimes \mathbb{C}1_{L^2(\mathbb{R})}. \end{aligned}$$

This is the conclusion. ■

### *Popa's intertwining theory*

As explained in Section 1, we refer the reader to [35, 37] for the origin of intertwining theory. Here we give a definition introduced in [15].

**Definition 2.4.** Let $M$ be a $\sigma$-finite von Neumann algebra and $A, B \subset M$ (possibly non-unital) von Neumann subalgebras with expectation. We will say that *a corner of $A$ embeds with expectation into $B$ inside $M$* and write $A \preceq_M B$ if there exist projections $e \in A$, $f \in B$, a partial isometry $v \in eMf$ and a unital normal $*$-homomorphism $\theta\colon eAe \to fBf$ such that

- $\theta(eAe) \subset fBf$ is with expectation;
- $v\theta(a) = av$ for all $a \in eAe$.

In this case, we will say that $(e, f, \theta, v)$ *witnesses $A \preceq_M B$*.

We recall known characterizations of the intertwining condition $A \preceq_M B$. For this, we borrow notation from [15, Section 4]. The same notation will be used in Section 3.

Let $M$ be a $\sigma$-finite von Neumann algebra and $A, B \subset M$ (possibly non-unital) von Neumann subalgebras with expectations. Fix a faithful normal conditional expectation $E_B$ for $B \subset 1_B M 1_B$. Put $\widetilde{B} := B \oplus \mathbb{C}(1_M - 1_B)$ and let $E_{\widetilde{B}}\colon M \to \widetilde{B}$ be a faithful normal conditional expectation which extends $E_B$. Let $B = B_1 \oplus B_2$ be the unique decomposition such that $B_1$ is finite and $B_2$ is properly infinite. Fix a faithful normal trace $\tau_{B_1}$ on $B_1$ and choose a faithful normal state $\varphi \in M_*$ such that $\varphi$ is preserved by $E_B$ and $E_{\widetilde{B}}$, and $\varphi|_{B_1} = \tau_{B_1}$ (up to scalar multiples). Fix a standard representation $L^2(M) := L^2(M, \varphi)$ and its modular conjugation $J := J_\varphi$. We write $e_{\widetilde{B}}$ and $e_B$ for the corresponding Jones projections (note that $e_{\widetilde{B}} 1_B = e_{\widetilde{B}} J 1_B J = e_B$), and $\widehat{E}_{\widetilde{B}}$ for the canonical operator valued weight from $\langle M, \widetilde{B} \rangle$ to $M$ given by $\widehat{E}_{\widetilde{B}}(x e_{\widetilde{B}} x^*) = xx^*$ for all $x \in M$. Denote by Tr the unique trace on $\langle M, \widetilde{B} \rangle J 1_{B_1} J$ satisfying $\mathrm{Tr}((x^* e_{\widetilde{B}} x) J 1_{B_1} J) = \tau_{B_1}(E_B(1_{B_1} x x^* 1_{B_1}))$ for all $x \in M$. Since $\mathcal{Z}(\langle M, \widetilde{B} \rangle J 1_{B_1} J) = J \mathcal{Z}(B_1) J$, there is a unique operator valued weight ctr: $\langle M, \widetilde{B} \rangle J 1_{B_1} J \to J \mathcal{Z}(B_1) J$ such that $\mathrm{Tr} = \overline{\tau_{B_1}(J \cdot J)} \circ \mathrm{ctr}$. Since Tr is a trace, ctr is an extended center valued trace. Let $\mathrm{ctr}_{B_1}$ be the center valued trace for $B_1$ and recall that $\tau_{B_1} \circ \mathrm{ctr}_{B_1} = \tau_{B_1}$. We have

$$\mathrm{ctr}((x^* e_{\widetilde{B}} x) J 1_{B_1} J) = J \, \mathrm{ctr}_{B_1} \circ E_B(1_{B_1} x x^* 1_{B_1}) J \quad \text{for all } x \in M.$$

We mention that the decomposition $B = B_1 \oplus B_2$ here is slightly different from the one in [15], and that ctr was not used in [15]. However the proof of [15, Theorem 4.3] works without any change if we use ctr and our decomposition for $B$. Our items introduced here are more appropriate in the context of intertwining conditions with actions, which will be discussed in the next section.

Now we introduce Popa's intertwining theorem. We refer the reader to [15, Theorem 4.3] and [2, Theorem 2] for the proof of this version.

**Theorem 2.5.** *The following conditions are equivalent:*

(1) $A \preceq_M B$.

(2) *There exists a non-zero positive element* $d \in A' \cap 1_A \langle M, \widetilde{B} \rangle 1_A$ *such that*

$$d = d J 1_B J \quad \text{and} \quad \widehat{E}_{\widetilde{B}}(d) \in M.$$

*If $A$ is finite, then for any $\sigma$-strongly dense subgroup $\mathcal{G} \subset \mathcal{U}(A)$, conditions* (1) *and* (2) *are also equivalent to*

(3) *There is no net $(u_i)_i$ in $\mathcal{G}$ such that $E_B(b^* u_i a) \to 0$ $\sigma$-strongly for all $a, b \in M 1_B$.*

Using the next lemma, we can replace the map $\theta$ for the condition $A \preceq_M B$ with a unital $*$-homomorphism on $A$.

**Lemma 2.6.**

(1) $A \preceq_M B$ *if and only if there exist a separable Hilbert space $H$, a projection $f \in B \mathbin{\overline{\otimes}} \mathbb{B}(H)$, a partial isometry $w \in (1_A \otimes e_{1,1})(M \mathbin{\overline{\otimes}} \mathbb{B}(H)) f$, where $e_{1,1}$ is a minimal projection, and a unital normal $*$-homomorphism $\pi \colon A \to f(B \mathbin{\overline{\otimes}} \mathbb{B}(H)) f$ such that*

- $\pi(A) \subset f(B \mathbin{\overline{\otimes}} \mathbb{B}(H)) f$ *is with expectation;*
- $w\pi(a) = (a \otimes e_{1,1}) w$ *for all $a \in A$.*

*In this case* (*to distinguish it from $A \preceq_M B$*) *we will say that* $(H, f, \pi, w)$ witnesses $A \preceq_M^{\mathrm{uni}} B$.

(2) *Assume that either*

- *$A$ does not have any direct summand which is semifinite and properly infinite, or*
- *$B$ is properly infinite.*

*If $A \preceq_M B$, then the Hilbert space $H$ in item* (1) *can be taken finite-dimensional.*

*Proof.* Since we will prove a very similar but more complicated statement in Lemma 3.6, we omit the proof. Indeed, to prove this lemma, one can follow the proof of Lemma 3.6 by regarding actions as trivial (and by using [15, Theorem 4.3 and Lemma 4.10]). ■

## 3. Intertwining theory with modular actions

In this section, we introduce several variants of Popa's intertwining condition. We investigate these conditions as well as relations between them. At the end of this section, we prove Theorem A. Throughout this section, we always fix (possibly non-unital) inclusions $A, B \subset M$ of $\sigma$-finite von Neumann algebras with expectations $E_A, E_B$ respectively.

### *Intertwining theory with group actions*

We first consider the intertwining condition $A \preceq_M B$ when a locally compact group acts on them. This idea was first used in [38, 39] to study cocycle superrigidity for discrete group actions. Although our main interest is the case of modular actions, we first study this condition by assuming that a general locally compact group acts on $A, B \subset M$.

We fix the following setting (which will be used in Definition 3.1 and Theorem 3.2). We use notation introduced before Theorems 2.5, so we use $A \subset 1_A M 1_A$, $B \subset 1_B M 1_B$, $B = B_1 \oplus B_2$, $\widetilde{B}$, $E_B$, $E_{\widetilde{B}}$, $L^2(M)$, $\varphi$, $J$, $e_B$, $e_{\widetilde{B}}$, $\tau_{B_1}$, Tr, $\widehat{E}_{\widetilde{B}}$, and ctr. Let $G$ be a locally compact second countable group, and consider continuous actions $\alpha$ and $\beta$ of $G$ on $M$ such that

- $\alpha_g(A) = A$ and $\beta_g(B) = B$ for all $g \in G$;
- $\alpha_g \circ E_A = E_A \circ \alpha_g$ on $1_A M 1_A$ and $\beta_g \circ E_B = E_B \circ \beta_g$ on $1_B M 1_B$ for all $g \in G$;
- $\alpha$ and $\beta$ are cocycle conjugate: there exists a $\beta$-cocycle $\omega\colon G \to M$ such that $\alpha_g = \mathrm{Ad}(\omega_g) \circ \beta_g (=: \beta_g^{\omega})$ for all $g \in G$.

In this setting, based on the viewpoint of Lemma 2.6(1), we define intertwining conditions *with group actions* as follows.

**Definition 3.1.** Keep the above setting. We say that $(A, \alpha)$ *embeds with expectation into* $(B, \beta)$ *inside* $M$ and write $(A, \alpha) \preceq_M^{\mathrm{uni}} (B, \beta)$ if there exists $(H, f, \pi, w)$ which witnesses $A \preceq_M^{\mathrm{uni}} B$ (in the sense of Lemma 2.6(1)) and a generalized cocycle $(u_g)_{g \in G}$ for $\beta \otimes \mathrm{id}_H$ with values in $B \mathbin{\overline{\otimes}} \mathbb{B}(H)$ and with support projection $f$ such that

- $w u_g = (\omega_g \otimes 1_H)(\beta_g \otimes \mathrm{id}_H)(w)$ for all $g \in G$;
- $u_g (\beta_g \otimes \mathrm{id}_H)(\pi(a)) u_g^* = \pi(\alpha_g(a))$ for all $g \in G$ and $a \in A$.

In this case, we will say that $(H, f, \pi, w)$ *and* $(u_g)_{g \in G}$ *witness* $(A, \alpha) \preceq_M^{\mathrm{uni}} (B, \beta)$.

Before proceeding, we record the following observations.

- In the definition, we may drop the assumption that $w$ is a partial isometry by considering its polar decomposition (e.g. [15, Remark 4.2(1)]).
- We can define a $*$-isomorphism $\Pi^{\omega}_{\beta,\alpha}\colon M \rtimes_\alpha G \to M \rtimes_\beta G$ such that $\Pi^{\omega}_{\beta,\alpha}(a) = a$ for $a \in M$ and $\Pi^{\omega}_{\beta,\alpha}(\lambda^{\alpha}_g) = \omega_g \lambda^{\beta}_g$ for $g \in G$. There exist unital inclusions $A \rtimes_\alpha G \subset 1_A(M \rtimes_\alpha G)1_A$ and $B \rtimes_\beta G \subset 1_B(M \rtimes_\beta G)1_B$.
- Using compression maps by $e_B \otimes 1$ and $e_A \otimes 1$, faithful normal conditional expectations $E_{B \rtimes_\beta G}\colon 1_B(M \rtimes_\beta G)1_B \to B \rtimes_\beta G$ and $E_{A \rtimes_\alpha G}\colon 1_A(M \rtimes_\alpha G)1_A \to A \rtimes_\alpha G$ are defined.
- For each $g \in G$, let $u^{\beta}_g \in \mathcal{U}(L^2(M))$ be the canonical implementing unitary for $\beta_g$. Then putting $\widehat{\beta}_g := \mathrm{Ad}(u^{\beta}_g)$, one can extend the action $\beta$ on $\langle M, \widetilde{B}\rangle$.
- Putting $\widehat{\alpha}_g := \mathrm{Ad}(\omega_g u^{\beta}_g) = \mathrm{Ad}(\omega_g) \circ \widehat{\beta}_g$ for $g \in G$, we can also extend $\alpha$ on $\langle M, \widetilde{B}\rangle$. Note that $\widehat{\alpha}_g(1_A) = 1_A$ and $\widehat{\alpha}_g(J1_BJ) = J1_BJ$ for all $g \in G$.
- For each $g \in G$, since $\beta_g$ commutes with $E_B$, we have $\widehat{E}_{\widetilde{B}} \circ \widehat{\beta}_g = \beta_g \circ \widehat{E}_{\widetilde{B}}$ on $(\langle M, \widetilde{B}\rangle J1_BJ)^+$. This implies that $\widehat{E}_{\widetilde{B}} \circ \widehat{\alpha}_g = \alpha_g \circ \widehat{E}_{\widetilde{B}}$ on $(\langle M, \widetilde{B}\rangle J1_BJ)^+$.

Our first goal in this section is to prove the following theorem, which gives fundamental characterizations of the condition $(A,\alpha) \preceq_M (B,\beta)$. We mention that the origins of these conditions can be found in [38,39] (see also [19]).

**Theorem 3.2.** *Consider the following conditions:*

(1) $(A,\alpha) \preceq_M^{\mathrm{uni}} (B,\beta)$.

(2) $\Pi^{\omega}_{\beta,\alpha}(A \rtimes_\alpha G) \preceq_{M \rtimes_\beta G} B \rtimes_\beta G$.

(3) *There exist no nets* $(u_i)_i$ *of unitaries in* $\mathcal{U}(A)$ *and* $(g_i)_i$ *in* $G$ *such that*

$$E_B(\beta_{g_i}(b^*)\omega_{g_i}^* u_i a) \to 0 \quad \sigma\textit{-strongly for all } a, b \in M1_B.$$

(4) *There exists a non-zero positive element* $d \in A' \cap 1_A\langle M, \widetilde{B}\rangle^{\widehat{\alpha}} 1_A$ *such that*

$$d = dJ1_BJ \quad \textit{and} \quad \widehat{E}_{\widetilde{B}}(d) \in M.$$

*Then we have* (4)⇔(1)⇒(2). *Moreover the following assertion holds true:*

- *Assume further that* $A \rtimes_\alpha G$ *is finite. Then* (2)⇔(3)⇒(4), *hence all conditions are equivalent. In this case, we can choose a Hilbert space* $H$ *in item* (1) *to be finite-dimensional.*

**Remark 3.3.** In the case $A = \mathbb{C}$, combined with Theorem 3.9 below, this theorem generalizes [19, Theorem 3.1]. When $A$ is not finite, the implication (2)⇒(1) does not hold since there is a counterexample [16, Theorem 4.9]. We will nevertheless use this theorem for general $A$ by taking tensor products with a type $\mathrm{III}_1$ factor (see Lemma 3.12).

*Proof of Theorem* 3.9. Throughout the proof, we will write a tensor product with $\mathbb{B}(H)$ and associated maps to the tensor product by adding the symbol $H$ as a superscript, such as $M^H := M \mathbin{\overline{\otimes}} \mathbb{B}(H)$, $\alpha^H_g := \alpha_g \otimes \mathrm{id}_H$, $\omega^H_g := \omega_g \otimes 1_H$ etc.

(1)⇒(2). Fix $(H, f, \pi, w)$ and $(u_g)_{g\in G}$. The generalized cocycle $(u_g)_{g\in G}$ gives a $*$-isomorphism

$$\Pi^{u}_{\beta^H,(\beta^H)^u}\colon f(M^H \rtimes_{(\beta^H)^u} G)f \to f(M^H \rtimes_{\beta^H} G)f$$

satisfying $\Pi^{u}_{\beta^H,(\beta^H)^u}(faf) = faf$ for $a \in M^H$ and $\Pi^{u}_{\beta^H,(\beta^H)^u}(f\lambda_g^{(\beta^H)^u} f) = fu_g\lambda_g^{\beta^H} f = u_g\lambda_g^{\beta^H}$ for $g \in G$. Note that this restricts to a $*$-isomorphism between $f(B^H \rtimes_{(\beta^H)^u} G)f$ and $f(B^H \rtimes_{\beta^H} G)f$. The equivariance property $(\beta^H)^u_g(\pi(a)) = u_g\beta^H_g(\pi(a))u_g^* = \pi(\alpha_g(a))$ for $a \in A$ and $g \in G$ implies that there is a $*$-homomorphism

$$A \rtimes_\alpha G \to \pi(A) \rtimes_{(\beta^H)^u} G \subset f(B^H \rtimes_{(\beta^H)^u} G)f.$$

Composing this map with $\Pi^{u}_{\beta^H,(\beta^H)^u}$, we get a $*$-homomorphism

$$\tilde{\pi}\colon A \rtimes_\alpha G \to f(B^H \rtimes_{\beta^H} G)f$$

such that $\tilde{\pi}(a) = \pi(a)$ for $a \in A$ and $\tilde{\pi}(\lambda_g^\alpha) = u_g\lambda_g^{\beta^H}$ for $g \in G$. The partial isometry $w$ then satisfies, inside $M^H \rtimes_{\beta^H} G$, for all $a \in A$ and $g \in G$,

$$\begin{aligned}\Pi^{\omega^H}_{\beta^H,\alpha^H}(a \otimes e_{1,1})w &= w\tilde{\pi}(a),\\ \Pi^{\omega^H}_{\beta^H,\alpha^H}(\lambda_g^{\alpha^H})w &= \omega^H_g\beta^H_g(w)\lambda_g^{\beta^H} = wu_g\lambda_g^{\beta^H} = w\tilde{\pi}(\lambda_g^\alpha).\end{aligned}$$

Hence using the isomorphism $M^H \rtimes_{\beta^H} G = (M \rtimes_\beta G) \mathbin{\overline{\otimes}} \mathbb{B}(H)$ and the fact that $\Pi^{\omega^H}_{\beta^H,\alpha^H} = \Pi^{\omega}_{\beta,\alpha} \otimes \mathrm{id}_H$, we see that $(H, \tilde{\pi}, f, w)$ witnesses $\Pi^{\omega}_{\beta,\alpha}(A \rtimes_\alpha G) \preceq^{\mathrm{uni}}_{M\rtimes_\beta G} B \rtimes_\beta G$. This is equivalent to item (2) by Lemma 2.6.

(1)⇒(4). Take $(H, \pi, f, w)$ and $(u_g)_{g\in G}$ witnessing item (1). Write $w = \sum_j w_j \otimes e_{1,j}$, where $(e_{i,j})_{i,j}$ is a matrix unit of $\mathbb{B}(H)$, and put $W := \sum_j w_j e_{\tilde{B}} \otimes e_{1,j} = we^H_{\tilde{B}}$ (where $e^H_{\tilde{B}} := e_{\tilde{B}} \otimes 1_H$). Then for any $a \in A$,

$$(a \otimes e_{1,1})WW^* = (a \otimes e_{1,1})we^H_{\tilde{B}}w^* = w\pi(a)e^H_{\tilde{B}}w^* = WW^*(a \otimes e_{1,1}),$$

so $WW^* \in (A \otimes \mathbb{C}e_{1,1})' \cap (1_A \otimes e_{1,1})\langle M^H, \tilde{B}^H\rangle(1_A \otimes e_{1,1}) = (A' \cap 1_A\langle M, \tilde{B}\rangle 1_A) \otimes \mathbb{C}e_{1,1}$. Moreover, for any $g \in G$, by the intertwining condition of $w$,

$$\begin{aligned}\hat{\alpha}^H_g(WW^*) &= \omega^H_g\hat{\beta}^H_g(we^H_{\tilde{B}}w^*)(\omega^H_g)^* = \omega^H_g\beta^H_g(w)e^H_{\tilde{B}}\beta^H_g(w^*)(\omega^H_g)^*\\ &= wu_ge^H_{\tilde{B}}u_g^*w^* = we^H_{\tilde{B}}fw^* = WW^*,\end{aligned}$$

so $WW^* \in (1_A\langle M, \tilde{B}\rangle 1_A)^{\hat{\alpha}} \otimes \mathbb{C}e_{1,1}$. Using the equality $\hat{E}_{\tilde{B}\mathbin{\overline{\otimes}}\mathbb{B}(H)} = \hat{E}_{\tilde{B}} \otimes \mathrm{id}_H$, we find that

$$(\hat{E}_{\tilde{B}} \otimes \mathrm{id}_H)(WW^*) = \hat{E}_{\tilde{B}\mathbin{\overline{\otimes}}\mathbb{B}(H)}(WW^*) = ww^* \in M \otimes \mathbb{C}e_{1,1} < \infty.$$

Thus by using the element $d$ such that $d \otimes e_{1,1} = WW^*$, we get (4).

