Studies on Discrete Integrable Systems with Positivity and Their Applications

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Chapter 1

Introduction

In this thesis, we study the (ultra)discretization of integrable systems and their application. We develop a theory of spectral transformations of the biorthogonal Laurent polynomials, and use it to derive and analyze nonautonomous discrete integrable systems. Using the obtained discrete integrable systems, we derive a family of box-ball systems (BBS) by ultradiscretization, and give their time evolution rules and conserved quantities. As an application of the obtained BBS, we formulate an algorithm for computing invariant factors of a certain tridiagonal matrix over a principal ideal domain.

In the following sections, we briefly review the theory of integrable systems.

1.1 Integrable systems

1.1.1 Infinite-dimensional integrable systems

The definition of the integrable system for the finite-dimensional Hamiltonian system is the following:

Definition 1.1.1. The $2n$-dimensional Hamiltonian system is *Liouville integrable* if there are n constants of the motion $H_1, H_2, ..., H_n$ that are functionally independent and Poisson-commute each other.

If a Hamiltonian system is Liouville integrable, then the system can be, in principle, solved by quadratures. Many examples of Liouville integrable systems has been discovered, such as the harmonic oscillator, the Kepler problem, spinning tops (the Euler, the Lagrange and the Kovalevskaya top), and so on. Since the 1960's, a research on dynamical systems which are nowadays called *infinite-dimensional integrable systems* has been conducted actively. These dynamical systems, including the famous KdV equations [43], Toda lattice [71], etc., are no longer integrable in the sense of Definition 1.1.1, but have the following remarkable properties in common:

- Existence of infinitely many conserved quantities.
- Existence of explicit solutions with arbitrary number of parameters.
- Existence of the Lax formalism [46], which enable us to analyze the structure of the system, e.g., derivation of the conserved quantities, solving the initial value problems through inverse scattering method [18], etc.
- We can use the bilinearization method [24] (also known as Hirota's method) to obtain the N-soliton solution. This leads to Sato's theory of the KP-hierarchy [61, 62], which state that the bilinear form of the KP equation is nothing but a Plücker relation of the infinite-dimensional Grassmannian.

It is a difficult problem to characterize integrability for general dynamical systems, and there is no rigorous definition that is widely accepted by researchers. In this thesis, we do not go deeply into the definition of integrability, but discuss the various properties and applications of integrable systems. The theory of integrable systems has developed rapidly since the late twentieth century. In the following sections, we will introduce three topics among these developments: discrete integrable systems, relations with orthogonal polynomials, and integrable cellular automata.

1.1.2 Discrete integrable systems

Discrete integrable systems are integrable systems in which all independent variables are discrete. As in the case of continuous systems, there is no rigorous definition of discrete integrable systems. The systematic study of the discrete integrable systems began with the pioneering works by Hirota [25, 26, 27, 28, 29], where the important method to obtain discrete integrable systems called the *integrable discretization* is introduced. Difference schemes used in numerical integration could destroy structures of the system (e.g., conserved quantities, particular solutions, etc.). However, for integrable systems, it is possible to discretize the system without destroying the properties of integrable systems mentioned in Section 1.1.1. Hirota used this method to obtain the discrete analogues of important integrable systems, including the KdV equation and the Toda lattice. Discrete integrable systems are not only interesting as dynamical systems but also have connections with various fields. One of the most remarkable connections to informatics is the application to numerical algorithms.

Let us explain the relationship between discrete integrable systems and numerical algorithms in the case of the discrete Toda lattice [31]:

$$q_{n+1}^{(t+1)} + e_n^{(t+1)} = q_{n+1}^{(t)} + e_{n+1}^{(t)}, \tag{1.1}$$

$$q_n^{(t+1)} e_n^{(t+1)} = q_{n+1}^{(t)} e_n^{(t)}, \tag{1.2}$$

with the finite lattice boundary condition $e_{-1}^{(t)} = e_{N-1}^{(t)} \equiv 0$. The system (1.1)–(1.2) has the following matrix form:

$$L^{(t+1)}R^{(t+1)} = R^{(t)}L^{(t)} \tag{1.3}$$

where

$$L^{(t)} = \begin{pmatrix} 1 & & & & \\ e_0^{(t)} & 1 & & & \\ & \ddots & \ddots & & \\ & & e_{N-3}^{(t)} & 1 & \\ & & & e_{N-2}^{(t)} & 1 \end{pmatrix}, \quad R^{(t)} = \begin{pmatrix} q_0^{(t)} & 1 & & & \\ & q_1^{(t)} & 1 & & \\ & & \ddots & \ddots & \\ & & & q_{N-2}^{(t)} & 1 \\ & & & & q_{N-1}^{(t)} \end{pmatrix}.$$

The equation (1.3) is called the *Lax form* of the discrete Toda lattice. It is easily shown that the matrix $X^{(t)} := L^{(t)}R^{(t)}$ satisfies $X^{(t+1)} = (L^{(t)})^{-1}X^{(t)}L^{(t)}$, thus the eigenvalues of the matrix $X^{(t)}$ is conserved by the system (1.1)–(1.2). Therefore, if limits $q_n^{(t)} \to \lambda_n < +\infty$, $e_n^{(t)} \to 0$ as $t \to \infty$ hold, the system (1.1)–(1.2) can be used to compute eigenvalues of the matrix $X^{(t)}$. In fact, the system (1.1)–(1.2) coincide with recurrences of the qd-algorithm by Rutishauser [59, 60]. This is one of the earliest examples of the connection between integrable systems and numerical algorithms.