(4)⇒(1). Take a non-zero spectral projection $p$ of $d$ such that $p \leq \lambda d$ for some $\lambda > 0$. Then $p$ satisfies exactly the same assumption as $d$. Fix a countably-infinite-dimensional Hilbert space $H$ (with a matrix unit $(e_{i,j})_{i,j}$ in $\mathbb{B}(H)$), and consider the inclusion

$$A \otimes \mathbb{C}e_{1,1} \subset \langle M, \widetilde{B}\rangle \mathbin{\overline{\otimes}} \mathbb{B}(H) = \langle M^H, \widetilde{B}^H\rangle.$$

Then the projection $p \otimes e_{1,1}$ satisfies

$$\widehat{E}_{\widetilde{B}^H}(p \otimes e_{1,1}) = \widehat{E}_{\widetilde{B}}(p) \otimes e_{1,1} < \infty.$$

Since the projection $e_{\widetilde{B}}^H(1_B \otimes 1_H) = (e_{\widetilde{B}} 1_B) \otimes 1_H$ is properly infinite, we can follow [15, Theorem 4.3, proof of (6)⇒(2-b)] (we do not need the finiteness of $A$). We can find a partial isometry $W \in \langle M^H, \widetilde{B}^H\rangle$ (of the form $we_{\widetilde{B}}^H = W$), a projection $f \in B^H$, a $*$-homomorphism $\pi\colon A \to fB^H f$ such that $\pi(a)e_{\widetilde{B}}^H = W^*(a \otimes e_{1,1})W$ and $w\pi(a) = (a \otimes e_{1,1})w$ for all $a \in A$, and $WW^* = p \otimes e_{1,1} \in (1_A\langle M, \widetilde{B}\rangle 1_A)^{\widehat{\alpha}} \mathbin{\overline{\otimes}} \mathbb{B}(H)$. Note that $(H, f, \pi, w)$ witnesses $A \preceq_M^{\mathrm{uni}} B$ (up to taking the polar decomposition of $w$).

We next construct a generalized cocycle. For any $g \in G$, since $W^*\omega_g^H \widehat{\beta}_g^H(W) \in 1_B e_{\widetilde{B}}^H \langle M, \widetilde{B}\rangle^H 1_B e_{\widetilde{B}}^H = B^H e_{\widetilde{B}}^H$, there is a unique $u_g \in B^H$ such that $u_g e_{\widetilde{B}}^H = W^*\omega_g^H \widehat{\beta}_g^H(W)$. Since $g \mapsto \omega_g^H$ and $g \mapsto \widehat{\beta}_g^H(W)$ are $*$-strongly continuous, so is the map $G \ni g \mapsto u_g$. Observe that

$$e_{\widetilde{B}}^H u_g u_g^* = W^*\omega_g^H \widehat{\beta}_g^H(WW^*)(\omega_g^H)^* W = W^*\widehat{\alpha}_g^H(WW^*)W = fe_{\widetilde{B}}^H$$

and similarly $e_{\widetilde{B}}^H u_g^* u_g = \beta_g^H(f)e_{\widetilde{B}}^H$ for all $g \in G$. For $g, h \in G$, we compute that

$$\begin{aligned} u_g \beta_g^H(u_h)e_{\widetilde{B}}^H &= W^*\omega_g^H \widehat{\beta}_g^H(W)\widehat{\beta}_g^H(W^*\omega_h^H \widehat{\beta}_h^H(W)) \\ &= W^*\widehat{\alpha}_g^H(WW^*)\omega_g^H \widehat{\beta}_g^H(\omega_h^H)\widehat{\beta}_{gh}^H(W) \\ &= W^*\omega_{gh}^H \widehat{\beta}_{gh}^H(W) = u_{gh}e_{\widetilde{B}}^H. \end{aligned}$$

Since $u_g$ is defined via $B^H e_{\widetilde{B}}^H \simeq B^H$, we can remove $e_{\widetilde{B}}^H$ from the conclusions of the above computations. Thus $(u_g)_{g\in G}$ is a generalized cocycle for $\beta^H$ with support projection $f$. Using the equation $(\omega_g^H)^* W u_g = \widehat{\beta}_g^H(W)$, we find that for any $a \in A$ and $g \in G$,

$$\begin{aligned} \beta_g^H(\pi(a))e_{\widetilde{B}}^H &= \widehat{\beta}_g^H(W^*(a \otimes e_{1,1})W) = u_g^* W^* \alpha_g^H(a \otimes e_{1,1})W u_g \\ &= u_g^* \pi(\alpha_g(a))u_g e_{\widetilde{B}}^H. \end{aligned}$$

We get the equivariance property $u_g \beta_g^H(\pi(a))u_g^* = \pi(\widehat{\alpha}_g(a))$ for all $a \in A$. Finally, since $W = we_{\widetilde{B}}^H$, the equation $(\omega_g^H)^* W u_g = \widehat{\beta}_g^H(W)$ for $g \in G$ implies $(\omega_g^H)^* w u_g e_{\widetilde{B}}^H = \beta_g^H(w)e_{\widetilde{B}}^H$. We get $wu_g = \omega_g^H \beta_g^H(w)$ for all $g \in G$, and thus $(u_g)_{g\in G}$ is the desired cocycle. We get item (1).

*From now on, we assume that $A \rtimes_\alpha G$ is finite.*

(2)⇔(3). Suppose first that (3) does not hold, hence there exist nets $(u_i)_i$ of unitaries in $\mathcal{U}(A)$ and $(g_i)_i$ in $G$ such that

$$E_B(\beta_{g_i}(b^*)\omega_{g_i}^* u_i a) \to 0 \quad \sigma\text{-strongly for all } a, b \in M1_B.$$

Then for any $a, b \in M1_B$ and $s, s' \in G$, we have

$$\begin{aligned} E_{B\rtimes_\beta G}(\lambda_s^\beta b^* \Pi_{\beta,\alpha}^\omega(\lambda_{g_i^{-1}}^\alpha) u_i a \lambda_{s'}^\beta) &= \lambda_s^\beta E_{B\rtimes_\beta G}(b^* \lambda_{g_i^{-1}}^\beta \omega_{g_i}^* u_i a)\lambda_{s'}^\beta \\ &= \lambda_{sg_i^{-1}}^\beta E_B(\beta_{g_i}(b^*)\omega_{g_i}^* u_i a)\lambda_{s'}^\beta. \end{aligned}$$

The last term converges to 0 in the $\sigma$-strong topology for all $a, b \in M1_B$ and $s, s' \in G$. By Theorem 2.5(3) (see also [15, Theorem 4.3(5)]), this means $\Pi_{\beta,\alpha}^\omega(A \rtimes_\alpha G) \not\preceq_{M\rtimes_\beta G} B \rtimes_\beta G$.

Conversely, suppose that $\Pi_{\beta,\alpha}^\omega(A \rtimes_\alpha G) \not\preceq_{M\rtimes_\beta G} B \rtimes_\beta G$. Then by Theorem 2.5(3), there exist nets $(u_i)_i$ of unitaries in $\mathcal{U}(A)$ and $(g_i)_i$ in $G$ such that

$$E_{B\rtimes_\beta G}(y^* \Pi_{\beta,\alpha}^\omega(\lambda_{g_i^{-1}}^\alpha) u_i x) \to 0 \quad \sigma\text{-strongly for all } x, y \in (M \rtimes_\beta G)1_B.$$

Using the same computation as above, we conclude that (3) does not hold.

(3)⇒(4). Let $\psi$ be a faithful normal state on $M \rtimes_\alpha G$ which is preserved by $E_{A\rtimes_\alpha G}$ such that $\psi|_{A\rtimes_\alpha G}$ is a trace. Observe that $\psi|_{1_A M 1_A}$ is $\alpha$-preserving, since $1_A\lambda_g^\alpha \in (1_A M 1_A)_\psi$ for all $g \in G$. It then follows that $\widehat{\psi} \circ \widehat{\alpha}_g = \widehat{\psi}$ on $(1_A\langle M, \widetilde{B}\rangle 1_A J 1_B J)^+$ for all $g \in G$.

By assumption, there exist $\delta > 0$ and a finite subset $\mathcal{F} \subset 1_A M 1_B$ such that

$$\sum_{a,b\in\mathcal{F}} \|E_B(\beta_g(b^*)w_g^* u a)\|_{2,\varphi}^2 > \delta \quad \text{for all } u \in \mathcal{U}(A),\ g \in G.$$

Put $d_0 := \sum_{y\in\mathcal{F}} y e_{\widetilde{B}} y^* \in (1_A\langle M, \widetilde{B}\rangle 1_A)^+$ and observe that $d_0 = d_0 J 1_B J$, $\widehat{E}_{\widetilde{B}}(d_0) = \sum_{y\in\mathcal{F}} yy^* \in 1_A M 1_A$ and $\mathrm{ctr}(d_0\, J1_{B_1}J) = \sum_{y\in\mathcal{F}} J\, \mathrm{ctr}_{B_1}(E_B(1_{B_1} y^* y 1_{B_1}))J < \infty$. Define

$$\mathcal{K} := \overline{\mathrm{co}}^{\mathrm{weak}} \left\{u^* \widehat{\alpha}_g(d_0) u \mid u \in \mathcal{U}(A),\ g \in G\right\} \subset 1_A\langle M, \widetilde{B}\rangle 1_A.$$

Following the proof of (5)⇒(6) of [15, Theorem 4.3], there exists a unique element $d \in \mathcal{K}$ of minimum $\|\cdot\|_{2,\widehat{\psi}}$-norm. Since $\widehat{\psi}$ is preserved by $\widehat{\alpha}$ and since $A$ is contained in the centralizer of $\widehat{\psi}$, we deduce that $d \in A' \cap (1_A\langle M, \widetilde{B}\rangle 1_A)^{\widehat{\alpha}}$. Note that $d = dJ1_BJ$, since $d_0 = d_0 J 1_B J$.

We prove that $d \neq 0$. For all $u \in \mathcal{U}(A)$ and $g \in G$, we have

$$\begin{aligned} \sum_{a\in\mathcal{F}} \langle u^* \widehat{\alpha}_g(d_0) u \Lambda_\varphi(a), \Lambda_\varphi(a)\rangle_\varphi &= \sum_{a,b\in\mathcal{F}} \langle u^* \widehat{\alpha}_g(b e_{\widetilde{B}} b^*) u \Lambda_\varphi(a), \Lambda_\varphi(a)\rangle_\varphi \\ &= \sum_{a,b\in\mathcal{F}} \langle u^* w_g \beta_g(b) e_B \beta_g(b^*) w_g^* u \Lambda_\varphi(a), \Lambda_\varphi(a)\rangle_\varphi \\ &= \sum_{a,b\in\mathcal{F}} \|E_B(\beta_g(b^*) w_g^* u a)\|_{2,\varphi_B}^2 > \delta. \end{aligned}$$

By taking convex combinations and a $\sigma$-weak limit, we obtain $\sum_{a\in\mathcal{F}}\langle d\Lambda_\varphi(a),\Lambda_\varphi(a)\rangle_\varphi \geq \delta$. This implies $d\neq 0$.

We prove $\widehat{E}_{\widetilde{B}}(d)\in M$. Observe that for any $g\in G$,

$$\begin{aligned}\widehat{E}_{\widetilde{B}}(u^*\widehat{\alpha}_g(d_0)u) &= \sum_{y\in\mathcal{F}}\widehat{E}_{\widetilde{B}}(u^*\alpha_g(y)\omega_g e_{\widetilde{B}}\omega_g^*\alpha_g(y^*)u)\\ &= \sum_{y\in\mathcal{F}} u^*\alpha_g(y)\alpha_g(y^*)u = u^*\alpha_g\Big(\sum_{y\in\mathcal{F}} yy^*\Big)u.\end{aligned}$$

Combining this with the normality of $\widehat{E}_{\widetilde{B}}$, we conclude that $\|\widehat{E}_{\widetilde{B}}(x)\|_\infty \leq \|\sum_{y\in\mathcal{F}} yy^*\|_\infty$ for all $x\in\mathcal{K}$, hence $\widehat{E}_{\widetilde{B}}(d)\in M$. We get item (4).

Finally, we prove that the Hilbert space $H$ in item (1) can be taken finite-dimensional. For this, we continue to use $d_0, d, \mathcal{K}$ and claim $\mathrm{ctr}(dJ1_{B_1}J)<\infty$. Using the formula for ctr given in Section 2 and using $\mathrm{ctr}_{B_1}\circ\beta_g = \beta_g\circ\mathrm{ctr}_{B_1}$ on $B_1$ for all $g\in G$, we compute that for any $g\in G$ and $u\in\mathcal{U}(A)$,

$$\begin{aligned}\mathrm{ctr}(u^*\widehat{\alpha}_g(d_0)uJ1_{B_1}J) &= \sum_{y\in\mathcal{F}}\mathrm{ctr}([u^*\omega_g\beta_g(y)]e_{\widetilde{B}}[\beta_g(y^*)\omega_g^*u]\,J1_{B_1}J)\\ &= \sum_{y\in\mathcal{F}} J\,\mathrm{ctr}_{B_1}\circ E_B(1_{B_1}[\beta_g(y^*)\omega_g^*u][\beta_g(y^*)\omega_g^*u]^*1_{B_1})J\\ &= \sum_{y\in\mathcal{F}} J\,\mathrm{ctr}_{B_1}\circ E_B(1_{B_1}\beta_g(y^*y)1_{B_1})J\\ &= J\beta_g\circ\mathrm{ctr}_{B_1}\circ E_B\Big(\sum_{y\in\mathcal{F}}1_{B_1}y^*y1_{B_1}\Big)J.\end{aligned}$$

Combined with the normality of ctr, this yields

$$\|\mathrm{ctr}(xJ1_{B_1}J)\|_\infty \leq \Big\|\mathrm{ctr}_{B_1}\Big(E_B\Big(\sum_{y\in\mathcal{F}}1_{B_1}y^*y1_{B_1}\Big)\Big)\Big\|_\infty$$

for all $x\in\mathcal{K}$. Thus we get $\mathrm{ctr}(dJ1_{B_1}J)<\infty$.

We next follow the proof of (4)⇒(1) above. Take a non-zero spectral projection $p$ of $d$ such that $p\leq\lambda d$ for some $\lambda>0$, so that $\mathrm{ctr}(pJ1_{B_1}J)<\infty$ and $\widehat{E}_{\widetilde{B}}(p)\in M$. We have either $pJ1_{B_1}J\neq 0$ or $pJ1_{B_2}J\neq 0$.

Assume that $pJ1_{B_2}J\neq 0$. We may assume $pJ1_{B_2}J=p$. Then since $B_2$ is properly infinite, we can follow the proof above (with $H=\mathbb{C}$ and $B=B_2$), so we get item (1) with $H=\mathbb{C}$.

Assume that $pJ1_{B_1}J\neq 0$ and we may assume $pJ1_{B_1}J=p$. Then using $\widehat{E}_{\widetilde{B}}(p)<\infty$ and $\mathrm{ctr}(p)<\infty$, we find that there is a family $\{w_i\}_{i=1}^n\subset M1_{B_1}$ such that $W_i := w_ie_{\widetilde{B}}$ are partial isometries for all $i$, $p=\sum_{i=1}^n w_ie_{\widetilde{B}}w_i^* = \sum_{i=1}^n W_iW_i^*$, and $E_B(w_i^*w_j)=\delta_{i,j}p_j$ for all $i,j$, where $p_j\in B_1$ are projections. This fact is well known to experts, but we include a short proof for the reader's convenience (but for the case $B_1=B=\widetilde{B}$). First, by a maximality argument, there exists a pair $(Q,q)$ of projections with $Q\in\langle M,B\rangle$ and

$q \in B$, which is maximal for the condition $p \geq Q \sim qe_B \leq e_B$. It follows that $z_B(r) \leq q$ for any other pair $(R, r)$ such that $p - Q \geq R \sim re_B \leq e_B$. Then we can construct inductively a family $(Q_i, q_i)_{i=1}^m$, where $m \in \mathbb{N} \cup \{\infty\}$, of pairs of projections such that $Q_i Q_j = 0$ for all $i \neq j$, $p \geq Q_i \sim q_i e_B$ for all $i$, and $(Q_i, q_i)$ is maximal with respect to the condition $p - \sum_{j=1}^{i-1} Q_j \geq Q_i \sim q_i e_B \leq e_B$ for all $i$. Then the maximality implies $z_B(q_{i+1}) \leq q_i$ for all $i$, hence $m < \infty$ (because $\mathrm{ctr}(p) < \infty$) and $p = \sum_{i=1}^m Q_i$. Take partial isometries $W_i \in \langle M, B\rangle$ such that $Q_i = W_i W_i^*$ and $q_i = W_i^* W_i$ for all $i$. Since $\widehat{E}_B(p) < \infty$, by the push down lemma (e.g. [15, Lemma 2.5]), one can write $W_i = w_i e_B$ for some $w_i \in M$, as desired.

Consider the $*$-homomorphism $\pi\colon p\langle M, \widetilde{B}\rangle p \to B_1 \mathbin{\overline{\otimes}} \mathbb{M}_n$ given by

$$pxp = \sum_{i,j=1}^n W_i (W_i^* x W_j) W_j^* \mapsto \sum_{i,j=1}^n E_B(w_i^* x w_j) \otimes e_{i,j}, \qquad x \in \langle M, \widetilde{B}\rangle.$$

Then using the identification $p\langle M, \widetilde{B}\rangle p \simeq p\langle M, \widetilde{B}\rangle p \otimes \mathbb{C}e_{1,1}$ and the partial isometry $W := \sum_j W_j \otimes e_{1,j}$, we see that $\pi$ satisfies $\pi(x)(e_{\widetilde{B}} \otimes 1_n) = W^*(x \otimes e_{1,1})W$ for all $x \in p\langle M, \widetilde{B}\rangle p$. Define $f := \pi(1_A) \in B_1 \otimes \mathbb{M}_n$ and $w := \sum_j w_j \otimes e_{1,j} \in M \otimes \mathbb{M}_n$, so that $W^*W = f(e_{\widetilde{B}} \otimes 1_n)$ and $W = w(e_{\widetilde{B}} \otimes 1_n)$. By restricting $\pi$ to $Ap$ and composing with the map $A \to Ap$, we have a unital normal $*$-homomorphism $\pi\colon A \to f(B_1 \otimes \mathbb{M}_n)f$ such that $(a \otimes e_{1,1})W = W\pi(a)$ for all $a \in A$. Thus we are exactly in the same situation as in the proof of (4)$\Rightarrow$(1) but with $H = \mathbb{C}^n$ and $B = B_1$. Following the same proof, we get item (1) with $H = \mathbb{C}^n$ as desired. ■

### *Intertwining theory with modular actions*

We next focus on the case of modular actions. We continue to use $A, B \subset M$ and fix faithful normal conditional expectations $E_A, E_B$ for $A, B$ respectively. Let $\psi, \varphi \in M_*$ be faithful normal positive functionals which are preserved by $E_A, E_B$ respectively. Then since $\sigma_t^\psi(A) = A$, $\sigma_t^\varphi(B) = B$ for all $t \in \mathbb{R}$, and $\sigma^\psi$ and $\sigma^\varphi$ are cocycle conjugate by $([D\psi, D\varphi]_t)_{t\in\mathbb{R}}$, the condition $(A, \sigma^\psi) \preceq_M^{\mathrm{uni}} (B, \sigma^\varphi)$ can be defined. In this setting, the extended actions of $\sigma^\psi$ and $\sigma^\varphi$ on $\langle M, \widetilde{B}\rangle$ are exactly the modular actions of $\widehat{\psi} := \psi \circ \widehat{E}_{\widetilde{B}}$ and $\widehat{\varphi} := \varphi \circ \widehat{E}_{\widetilde{B}}$ respectively.

As in the usual intertwining condition, we introduce intertwining conditions with modular actions at the level of *corners*.

**Definition 3.4.** In the above setting, we will say that *a corner of* $(A, \sigma^\psi)$ *embeds with expectation into* $(B, \sigma^\varphi)$ *inside* $M$ and write $(A, \sigma^\psi) \preceq_M (B, \sigma^\varphi)$ if there exists $(e, f, \theta, v)$ which witnesses $A \preceq_M B$ with $e \in A_\psi$, and a generalized cocycle $(u_t)_{t\in\mathbb{R}}$ for $\sigma^\varphi$ with values in $B$ and with support projection $f$ such that, with $\omega_t := [D\psi, D\varphi]_t$,

- $vu_t = \omega_t \sigma_t^\varphi(v)$ for all $t \in \mathbb{R}$;
- $u_t \sigma_t^\varphi(\theta(a)) u_t^* = \theta(\sigma_t^\psi(a))$ for all $a \in eAe$ and $t \in \mathbb{R}$.

In this case, we will say that $(e, f, \theta, u)$ and $(u_g)_{g\in G}$ *witness* $(A, \sigma^\psi) \preceq_M (B, \sigma^\varphi)$.

Below we collect elementary lemmas. We omit proofs since they are straightforward.

**Lemma 3.5.** *Assume* $(A,\sigma^{\psi}) \preceq_M (B,\sigma^{\varphi})$ *and fix* $(e,f,\theta,v)$ *and* $(u_t)_{t\in\mathbb{R}}$ *which witness* $(A,\sigma^{\psi}) \preceq_M (B,\sigma^{\varphi})$ *in the sense of Definition* 3.4.

(1) *For any projection* $e_0 \in eA_{\psi}e$ *with* $e_0v = v\theta(e_0) \neq 0$, $(e_0, \theta(e_0), \theta|_{e_0Ae_0}, e_0v)$ *and* $(\theta(e_0)u_t)_{t\in\mathbb{R}}$ *witness* $(A,\sigma^{\psi}) \preceq_M (B,\sigma^{\varphi})$ *(up to the polar decomposition of* $e_0v$*).*

(2) *For any projection* $z \in B \cap \theta(eAe)' \cap \{u_t \mid t \in \mathbb{R}\}'$ *(e.g.* $z \in \mathcal{Z}(B)$*) with* $vz \neq 0$, $(e, fz, \theta(\cdot)z, vz)$ *and* $(u_tz)_{t\in\mathbb{R}}$ *witness* $(A,\sigma^{\psi}) \preceq_M (B,\sigma^{\varphi})$ *(up to the polar decomposition of* $vz$*).*

(3) *Let* $u \in A$ *and* $w \in B$ *be partial isometries such that* $e = u^*u$ *and* $f = ww^*$*. Then* $(uu^*, w^*w, \mathrm{Ad}(w^*)\circ\theta\circ\mathrm{Ad}(u^*), uvw)$ *and the generalized cocycle* $(w^*u_t\sigma_t^{\varphi}(w))_{t\in\mathbb{R}}$ *witness* $(A,\sigma^{\psi'}) \preceq_M (B,\sigma^{\varphi})$*, where* $\psi' \in M_*^+$ *is any faithful element which is preserved by* $E_A$ *such that* $uu^*\psi'uu^* = u\psi u^*$ *and* $uu^* \in A_{\psi'}$*.*

(4) *Let* $\psi'$ *and* $\varphi'$ *be any faithful normal positive functionals on* $M$ *which are preserved by* $E_A$ *and* $E_B$ *respectively and have the property that* $e \in A_{\psi'}$*. Then* $(e,f,\theta,v)$ *and* $(\theta(e[D\psi', D\psi]_te)u_t[D\varphi, D\varphi']_t)_t$ *witness* $(A,\sigma^{\psi'}) \preceq_M (B,\sigma^{\varphi'})$*.*

*Moreover all these statements hold if we consider* $(H,f,\pi,w)$ *and* $(u_t)_{t\in\mathbb{R}}$ *which witness* $(A,\sigma^{\psi}) \preceq_M^{\mathrm{uni}} (B,\sigma^{\varphi})$ *in the sense of Definition* 3.1*. (In this case, we use* $\mathcal{Z}(A)$ *and* $B \mathbin{\overline{\otimes}} \mathbb{B}(H)$ *instead of* $A_{\psi}$ *and* $B$ *in items* (1)–(3)*, and item* (4) *holds without the assumption* $e \in A_{\psi'}$*).*

The next lemma clarifies the relation between $\preceq$ and $\preceq^{\mathrm{uni}}$ for modular actions. It should be compared to Lemma 2.6.

**Lemma 3.6.**

(1) $(A,\sigma^{\psi}) \preceq_M (B,\sigma^{\varphi})$ *if and only if* $(A,\sigma^{\psi}) \preceq_M^{\mathrm{uni}} (B,\sigma^{\varphi})$*. In particular, these notions do not depend on the choice of* $\psi$ *and* $\varphi$ *(as long as they are preserved by* $E_A$ *and* $E_B$ *respectively).*

(2) *Assume either*

- *A does not have any direct summand which is semifinite and properly infinite, or*
- *B is properly infinite.*

*If* $(A,\sigma^{\psi}) \preceq_M^{\mathrm{uni}} (B,\sigma^{\varphi})$*, then the Hilbert space* $H$ *in Definition* 3.1 *can be taken finite-dimensional.*

*Proof.* We decompose $A = A_1 \oplus A_2 \oplus A_3$ and $B = B_1 \oplus B_2 \oplus B_3$, where $A_1, B_1$ are finite, $A_2, B_2$ are semifinite and properly infinite, and $A_3, B_3$ are of type III. Then by Lemma 3.5(1, 2) and [15, Remark 4.2(2)], we know that $(A,\sigma^{\psi}) \preceq_M (B,\sigma^{\varphi})$ if and only if $(A_i,\sigma^{\psi}) \preceq_M (B_j,\sigma^{\varphi})$ for some $i, j$. Hence we can always assume that $A = A_i$ and $B = B_j$ for some $i, j$. The same is true for $(A,\sigma^{\psi}) \preceq_M^{\mathrm{uni}} (B,\sigma^{\varphi})$.

(1) By Lemma 3.5(4), the condition $(A,\sigma^{\psi}) \preceq_M^{\mathrm{uni}} (B,\sigma^{\varphi})$ does not depend on the choice of $\psi, \varphi$. Hence if this statement is proven, then $(A,\sigma^{\psi}) \preceq_M (B,\sigma^{\varphi})$ does not depend on $\psi, \varphi$ either.

Assume that $(A_i,\sigma^{\psi}) \preceq_M^{\mathrm{uni}} (B_j,\sigma^{\varphi})$ for some $i,j$ and take $(H,f,\pi,w)$ and $(u_t)_t$ as in the definition. Let $z\in\mathcal{Z}(A)$ be a non-zero projection such that $Az\ni a\mapsto \pi(a)w^*w$ is injective. Since $z\in A_\psi$, up to replacing $Az$ by $A$, we may assume that $A\ni a\mapsto\pi(a)w^*w$ is injective. In particular $w\pi(e)\neq 0$ for any non-zero projection $e\in A$.

Assume that $B=B_2$ or $B=B_3$. Then since $1_B\otimes e_{1,1}$ is properly infinite, one has $f\prec 1_B\otimes e_{1,1}$. Up to equivalence of projections, using Lemma 3.5(3), we may assume that $f$ is contained in $B\otimes\mathbb{C}e_{1,1}$. So using $M=M\otimes\mathbb{C}e_{1,1}$, we get $(A,\sigma^{\psi})\preceq_M(B,\sigma^{\varphi})$.

Assume that $B=B_1$. Then $A=A_1$ or $A_2$. If $A=A_2$, then by using $eAe$ for any fixed finite projection $e\in A_\psi$ (note that $A_\psi$ contains many finite projections, e.g. by the first part of the proof of [21, Lemma 2.1]) and using Lemma 3.5(1), we may assume that $A$ is finite. By the last statement of Theorem 3.2, we may assume that $A$ is finite and $H$ is finite-dimensional. We can still assume that $A\ni a\mapsto\pi(a)w^*w$ is injective.

Write $H=\mathbb{C}^n$ for some $n\in\mathbb{N}$. As in the proof of [5, Proposition F.10] or [48, Proposition 3.1(ii)⇒(iii)], there is a projection $e\in A$ such that $\pi(e)$ is equivalent to a projection $f_0\otimes e_{1,1}$ for some $f_0\in B$. By [21, Lemma 2.1], $e$ is equivalent to a projection in $A_\psi$, so we may assume $e\in A_\psi$. Observe that, regarding $\pi$ as a map from $A\otimes\mathbb{C}e_{1,1}$, $(1_A\otimes e_{1,1},f,\pi,w)$ and $(u_t)_t$ witness $(A\otimes\mathbb{C}e_{1,1},\sigma^{\psi})\preceq_{M\otimes\mathbb{M}_n}(B\otimes\mathbb{M}_n,\sigma^{\varphi\otimes\mathrm{tr}_n})$. Since $\pi(e)w^*w\neq 0$, by Lemma 3.5(1), $(e\otimes e_{1,1},\pi(e),\pi|_{eAe\otimes e_{1,1}},(e\otimes e_{1,1})w)$ witnesses $(A\otimes\mathbb{C}e_{1,1},\sigma^{\psi})\preceq_{M\otimes\mathbb{M}_n}(B\otimes\mathbb{M}_n,\sigma^{\varphi\otimes\mathrm{tr}_n})$ as well. We then apply Lemma 3.5(3) for $\pi(e)\sim f_0\otimes e_{1,1}$, and find that $(e\otimes e_{1,1},f_0\otimes e_{1,1},\pi',w')$ and some generalized cocycle witness $(A\otimes\mathbb{C}e_{1,1},\sigma^{\psi})\preceq_{M\otimes\mathbb{M}_n}(B\otimes\mathbb{M}_n,\sigma^{\varphi\otimes\mathrm{tr}_n})$ for some $\pi'$ and $w'$. Finally, since $f_0\otimes e_{1,1}$ and $w'$ are contained in $M\otimes\mathbb{C}e_{1,1}$, by identifying $M\otimes\mathbb{C}e_{1,1}=M$, we get $(A,\sigma^{\psi})\preceq_M(B,\sigma^{\varphi})$.

We next show the ‘only if’ direction. Assume that $(A,\sigma^{\psi})\preceq_M(B,\sigma^{\varphi})$ and take $(e,f,\theta,v)$ and $(u_t)_t$ as in the definition. As in the proof above, we can assume $eAe\ni a\mapsto v^*v\theta(a)$ is injective and hence $v\theta(e_0)\neq 0$ for any non-zero projection $e_0\in eAe$.

Let $z$ be the central support projection of $e$ in $A$, and take partial isometries $(w_i)_{i\in I}$ in $A$ such that $w_0=e$, $e_i:=w_i^*w_i\leq e$ for all $i\in I$, and $\sum_{i\in I}w_iw_i^*=z$. Note that $I$ is a countable set, so we regard $I\subset\mathbb{N}$. We put $v_n:=w_nv$ for all $n\in I$ and $d=\sum_{n\in I}v_ne_{\widetilde{B}}v_n^*$, and then it is easy to see that $d=dJ1_BJ$ and $\widehat{E}_{\widetilde{B}}(d)\in M$. We note that $d\neq 0$, since each $v_n$ is non-zero by $w_n^*v_n=w_n^*w_nv=v\theta(w_n^*w_n)\neq 0$. Then for any $a\in A$, we have

$$\begin{aligned} ad=zad&=\sum_{i\in I}w_iw_i^*a\sum_{j\in I}v_je_{\widetilde{B}}v_j^*=\sum_{i,j\in I}w_i(w_i^*aw_j)ve_{\widetilde{B}}v^*w_j^*\\ &=\sum_{i,j\in I}w_iv\theta(w_i^*aw_j)e_{\widetilde{B}}v^*w_j^*=\sum_{i,j\in I}w_ive_{\widetilde{B}}v^*(w_i^*aw_j)w_j^*=daz=da. \end{aligned}$$

It follows that $d\in A'\cap 1_A\langle M,\widetilde{B}\rangle 1_A$. Define a faithful normal positive functional $\psi'$ on $M$ by

$$\psi':=\sum_{n\in I}\frac{1}{2^n}w_n\psi w_n^*+(1-z)\psi(1-z).$$

Note that $\psi'$ is preserved by $E_A$. By Lemma 2.2, the equality $e_n\psi'e_n=2^{-n}w_n\psi w_n^*$ implies $\sigma_t^{\psi}(w_n)=2^{-itn}[D\psi',D\psi]_t^*w_n$ for all $t\in\mathbb{R}$ and $n\in I$. An easy computation shows that

$$\sigma_t^{\widehat{\psi}}(d) = [D\psi, D\varphi]_t \sigma_t^{\widehat{\varphi}}(d)[D\psi, D\varphi]_t^* = [D\psi', D\psi]_t^*\, d\, [D\psi', D\psi]_t \quad \text{for all } t \in \mathbb{R}.$$

We see that $\sigma_t^{\widehat{\psi}'}(d) = d$ for all $t \in \mathbb{R}$ and hence $d \in A' \cap (1_A \langle M, \widetilde{B}\rangle 1_A)_{\widehat{\psi}'}$. By Theorem 3.2, this means $(A, \sigma^{\psi'}) \preceq_M^{\mathrm{uni}} (B, \sigma^{\varphi})$. By Lemma 3.5(4), this is equivalent to $(A, \sigma^{\psi}) \preceq_M^{\mathrm{uni}} (B, \sigma^{\varphi})$.

(2) Assume that $(A_i, \sigma^{\psi}) \preceq_M^{\mathrm{uni}} (B_j, \sigma^{\varphi})$ for some $i, j$. If $B = B_2$ or $B_3$, then the first half of the proof of item (1) shows that one can assume $H = \mathbb{C}$. So we get the conclusion. If $A = A_3$, then we must have $B = B_3$, which we proved. Finally, if $A = A_1$, then the last part of Theorem 3.2 gives the conclusion. ■

### *Intertwining theory with conditional expectations*

In [19], a notion of intertwining conditions for *states* was introduced. Inspired by this, we introduce a notion of intertwining conditions for *conditional expectations*. We still fix $A, B \subset M$ with expectations $E_A, E_B$.

**Definition 3.7.** We say that *a corner of* $(A, E_A)$ *embeds with expectation into* $(B, E_B)$ *inside* $M$ and write $(A, E_A) \preceq_M (B, E_B)$ if there exists $(e, f, \theta, v)$ which witnesses $A \preceq_M B$ and faithful normal positive functionals $\psi, \varphi \in M_*$ which are preserved by $E_A, E_B$ respectively such that

$$vv^* \in (1_A M 1_A)_{\psi}, \qquad v^*v \in (1_B M 1_B)_{\varphi}, \qquad \text{and} \qquad vv^*\psi vv^* = v\varphi v^*.$$

In this case, we say that $(e, f, \theta, v)$ and $\psi, \varphi$ *witness* $(A, E_A) \preceq_M (B, E_B)$.

The next lemma clarifies relations between $A \preceq_M B$ and $(A, E_A) \preceq_M (B, E_B)$. Note that, as in the statement of Theorem A, one can actually take $q = 1_A$ in the next lemma (this will be proved later).

**Lemma 3.8.** *The condition $A \preceq_M B$ holds if and only if there is a non-zero projection $q \in A' \cap 1_A M 1_A$ and a faithful normal conditional expectation $E_{Aq}\colon qMq \to Aq$ such that $(Aq, E_{Aq}) \preceq_M (B, E_B)$.*

*Proof.* The 'if' direction is trivial, so we prove the 'only if' direction. Take $(e, f, \theta, v)$ which witnesses $A \preceq_M B$. By [15, Remark 4.2(2, 3)], we may assume that $A$ is finite or of type III, and that $eAe \ni a \mapsto \theta(a)v^*v$ is injective. Up to replacing $e$ with a smaller projection if necessary, we may assume that there exist finitely many orthogonal and equivalent projections $(e_i)_{i=1}^n$ in $A$ such that $\sum_{i=1}^n e_i =: z_A(e) \in \mathcal{Z}(A)$. Fix a faithful normal conditional expectation $E_\theta$ for the inclusion $\theta(eAe) \subset fBf$, and take a faithful normal state $\varphi_B$ on $B$ such that $\varphi_B \circ E_\theta = \varphi_B$ on $fBf$. Put $\varphi := \varphi_B \circ E_B$ on $1_B M 1_B$ and observe that the modular action of $\varphi$ globally preserves $\theta(eAe)$ and $fBf$. In particular it also preserves $\theta(eAe)' \cap fMf$, so by [21, Lemma 2.1], there is a partial isometry $w \in \theta(eAe)' \cap fMf$ such that $w^*w = v^*v$ and $ww^* \in (\theta(eAe)' \cap fMf)^{\sigma^{\varphi}}$. Up to replacing $vw^*$ by $v$, we may assume that $v^*v$ is in $(fMf)^{\sigma^{\varphi}}$.