The above observation leads to the following general strategy for developing numerical algorithms from discrete integrable systems:

- Identify conserved quantities of a discrete integrable system with some characteristics that are important in numerical computations (eigenvalues of a tridiagonal matrix in the above example).
- Prove the convergence of dependent variables in order to extract these characteristics.

From the viewpoint of the integrable systems, various numerical algorithms have been developed [14, 34, 53, 57]. By taking this strategy, we give a method to compute invariant factors of a matrix over a principal ideal domain in Chapter 4.

1.1.3 Relation with the orthogonal polynomials

In this section, we explain the relationship between integrable systems and orthogonal polynomials. The orthogonal polynomial sequence is a family of polynomials which form an orthogonal basis with respect to some linear functional. Let us give the precise definition. Let $\mathcal{L}\colon \mathbb{C}[x] \to \mathbb{C}$ be a linear functional, where $\mathbb{C}[x]$ is a vector space consisting of all polynomials in one variable x with complex coefficients.

Definition 1.1.2. Let N be a positive integer or $+\infty$. A sequence of polynomials $p_0(x), p_1(x), ..., p_N(x) \in \mathbb{C}[x]$ is called the (monic) orthogonal polynomial sequence (OPS) with respect to the functional $\mathcal{L}$ if

- the polynomial $p_n(x)$ is monic and of degree n for all $0 \le n \le N$, and
- the sequence $p_0(x), p_1(x), ..., p_N(x)$ satisfies

$$\mathcal{L}[p_i(x)p_j(x)] = h_i \delta_{ij}, \quad 0 \le i, j < N,$$

  for some non-zero constant $h_i \in \mathbb{C}$. If N is finite, we impose the additional condition that

$$\mathcal{L}[p_N(x)\pi(x)] = 0$$

  for all polynomials $\pi(x) \in \mathbb{C}[x]$.

The classical examples are the Hermite polynomials, the Laguerre polynomials and the Jacobi polynomials, which arise in a variety of situations in mathematics, physics, and engineering. We will see below that the deformation of the functional yields the Toda equation.

Let us first consider the linear functional $\mathcal{L}^{(t)}$ of the form

$$\mathcal{L}^{(t)}[p(x)] = \sum_{i=0}^{N-1} w_i e^{x_i t} p(x_i),$$

where $w_0, w_1, ..., w_{N-1}$ are positive real numbers and $x_0, x_1, ..., x_{N-1}$ are distinct positive real numbers. Then, there exists the OPS $\{p_i(x;t)\}_{i=0}^N$ with respect to the functional $\mathcal{L}^{(t)}$. Note that the variable x does not depend on t. We can see that the $p_N(x;t)$ depends only on the support $\{x_0, x_1, ..., x_{N-1}\}$ of the functional $\mathcal{L}^{(t)}$, thus $dp_N(x;t)/dt = 0$. In what follows, we omit the t-dependence of the OPS as $p_i(x) := p_i(x;t)$. The OPS $\{p_i(x)\}_{i=0}^N$ satisfies relations

$$xp_n(x) = p_{n+1}(x) + (q_n + e_{n-1})p_n(x) + e_{n-1}q_{n-1}p_{n-1}(x), \tag{1.4}$$

$$\frac{dp_n(x)}{dt} = -e_{n-1}q_{n-1}p_{n-1}(x), \tag{1.5}$$

where $e_i = e_i(t), q_i = q_i(t)$ are some constants depending only on t and i. Let us write (1.4) and (1.5) in matrix-vector form. We introduce matrices

$R(t)$ and $L(t)$ as

$$L(t) = \begin{pmatrix} 1 & & & & \\ e_0(t) & 1 & & & \\ & \ddots & \ddots & & \\ & & e_{N-3}(t) & 1 & \\ & & & e_{N-2}(t) & 1 \end{pmatrix},$$

$$R(t) = \begin{pmatrix} q_0(t) & 1 & & & \\ & q_1(t) & 1 & & \\ & & \ddots & \ddots & \\ & & & q_{N-2}(t) & 1 \\ & & & & q_{N-1}(t) \end{pmatrix}.$$

Let $X(t) := L(t)R(t)$ and define $X(t)_{<0}, X(t)_{\geq 0}$ as

$$((X(t))_{<0})_{ij} = \begin{cases} X(t)_{ij} & (i > j) \\ 0 & (i \leq j) \end{cases}, \quad X(t)_{\geq 0} = X - (X(t))_{<0}.$$