We put $e_0 := vv^* \in (eAe)' \cap eMe$ and $f_0 := v^*v \in (\theta(eAe)' \cap fMf)^{\sigma^\varphi}$. Since $\theta(eAe)f_0 \subset f_0Mf_0$ is globally preserved by $\sigma^\varphi$, it is with expectation, say $E\colon f_0Mf_0 \to \theta(eAe)f_0$, which satisfies $\varphi \circ E = \varphi$ on $f_0Mf_0$. Observe that $\mathrm{Ad}(v)$ gives a spatial isomorphism from $\theta(eAe)f_0$ onto $(eAe)e_0$. Hence we can define a conditional expectation by

$$E'_A := \mathrm{Ad}(v) \circ E \circ \mathrm{Ad}(v^*)\colon e_0Me_0 \to (eAe)e_0.$$

Define a positive functional $\psi'_A := v\varphi v^*$ on $(eAe)e_0$ and put $\psi' := \psi'_A \circ E'_A$ on $e_0Me_0$. We have $v^*v = f_0 \in (1_BM1_B)_\varphi$ and $vv^* = e_0 \in (e_0Me_0)_{\psi'}$. By using $\psi'_A = v\varphi v^*$ on $(eAe)e_0$ and $\varphi \circ E = \varphi$ on $f_0Mf_0$, we compute that, for any $x \in M$,

$$\begin{aligned} vv^*\psi'(x)vv^* &= \psi'_A \circ E'_A(vv^*xvv^*) = (v\varphi v^*)(vE(v^*vv^*xvv^*v)v^*) \\ &= \varphi(f_0E(v^*xv)f_0) = \varphi \circ E(v^*xv) = \varphi(v^*xv). \end{aligned}$$

We get $vv^*\psi'vv^* = v\varphi v^*$. Since they satisfy $\varphi = \varphi \circ E_B$ on $1_BM1_B$ and $\psi' = \psi' \circ E'_A$ on $e_0Me_0$, we can extend $\varphi$ and $\psi'$ to normal states on $M$ which are preserved by $E_B$ and $E'_A$ respectively. In this case, we still have $f_0 \in M_\varphi$, $e_0 \in M_{\psi'}$, and $vv^*\psi'vv^* = v\varphi v^*$.

We claim $((eAe)e_0, E'_A) \preceq_M (B, E_B)$. Let $z \in \mathcal{Z}(eAe)$ be the central support projection of $e_0$ in $(eAe)'$ and observe that $(eAe)e_0 \simeq eAez$. Since we have assumed $eAe \ni a \mapsto v^*v\theta(a) = v^*av$ is injective, the map $eAe \ni a \mapsto \mathrm{Ad}(v)(v^*v\theta(a)) = ae_0$ is also injective. In particular we get $z = e$ and $(eAe)e_0 \simeq eAe$. Consider $\theta_0\colon (eAe)e_0 \simeq eAe \to^\theta fBf$ given by $\theta_0(ae_0) := \theta(a)$ for $a \in eAe$. Then $(ee_0, f, \theta_0, v)$ witnesses $(eAe)e_0 \preceq_M B$. Together with $\varphi$ and $\psi'$, this witnesses $((eAe)e_0, E'_A) \preceq_M (B, E_B)$.

Since $e_0 \in (eAe)' \cap (eMe) = (A' \cap 1_AM1_A)e$, there is a projection $q \in A' \cap 1_AM1_A$ such that $qe = e_0$ and $q = z_A(e)q$. Using projections $(e_i)_{i=1}^n$ which we fixed in the first paragaph of the proof, we have an identification $qMq \simeq e_0Me_0 \otimes \mathbb{M}_n$ which restricts $Aq \simeq eAeq \otimes \mathbb{M}_n$. In particular, there is a faithful normal conditional expectation $E_{Aq}\colon qMq \to Aq$ such that $E_{Aq}|_{e_0Me_0} = E'_A$. Since we chose $\psi'$ as any extension of $\psi'|_{e_0Me_0}$ which is preserved by $E'_A$, we can in particular choose $\psi'$ as the one which is preserved by $E'_A$ and $E_{Aq}$. Then it is easy to see that the same $(ee_0, f, \theta_0, v)$ as above and $\psi', \varphi$ witness $(Aq, E_{Aq}) \preceq_M (B, E_B)$. ■

The next theorem clarifies the relation between $(A, E_A) \preceq_M (B, E_B)$ and $(A, \sigma^\psi) \preceq_M (B, \sigma^\varphi)$. The proof uses Connes cocycles to construct a positive functional. Note that the case $A = \mathbb{C}$ was proved in [19, proof of Theorem 3.1].

**Theorem 3.9.** $(A, E_A) \preceq_M (B, E_B)$ *if and only if there exist faithful normal states* $\psi, \varphi \in M_*$ *which are preserved by* $E_A, E_B$ *respectively such that* $(A, \sigma^\psi) \preceq_M (B, \sigma^\varphi)$.

**Remark 3.10.** Combined with Lemma 3.6(1), characterizations given in Theorem 3.2 can be adapted to $(A, E_A) \preceq_M (B, E_B)$ and $(A, \sigma^\psi) \preceq_M (B, \sigma^\varphi)$. Moreover $\psi$ and $\varphi$ for $(A, \sigma^\psi) \preceq_M (B, \sigma^\varphi)$ can be taken arbitrary as long as they are preserved by $E_A$ and $E_B$ respectively.

*Proof of Theorem* 3.9. Suppose $(A, E_A) \preceq_M (B, E_B)$ and take $(e, f, \theta, v)$ and $\psi, \varphi$. We put $d := ve_{\widetilde{B}}v^*$ and observe that $d \in (eAe)' \cap (e\langle M, \widetilde{B}\rangle e)$, $d = dJ1_BJ$, and

$\widehat{E}_{\widetilde{B}}(d) < \infty$. By Lemma 2.2, the equation $vv^*\psi vv^* = v\varphi v^*$ implies $[D\psi, D\varphi]_t\sigma_t^\varphi(v) = v$ for all $t \in \mathbb{R}$. Then $\sigma_t^{\widehat{\psi}}(d) = d$ for any $t \in \mathbb{R}$, hence $d \in A' \cap (1_A\langle M, \widetilde{B}\rangle 1_A)_{\widehat{\psi}}$. We get $(eAe, \sigma^\psi) \preceq_M^{\mathrm{uni}} (B, \sigma^\varphi)$ by Theorem 3.2. This implies $(eAe, \sigma^\psi) \preceq_M (B, \sigma^\varphi)$ by Lemma 3.6, and hence $(A, \sigma^\psi) \preceq_M (B, \sigma^\varphi)$.

Suppose $(A, \sigma^\psi) \preceq_M (B, \sigma^\varphi)$ and take $(e, f, \theta, v)$ and $(u_t)_{t\in\mathbb{R}}$. Since $(u_t)_{t\in\mathbb{R}}$ is a generalized cocycle for $\sigma^\varphi$ with support projection $f$, by Theorem 2.1 there is a unique faithful normal semifinite weight $\mu_B$ on $fBf$ such that $[D\mu_B, D\varphi_B]_t = u_t$ for all $t \in \mathbb{R}$. Put $\mu := \mu_B \circ E_B$ on $fMf$ and observe $[D\mu, D\varphi]_t = u_t$ for all $t \in \mathbb{R}$. For any $t \in \mathbb{R}$ and $a \in eAe$, using the equation $vu_t = \omega_t\sigma_t^\varphi(v)$ where $\omega_t = [D\psi, D\varphi]_t$, it is easy to compute that

$$\sigma_t^\psi(vv^*) = vv^*, \quad \sigma_t^\mu(v^*v) = v^*v, \quad \text{and} \quad \sigma_t^\mu(\theta(a)) = \theta(\sigma_t^\psi(a)).$$

We find that $vv^* \in eM_\psi e$ and $v^*v \in (fMf)_\mu$. We extend $\mu$ by $f\mu f + (1-f)\varphi(1-f)$ and still denote it by $\mu$. It satisfies $\mu = \mu \circ E_B$ on $1_BM1_B$ and $1_B, f \in M_\mu$. We put $e_0 := vv^* \in eM_\psi e$ and $f_0 := v^*v \in fM_\mu f$. For any $t \in \mathbb{R}$, using Lemma 2.2, we have

$$\begin{aligned}[D(v\mu v^*), D\varphi]_t &= [D(v\mu v^*), D\mu]_t[D\mu, D\varphi]_t = v\sigma_t^\mu(v^*)[D\mu, D\varphi]_t \\ &= v[D\mu, D\varphi]_t\sigma_t^\varphi(v^*) = vu_t\sigma_t^\varphi(v^*) = \omega_t\sigma_t^\varphi(vv^*) \\ &= \sigma_t^\psi(vv^*)\omega_t = vv^*\omega_t = [D(e_0\psi e_0), D\varphi]_t.\end{aligned}$$

We get $e_0\psi e_0 = v\mu v^*$. Hence $(e, f, \theta, v)$ and $\psi, \mu$ witness $(A, E_A) \preceq_M (B, E_B)$, but $\mu$ is not necessarily bounded. So we have to replace $\mu$ by a bounded one.

Since $e_0\psi e_0 = v\mu v^*$, it follows that $\mu_B(E_B(f_0)) = \mu(v^*v) = \psi(e_0) < \infty$. Since $\sigma_t^{\mu_B}(E_B(f_0)) = E_B(\sigma_t^\mu(f_0)) = E_B(f_0)$ for all $t \in \mathbb{R}$, and since $f_0 = v^*v \in \theta(eAe)'$, $E_B(f_0)$ is contained in $(fBf)_{\mu_B} \cap \theta(eAe)'$. Combined with the fact that $v^*vE_B(f_0) \neq 0$ (because $E_B(v^*vE_B(f_0)) = E_B(f_0)^2 \neq 0$), this shows that there is a non-zero spectral projection $f' \in (fBf)_{\mu_B} \cap \theta(eAe)'$ of $E_B(f_0)$ such that $vf' \neq 0$ and $\mu_B(f') < \infty$. Put $v' := vf'$, $\theta'(a) := \theta(a)f'$ for $a \in eAe$ and $u'_t := f'u_t$ for $t \in \mathbb{R}$. We claim that, up to the polar decomposition of $v'$, $(e, f', \theta', v')$ and $(u'_t)_{t\in\mathbb{R}}$ witness $(A, \sigma^\psi) \preceq_M (B, \sigma^\varphi)$.

It is easy to see that $v'\theta'(a) = av'$ for all $a \in eAe$, hence $(e, f', \theta', v')$ witnesses $A \preceq_M B$. For any $t \in \mathbb{R}$, since $f' = \sigma_t^\mu(f')$, one has

$$(u'_t)^*u'_t = u_t^*f'u_t = u_t^*\sigma_t^\mu(f')u_t = \sigma_t^\varphi(f').$$

This means $u'_t = f'u_t = u_t\sigma_t^\varphi(f')$ for all $t \in \mathbb{R}$. Using this, it is easy to compute that for any $a \in eAe$ and $t, s \in \mathbb{R}$,

$$u'_{t+s} = u'_t\sigma_t^\varphi(u'_s), \quad v'u'_t = \omega_t\sigma_t^\varphi(v'), \quad \text{and} \quad u'_t\sigma_t^\varphi(\theta'(a))(u'_t)^* = \theta'(\sigma_t^\psi(a)).$$

Thus $(e, f', \theta', v')$ and $(u'_t)_{t\in\mathbb{R}}$ witness $(A, \sigma^\psi) \preceq_M (B, \sigma^\varphi)$.

We replace $v'$ with its polar part. Then by using $(e, f', \theta', v')$ and $(u'_t)_{t\in\mathbb{R}}$, and by following the same construction as we did for $\mu$, we again construct a faithful normal semifinite weight $\mu'$ on $M$ such that $u'_t = [Df'\mu' f', D\varphi]_t$ for all $t \in \mathbb{R}$, and $e'_0\psi e'_0 = v'\mu' v'^*$, where $e'_0 := v'v'^*$. Since

$$[Df'\mu' f', D\varphi]_t = u'_t = f'u_t = f'[Df\mu f, D\varphi]_t = [Df'\mu f', D\varphi]_t$$

for all $t \in \mathbb{R}$, it follows that $f'\mu' f' = f'\mu f'$. In particular, since $\mu(f') < \infty$, $f'\mu' f'$ is bounded. By construction, $\mu'$ is bounded on $M$ and hence $(e, f', \theta', v')$ and $\psi, \mu'$ witness $(A, E_A) \preceq_M (B, E_B)$. ∎

We record the following permanence property.

**Lemma 3.11.** *Let $D \subset A$ be a unital von Neumann subalgebra with expectation $E_D$.*

(1) *If $(A, \sigma^\psi) \preceq_M (B, \sigma^\varphi)$, then $(D, \sigma^{\psi'}) \preceq_M (B, \sigma^\varphi)$ for any faithful $\psi' \in M_*^+$ which is preserved by $E_D \circ E_A$.*

(2) *If $(A, E_A) \preceq_M (B, E_B)$, then $(D, E_D \circ E_A) \preceq_M (B, E_B)$.*

*Proof.* These are immediate by Lemma 3.6(1) and Theorem 3.9. ∎

### *Proof of Theorem* A

Now we prove Theorem A. We continue to use $A, B \subset M$ with expectations, and we only fix $E_B$. We also fix a type III$_1$ factor $(N, \omega)$ as in the statement of Theorem A.

The next lemma is the key observation to prove Theorem A.

**Lemma 3.12.** *Let $E_A\colon 1_A M 1_A \to A$ be a faithful normal conditional expectation, and let $\psi, \varphi \in M_*$ be faithful states which are preserved by $E_A, E_B$ respectively. The following conditions are equivalent:*

(1) $(A, E_A) \preceq_M (B, E_B)$.

(2) $(A \mathbin{\overline{\otimes}} N, E_A \otimes \mathrm{id}_N) \preceq_{M \overline{\otimes} N} (B \mathbin{\overline{\otimes}} N, E_B \otimes \mathrm{id}_N)$.

(3) $\Pi_{\varphi\otimes\omega, \psi\otimes\omega}(C_{\psi\otimes\omega}(A \mathbin{\overline{\otimes}} N)) \preceq_{C_{\varphi\otimes\omega}(M \overline{\otimes} N)} C_{\varphi\otimes\omega}(B \mathbin{\overline{\otimes}} N)$.

*Proof.* (1)⇒(2). This is trivial (one only needs to take tensor products with $1_N$ or $\mathrm{id}_N$).

(2)⇒(3). By Theorem 3.9 and Lemma 3.6(1), item (2) is equivalent to $(A \mathbin{\overline{\otimes}} N, \sigma^{\psi\otimes\omega}) \preceq^{\mathrm{uni}}_{M \overline{\otimes} N} (B \mathbin{\overline{\otimes}} N, \sigma^{\varphi\otimes\omega})$. By Theorem 3.2, we get item (3).

(3)⇒(1). We first recall the following general facts (some of which were mentioned in Section 2). Since $\langle C_\varphi(M), C_\varphi(\widetilde{B})\rangle$ is generated by $\langle M, \widetilde{B}\rangle$ and $L_\varphi\mathbb{R}$, and since $\sigma_t^{\widehat{\varphi}} = \mathrm{Ad}(\Delta_\varphi^{it})$, where $\widehat{\varphi} = \varphi \circ \widehat{E}_{\widetilde{B}}$, $\langle C_\varphi(M), C_\varphi(\widetilde{B})\rangle$ is canonically identified as $C_{\widehat{\varphi}}(\langle M, \widetilde{B}\rangle)$. Put $\widehat{\psi} := \psi \circ \widehat{E}_{\widetilde{B}}$. Since $[D\widehat{\psi}, D\widehat{\varphi}]_t = [D\psi, D\varphi]_t$ for all $t \in \mathbb{R}$, the map $\Pi_{\widehat{\varphi},\widehat{\psi}}\colon C_{\widehat{\psi}}(\langle M, \widetilde{B}\rangle) \to C_{\widehat{\varphi}}(\langle M, \widetilde{B}\rangle)$ restricts to $\Pi_{\varphi,\psi}\colon C_\psi(M) \to C_\varphi(M)$. Since $1_B = \pi_{\sigma^\varphi}(1_B)$ is the unit of $C_\varphi(B)$, the modular conjugation $J_{C_\varphi(M)}$ on $L^2(C_\varphi(M)) = L^2(M) \otimes L^2(\mathbb{R})$ (with respect to the dual weight of $\varphi$) satisfies

$$J_{C_\varphi(M)} 1_{C_\varphi(B)} J_{C_\varphi(M)} = J_{C_\varphi(M)} 1_B J_{C_\varphi(M)} = J 1_B J \otimes 1_{L^2(\mathbb{R})}.$$

We note that the unitization of $C_\varphi(B)$ is contained in $C_\varphi(\widetilde{B})$, but they are different in general. We will use these observations for $A \mathbin{\overline{\otimes}} N, B \mathbin{\overline{\otimes}} N \subset M \mathbin{\overline{\otimes}} N$.

Now we start the proof. We put $\mathcal{B} := C_{\varphi\otimes\omega}(B \mathbin{\overline{\otimes}} N)$, $\mathcal{B}_1 := C_{\varphi\otimes\omega}(\widetilde{B} \mathbin{\overline{\otimes}} N)$, $\mathcal{M} := C_{\varphi\otimes\omega}(M \mathbin{\overline{\otimes}} N)$, $\mathcal{A} := C_{\psi\otimes\omega}(A \mathbin{\overline{\otimes}} N)$, and $\Pi := \Pi_{\widehat{\varphi\otimes\omega},\widehat{\psi\otimes\omega}}$, so that our assumption is written as $\Pi(\mathcal{A}) \preceq_{\mathcal{M}} \mathcal{B}$. Note that the unitization of $\mathcal{B}$ is contained in $\mathcal{B}_1$. Take $(e, f, \theta, v)$ which witnesses $\Pi(\mathcal{A}) \preceq_{\mathcal{M}} \mathcal{B}$. Let $w_i \in \mathcal{A}$ be partial isometries such that $w_i^* w_i \leq e$ and $\sum_i w_i w_i^* = z_{\mathcal{A}}(e)$, where $z_{\mathcal{A}}(e)$ is the central support of $e$ in $\mathcal{A}$. Put $d := \sum_i \Pi(w_i) v e_{\mathcal{B}_1} v^* \Pi(w_i^*)$ and observe that

$$d \in \Pi(\mathcal{A})' \cap 1_{\Pi(\mathcal{A})}\langle \mathcal{M}, \mathcal{B}_1\rangle 1_{\Pi(\mathcal{A})}, \quad d = d\mathcal{J}1_{\mathcal{B}}\mathcal{J}, \quad \text{and} \quad \widehat{E}_{\mathcal{B}_1}(d) < \infty,$$

where $\mathcal{J}$ is the modular conjugation for $L^2(\mathcal{M})$. Note that $\mathcal{J}1_{\mathcal{B}}\mathcal{J} = J1_BJ \otimes 1_N \otimes 1_{L^2(\mathbb{R})}$ as we have explained.

**Claim.** *The element $d$ is contained in*

$$[A' \cap 1_A\langle M, \widetilde{B}\rangle J1_BJ1_A]_{\widehat{\psi}} \otimes \mathbb{C}1_N \otimes \mathbb{C}1_{L^2(\mathbb{R})}.$$

*Proof.* Observe that

$$\Pi^{-1}(d) \in \mathcal{A}' \cap 1_{\mathcal{A}}\Pi^{-1}(\langle \mathcal{M}, \mathcal{B}_1\rangle \mathcal{J}1_{\mathcal{B}}\mathcal{J})1_{\mathcal{A}},$$

and $\Pi^{-1}(\langle \mathcal{M}, \mathcal{B}_1\rangle) = C_{\widehat{\psi\otimes\omega}}(\langle M \mathbin{\overline{\otimes}} N, \widetilde{B} \mathbin{\overline{\otimes}} N\rangle)$ and $\widehat{\psi\otimes\omega} = (\psi \otimes \omega) \circ \widehat{E}_{\widetilde{B}\mathbin{\overline{\otimes}} N} = \widehat{\psi} \otimes \omega$. Then using $\widehat{\psi} = \psi \circ E_A \circ \widehat{E}_{\widetilde{B}}$ on $1_A\langle M, \widetilde{B}\rangle 1_A$, we can apply Lemma 2.3 (to the inclusion $A \subset 1_A\langle M, \widetilde{B}\rangle 1_A$ with the operator valued weight $E_A \circ \widehat{E}_{\widetilde{B}}$) to get

$$\mathcal{A}' \cap 1_{\mathcal{A}}\Pi^{-1}(\langle \mathcal{M}, \mathcal{B}_1\rangle)1_{\mathcal{A}} = [A' \cap 1_A\langle M, \widetilde{B}\rangle 1_A]_{\widehat{\psi}} \otimes \mathbb{C}1_N \otimes \mathbb{C}1_{L^2(\mathbb{R})}.$$

Since $\Pi$ is the identity on $\langle M \mathbin{\overline{\otimes}} N, \widetilde{B} \mathbin{\overline{\otimes}} N\rangle$, $d$ is also contained in this set. Finally, by multiplying by $\mathcal{J}1_{\mathcal{B}}\mathcal{J} = J1_BJ \otimes 1_N \otimes 1_{L^2(\mathbb{R})}$, we get the conclusion of the claim. ■

By the claim, we can regard that $d$ is in $[A' \cap 1_A\langle M, \widetilde{B}\rangle J1_BJ1_A]_{\widehat{\psi}}$. As mentioned in Section 2, $\widehat{E}_{\mathcal{B}_1}$ coincides with $\widehat{E}_{\widetilde{B}\mathbin{\overline{\otimes}} N} \rtimes \mathbb{R}$ (the natural crossed product extension of $\widehat{E}_{\widetilde{B}\mathbin{\overline{\otimes}} N}$), hence the restriction of $\widehat{E}_{\mathcal{B}_1}$ on $\langle M \mathbin{\overline{\otimes}} N, \widetilde{B} \mathbin{\overline{\otimes}} N\rangle$ coincides with $\widehat{E}_{\widetilde{B}\mathbin{\overline{\otimes}} N}$. It then follows that

$$\infty > \widehat{E}_{\mathcal{B}_1}(d) = \widehat{E}_{\widetilde{B}\mathbin{\overline{\otimes}} N}(d) = (\widehat{E}_{\widetilde{B}} \otimes \mathrm{id}_N)(d) = \widehat{E}_{\widetilde{B}}(d).$$

Thus $d$ satisfies the condition in Theorem 3.2(4) and we get $(A, \sigma^\psi) \preceq_M^{\mathrm{uni}} (B, \sigma^\varphi)$. By Lemma 3.6(1) and Theorem 3.9, this is equivalent to item (1). ■

*Proof of Theorem A.* We first prove the equivalence of the first two conditions. Assume that $A \preceq_M B$. By Lemma 3.8, there is a projection $q \in A' \cap 1_AM1_A$ and a faithful normal conditional expectation $E_{Aq}\colon qMq \to Aq$ such that $(Aq, E_{Aq}) \preceq_M (B, E_B)$. Put $A^q := W^*\{A, q\} = Aq \oplus Aq^\perp$, where $q^\perp := 1_A - q$. Observe that $Aq^\perp \subset q^\perp Mq^\perp$ is with expectation, say $E_{Aq^\perp}$. Then by definition, the condition $(Aq, E_{Aq}) \preceq_M (B, E_B)$ implies

$(A^q, E_{A^q} \oplus E_{A^{q^\perp}}) \preceq_M (B, E_B)$. Since $A \subset 1_A M 1_A$ is with expectation, so is $A \subset A^q$. By Lemma 3.11, we have $(A, E_A) \preceq_M (B, E_B)$ for some faithful normal conditional expectation $E_A\colon 1_A M 1_A \to A$. By Theorem 3.9, $(A, \sigma^\psi) \preceq_M (B, \sigma^\varphi)$ for any faithful $\psi \in M_*^+$ which is preserved by $E_A$. This finishes the proof of the first part of the theorem.

We next prove the equivalence of (1)–(3). The equivalence of items (1) and (2) is proved in Theorem 3.9. By Lemma 3.12, item (3) is also equivalent. ■

## 4. Crossed products with groups in the class $\mathscr{C}$

In this section we prove Theorem D. Throughout this section, we will fix an outer action $\Gamma \curvearrowright^{\beta} B$ of a discrete group $\Gamma$ on a $\sigma$-finite diffuse factor $B$. We put $M := B \rtimes_\beta \Gamma$.

### *General facts on outer actions*

We first recall several well known facts on outer actions and associated crossed products.

**Lemma 4.1.** *Let $\varphi$ be a faithful normal state on $M$ which is preserved by $E_B$. Then one can define a $\Gamma$-action $\widetilde{\beta}$ on $C_\varphi(B)$ by setting, for all $g \in \Gamma$, $b \in B$, $t \in \mathbb{R}$,*

$$\widetilde{\beta}_g(b) = \beta_g(b) \quad \text{and} \quad \widetilde{\beta}_g(\lambda_t^\varphi) = [D(\varphi \circ \beta_{g^{-1}}), D\varphi]_t \lambda_t^\varphi.$$

*We have a canonical identification*

$$(B \rtimes_\beta \Gamma) \rtimes_{\sigma^\varphi} \mathbb{R} \simeq (B \rtimes_{\sigma^\varphi} \mathbb{R}) \rtimes_{\widetilde{\beta}} \Gamma,$$

*which is the identity on $B$, $L\Gamma$, and $L_\varphi\mathbb{R}$.*

*Proof.* This follows by direct computations using $\mathrm{Ad}(\Sigma)$, where $\Sigma$ is the flip map on $L^2(B) \otimes \ell^2(\Gamma) \otimes L^2(\mathbb{R})$ for the second and third components. ■

Recall that an inclusion of factors $P \subset N$ is called *irreducible* if $P' \cap N = \mathbb{C}$.

**Lemma 4.2.** *Let $p \in B$ be a projection, $B_0 \subset pBp$ an irreducible subfactor, $q, r \in B_0$ projections, and $\sigma\colon qB_0q \to rB_0r$ a $*$-homomorphism such that $\sigma(qB_0q)' \cap rBr = \mathbb{C}r$. Let $x \in rMq$ be any element with Fourier decomposition $x = \sum_{g\in\Gamma} x_g\lambda_g$. Assume that $xy = \sigma(y)x$ for all $y \in qB_0q$. Then*

- *$x_g\lambda_g y = \sigma(y)x_g\lambda_g$ and $x_g\beta_g(y) = \sigma(y)x_g$ for all $y \in qB_0q$ and $g \in \Gamma$;*
- *$x_g x_g^* \in \mathbb{C}r$ and $x_g^* x_g \in \mathbb{C}\beta_g(q)$;*
- *if $x^*x = q$, $xx^* = r$, and $(qB_0q)' \cap qMq = \mathbb{C}q$, there is a unique $g \in \Gamma$ such that $x = x_g\lambda_g$.*

*Proof.* For all $y \in qB_0q$, we have

$$\sum_{g\in\Gamma} x_g\lambda_g y = xy = \sigma(y)x = \sum_{g\in\Gamma} \sigma(y)x_g\lambda_g.$$

By comparing coefficients, one has $x_g\lambda_g y = \sigma(y)x_g\lambda_g$ and $x_g\beta_g(y) = \sigma(y)x_g$ for all $y \in qB_0q$ and $g \in \Gamma$. It follows that $x_g x_g^* = x_g\lambda_g(x_g\lambda_g)^* \in \sigma(qB_0q)' \cap rBr = \mathbb{C}r$, and $\beta_{g^{-1}}(x_g^* x_g) = (x_g\lambda_g)^* x_g\lambda_g \in (qB_0q)' \cap qBq = \mathbb{C}q$ for all $g \in \Gamma$. Assume further that $x^*x = q$, $xx^* = r$, and $(qB_0q)' \cap qMq = \mathbb{C}q$. Fix $g \in \Gamma$ such that $x_g \neq 0$. Then

$$x_g\lambda_g y = \sigma(y)x_g\lambda_g = \sigma(y)xx^*x_g\lambda_g = xyx^*x_g\lambda_g \quad \text{for all } y \in qB_0q,$$

hence $x^*x_g\lambda_g \in (qB_0q)' \cap qMq = \mathbb{C}q$. We conclude that $x = x_g\lambda_g$. ■

**Lemma 4.3.** *Let $\Lambda \curvearrowright^{\alpha} A$ be any outer action of a discrete group on a factor. Assume that $M = A \rtimes_\alpha \Lambda$ and $A \subset B$. Then there is a surjective homomorphism $\pi\colon \Lambda \to \Gamma$ such that*

- *for any $h \in \Lambda$ there is a unique $u_h \in \mathcal{U}(B)$ such that $\lambda_h^\Lambda = u_h\lambda_{\pi(h)}^\Gamma$;*
- *$B = A \rtimes_\alpha \ker(\pi)$.*

*In particular, $\alpha$ induces a cocycle action $\Lambda/\ker(\pi) \curvearrowright A \rtimes_\alpha \ker(\pi)$, and it is cocycle conjugate to $\beta$ via $A \rtimes_\alpha \ker(\pi) = B$ and $\pi\colon \Lambda/\ker(\pi) \simeq \Gamma$.*

*Proof.* Since $A' \cap M = \mathbb{C}$, by Lemma 4.2, any $\lambda_h^\Lambda$ for $h \in \Lambda$ can be uniquely written as $\lambda_h^\Lambda = u_h\lambda_g^\Gamma$ for some $g \in \Gamma$ and some $u_h \in \mathcal{U}(B)$. By the uniqueness, if we put $g = \pi(h)$, then $\pi\colon \Lambda \to \Gamma$ defines a homomorphism. Since $A$ and $\lambda_h^\Lambda$ $(h \in \Lambda)$ generate $M$, it follows that $B$ and $\pi(\Gamma)$ generate $M$ as well. This implies that $\pi(\Lambda) = \Gamma$ and $\pi$ is surjective.

Put $\Lambda_0 := \ker(\pi)$. By construction, $\lambda_h = u_h$ for all $h \in \Lambda_0$ and hence

$$B_0 := A \rtimes_\alpha \Lambda_0 \subset B.$$

We have to show the opposite inclusion. Let $E_B\colon M \to B$ and $E_{B_0}\colon M \to B_0$ be canonical conditional expectations. Observe that $E_{B_0} \circ E_B = E_{B_0}$. Fix any faithful normal state $\varphi$ on $B_0$ and extend it by $\varphi \circ E_{B_0}$. Then $E_B$ and $E_{B_0}$ extend to Jones projections $e_B$ and $e_{B_0}$ on $L^2(M, \varphi)$. Let $x = \sum_{h\in\Lambda} x_h\lambda_h^\Lambda \in A \rtimes_\alpha \Lambda$ be any element with its Fourier decomposition. Then

$$\begin{aligned} e_B\Lambda_\varphi(x) &= \sum_{h\in\Lambda} e_B\Lambda_\varphi(x_h\lambda_h^\Lambda) = \sum_{h\in\Lambda} e_B\Lambda_\varphi(x_h u_h\lambda_{\pi(h)}^\Gamma) = \sum_{h\in\Lambda_0} \Lambda_\varphi(x_h u_h) \\ &= \sum_{h\in\Lambda_0} \Lambda_\varphi(x_h\lambda_h^\Lambda). \end{aligned}$$

Since the last element is in $A \rtimes_\alpha \Lambda_0$, we see that $B \subset A \rtimes_\alpha \Lambda_0$.

Put $\widetilde{\Lambda} := \Lambda/\Lambda_0$ and $\widetilde{A} := A \rtimes_\alpha \Lambda_0$, and fix any section $s\colon \widetilde{\Lambda} \to \Lambda$ such that $s(\Lambda) = e$. For any $g, h \in \widetilde{\Lambda}$, we define $\lambda_g^{\widetilde{\Lambda}} := \lambda_{s(g)}^\Lambda$, $\widetilde{\alpha}_g := \mathrm{Ad}(\lambda_{s(g)}^\Lambda) \in \mathrm{Aut}(\widetilde{A})$, $\widetilde{u}_g := u_{s(g)}$, and $c(g,h) := \lambda_{s(g)s(h)s(gh)^{-1}}^\Lambda \in L\Lambda_0$. Then it is easy to check that $(\widetilde{\alpha}, c)$ defines a cocycle action of $\widetilde{\Lambda}$ on $\widetilde{A}$, and that $\widetilde{\alpha}_g = \mathrm{Ad}(\widetilde{u}_{s(g)}) \circ \beta_{\pi(g)}$ and $1 = \widetilde{u}_g^*\widetilde{\alpha}_g(\widetilde{u}_h^*)c(g,h)\widetilde{u}_{gh}$ for all $g, h \in \widetilde{\Lambda}$. Thus using $\widetilde{A} = B$ and $\pi\colon \widetilde{\Lambda} \simeq \Gamma$, we find that $(\widetilde{u}_g)_{g\in\widetilde{\Lambda}}$ gives a cocycle conjugacy between $\widetilde{\Lambda} \curvearrowright^{(\widetilde{\alpha},c)} \widetilde{A}$ and $\Gamma \curvearrowright^{\beta} B$. ■

### *Actions of groups in the class $\mathscr{C}$*

We continue to use the outer action $\Gamma \curvearrowright^{\beta} B$ on a $\sigma$-finite diffuse factor and $M = B \rtimes \Gamma$. Note that if $B$ is a $\mathrm{II}_1$ factor, then $\beta$ preserves the canonical trace, so $M$ is also a $\mathrm{II}_1$ factor. The next proposition is a generalization of [27, Lemma 8.4].

**Proposition 4.4.** *Let $p \in B$ be a projection and $A \subset pMp$ be a subfactor with expectation such that $A' \cap pMp = \mathbb{C}p$ and $s\mathcal{N}_{pMp}(A)'' = pMp$.*

(1) *If $A \preceq_M B$, then there exist $(e, f, \theta, v)$ witnessing $A \preceq_M B$ and a finite normal subgroup $K \leq \Gamma$ such that*

$$\theta(eAe)' \cap fBf = \mathbb{C}f, \qquad vv^* = e, \qquad v^*v \in \theta(eAe)' \cap f(B \rtimes K)f.$$

*Assume further that $\Gamma$ has no finite normal subgroups, and that either $B$ is of type $II_1$ or both $A$ and $B$ are properly infinite. Then we can choose $e = f = p$ and $v \in \mathcal{U}(pMp)$.*

(2) *Assume that $p = 1$ and $A$ has a decomposition $M = A \rtimes \Lambda$ for some outer action of a discrete group $\Lambda$ on $A$. Assume that $\Gamma$ and $\Lambda$ are ICC. If $A \preceq_M B$ and $B \preceq_M A$, then $A$ and $B$ are unitarily conjugate in $M$.*

*Proof.* (1) Since $B$ is a factor, using [15, Remark 4.5] we may assume that $A \preceq_M pBp$. We first show, using the assumption $A' \cap pMp = \mathbb{C}p$, that there is $(e, f, \theta, v)$ which witnesses $A \preceq_M pBp$ such that $\theta(eAe) \subset fBf$ is irreducible.

To see this, we fix any $(e, f, \theta, v)$ which witnesses $A \preceq_M pBp$ and we will modify it. Since $vv^* \in (eAe)' \cap eMe = \mathbb{C}e$, one has $vv^* = e$ and moreover $v^*v$ is a minimal projection in $\theta(eAe)' \cap fMf$. Indeed, for any projection $r \leq v^*v$ in $\theta(eAe)' \cap fMf$, $vrv^* \in (eAe)' \cap eMe = \mathbb{C}e$ is again $e$, hence $r = vv^*$. We may assume that the support projection of $E_B(v^*v)$, which is contained in $\theta(eAe)' \cap fBf$, coincides with $f$. Let $z$ be the central support projection of $v^*v$ in $\theta(eAe)' \cap fMf$. Then since $v^*v$ is minimal, $(\theta(eAe)' \cap fMf)z$ is a type I factor. Since $\theta(eAe) \subset fBf$ is with expectation, so is the inclusion $\theta(eAe)' \cap fBf \subset \theta(eAe)' \cap fMf$. In particular, $(\theta(eAe)' \cap fBf)z$ is an atomic von Neumann algebra. Since $z$ commutes with $\theta(eAe)' \cap fBf$, there is a unique projection $w \in \mathcal{Z}(\theta(eAe)' \cap fBf)$ such that $(\theta(eAe)' \cap fBf)w \ni aw \mapsto az \in (\theta(eAe)' \cap fBf)z$ is isomorphic. Thus there is a minimal projection $q$ in $\theta(eAe)' \cap fBf$. Since $q \leq f$, $q$ is smaller than the support of $E_B(v^*v)$, hence $vq \neq 0$. Now $(e, q, \theta(\cdot)q, vq)$ witnesses $A \preceq_M pBp$ (up to the polar decomposition of $vq$) and has the property that $\theta(eAe)q \subset qBq$ is an irreducible inclusion.

Thus we can start the proof by assuming $\theta(eAe)' \cap fBf = \mathbb{C}f$. Put $B_0 := \theta(eAe) \subset fBf$ and note that $B_0' \cap fBf = \mathbb{C}f$. Consider the Fourier decomposition $z := v^*v = \sum_{g \in \Gamma} x_g \lambda_g \in B \rtimes \Gamma$. Since $z \in B_0' \cap fMf$, by Lemma 4.2 (for the case $\sigma = \mathrm{id}$) we have $x_g\lambda_g \in B_0' \cap fMf$, $x_g x_g^* = \mathbb{C}f$, and $x_g^* x_g \in \mathbb{C}\beta_g(f)$. Define a subgroup $K \leq \Gamma$ and a subset $\Gamma_0 \subset \Gamma$ by

$$K := \{g \in \Gamma \mid \mathrm{Ad}(w_g) \circ \beta_g|_{B_0} = \mathrm{id}_{B_0} \text{ for some } w_g \in B \\ \text{with } w_g w_g^* = f,\ w_g^* w_g = \beta_g(f)\},$$

$$\Gamma_0 := \{g \in \Gamma \mid \mathrm{Ad}(w_g) \circ \beta_g(r_g B_0 r_g) = q_g B_0 q_g \text{ for some } w_g \in B,\ q_g, r_g \in B_0 \\ \text{with } w_g w_g^* = q_g,\ w_g^* w_g = \beta_g(r_g)\}.$$

By definition, $z$ is in $B \rtimes K$. We will prove that $|K| < \infty$, $\Gamma_0$ is a group, $K$ is normal in $\Gamma_0$, and $\Gamma_0 = \Gamma$. This will finish the proof of the first half of item (1).

We claim that $K$ is a finite group. Fix $(w_g)_{g \in K}$ which appeared in the definition of $K$ such that $w_e = 1$. For all $g, h \in K$, define

$$\beta_g^w := \mathrm{Ad}(w_g) \circ \beta_g \quad \text{and} \quad \mu_{g,h} := w_g \beta_g(w_h) w_{gh}^* \in \mathcal{U}(fBf)$$

and observe that $(\beta^w, \mu)$ gives a cocycle action of $K$ on $fBf$, so that $f(B \rtimes_\beta K)f = fBf \rtimes_{(\beta^w,\mu)} K$. The condition $\beta^w|_{B_0} = \mathrm{id}_{B_0}$ implies that $\mu_{g,h} \in \mathbb{C}f$ for all $g, h \in K$, hence we can regard $\mu$ as a scalar 2-cocycle. In particular $fBf \rtimes_{(\beta^w,\mu)} K$ contains a finite von Neumann algebra $(\mathbb{C}f) \rtimes_{(\beta^w,\mu)} K$. Since $B_0' \cap fBf = \mathbb{C}f$ and $\beta^w|_{B_0} = \mathrm{id}_{B_0}$, using Fourier decompositions it is easy to see that

$$B_0' \cap [fBf \rtimes_{(\beta^w,\mu)} K] = (\mathbb{C}f) \rtimes_{(\beta^w,\mu)} K.$$

The left hand side contains the minimal projection $z$, and hence so does the right hand side. This implies that $K$ is a finite group. (Indeed, if it is infinite, one has a sequence of unitaries which converges weakly to 0, but this is impossible in a finite von Neumann algebra with a minimal projection.)