Then relations (1.4) and (1.5) can be written as

$$x\boldsymbol{p} = X(t)\boldsymbol{p} + \boldsymbol{p}_N, \tag{1.6}$$

$$\frac{d\boldsymbol{p}}{dt} = -X(t)_{<0}\boldsymbol{p}, \tag{1.7}$$

where $\boldsymbol{p} = (p_0(x), p_1(x), ..., p_{N-1}(x))^T$ and $\boldsymbol{p}_N = (0, 0, ..., 0, p_N(x))^T$. From the compatibility condition of (1.6) and (1.7), we obtain

$$\frac{dX(t)}{dt} = [X(t), X(t)_{<0}] = [X(t)_{\geq 0}, X(t)], \tag{1.8}$$

where $[A, B] := AB - BA$ for matrices A, B. The equation (1.8) is the Lax form of the Toda lattice. In summary, we see that the (continuous) Toda equation is naturally derived from the continuous deformation of the functional.

To derive discrete Toda lattice, we introduce a discrete parameter t into the functional as

$$\mathcal{L}^{(t)}[p(x)] = \sum_{i=0}^{N-1} w_i x_i^t p(x_i).$$

Suppose that there exists the OPS $\{p_n^{(t)}(x)\}_{n=0}^N$ with respect to the functional $\mathcal{L}^{(t)}$ for all $t \geq 0$. Then, the following contiguous relations for the OPS $\{p_n^{(t)}(x)\}_{n=0}^N$ hold.

Proposition 1.1.1. There exist constants $q_n^{(t)}, e_n^{(t)} \in \mathbb{C}$ such that

$$xp_n^{(t+1)}(x) = p_{n+1}^{(t)}(x) + q_n^{(t)} p_n^{(t)}(x), \tag{1.9}$$

$$p_{n+1}^{(t)}(x) = p_{n+1}^{(t+1)}(x) + e_n^{(t)} p_n^{(t+1)}(x). \tag{1.10}$$

Transformations (1.9) and (1.10) are called the *Christoffel transformation* and the *Geronimus transformation*, respectively. The both (1.9) and (1.10) are also called the *spectral transformations*. From (1.9) and (1.10), we obtain the following relations for $q_n^{(t)}$'s and $e_n^{(t)}$'s:

$$q_{n+1}^{(t+1)} + e_n^{(t+1)} = q_{n+1}^{(t)} + e_{n+1}^{(t)}, \tag{1.11}$$

$$q_n^{(t+1)} e_n^{(t+1)} = q_{n+1}^{(t)} e_n^{(t)}, \tag{1.12}$$

where $e_{-1}^{(t)}, e_{N-1}^{(t)} = 0$ for all $t \geq 0$. The relations (1.11) and (1.12) coincide with the system (1.1)–(1.2). The connection between the discrete Toda lattice and the spectral transformation of orthogonal polynomials is investigated in [63]. This method provide a powerful tool to derive and analyze discrete integrable systems. There are many relationships between various (bi)orthogonal functions and integrable systems, such as orthogonal polynomials on the unit circle (OPUC) and Schur flows [20, 54], Laurent biorthogonal polynomials (LBP) and the relativistic Toda lattice [67, 75], skew orthogonal polynomials (SOP) and the Pfaff lattice [1, 2], and so on.

1.2 Box-ball systems

The study of integrable cellular automata has been continued since the discovery of Takahashi-Satsuma's box-ball system [68] (BBS) in 1990. Let us first review the BBS and then give a brief summary of the further developments.

The BBS is a cellular automaton on a one-dimensional lattice. The state of the BBS is represented by a semi-infinite 01-sequence $u^{(t)} = (u_n^{(t)})_{n=0}^{\infty}$, $u_n \in \{0, 1\}$. Integers '0' and '1' are called an *empty box* and a *ball*, respectively. We define the time evolution $T\colon u^{(t)} \mapsto u^{(t+1)}$ by the following rule:

1. Move the leftmost ball to the nearest empty box on the right.
2. Repeat Step 1 until all the balls have been moved exactly once.

An example of time evolutions of the BBS is as follows:

$$\begin{aligned}
u &: 01111000111001000000000000000000\cdots \\
T^1 u &: 00000111000110111000000000000000\cdots \\
T^2 u &: 00000000111001000111100000000000\cdots \\
T^3 u &: 00000000000110110000011110000 0\cdots \\
T^4 u &: 00000000000001001110000001111 0\cdots
\end{aligned}$$

In this thesis, we assume that the number of balls is always finite. When the number of balls is finite, the dynamics is always well-defined and reversible, that is, the inverse of the time evolution is uniquely determined. A state with an infinite number of balls can be studied through the method of probability theory (cf. [7]). The BBS exhibits soliton property analogous to the KdV equation, which is explained below. If there are sufficiently many empty boxes before and after a block of consecutive balls, then the block of balls propagates to the right at the same speed as the block length. We call such a block of consecutive balls a *soliton*. As seen in the example above the larger soliton overtakes the smaller soliton, and the soliton amplitudes do not change after collisions. The length of solitons remains unchanged over time (see proof in [73]).

There are several ways to understand the integrability of the BBS. One way is the *ultradiscretization*. It is a limiting procedure to obtain a piecewise-linear equation from discrete equation which does not contain subtractions. Let us explain by an example. Suppose we have positive variables a_1, a_2, a_3, a_4 and a_5 and the relation

$$a_1 = \frac{a_2(a_3 + a_4)}{a_5}. \tag{1.13}$$

We substitute $a_i = e^{-A_i/\varepsilon}$ into the relation (1.13), and take the limit as $\varepsilon \to +0$. Then we obtain

$$A_1 = A_2 + \min(A_3, A_4) - A_5. \tag{1.14}$$

We see that the transition from equation (1.13) to (1.14) is the same as replacing $(\times, /, +)$ with $(+, -, \min)$, respectively. The BBS is obtained by the ultradiscretization of an integrable system. The time evolution of the BBS can be described by the following piecewise-linear equation:

$$(Tu)_n = \min\left\{1 - u_n, \sum_{m=-\infty}^{n-1} (u_m - (Tu)_m)\right\}. \tag{1.15}$$

The equation (1.15) is called the *ultradiscrete KdV equation*. The ultradiscrete KdV equation, as the name suggests, can be derived from the KdV equation by the ultradiscretization (see [72]). The N-soliton solution to the equation (1.15) is given in [47]. Thus, we see that the integrability of the KdV equation is preserved by the ultradiscretization. Interestingly, the BBS can also be obtained by the ultradiscretization of the Toda lattice (see [55]). Not only Takahashi-Satsuma's BBS, various extensions of the BBS can be obtained from the ultradiscretization of discrete integrable systems [34, 49, 50, 73].

The formulation of the BBS by the *crystal basis theory* provides another way to understand its integrability. The crystal basis, introduced by Kashiwara [38, 39], is a notion from the representation theory of quantum groups.

It is a combinatorial object obtained from a basis of representation of a quantum group by specializing deformation parameter $q = 0$. Originally, the introduction of the concept of crystal bases was motivated by the representation theory, but it turned out that the BBS can be formulated using the theory of the crystal basis [16, 21, 22]. As an application of the crystal formulation, it is shown in [44] that Takahashi-Satsuma's BBS and its multicolor extension [69, 73] can be linearized by using combinatorial bijection called the Kerov-Kirillov-Rheshetikhin bijection (also called the rigged configuration bijection) in [40, 41]. Though this topic is not directly related to the thesis, this is one of the important developments of the BBS. See [32] for detailed exposition on this topic.

1.3 Outline of the thesis

In this thesis, we study an (ultra)discretization of an integrable system called the *elementary Toda orbits*, which was introduced by Faybusovich and Gekhtman in [9]. The elementary Toda orbits is not a single integrable system, but a family of integrable systems. Let us give the definition of the elementary Toda orbits. Define $\epsilon = (\epsilon_0, \epsilon_1, ..., \epsilon_{N-1}) \in \{0, 1\}^N$. Let $L_1(t), L_2(t), R(t)$ be $N \times N$ matrices of the forms

$$\begin{aligned}
(L_1(t))^{-1} &= I_N + \sum_{i=1}^{N-1} -\epsilon_{i-1} e_{i-1}(t) E_{i+1,i}, \\
L_2(t) &= I_N + \sum_{i=1}^{N-1} (1 - \epsilon_{i-1}) e_{i-1}(t) E_{i+1,i}, \\
R(t) &= \sum_{i=1}^{N} q_{i-1}(t) E_{i,i} + \sum_{i=1}^{N-1} E_{i,i+1},
\end{aligned}$$

where I_N is an $N \times N$ identity matrix, and $E_{i,j} = (\delta_{ik}\delta_{jl})_{k,l=1}^N$. Let $X(t) = L_1(t)L_2(t)R(t)$. The following equation is called the *elementary Toda orbits*:

$$\frac{dX}{dt} = [(X)_{\geq 0}, X], \tag{1.16}$$

where $(X)_{\geq 0}$ denotes the upper-triangular part of the matrix X, namely,

$$((X)_{\geq 0})_{ij} = \begin{cases} X_{ij} & i \leq j, \\ 0 & i > j. \end{cases}$$

When $\epsilon = (0, 0, ..., 0)$ and $\epsilon = (1, 1, ..., 1)$, the equation (1.16) coincide with the Lax form of the Toda equation and the relativistic Toda equation, respectively. Let us state several motivations to work on this integrable systems:

- It contains the Toda lattice and the relativistic Toda lattice as special cases, both of which are important and well investigated in mathematical physics.

- The ultradiscretization of the Toda lattice and the relativistic Toda lattice are related to important combinatorial operations [13, 35, 56]. Therefore, it is expected that one can find an interesting combinatorial structure in the elementary Toda orbits.

- The invariant factor computation algorithm by the BBS (see Chapter 4) is naturally extended to the elementary Toda orbits.