We next claim that $\Gamma_0$ is a group and $K$ is normal in $\Gamma_0$. For this, take $g \in \Gamma_0$ and pick any $(w_g, q_g, r_g)$ as in the definition of $\Gamma_0$. Observe that if we replace $q_g$ by a projection $q_g^0 \in B_0$ which satisfies $q_g^0 \preceq q_g$ in $B_0$, then $q_g^0$ satisfies the same condition as $q_g$ (with some appropriate $w_g, r_g$). The same holds for $r_g$. Take another $h \in \Gamma_0$ and $(w_h, q_h, r_h)$. Then since $B_0$ is a factor, up to replacing $r_g$ or $q_h$ with a smaller and equivalent projection in $B_0$, we may assume $r_g = q_h$. Then it is easy to see $gh \in \Gamma_0$. We also have $g^{-1} \in \Gamma_0$, because $(w_{g^{-1}}, q_{g^{-1}}, r_{g^{-1}}) := (\beta_g^{-1}(w_g^*), r_g, q_g)$ works. Using this family for $g^{-1}$, for $h := gkg^{-1}$ for any fixed $k \in K$, the family $(w_h, q_h, r_h)$ can be taken so that $q_h = r_h$ and $\mathrm{Ad}(w_h) \circ \beta_h = \mathrm{id}$ on $q_r B_0 q_h$. Since $fB_0 f$ is a diffuse factor, we can apply the usual patching method and obtain $(w_h, q_h, r_h)$ such that $q_h = r_h = f$ and $\mathrm{Ad}(w_h) \circ \beta_h = \mathrm{id}$ on $B_0$. This means $h \in K$, hence $K$ is normal in $\Gamma_0$.

We show $\Gamma = \Gamma_0$. Observe that $eAe$ is a diffuse factor and $s\mathcal{N}_{e(B \rtimes \Gamma)e}(eAe)'' = e(B \rtimes \Gamma)e$. Since $\mathrm{Ad}(v^*)$ is an isomorphism between $eAe \subset e(B \rtimes \Gamma)e$ and $B_0 z \subset z(B \rtimes \Gamma)z$, it follows that $s\mathcal{N}_{z(B \rtimes \Gamma)z}(B_0 z)'' = z(B \rtimes \Gamma)z$. Fix any partial isometry $u \in s\mathcal{N}_{z(B \rtimes \Gamma)z}(B_0 z)$ with $u^* u = qz$, $uu^* = rz$ for $q, r \in B_0$, and consider the Fourier decomposition $u = \sum_{g \in \Gamma} x_g \lambda_g \in B \rtimes \Gamma$. Since $\mathrm{Ad}(u)$ is an isomorphism from $qB_0 qz$ to $rB_0 rz$, using $B_0 z \simeq B_0$ we can define an isomorphism $\alpha^u \colon qB_0 q \to rB_0 r$ by $\alpha^u(y)z = uyu^*$ for all $y \in qB_0 q$. By Lemma 4.2, for all $y \in qB_0 q$ and $g \in \Gamma$,

$$x_g \lambda_g y = \alpha^u(y) x_g \lambda_g, \quad x_g x_g^* \in \mathbb{C}r, \quad \text{and} \quad x_g^* x_g \in \mathbb{C}\beta_g(q).$$

So each $x_g \in rB\beta_g(q)$ is a scalar multiple of a partial isometry. We can write $x_g = a_g \omega_g$ for some $a_g \in \mathbb{C}$, where $\omega_g$ is a partial isometry. Observe that if $x_g \neq 0$, then $\mathrm{Ad}(\omega_g \lambda_g)(y) = \alpha^u(y) r \in rB_0 r$ for all $y \in qB_0 q$, so $g$ is contained in $\Gamma_0$. It follows that

$u \in z(B \rtimes \Gamma_0)z$. Now take any $x \in s\mathcal{N}_{z(B\rtimes\Gamma)z}(B_0z)$ and consider its polar decomposition $x = v|x|$. Then since $|x| \in B_0z$ and since $v$ is a partial isometry in $s\mathcal{N}_{z(B\rtimes\Gamma)z}(B_0z)$, we find that $x \in z(B \rtimes \Gamma_0)z$. Since $s\mathcal{N}_{z(B\rtimes\Gamma)z}(B_0z)'' = z(B \rtimes \Gamma)z$, we conclude that $z(B \rtimes \Gamma)z = z(B \rtimes \Gamma_0)z$. Since $z \in B \rtimes \Gamma_0$ and $B \rtimes \Gamma_0$ is a diffuse factor, we indeed have $B \rtimes \Gamma = B \rtimes \Gamma_0$. This means $\Gamma = \Gamma_0$.

We next assume that $\Gamma$ has no finite normal subgroups. Then $K$ must be trivial, so $v^*v \in B$ and we may assume $f = v^*v$. There is a partial isometry $v \in pMp$ such that $vv^* = e \in A$, $v^*v = f \in pBp$, and $v^*Av \subset fBf$. If $B$ is of type $\mathrm{II}_1$ (so that $M, A$ are $\mathrm{II}_1$ factors) or if both $A$ and $B$ are properly infinite, then (up to replacing $e, f$ by smaller ones if necessary) we can apply the patching method and obtain $e = f = p$ and $v \in \mathcal{U}(pMp)$. This is the conclusion.

(2) Observe that $B$ is a $\mathrm{II}_1$ factor (hence so is $M$) if and only if $A$ is. Hence using item (1) of this proposition, we can find $v, w \in \mathcal{U}(M)$ with $vAv^* \subset B$ and $wBw^* \subset A$. Put $u := vw$ and observe that $uBu^* \subset B$ and $(uBu^*)' \cap B \subset (uBu^*)' \cap M = u(B' \cap M)u^* = \mathbb{C}$. By Lemma 4.2, we can write $u = x_g\lambda_g$ for some $g \in \Gamma$ and $x_g \in \mathcal{U}(B)$. In particular we have $B = uBu^* = vwBw^*v^* \subset vAv^* \subset B$. We conclude that $vAv^* = B$. ■

The next lemma explains how we use the properties of the class $\mathscr{C}$ for actions on type III factors. This uses our Theorem A.

**Lemma 4.5.** *Let $p \in M$ be a projection, and $A \subset pMp$ be a subfactor with expectation $E_A$. Assume that $\Gamma$ is in the class $\mathscr{C}$, $A' \cap pMp = \mathbb{C}$, $A$ is amenable, and $\mathcal{N}_{pMp}(A)''$ has finite index in $pMp$. Then $A \preceq_M B$.*

*Proof.* Put $P := \mathcal{N}_{pMp}(A)''$ and let $N$ be the hyperfinite type $\mathrm{III}_1$ factor and $\omega$ a faithful normal state such that $N'_\omega \cap N = \mathbb{C}$. Let $E_A, E_P$ be any faithful normal conditional expectations for $A, P$ respectively. Observe that the condition $A' \cap pMp \subset A$ implies that normal expectations onto $A$ and $P$ are unique, hence $E_A \circ E_P = E_A$. Fix any faithful states $\psi, \varphi \in M_*^+$ which are preserved by $E_A, E_B$ respectively. Then, by the uniqueness of $E_A$ and by Theorem A, $A \preceq_M B$ is equivalent to

$$\Pi_{\varphi\otimes\omega,\psi\otimes\omega}(C_{\psi\otimes\omega}(A \mathbin{\overline{\otimes}} N)) \preceq_{C_{\varphi\otimes\omega}(M\mathbin{\overline{\otimes}}N)} C_{\varphi\otimes\omega}(B \mathbin{\overline{\otimes}} N).$$

There is a canonical inclusion $C_{\psi\otimes\omega}(A \mathbin{\overline{\otimes}} N) \subset C_{\psi\otimes\omega}(P \mathbin{\overline{\otimes}} N)$, which is regular by [3, Lemma 4.1]. For notational simplicity, we omit $\Pi_{\varphi\otimes\omega,\psi\otimes\omega}$ and write $\mathcal{M} := C_{\varphi\otimes\omega}(M \mathbin{\overline{\otimes}} N)$, $\mathcal{B} := C_{\varphi\otimes\omega}(B \mathbin{\overline{\otimes}} N)$, $\mathcal{A} := C_{\psi\otimes\omega}(A \mathbin{\overline{\otimes}} N)$, and $\mathcal{P} := C_{\psi\otimes\omega}(P \mathbin{\overline{\otimes}} N)$. Observe that $\mathcal{A}$ is amenable and $\mathcal{P} \subset \mathcal{M}$ has finite index.

By Lemma 4.1, there is an identification $\mathcal{M} = \mathcal{B} \rtimes_{\widetilde{\beta}} \Gamma$. Let $r \in L_{\varphi\otimes\omega}\mathbb{R}$ be any projection such that $\mathrm{Tr}_{\varphi\otimes\omega}(r) < \infty$. Then since $\mathcal{B}$ is a type $\mathrm{II}_\infty$ factor and since $\widetilde{\beta}$ preserves the canonical trace on $\mathcal{B}$, $r\mathcal{M}r$ is realized as a cocycle crossed product $r\mathcal{B}r \rtimes_{(\widetilde{\beta}^r,u)} \Gamma$ for some 2-cocycle $u\colon \Gamma \times \Gamma \to r\mathcal{B}r$ (because $r \sim \widetilde{\beta}_g(r)$ for all $g \in \Gamma$, see Section 2). Since $\mathcal{M}$ is a $\mathrm{II}_\infty$ factor, and $p$ is infinite while $r$ is finite, there is $v \in \mathcal{M}$ such that $vv^* = r$ and $p_0 := v^*v \in p\mathcal{A}p$. Put $\mathcal{A}^v := v\mathcal{A}v^*$. Observe that $\mathcal{A}^v$ is amenable and $(\mathcal{A}^v)' \cap r\mathcal{M}r = \mathbb{C}r$ (use Lemma 2.3). Since $\mathcal{A}$ is a $\mathrm{II}_\infty$ factor, we have $p_0 \mathcal{N}_{p\mathcal{M}p}(\mathcal{A})'' p_0 = \mathcal{N}_{p_0\mathcal{M}p_0}(p_0\mathcal{A}p_0)''$. In particular $\mathcal{N}_{r\mathcal{M}r}(\mathcal{A}^v)''$ in $r\mathcal{M}r$ has finite index. Hence by the definition of the class $\mathcal{C}$, we have $\mathcal{A}^v \preceq_{r\mathcal{M}r} r\mathcal{B}r$. This implies $\mathcal{A} \preceq_{\mathcal{M}} \mathcal{B}$ and hence $A \preceq_M B$ as we explained. ■

*Proof of Theorem* D. By Lemma 4.5, we have $A \preceq_M B$. Note that $A$ is a type $\mathrm{II}_1$ factor if and only if $B$ is. Hence we can apply Proposition 4.4 and find a unitary $u \in \mathcal{U}(M)$ such that $uAu^* \subset B$. Thus we may assume that $A \subset B$. We then apply Lemma 4.3 to get the conclusion. Note that $\ker(\pi)$ is amenable since $A \rtimes \ker(\pi)$ is amenable and $A$ is a factor. ■

## 5. Rigidity of Bernoulli shift actions

In this section, we will study Bernoulli shift actions with type III base algebras. In particular we prove Theorem C and Proposition F.

### *Popa's criterion for cocycle superrigidity*

The next proposition is a variant of Popa's theorem which was used to prove cocycle superrigidity [36, 38, 39]. See also [52, Theorem 7.1].

**Proposition 5.1.** *Let $G$ be a locally compact second countable group, $G_1 \leq G$ a closed normal subgroup, and $(P, \varphi)$ a von Neumann algebra with a faithful normal state. Let $G \curvearrowright^{\beta} (P, \varphi)$ be a state preserving continuous action. Let $\omega\colon G \to \mathcal{U}(P)$ be a $\sigma$-strongly continuous map such that $\alpha_g := \mathrm{Ad}(\omega_g) \circ \beta_g$ and $v(g,h) := \omega_g \beta_g(\omega_h)\omega_{gh}^*$ for $g, h \in G$ define a cocycle action of $G$. Assume that*

- *$v(g,h) = 1 = v(h,g)$ for all $g \in G_1$ and $h \in G$ (hence $\alpha|_{G_1}$ is a genuine action);*
- *there is a faithful state $\psi \in P_*$ which is preserved by $\alpha|_{G_1}$;*
- *$(\mathbb{C}p, \alpha|_{G_1}) \preceq_P^{\mathrm{uni}} (\mathbb{C}1_P, \beta|_{G_1})$ for all projections $p \in P^{\alpha}$;*
- *$\beta|_{G_1}$ is weakly mixing.*

*Then there exist a separable Hilbert space $H$, a projection $f \in \mathbb{B}(H)$, a $\sigma$-strongly continuous map $u\colon G \to \mathcal{U}(f\mathbb{B}(H)f)$, and a partial isometry $w \in P \mathbin{\overline{\otimes}} \mathbb{B}(H)$ such that*

$$w^*w = f, \quad ww^* = 1 \otimes e_{1,1}, \quad \text{and} \quad wu_g = (w_g \otimes 1_H)(\beta_g \otimes \mathrm{id}_H)(w) \quad \text{for all } g \in G,$$

*where $e_{1,1}$ is a minimal projection in $\mathbb{B}(H)$. In particular, $(\mathrm{Ad}(u_g))_{g\in G}$ and $(u_g u_h u_{gh}^*)_{g,h\in G}$ define a cocycle action on $f\mathbb{B}(H)f$, and $\alpha$ is conjugate to the cocycle action $(\beta_g \otimes \mathrm{Ad}(u_g))_{g\in G}$ by $w$:*

$$\alpha_g(wxw^*) = \beta_g^{\omega}(wxw^*) = w(\beta_g \otimes \mathrm{Ad}(u_g))(x)w^* \quad \text{for all } x \in P \mathbin{\overline{\otimes}} f\mathbb{B}(H)f.$$

*Proof.* Since most of the arguments are straightforward adaptations of [52, proof of Theorem 7.1], we give only a sketch of the proof. Take $(H, f, \pi, w)$ and $(u_g)_{g\in G_1}$ which witness $(\mathbb{C}p, \alpha|_{G_1}) \preceq_P (\mathbb{C}1_P, \beta|_{G_1})$ (and $H$ can be finite-dimensional). Observe that

$w^*w \in (P \mathbin{\overline{\otimes}} \mathbb{B}(H))^{\beta\otimes \mathrm{Ad}(u)|_{G_1}} = \mathbb{C}1_P \mathbin{\overline{\otimes}} \mathbb{B}(H)$ (because $\beta|_{G_1}$ is weakly mixing), hence up to replacing $f$ by $w^*w$, we may assume that $w^*w = f$.

Thus the condition $(\mathbb{C}p, \alpha|_{G_1}) \preceq_P (\mathbb{C}1_P, \beta|_{G_1})$ means that there exist a projection $f \in \mathbb{M}_n$, a continuous homomorphism $u\colon G_1 \to \mathcal{U}(f\mathbb{M}_n f)$, and a partial isometry $w \in (p \otimes e_{1,1})(P \otimes \mathbb{M}_n)f$ such that $wu_g = (\omega_g \otimes 1_n)(\beta_g \otimes \mathrm{id}_n)(w)$ for all $g \in G_1$.

**Claim.** *There exist a separable Hilbert space $H$, a projection $f \in \mathbb{B}(H)$, a partial isometry $w \in P \mathbin{\overline{\otimes}} \mathbb{B}(H)$, and a continuous homomorphism $u\colon G_1 \to \mathcal{U}(f\mathbb{B}(H)f)$ such that*

- $wu_g = (\omega_g \otimes 1_H)(\beta_g \otimes \mathrm{id}_H)(w)$ *for all* $g \in G_1$*;*
- $w^*w = f$ *and* $ww^* \in pP^\alpha p \mathbin{\overline{\otimes}} \mathbb{C}e_{1,1}$*, where* $e_{1,1}$ *is a fixed minimal projection;*
- *there exist finite rank projections* $(P_k)_{k\in\mathbb{N}}$ *in* $\mathbb{B}(H)$ *such that* $P_k \to 1_H$ *as* $k \to \infty$ *and each* $P_k$ *commutes with* $u_g$ *for all* $g \in G_1$.

*Proof.* Let $\mathcal{E}$ denote the set of all non-zero projections $e \in P$ $(= P \otimes \mathbb{C}e_{1,1})$ such that there exists $(n, f, w, u)$ which witnesses $(\mathbb{C}p, \alpha|_{G_1}) \preceq_P (\mathbb{C}1_P, \beta|_{G_1})$ with $e = ww^*$. Then it is straightforward to check that $\mathcal{E}$ is closed under the following operations: $\beta_h(e) \in \mathcal{E}$ for all $h \in G$ and all $e \in \mathcal{E}$; $e \vee f \in \mathcal{E}$ for all $e, f \in \mathcal{E}$; and $e_0 \in \mathcal{E}$ for all projections $e_0 \in eP^{\alpha|_{G_1}}e$ and $e \in \mathcal{E}$.

Fix any countable dense subset $X \subset G$. Observe that $\sup_{h\in X} \beta_h(e) \in pP^\alpha p$ is realized as a (countably) infinite direct sum of projections in $\mathcal{E}$, that is, there is a family $(n_i, f_i, w_i, u^i)_{i\in I}$ such that $\sum_{i\in I} w_i w_i^* = \sup_{h\in X} \beta_h(e)$, where $I$ is a countable set. By defining $H := \bigoplus_{i\in I} \mathbb{C}^{n_i}$, $f := \bigoplus_{i\in I} f_i$, $w = [w_i]_{i\in I} \in (p \otimes e_{1,1})(B \mathbin{\overline{\otimes}} \mathbb{B}(H))f$, and $u := \bigoplus_{i\in I} u^i$, we get the conclusion. ■

Now we define $\mathcal{F}$ as the set of all non-zero projections $e \in P^\alpha$ $(= P^\alpha \otimes \mathbb{C}e_{1,1})$ such that there exists $(H, f, w, u)$ which witnesses the conclusion of the claim above with $e = ww^*$. Now using the assumption $(\mathbb{C}p, \alpha|_{G_1}) \preceq_P (\mathbb{C}1_P, \beta|_{G_1})$ for all $p \in P^\alpha$ and applying a maximality argument, there is a family $(H_i, f_i, w_i, u^i)_{i\in I}$ such that $\sum_{i\in I} w_i w_i^* = 1_P$ $(= 1_P \otimes e_{1,1})$, where $I$ is a countable set. Define $(H, f, w, u)$ as a direct sum of all $(H_i, f_i, w_i, u^i)_{i\in I}$ (with $w = [w_i]_{i\in I} \in (1 \otimes e_{1,1})(B \mathbin{\overline{\otimes}} \mathbb{B}(H))$); then it satisfies all the conditions in the claim above with $ww^* = 1 \otimes e_{1,1}$. Hence $(H, f, w, u)$ satisfies the conclusion of this theorem but only for $G_1$.