The purpose of this thesis is to derive the nonautonomous discrete analogue of the elementary Toda orbits, and give its applications. First, we derive the nonautonomous discrete elementary Toda orbits using the theory of spectral transformations of the biorthogonal Laurent polynomials which is developed in this thesis. Then we derive a new family of the BBS, which we call the ϵ-BBS, by ultradiscretizing the nonautonomous discrete elementary Toda orbits. Furthermore, we introduce a multicolor extension of the ϵ-BBS and show that its conserved quantities are given by the combinatorial algorithm called the Schensted insertion. Finally, we show that the BBS computes the matrix characteristic called the *invariant factor*. Any matrix over a principal ideal domain R can be transformed into a particular form of diagonal matrix by unimodular transformations. That is, for any matrix $A \in R^{m\times n}$, there exist invertible matrices $P \in R^{m\times m}$ and $Q \in R^{n\times n}$ such that the matrix $S = PAQ$ vanishes off the main diagonal, $(e_1, e_2, ..., e_r, 0, ..., 0)$, where e_i divides e_{i+1} for $1 \leq i \leq r-1$. The matrix S is called the *Smith normal form* of A and the quantities $e_1, e_2, ..., e_r$ are the *invariant factors* of A. The Smith normal form has applications in many areas, including integer programming [19], combinatorics [65] and computations of homology groups [8]. There are many algorithms for computing the Smith normal form [37, 67]. The method introduced in this thesis is the first instance of the usage of ultradiscrete integrable systems in the computation of invariant factors. We also show that the ϵ-BBS can also compute invariant factors of a certain tridiagonal matrix.

This thesis is organized as follows.

In Chapter 2, we derive the nonautonomous discrete analogue of the elementary Toda orbits and construct its particular solutions and conserved quantities. To this end, we develop a theory of spectral transformations of the biorthogonal Laurent polynomials introduced in [11]. We also show that the obtained system can be transformed into a subtraction-free form. As an application, we derive a family of integrable cellular automata that we call the ϵ-BBS, which unifies the BBS with a carrier capacity [70] and the BBS associated with the discrete relativistic Toda lattice [34].

In Chapter 3, we show that the P-symbol of the RSK correspondence is a conserved quantity of a generalization of the ϵ-BBS. We consider what is known as the "hungry" extension of the discrete elementary Toda orbits, by which we can obtain a BBS with several kind of balls. In order to prove the conservation of the P-symbol, we use birational transformations among the elementary Toda orbits [9]. It was shown in [9] that transformations commute with the time evolution of the (continuous) elementary Toda orbits. In this thesis, we show that same transformations even commute with the time evolution of the discrete hungry elementary Toda orbits, which generalizes the result of [9]. We also show that Noumi-Yamada's geometric Schensted insertion [56] is invariant under this transformation.

In Chapter 4, we show that the ultradiscrete Toda lattice can compute invariant factors of bidiagonal matrices with elements of a principal ideal domain as entries. The key observation is that the greatest common divisor (gcd) operation in principal ideal domain is equivalent to applying min operation for each irreducible factors, which allows us to run multiple ultradiscrete Toda lattices simultaneously. Using the main result of Chapter 4, we present a new method for computing the Smith normal form of a given matrix. We also show that the main theorem of Chapter 4 (Theorem 4.2.1) naturally extends to the ultradiscrete elementary Toda orbits.

In Chapter 5, we give concluding remarks.

Chapter 2

Nonautonomous discrete elementary Toda orbits and their ultradiscretization

In this chapter, we derive the nonautonomous discrete elementary Toda orbits and ultradiscretize it to define the ϵ-BBS. Particular solutions and conserved quantities of the nonautonomous discrete elementary Toda orbits are also given. To this end, we develop the theory of spectral transformation of a certain biorthogonal Laurent polynomials.

2.1 Biorthogonal Laurent polynomials and integrable systems

In this section, we review the class of biorthogonal Laurent polynomials introduced in [11] and derive a nonautonomous discrete analogue of the elementary Toda orbits from their spectral transformations.

2.1.1 Basic definitions

We start from the class of biorthogonal Laurent polynomials introduced in [11]. Let N be a positive integer and define $\epsilon = (\epsilon_0, ..., \epsilon_{N-1}) \in \{0, 1\}^N$ and $\nu = (\nu_0, ..., \nu_{N-1})$ as

$$\nu_i = \begin{cases} i - \sum_{j=1}^{i} \epsilon_j & \epsilon_i = 0, \\ -\sum_{j=1}^{i} \epsilon_j & \epsilon_i = 1. \end{cases}$$

For any i, the set $\{\nu_0, \nu_1, ..., v_i\}$ consists of $(i+1)$ consecutive integers. For example, if we choose $N = 6$ and $\epsilon = (1, 0, 0, 1, 0, 0)$, then $\nu = (0, 1, 2, -1, 3, 4)$. Note that the value of ϵ_{N-1} does not affect results of Chapter 2 (in particular Proposition 2.1.4 and the nde-Toda orbits). Let $\mathcal{L} : \mathbb{C}[x, x^{-1}] \to \mathbb{C}$ be a linear functional. Let us consider polynomials $\{p_i(x)\}_{i=0}^{N}$ and Laurent polynomials $\{r_i(x)\}_{i=0}^{N-1}$ satisfying the followings:

1. $\deg p_i(x) = i$ and
2. $r_i(x)$ has the form

$$r_i(x) = x^{\nu_i} + \sum_{\beta=0}^{i-1} r_{i\beta} x^{\nu_\beta}, \quad r_{i\beta} \in \mathbb{C}.$$

Definition 2.1.1. We call the (Laurent) polynomial sequences $\{p_i(x)\}_{i=0}^{N}$ and $\{r_i(x)\}_{i=0}^{N-1}$ the pair of (finite) monic *ϵ-biorthogonal Laurent polynomial sequences* (ϵ-BLP) with respect to $\mathcal{L}$ if

$$\begin{aligned} &\mathcal{L}[p_i(x)r_j(x)] = h_i\delta_{ij}, \quad h_i \neq 0, \quad i,j = 0,1,...,N-1, \\ &\mathcal{L}[p_N(x)\pi(x)] = 0, \quad {}^{\forall}\pi(x) \in \mathbb{C}[x,x^{-1}], \end{aligned} \tag{2.1}$$

holds.

Remark 2.1.1. Special cases of the orthogonal relations (2.1) are the relations for the orthogonal polynomials [6] and the Laurent biorthogonal polynomials (LBP) [75] when $\epsilon = (0,0,...,0)$ and $\epsilon = (1,1,...,1,1)$, respectively. The ϵ-BLP is a special case of the classes of orthogonal polynomials called the *type R_I polynomials* and the *type R_{II} polynomials* which are introduced by Ismail and Masson [33]. One can specialize the type R_{II} polynomials to obtain the type R_I polynomials (see [76]). From the spectral transformations of the R_{II} polynomials, a discrete integrable system called the *R_{II} chain* was derived [64], and further investigated in [51, 52].

Let us give several basic properties of the ϵ-BLP.

Lemma 2.1.1. If $\{p_i(x)\}_{i=0}^{N}$ and $\{r_i(x)\}_{i=0}^{N-1}$ are a pair of monic ϵ-BLP sequences, then

$$\mathcal{L}[p_i(x)x^{\nu_j}] = \mathcal{L}[x^j r_i(x)] = h_i\delta_{ij}, \quad i = 0,1,...,N-1, \ j = 0,1,...,i. \tag{2.2}$$

The proof of Lemma 2.1.1 goes the same as OPS (see proof in [6]). Using Lemma 2.1.1, we can prove the following three-term recurrence relation for ϵ-BLP.

Proposition 2.1.1 ([11])**.** There exist constants a_i, b_i and c_i such that

$$p_{i+1}(x) + b_ip_i(x) + (1-\epsilon_{i-1})a_{i-1}p_{i-1}(x) = x(p_i(x) - \epsilon_{i-1}c_ip_{i-1}(x)), \tag{2.3}$$

for $i = 0,1,...,N-1$, where $p_{-1}(x) = 0$.

A determinant expression of the ϵ-BLP can be obtained in the same way as the OPS, which is used for the determinant expression of the dependent variable of the nd-eToda orbits.

Proposition 2.1.2. The pair of monic ϵ-biorthogonal Laurent polynomial sequences $\{p_i(x)\}_{i=0}^{N}$ and $\{r_i(x)\}_{i=0}^{N-1}$ have the determinant expressions

$$p_i(x) = \frac{1}{\Delta_i} \begin{vmatrix} \mu_{\nu_0} & \mu_{\nu_0+1} & \cdots & \mu_{\nu_0+i} \\ \mu_{\nu_1} & \mu_{\nu_1+1} & \cdots & \mu_{\nu_1+i} \\ \vdots & \vdots & & \vdots \\ \mu_{\nu_{i-1}} & \mu_{\nu_{i-1}+1} & \cdots & \mu_{\nu_{i-1}+i} \\ 1 & x & \cdots & x^i \end{vmatrix}, \tag{2.4}$$

$$r_i(x) = \frac{1}{\Delta_i} \begin{vmatrix} \mu_{\nu_0} & \mu_{\nu_0+1} & \cdots & \mu_{\nu_0+i-1} & x^{\nu_0} \\ \mu_{\nu_1} & \mu_{\nu_1+1} & \cdots & \mu_{\nu_1+i-1} & x^{\nu_1} \\ \vdots & \vdots & & \vdots & \\ \mu_{\nu_i} & \mu_{\nu_i+1} & \cdots & \mu_{\nu_i+i-1} & x^{\nu_i} \end{vmatrix}, \tag{2.5}$$

where μ_l is the moment of $\mathcal{L}$ defined by

$$\mu_l := \mathcal{L}[x^l], \quad l \in \mathbb{Z}$$

and Δ_l is

$$\Delta_0 := 1, \quad \Delta_l := |\mu_{\nu_i+j}|_{i,j=0}^{l-1}, \quad l = 0, 1, ..., N,$$

and h_i in (2.1) is given by

$$h_i = \frac{\Delta_{i+1}}{\Delta_i}.$$

Proof. Let us express $p_i(x)$ as

$$p_i(x) = x^i + \sum_{l=0}^{i-1} c_{i,l} x^l, \quad c_{i,l} \in \mathbb{C}.$$

Then the biorthogonal relation (2.2) gives

$$\begin{pmatrix} \mu_{\nu_0} & \mu_{\nu_0+1} & \cdots & \mu_{\nu_0+i} \\ \mu_{\nu_1} & \mu_{\nu_1+1} & \cdots & \mu_{\nu_1+i} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{\nu_i} & \mu_{\nu_i+1} & \cdots & \mu_{\nu_i+i} \end{pmatrix} \begin{pmatrix} c_{i,0} \\ c_{i,1} \\ \vdots \\ c_{i,i-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h_i \end{pmatrix}.$$

Using Cramer's rule, we have $1 = \Delta_i h_i / \Delta_{i+1}$ and (2.4). The same argument applies to (2.5). □

For later discussion, we rewrite the determinantal expression of $p_k(x)$ in terms of Hankel determinants $\tau_k^{(l)} := |\mu_{i+j+l}|_{i,j=0}^{k-1}$. Let $\eta_k := \min_{0\le l\le k} \nu_l$. Then we have

$$\Delta_i = (-1)^{J_i}\tau_i^{(\eta_{i-1})},$$

where $J_k := \sum_{l=0}^{k-1} l\epsilon_l$. Then, we can rewrite the determinantal expression of p_k as

$$p_k(x) = \frac{1}{\tau_k^{(\eta_{k-1})}} \begin{vmatrix} \mu_{\eta_{k-1}} & \mu_{\eta_{k-1}+1} & \cdots & \mu_{\eta_{k-1}+k} \\ \mu_{\eta_{k-1}+1} & \mu_{\eta_{k-1}+2} & \cdots & \mu_{\eta_{k-1}+k+1} \\ \vdots & \vdots & & \vdots \\ \mu_{\eta_{k-1}+k-1} & \mu_{\eta_{k-1}+k} & \cdots & \mu_{\eta_{k-1}+2k-1} \\ 1 & x & \cdots & x^k \end{vmatrix}. \tag{2.6}$$

We use the expression (2.6) instead of (2.4) to discuss the positivity of solutions of the nd-eToda orbits, which we derive later. Next, we give an analogue of the Gauss quadrature formula for finite ϵ-BLP.

Proposition 2.1.3. Suppose $\{p_i(x)\}_{i=0}^N$ and $\{r_i(x)\}_{i=0}^{N-1}$ are a pair of finite monic ϵ-biorthogonal Laurent polynomial sequences with respect to a functional $\mathcal{L}$, and $p_N(x)$ has N distinct zeros $x_0, x_1, ..., x_{N-1}$, all of which are nonzero. Then, $\mathcal{L}$ has the expression

$$\mathcal{L}[\pi(x)] = \sum_{r=0}^{N-1} w_r \pi(x_r), \quad \forall \pi(x) \in \mathbb{C}[x, x^{-1}] \tag{2.7}$$

for some $w_0, w_1, ..., w_{N-1} \in \mathbb{C}$. Conversely, if a functional $\mathcal{L}$ is of the form (2.7), the polynomial $p_N(x)$ of corresponding ϵ-BLP sequences has N distinct zeros $x_0, x_1, ..., x_{N-1}$.

Proof. The converse statement can be directly proved by substituting the expression of moments

$$\mu_l = \mathcal{L}[x^l] = \sum_{r=0}^{N-1} w_r x_r^l, \quad l \in \mathbb{Z}$$

into (2.6). Let $L(x)$ be a polynomial of the form

$$L(x) = \sum_{r=0}^{N-1} \frac{\pi(x_r)p_N(x)}{p_N'(x)\big|_{x=x_r}(x-x_r)}.$$

We see that $\pi(x) - L(x) = \psi(x)p_N(x)$ for some Laurent polynomial $\psi(x)$. Then, we obtain

$$\begin{aligned} \mathcal{L}[\pi(x)] &= \mathcal{L}[L(x) + \psi(x)p_N(z)] \\ &= \mathcal{L}[L(x)] \\ &= \sum_{r=0}^{N-1} \mathcal{L}\left[\frac{p_N(x)}{p_N'(x)\big|_{x=x_r}(x-x_r)}\right]\pi(x_r). \end{aligned}$$

Therefore, we set $w_r := \mathcal{L}[\frac{p_N(x)}{p'_N(x)\big|_{x=x_r}(x-x_r)}]$ to conclude the proof. □