We have to extend the conditions on $G_1$ to those on $G$, using the weak mixing of $\beta|_{G_1}$. Put $\omega_g^H := \omega_g \otimes 1_H$, $\beta_g^H := \beta_g \otimes \mathrm{id}_H$, $\alpha_g^H := \alpha_g \otimes \mathrm{id}_H$, and $v^H(g,h) := v(g,h) \otimes 1_H$ for all $g, h \in G$. Extend the map $u$ to one on $G$ by

$$u_g := w^* \omega_g^H \beta_g^H(w) \quad \text{for all } g \in G.$$

It is easy to compute that for any $g, h \in G$,

$$u_g u_g^* = f = u_g^* u_g \quad \text{and} \quad u_g \beta_g^H(u_h) = w^* v^H(g,h) w u_{gh}.$$

In particular, $u\colon G \to \mathcal{U}(P \mathbin{\overline{\otimes}} f\mathbb{B}(H)f)$ is a cocycle for $\beta^H$ with a 2-cocycle $w^* v^H(\cdot,\cdot)w$. To finish the proof, we have only to show that $u$ is a map into $f\mathbb{B}(H)f$, so that $\beta_g^H(u_h) = u_h$ and $u_g u_h u_{gh}^* = w^* v^H(g,h) w \in f\mathbb{B}(H)f$ for all $g, h \in G$.

Fix $g \in G$ and $k \in \mathbb{N}$. Put $H_k := P_k H$ and $u_h^k := P_k u_h P_k$ for all $h \in G$, where $(P_n)_{n \in \mathbb{N}}$ is a family of finite rank projections as in the claim (and we regard $P_k = 1_P \otimes P_k$). Then since $P_k$ commutes with $u_h$ for all $h \in G_1$, putting $\beta_h^u := \mathrm{Ad}(u_h) \circ \beta_h$ we have

$$\beta_h^u(u_g^k) = P_k \beta_h^u(u_g) P_k = u_g^k u_{g^{-1}hg}^k (u_h^k)^* \in u_g^k \mathbb{B}(H_k) \quad \text{for all } h \in G_1.$$

Observe that $\beta_h^u$ is of the form $\beta_h \otimes \mathrm{Ad}(u_h)$ for all $h \in G_1$. Then combining the weak mixing of $\beta|_{G_1}$ with $(\beta_h \otimes \mathrm{Ad}(u_h^k))(u_g^k) \in u_g^k \mathbb{B}(H_k)$ for all $h \in G_1$, we find that $u_g^k \in \mathbb{B}(H_k)$. Since $k$ is arbitrary, we conclude that $u_g \in \mathbb{B}(H)$ as required. ∎

### *Rigidity of Bernoulli shifts for cocycle actions*

Let $\Gamma$ be a countable discrete group, $B_0$ an amenable von Neumann algebra with separable predual, $\varphi_0$ a faithful normal state on $B_0$, and $\Gamma \curvearrowright^{\beta} \bigotimes_\Gamma (B_0, \varphi_0) =: (B, \varphi)$ the Bernoulli shift action. Put $M := B \rtimes_\beta \Gamma$. Here we recall the following fact.

**Theorem 5.2.** *Let $p \in M$ be a projection and $A \subset pMp$ a von Neumann subalgebra with expectation $E_A$. Fix a faithful $\psi \in M_*$ which is preserved by $E_A$, and set $P := A' \cap pM_\psi p$. If $C_\psi(A) \not\preceq_{C_\varphi(M)} C_\varphi(L\Gamma)$, then $P$ has an amenable direct summand.*

*Proof.* This can be proved by applying arguments in [9, Theorem 4.1], which is based on the arguments in [37, 38, 41] (together with the deformation given in [23]). Actually one has to modify the spectral gap argument [41] as follows. Put $\widetilde{B} := \bigotimes_\Gamma (B_0 * L\mathbb{Z}, \varphi_0 * \tau_{L\mathbb{Z}})$ and extend $\varphi$ and $\beta$ on $\widetilde{B}$, so that there are canonical inclusions $M \subset \widetilde{B} \rtimes_\beta \Gamma =: \widetilde{M}$ and $C_\varphi(M) \subset C_\varphi(\widetilde{M})$. Then we can prove the following weak containment:

$${}_M L^2(C_\varphi(\widetilde{M})) \ominus L^2(C_\varphi(M))_{C_\varphi(M)} \prec {}_M L^2(C_\varphi(M)) \otimes L^2(C_\varphi(M))_{C_\varphi(M)}$$

(e.g. see [32, proof of Theorem 5.2]). Then using the spectral gap argument given in [32, Lemma 4.1], we can follow [9, proof of Theorem 4.1]. ∎

*Proof of Theorem C.* Put $M := B \rtimes_\beta \Gamma$ and regard $M = A \rtimes_\alpha \Lambda$ via the given isomorphism. We have $A \preceq_M B$ by Lemma 4.5, hence by Proposition 4.4, there is $u \in \mathcal{U}(M)$ such that $uAu^* \subset B$. Then up to replacing the initial isomorphism by the one with $\mathrm{Ad}(u)$, we may assume $A \subset B$. Then by Lemma 4.3, there is a surjective homomorphism $\pi \colon \Lambda \to \Gamma$ such that $A \rtimes_\alpha \Lambda_0 = B$, where $\Lambda_0 := \ker(\pi)$, and for any $h \in \Lambda$, there is a unique $u_h \in \mathcal{U}(B)$ such that $\lambda_h^\Lambda = u_h \lambda_{\pi(h)}^\Gamma$. Put $\widetilde{A} := A \rtimes_\alpha \Lambda_0$ and $\widetilde{\Lambda} := \Lambda / \Lambda_0$. Using a fixed section $s \colon \widetilde{\Lambda} \to \Lambda$ such that $s(\Lambda_0)$ is the unit, we will use the following notation: for all $g, h \in \widetilde{\Lambda}$, $\tilde{\alpha}_g := \mathrm{Ad}(\lambda_{s(g)}^\Lambda) \in \mathrm{Aut}(\widetilde{A})$, $c(g,h) := \lambda_{s(g)s(h)s(gh)^{-1}}^\Lambda$, $\lambda_g^{\widetilde{\Lambda}} := \lambda_{s(g)}^\Lambda$, and $u_g := u_{s(g)}$. We have a cocycle action $\widetilde{\Lambda} \curvearrowright^{(\tilde{\alpha}, c)} \widetilde{A}$ with the relations

$$\lambda_h^{\widetilde{\Lambda}} = u_g \lambda_{\pi(h)}^\Gamma, \quad \mathrm{Ad}(u_g) \circ \beta_{\pi(g)} = \tilde{\alpha}_g, \quad c(g,h) = \tilde{u}_g \beta_g(\tilde{u}_h) \tilde{u}_{gh}^* \quad \text{for all } g, h \in \widetilde{\Lambda}.$$

For simplicity we identify $C_\psi(M) = C_\varphi(M)$. Then by Lemma 4.1, there is an inclusion

$$L_\psi\mathbb{R} \subset C_\psi(\widetilde{A} \rtimes_{\widetilde{\alpha}} \widetilde{\Lambda}) = C_\varphi(M) = C_\varphi(B) \rtimes_\beta \Gamma.$$

Observe that, since $\widetilde{\alpha}$ is $\psi$-preserving, $(L_\psi\mathbb{R})' \cap C_\varphi(M)$ contains a copy of $L\widetilde{\Lambda}$ with expectation, hence $(L_\psi\mathbb{R})' \cap C_\varphi(M)$ has no amenable direct summand (because $L\widetilde{\Lambda}$ has no such summand).

**Claim.** *We have* $(\mathbb{C}p, \sigma^\psi) \preceq_B (\mathbb{C}1_B, \sigma^\varphi)$ *for all projections* $p \in B_\psi^{\widetilde{\alpha}}$.

*Proof of Claim.* Fix any projection $p \in B_\psi^{\widetilde{\alpha}}$. Since $L\widetilde{\Lambda}p$ has no amenable summand, by applying Theorem 5.2 to $L_\psi\mathbb{R}p$ we find that $L_\psi\mathbb{R}p \preceq_{C_\varphi(M)} C_\varphi(L\Gamma)$. By Theorem 3.2, to prove this claim, we have only to show that $L_\psi\mathbb{R}p \preceq_{C_\varphi(B)} L_\varphi\mathbb{R}$.

Suppose for contradiction that $L_\psi\mathbb{R}p \not\preceq_{C_\varphi(B)} L_\varphi\mathbb{R}$. Take a net $(u_i)_i$ in $\mathcal{U}(L_\psi\mathbb{R})$ such that

$$E_{L_\varphi\mathbb{R}}(b^* u_i pa) \to 0 \quad \text{for all } a, b \in C_\varphi(B).$$

Observe that for all $h \in \widetilde{\Lambda}$ and $u_i \in L_\psi\mathbb{R}$, since $u_i$ commutes with $\lambda_h^{\widetilde{\Lambda}}$,

$$\lambda_{\pi(h)}^\Gamma u_i p (\lambda_{\pi(h)}^\Gamma)^* = u_h^* \lambda_h^{\widetilde{\Lambda}} u_i p (\lambda_h^{\widetilde{\Lambda}})^* u_h = u_h^* u_i p u_h.$$

It follows that for all $a, b \in C_\varphi(B)$ and $g, h \in \widetilde{\Lambda}$,

$$\begin{aligned} E_{C_\varphi(L\Gamma)}(b\lambda_{\pi(h)}^\Gamma u_i pa\lambda_{\pi(g)}^\Gamma) &= E_{C_\varphi(L\Gamma)}(b[\lambda_{\pi(h)}^\Gamma u_i p(\lambda_{\pi(h)}^\Gamma)^*]\beta_{\pi(h)}(a)\lambda_{\pi(hg)}^\Gamma) \\ &= E_{C_\varphi(L\Gamma)}(b[u_h^* u_i p u_h]\beta_{\pi(h)}(a)\lambda_{\pi(hg)}^\Gamma) \\ &= E_{L_\varphi\mathbb{R}}(bu_h^* u_i p u_h \beta_{\pi(h)}(a))\lambda_{\pi(hg)}^\Gamma \to 0. \end{aligned}$$

By [15, Theorem 4.3(5)], we get $L_\psi\mathbb{R}p \not\preceq_{C_\varphi(M)} C_\varphi(L\Gamma)$, a contradiction. ■

Define $G := \Gamma \times \mathbb{R}$. Since $\beta$ and $\sigma^\varphi$ commute, we can define a continuous action $G \curvearrowright^{\beta^\varphi} (B, \varphi)$ by

$$\beta_{(g,t)}^\varphi := \beta_g \circ \sigma_t^\varphi = \sigma_t^\varphi \circ \beta_g \quad \text{for all } (g,t) \in G.$$

The condition $B_\varphi = \mathbb{C}$ then means that $\beta^\varphi|_\mathbb{R}$ is weakly mixing. In the same way, we can define a continuous cocycle action $\widetilde{\Lambda} \times \mathbb{R} \curvearrowright^{\widetilde{\alpha}^\psi} (\widetilde{A}, \psi)$ with the 2-cocycle $c^\psi((g,t),(h,s)) := c(g,h)$ for all $(g,t),(h,s) \in \widetilde{\Lambda} \times \mathbb{R}$.

**Claim.** *Identify* $\widetilde{\Lambda} = \Gamma$ *and* $\widetilde{A} = B$. *Define a* $\sigma$*-strongly continuous map* $\omega\colon G \to \mathcal{U}(B)$ *by*

$$\omega_{(g,t)} := [D\psi, D\varphi]_t \sigma_t^\varphi(u_g) = \sigma_t^\psi(u_g)[D\psi, D\varphi]_t, \qquad g \in \Gamma,\ t \in \mathbb{R}.$$

*Then* $\omega$ *gives a cocycle conjugacy between* $\beta^\varphi$ *and* $\widetilde{\alpha}^\psi$*: for all* $(g,t),(h,s) \in G$,

$$\operatorname{Ad}(\omega_{(g,t)}) \circ \beta_{(g,t)}^\varphi = \widetilde{\alpha}_{(g,t)}^\psi \quad \textit{and} \quad \omega_{(g,t)}\beta_{(g,t)}^\varphi(\omega_{(h,s)}) = c^\psi((g,t),(h,s))\omega_{(gh,t+s)}.$$

*Proof of Claim.* Observe that for any $(g,t)\in G$, since $\lambda_t^\varphi$ and $\lambda_g^\beta$ commute in $C_\varphi(M)$,

$$\begin{aligned}\lambda_g^\beta\lambda_t^\varphi &= u_g^*\lambda_g^{\tilde{\alpha}}[D\varphi,D\psi]_t\lambda_t^\psi = u_g^*\tilde{\alpha}_g([D\varphi,D\psi]_t)\lambda_g^{\tilde{\alpha}}\lambda_t^\psi\\ &= \lambda_t^\varphi\lambda_g^\beta = [D\varphi,D\psi]_t\lambda_t^\psi u_g^*\lambda_g^{\tilde{\alpha}} = [D\varphi,D\psi]_t\sigma_t^\psi(u_g^*)\lambda_t^\psi\lambda_g^{\tilde{\alpha}}.\end{aligned}$$

Since $\lambda_t^\psi\lambda_g^{\tilde{\alpha}} = \lambda_g^{\tilde{\alpha}}\lambda_t^\psi$, using $[D\varphi,D\psi]_t^* = [D\psi,D\varphi]_t$ we get

$$\omega_{(g,t)} = \sigma_t^\psi(u_g)[D\psi,D\varphi]_t = \tilde{\alpha}_g([D\psi,D\varphi]_t)u_g = u_g\beta_g([D\psi,D\varphi]_t).$$

Recall that we have the cocycle relations

$$\begin{aligned}&c(g,h) = u_g\beta_g(u_h)u_{gh}^* \quad \text{for all } g,h\in\Gamma;\\ &[D\psi,D\varphi]_{t+s} = [D\psi,D\varphi]_t\sigma_t^\varphi([D\psi,D\varphi]_s) \quad \text{for all } t,s\in\mathbb{R}.\end{aligned}$$

We then compute that for any $(g,t),(h,s)\in G$,

$$\begin{aligned}\omega_{(g,t)}\beta_{(g,t)}^\varphi(\omega_{(h,s)}) &= u_g\beta_g([D\psi,D\varphi]_t)\beta_g\circ\sigma_t^\varphi([D\psi,D\varphi]_s\sigma_s^\varphi(u_h))\\ &= u_g\beta_g([D\psi,D\varphi]_{t+s}\sigma_{t+s}^\varphi(u_h)) = u_g\beta_g(w_{(h,t+s)})\\ &= u_g\beta_g(u_h\beta_h([D\psi,D\varphi]_{t+s})) = c(g,h)u_{gh}\beta_{gh}([D\psi,D\varphi]_{t+s})\\ &= c^\psi((g,t),(h,s))\omega_{(gh,t+s)},\end{aligned}$$

and similarly $\mathrm{Ad}(\omega_{(g,t)})\circ\beta_{(g,t)}^\varphi = \tilde{\alpha}_{(g,t)}^\psi$. ■

Now we put $G_1 := \mathbb{R}\le G$. Then since we already have $(\mathbb{C}p,\sigma^\psi)\preceq_B(\mathbb{C},\sigma^\varphi)$ for all projections $p\in B_\psi^{\tilde{\alpha}} = B^{\tilde{\alpha}^\psi}$, we can apply Proposition 5.1. Thus there exist a separable Hilbert space $H$, a projection $f\in\mathbb{B}(H)$, a $\sigma$-strongly continuous map $v\colon G=\Gamma\times\mathbb{R}\to\mathcal{U}(f\mathbb{B}(H)f)$, and a partial isometry $w\in B\mathbin{\overline{\otimes}}\mathbb{B}(H)$ such that

- $wv_g = (\omega_g\otimes 1_H)(\beta_g^\varphi\otimes\mathrm{id}_H)(w)$ for all $g\in G$;
- $w^*w = f$ and $ww^* = 1\otimes e_{1,1}$, where $e_{1,1}\in\mathbb{B}(H)$ is a minimal projection;
- $(\mathrm{Ad}(v_g))_{g\in G}$ and $(v_gv_hv_{gh}^*)_{g,h\in G}$ define a cocycle action on $f\mathbb{B}(H)f$;
- $\tilde{\alpha}_g^\psi(wxw^*) = w(\beta_g^\varphi\otimes\mathrm{Ad}(v_g))(x)w^*$ for all $x\in B\mathbin{\overline{\otimes}}f\mathbb{B}(H)f$.

As in the proof of Proposition 5.1, the first equation implies $v_{t+s}=v_tv_s$ for all $t,s\in\mathbb{R}$, hence $(v_t)_{t\in\mathbb{R}}$ is a continuous homomorphism. By Stone's theorem, there is a unique infinitesimal generator $h$ on $fH$, so that $[\mathrm{Tr}_H(h\,\cdot), f\mathrm{Tr}_Hf]_t = h^{it} = v_t$ for all $t\in\mathbb{R}$, where $\mathrm{Tr}_H$ is a fixed semifinite trace on $\mathbb{B}(H)$ (with $\mathrm{Tr}_H(e_{1,1})=1$). We then compute that for all $t\in\mathbb{R}$, with $\varphi^H := \varphi\otimes\mathrm{Tr}_H$, $\psi^H := \psi\otimes\mathrm{Tr}_H$ and $h = 1_B\otimes h$, using Lemma 2.2,

$$\begin{aligned}[][Df\varphi^H(h\cdot)f, D\psi^H\circ\mathrm{Ad}(w)]_t &= [Df\varphi^H(h\cdot)f, Df\varphi^Hf]_t[Df\varphi^Hf, D\psi^H\circ\mathrm{Ad}(w)]_t\\ &= v_t[Df\varphi^Hf, Df\psi^Hf]_t[Df\psi^Hf, D\psi^H\circ\mathrm{Ad}(w)]_t\\ &= v_t([D\varphi,D\psi]_t\otimes 1_H)(\sigma_t^\psi\otimes\mathrm{id}_H)(w^*)w\\ &= v_t(\sigma_t^\varphi\otimes\mathrm{id}_H)(w^*)([D\varphi,D\psi]_t\otimes 1_H)w\\ &= w^*([D\psi,D\varphi]_t\otimes 1_H)([D\varphi,D\psi]_t\otimes 1_H)w = f.\end{aligned}$$

We find that $\varphi^H(h\cdot) = \psi^H \circ \mathrm{Ad}(w)$. In particular, putting $\mu := \mathrm{Tr}_H(h\cdot)$, we see that

$$\mathrm{Ad}(w^*)\colon B = B \otimes \mathbb{C}e_{1,1} \to B \mathbin{\overline{\otimes}} f\mathbb{B}(H)f$$

satisfies $\psi = (\varphi \otimes \mu) \circ \mathrm{Ad}(w^*)$. Since $\mathrm{Ad}(w^*)$ gives a conjugacy between $\beta^\varphi \otimes \mathrm{Ad}(u)$ and $\widetilde{\alpha}^\psi$, by restriction, it gives a state preserving conjugacy between $\beta \otimes \mathrm{Ad}(u)$ and $\widetilde{\alpha}$.

Finally, we show that $\Lambda_0$ is a finite group. Observe that $\mathrm{Tr}_H(h) = \psi(1) < \infty$, so $h$ is a compact operator on $fH$. We have

$$A_\psi \rtimes_\alpha \Lambda_0 = (A \rtimes_\alpha \Lambda_0)_\psi \simeq (B \mathbin{\overline{\otimes}} f\mathbb{B}(H)f)_{\varphi\otimes\mu}.$$

Since $h$ is a compact operator, there exist finite rank projections $r_n$ on $fH$ which commute with $h$ such that $r_n \to f$. Then since $\sigma^\varphi$ is weakly mixing, one has $r_n(B \mathbin{\overline{\otimes}} f\mathbb{B}(H)f)_{\varphi\otimes\mu}r_n = \mathbb{C} \otimes (r_n\mathbb{B}(H)r_n)_\mu$ for all $n$. In particular $(B \mathbin{\overline{\otimes}} f\mathbb{B}(H)f)_{\varphi\otimes\mu}$ is an atomic von Neumann algebra, so that $A_\psi \rtimes_\alpha \Lambda_0$ is one as well. This implies that $\Lambda_0$ is a finite group (and $A_\psi$ is atomic). ■

### *Rigidity of Bernoulli shifts for genuine actions*

We continue to use the Bernoulli shift action $\Gamma \curvearrowright^\beta \bigotimes_\Gamma(B_0, \varphi_0) = (B, \varphi)$ and $M = B \rtimes_\beta \Gamma$, assuming that $B_0$ is amenable. We recall the following fact.

**Theorem 5.3** ([32, Theorem A]). *Let $p \in M$ be a projection, and $A \subset pMp$ a finite von Neumann subalgebra with expectation.*

(1) *If $A \not\preceq_M L\Gamma$, then $A' \cap pMp$ has an amenable direct summand.*

(2) *If $A$ has relative property* (T) *in $pMp$, then $A \preceq_M L\Gamma$.*

*Proof of Proposition* F. By assumption, there are isomorphisms $\Gamma \simeq \Lambda$ and $A \simeq B$, and there is a cocycle $\omega\colon \Gamma \to \mathcal{U}(B)$ such that $\alpha = \beta^\omega$.

Assume that $\Gamma$ has a normal subgroup $\Gamma_1 \leq \Gamma$ with relative property (T). Let $\Lambda_1 \leq \Lambda$ be the image of $\Gamma_1$. For any projection $q \in L\Lambda_1' \cap B$, we apply Theorem 5.3(2) to $L\Lambda_1 q$ and find that $L\Lambda_1 q \preceq_M L\Gamma$.

Assume that $\Gamma$ is a direct product $\Gamma = \Gamma_1 \times \Gamma_2$ with $\Gamma_2$ non-amenable. We let $\Lambda_i \leq \Lambda$ be the images of $\Gamma_i$ for $i = 1, 2$. For any projection $q \in L\Lambda_1' \cap B$, we apply Theorem 5.3(1) to $L\Lambda_1 q$. We get $L\Lambda_1 q \preceq_M L\Gamma$.

Thus in both cases, one has $L\Lambda_1 q \preceq_M L\Gamma$ for any projection $q \in L\Lambda_1' \cap B$. Fix such $q \in L\Lambda_1' \cap B$; we claim that $(\mathbb{C}q, \alpha|_{\Lambda_1}) \preceq_B (\mathbb{C}, \beta|_{\Gamma_1})$. Indeed, suppose for contradiction that there is $(g_i)_{i\in I}$ in $\Lambda_1$ such that

$$\varphi(\beta_{g_i}(b^*)\omega_{g_i}^* qa) \to 0 \quad \sigma\text{-strongly for all } a, b \in B.$$

Then for any $a, b \in B$ and $s, s' \in \Gamma$, we have

$$\begin{aligned} E_{L\Gamma}(\lambda_s^\beta b^* \Pi^\omega_{\beta,\alpha}(\lambda^\alpha_{g_i^{-1}}) qa \lambda^\beta_{s'}) &= \lambda^\beta_s E_{L\Gamma}(b^* \lambda^\beta_{g_i^{-1}} \omega^*_{g_i} qa) \lambda^\beta_{s'} \\ &= \lambda^\beta_{sg_i^{-1}} \varphi(\beta_{g_i}(b^*) \omega^*_{g_i} qa) \lambda^\beta_{s'}. \end{aligned}$$

The last term converges to 0, hence $L\Lambda_1 q \not\preceq_M L\Gamma$, a contradiction.

Finally, since $\Lambda_1 \leq \Lambda$ is normal, we can apply Proposition 5.1 to get a cocycle action $(\mathrm{Ad}(u_g))_{g\in\Gamma}$ on a factor $\mathbb{B}$. By construction, this cocycle action is a genuine action, which finishes the proof. ■

## 6. Strong solidity of free product factors

For amalgamated free products von Neumann algebras and their modular theory, we refer the reader to [46, 54]. Throughout this section we fix the following setting.

Let $I$ be a set, $(M_i)_{i\in I}$ a family of $\sigma$-finite von Neumann algebras, $B \subset M_i$ a common unital von Neumann subalgebra, and $E_i \colon M_i \to B$ faithful normal conditional expectations for all $i \in I$. Denote by $M := *_B\,(M_i, E_i)_{i\in I}$ the amalgamated free product von Neumann algebra, and by $E_B \colon M \to B$ the canonical conditional expectation. For any subset $\mathcal{F} \subset I$, we denote $M_{\mathcal{F}} := *_B\,(M_i, E_i)_{i\in\mathcal{F}}$, and $E_{\mathcal{F}} \colon M \to M_{\mathcal{F}}$ is the canonical conditional expectation.

To prove Theorem G, we first prove the following special case. This is a variant of Ioana's theorem [25, Theorem 1.6] (see also [21, 51]), and the proof uses a theorem in [3].

**Lemma 6.1.** *Let* $I = \{1, 2\}$. *Assume that there is a semifinite trace* $\mathrm{Tr}_B$ *on* $B$ *such that* $\mathrm{Tr}_B \circ E_i$ *are tracial for all* $i \in I$. *Then the conclusion of Theorem* G *holds for any* $p \in M$ *and* $A \subset pMp$ *as in the statement, provided that* $\mathrm{Tr}_B \circ E_B(p) < \infty$.

*Proof.* Recall that for any semifinite von Neumann algebra, relative injectivity and relative semidiscreteness are the same condition (see [29, Theorem A.6]). To prove this lemma, we follow the argument in the paragraph just before [21, Theorem A.4]. In this argument, we can apply [3, Theorem 3.11] instead of [43, Theorem 1.6]. Then all other proofs work if we replace the normalizer algebra with the stable normalizer algebra. Thus the conclusion of [21, Theorem A.4] holds for the stable normalizer von Neumann algebra and the lemma is proven. ■

*Proof of Theorem* G. Suppose that $A \not\preceq_M B$ and $s\mathcal{N}_{pMp}(A)'' \not\preceq_M M_i$ for $i = 1, 2$. We will prove that $P := s\mathcal{N}_{pMp}(A)''$ is injective relative to $B$ in $M$.

Let $E_A$ and $E_P$ be faithful normal conditional expectations for $A$ and $P$ respectively, $N$ the hyperfinite type III$_1$ factor, and $\omega$ a faithful normal state such that $N_\omega' \cap N = \mathbb{C}$. Observe that $A' \cap pMp \subset A$ implies that $E_A$ and $E_P$ are unique normal expectations, hence $E_A \circ E_P = E_A$. From this uniqueness and Theorem A, there exist $\psi$ preserved by $E_A$, $E_P$, and $\varphi$ preserved by $E_B$, $E_{M_i}$ for $i = 1, 2$, such that

$$\Pi_{\varphi\otimes\omega,\psi\otimes\omega}(C_{\psi\otimes\omega}(A \mathbin{\overline{\otimes}} N)) \not\preceq_{C_{\varphi\otimes\omega}(M\overline{\otimes}N)} C_{\varphi\otimes\omega}(B \mathbin{\overline{\otimes}} N),$$
$$\Pi_{\varphi\otimes\omega,\psi\otimes\omega}(C_{\psi\otimes\omega}(P \mathbin{\overline{\otimes}} N)) \not\preceq_{C_{\varphi\otimes\omega}(M\overline{\otimes}N)} C_{\varphi\otimes\omega}(M_i \mathbin{\overline{\otimes}} N) \quad \text{for } i = 1, 2.$$

Observe that since $A \mathbin{\overline{\otimes}} N$ is properly infinite, by [12, Lemma 2.4] we have

$$A \mathbin{\overline{\otimes}} N \subset P \mathbin{\overline{\otimes}} N \subset s\mathcal{N}_{pMp\overline{\otimes}N}(A \mathbin{\overline{\otimes}} N)'' = \mathcal{N}_{pMp\overline{\otimes}N}(A \mathbin{\overline{\otimes}} N)''.$$

In particular the inclusion $A \mathbin{\overline{\otimes}} N \subset P \mathbin{\overline{\otimes}} N$ is regular, and hence by [3, Lemma 4.1] the inclusion $C_{\psi\otimes\omega}(A \mathbin{\overline{\otimes}} N) \subset C_{\psi\otimes\omega}(P \mathbin{\overline{\otimes}} N)$ is regular as well. For notational simplicity, we omit $\Pi_{\varphi\otimes\omega,\psi\otimes\omega}$ and write $\mathcal{M} := C_{\varphi\otimes\omega}(M \mathbin{\overline{\otimes}} N)$, $\mathcal{M}_i := C_{\varphi\otimes\omega}(M_i \mathbin{\overline{\otimes}} N)$ for $i = 1, 2$, $\mathcal{B} := C_{\varphi\otimes\omega}(B \mathbin{\overline{\otimes}} N)$, and $\mathcal{A} := C_{\psi\otimes\omega}(A \mathbin{\overline{\otimes}} N)$. Let $\mathcal{E}_i \colon \mathcal{M}_i \to \mathcal{B}$ be the faithful normal conditional expectation such that $\mathcal{E}_i|_{M_i \mathbin{\overline{\otimes}} N} = E_i \otimes \mathrm{id}_N$ and $\mathcal{E}|_{L\mathbb{R}_\varphi} = \mathrm{id}_{L\mathbb{R}_\varphi}$ and note that $\mathcal{M}$ has an amalgamated free product structure,

$$\mathcal{M} = (\mathcal{M}_1, \mathcal{E}_1) *_{\mathcal{B}} (\mathcal{M}_2, \mathcal{E}_2).$$

In this setting, our assumptions are translated to $\mathcal{A} \not\preceq_{\mathcal{M}} \mathcal{B}$, $\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})'' \not\preceq_{\mathcal{M}} \mathcal{M}_i$ for all $i = 1, 2$, and $\mathcal{A}$ is injective relative to $\mathcal{B}$ in $\mathcal{M}$ (use [29, Corollary 3.6 and Theorem 3.2]). Fix any projection $r \in L_{\psi\otimes\omega}\mathbb{R}$ such that $\mathrm{Tr}_{\psi\otimes\omega}(r) < \infty$, and observe that $r\mathcal{A}r \not\preceq_{\mathcal{M}} \mathcal{B}$ and $r\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})''r \not\preceq_{\mathcal{M}} \mathcal{M}_i$ for all $i = 1, 2$. Using the inclusion $r\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})''r \subset s\mathcal{N}_{pr\mathcal{M}pr}(r\mathcal{A}r)''$ (e.g. [12, Proposition 2.10]), by applying Lemma 6.1 to $r\mathcal{A}r \subset rp\mathcal{M}rp$, we find that $r\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})''r$ is injective relative to $\mathcal{B}$. Since $r$ is arbitrary, by [16, Lemma 3.3(v)] we conclude that $\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})''$ is injective relative to $\mathcal{B}$ in $\mathcal{M}$. Since $\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})''$ contains $C_{\psi\otimes\omega}(P \mathbin{\overline{\otimes}} N)$ with expectation, by [29, Theorem 3.2] we know that $P \mathbin{\overline{\otimes}} N$ is injective relative to $B \mathbin{\overline{\otimes}} N$ in $M \mathbin{\overline{\otimes}} N$. Finally, it is easy to see that $P$ is injective relative to $B$ in $M$. This is the conclusion. ∎

*Proof of Corollary* H. If $M$ is stably strongly solid, then since all $M_i$'s are von Neumann subalgebras with expectation, all $M_i$'s are stably strongly solid. We have to show the converse.

Let $p \in M$ be a projection and $A \subset pMp$ a diffuse amenable von Neumann subalgebra with expectation. We have to show that $P := s\mathcal{N}_{pMp}(A)''$ is amenable. Since $pMp$ is solid by [21, Theorem 6.1], $A' \cap pMp$ is amenable. Then as in [3, proof of Main Theorem], up to replacing $A \vee (A' \cap pMp)$ by $A$, we may assume that $A' \cap pMp \subset A$. Let $z \in P$ be the unique projection such that $P(p - z)$ is amenable and $Pz$ has no amenable direct summand. We will deduce a contradiction by assuming that $z \neq 0$. In this case, using $Pz \subset s\mathcal{N}_{zMz}(Az)''$, up to replacing $z$ by $p$ we may assume that $P$ has no amenable direct summand. Define $M^\infty := M \mathbin{\overline{\otimes}} \mathbb{B}(\ell^2)$, $M_i^\infty := M_i \mathbin{\overline{\otimes}} \mathbb{B}(\ell^2)$, $A^\infty := A \mathbin{\overline{\otimes}} \mathbb{B}(\ell^2)$, and $E_i^\infty := E_i \otimes \mathrm{id}_{\mathbb{B}(\ell^2)}$, and observe that $M^\infty = *_{\mathbb{B}(\ell^2)} (M_i^\infty, E_i^\infty)_{i\in I}$ and $s\mathcal{N}_{pM^\infty p}(A^\infty)'' = \mathcal{N}_{pM^\infty p}(A^\infty)''$ (since $A^\infty$ is properly infinite). Since $A^\infty$ is diffuse, we have $A^\infty \not\preceq_{M^\infty} \mathbb{B}(\ell^2)$.

Suppose first that $I = \{1, 2\}$. We can apply Theorem G to $A^\infty \subset pM^\infty p$, and find (ii) $\mathcal{N}_{pM^\infty p}(A^\infty)'' \preceq_{M^\infty} M_i^\infty$ for some $i \in \{1, 2\}$ or (iii) $\mathcal{N}_{pM^\infty p}(A^\infty)''$ is amenable. If (iii) holds, then since $P \mathbin{\overline{\otimes}} \mathbb{B}(\ell^2) \subset \mathcal{N}_{pM^\infty p}(A^\infty)''$ is with expectation, we infer that $P$ is amenable, a contradiction. Hence condition (ii) holds. Fix $i$ such that $\mathcal{N}_{pM^\infty p}(A^\infty)'' \preceq_{M^\infty} M_i^\infty$, and take $(H, f, \pi, w)$ witnessing this condition. Observe that $\pi(A^\infty) \subset f(M_i^\infty \otimes \mathbb{M}_n)f$ is a diffuse amenable von Neumann subalgebra with expectation and that $\pi(P \mathbin{\overline{\otimes}} \mathbb{B}(\ell^2)) \subset \mathcal{N}_{f(M_i^\infty \otimes \mathbb{M}_n)f}(\pi(A^\infty))''$ is with expectation. Since $M_i$ is assumed to be stably strongly solid, $M_i^\infty \otimes \mathbb{M}_n$ is strongly solid by [3, Corollary 5.2]. Thus $\pi(P \mathbin{\overline{\otimes}} \mathbb{B}(\ell^2))$ is amenable. Since $\pi$ is a normal $*$-homomorphism, $P$ has an amenable direct summand, a contradiction. We have thus proved this theorem in the case $I = \{1, 2\}$.

Now we prove the general case. Let $I$ be a general set and we put $M_{\mathcal{F}} := *_{i\in\mathcal{F}}\,(M_i, \varphi_i)$ for any subset $\mathcal{F} \subset I$. We fix any finite subset $\mathcal{F} \subset I$ and observe that $M_{\mathcal{F}}$ is stably strongly solid by the result in the last paragraph. We apply the same argument as in the case $I = \{1, 2\}$ to $A \subset pMp$ using the decomposition $M = M_{\mathcal{F}} * M_{\mathcal{F}^c}$. Then since $M_{\mathcal{F}}$ is stably strongly solid, the only possible condition is that $\mathcal{N}_{pM^\infty p}(A^\infty)'' \preceq_{M^\infty} M^\infty_{\mathcal{F}^c}$. By assuming that this condition holds for all finite subsets $\mathcal{F} \subset I$, we will deduce a contradiction.

Since $P \,\overline{\otimes}\, \mathbb{B}(\ell^2) \subset \mathcal{N}_{pM^\infty p}(A^\infty)''$, using [15, Lemma 4.8] we find that indeed $P \,\overline{\otimes}\, \mathbb{B}(\ell^2) \preceq_{M^\infty} M^\infty_{\mathcal{F}^c}$ for all finite subsets $\mathcal{F} \subset I$. Then as in the proof of Theorem G, by applying Theorem A (and using $N \simeq N \,\overline{\otimes}\, \mathbb{B}(\ell^2)$) one has $\mathcal{P} \preceq_{\mathcal{M}} \mathcal{M}_{\mathcal{F}^c}$ for all finite subsets $\mathcal{F} \subset I$, where we have used similar notations to ones in the proof of Theorem G, such as $\mathcal{P} := C_{\psi\otimes\omega}(P \,\overline{\otimes}\, N)$, $\mathcal{M}_{\mathcal{F}^c} := C_{\varphi\otimes\omega}(M_{\mathcal{F}^c} \,\overline{\otimes}\, N)$ for appropriate $E_P, \psi, \varphi$.

Fix any projection $r \in L_{\psi\otimes\omega}\mathbb{R}$ such that $\mathrm{Tr}_{\psi\otimes\omega}(r) < \infty$. Fix any projection $z \in \mathcal{P}' \cap p\mathcal{M}p = (P' \cap pMp)_\psi = \mathcal{Z}(P)$ (e.g. by Lemma 2.3). We will prove that $r\mathcal{P}rz \preceq_{\mathcal{M}} \mathcal{M}_{\mathcal{F}^c}$ for all finite subsets $\mathcal{F} \subset I$. Then [21, Proposition 4.2] will imply the amenability of $r\mathcal{P}r$ and hence the one of $\mathcal{P}$, a contradiction. To prove this condition, fix $\mathcal{F}$, $r$ and $z$. Observe that $Pz \subset s\mathcal{N}_{zMz}(Az)''$. Then since $Pz$ has no amenable direct summand, we can apply the same argument to $Az \subset Pz$ (as we applied to $A \subset P$), and get $\mathcal{P}z \preceq_{\mathcal{M}} \mathcal{M}_{\mathcal{F}^c}$. Since the central support of $rz$ in $\mathcal{P}z$ is $z$, by [15, Remark 4.2(3)] we get $r\mathcal{P}rz \preceq_{\mathcal{M}} \mathcal{M}_{\mathcal{F}^c}$. This is the desired condition. ■

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