2.1.2 Spectral transformations

Let $\mathcal{L}^{(0,0)}\colon \mathbb{C}[x,x^{-1}] \to \mathbb{C}$ be a functional. We introduce discrete parameters $t, k \in \mathbb{Z}_{\geq 0}$ into $\mathcal{L}^{(0,0)}$ as

$$\mathcal{L}^{(k+1,t)}[\cdot] = \mathcal{L}^{(k,t)}[x\,\cdot], \tag{2.8}$$
$$\mathcal{L}^{(k,t+1)}[\cdot] = \mathcal{L}^{(k,t)}[(x+s^{(t)})\,\cdot], \tag{2.9}$$

for a given $s^{(t)} \in \mathbb{C}$. A similar argument can be found in [49], which enables us to derive a BBS with a carrier capacity [48, 70] by ultradiscretization. Let us define the moment of $\mathcal{L}^{(k,t)}$ as

$$\mu_l^{(k,t)} := \mathcal{L}^{(k,t)}[x^l], \quad l \in \mathbb{Z}.$$

From (2.8) and (2.9), the moment satisfies the following linear relations:

$$\mu_l^{(k+1,t)} = \mu_{l+1}^{(k,t)},$$
$$\mu_l^{(k,t+1)} = \mu_{l+1}^{(k,t)} + s^{(t)}\mu_l^{(k,t)}.$$

Assume there exists a pair of monic ϵ-biorthogonal Laurent polynomial sequences $\{p_i^{(k,t)}(x)\}_{i=0}^N$ and $\{r_i^{(k,t)}(x)\}_{i=0}^{N-1}$ for all $k, t \in \mathbb{Z}_{\geq 0}$. We define $\Delta_l^{(k,t)} := |\mu_{\nu_i+j}^{(k,t)}|_{i,j=0}^{l-1}$ and $\tau_l^{(k,t)} = |\mu_{i+j}^{(k,t)}|_{i,j=0}^{l-1}$. Then $h_i^{(k,t)}$ can be written as

$$h_i^{(k,t)} = \Delta_{i+1}^{(k,t)}/\Delta_i^{(k,t)} = (-1)^{i\epsilon_i}\tau_{i+1}^{(k+\eta_i,t)}/\tau_i^{(k+\eta_{i-1},t)}. \tag{2.10}$$

By Jacobi's determinant identity (see [30] for a detailed illustration), we find that $\tau_i^{(k,t)}$ satisfy the following bilinear equations:

$$\tau_{i+1}^{(k+2,t)}\tau_{i+1}^{(k,t)} - (\tau_{i+1}^{(k+1,t)})^2 = \tau_i^{(k+2,t)}\tau_{i+2}^{(k,t)}, \tag{2.11}$$
$$\tau_i^{(k+1,t+1)}\tau_i^{(k,t)} - \tau_i^{(k+1,t)}\tau_i^{(k,t+1)} = \tau_{i-1}^{(k+1,t+1)}\tau_{i+1}^{(k,t)}. \tag{2.12}$$

Using (2.11), (2.12), and biorthogonal relation (2.2), we obtain the following contiguous relations of ϵ-biorthogonal Laurent polynomials.

Proposition 2.1.4. There exist constants $\widetilde{q}_i^{(k,t)}, \widetilde{e}_i^{(k,t)}, q_i^{(k,t)}, e_i^{(k,t)} \in \mathbb{C}$ such that

$$(x+s^{(t)})p_i^{(k,t+1)}(x) = p_{i+1}^{(k,t)}(x) + \widetilde{q}_i^{(k,t)}p_i^{(k,t)}(x), \tag{2.13}$$
$$p_{i+1}^{(k,t+1)}(x) + (1-\epsilon_i)\widetilde{e}_i^{(k,t)}p_i^{(k,t+1)}(x) = p_{i+1}^{(k,t)}(x) - \epsilon_i\widetilde{e}_i^{(k,t)}p_i^{(k-1,t+1)}(x), \tag{2.14}$$
$$xp_i^{(k+1,t)}(x) = p_{i+1}^{(k,t)}(x) + q_i^{(k,t)}p_i^{(k,t)}(x), \tag{2.15}$$
$$p_{i+1}^{(k+1,t)}(x) + (1-\epsilon_i)e_i^{(k,t)}p_i^{(k+1,t)}(x) = p_{i+1}^{(k,t)}(x) - \epsilon_i e_i^{(k,t)}p_i^{(k,t)}(x), \tag{2.16}$$

hold. Furthermore, the constants have the following determinantal expressions:

$$\widetilde{q}_i^{(k,t)} = \frac{\tau_{i+1}^{(k+\eta_i,t+1)}\tau_i^{(k+\eta_{i-1},t)}}{\tau_i^{(k+\eta_{i-1},t+1)}\tau_{i+1}^{(k+\eta_i,t)}}, \qquad \widetilde{e}_i^{(k,t)} = \frac{\tau_{i+2}^{(k+\eta_i,t)}\tau_i^{(k+\eta_i,t+1)}}{\tau_{i+1}^{(k+\eta_i,t)}\tau_{i+1}^{(k+\eta_i,t+1)}}, \tag{2.17}$$

$$q_i^{(k,t)} = \frac{\tau_i^{(k+\eta_{i-1},t)}\tau_{i+1}^{(k+\eta_i+1,t)}}{\tau_i^{(k+\eta_{i-1}+1,t)}\tau_{i+1}^{(k+\eta_i,t)}}, \qquad e_i^{(k,t)} = \frac{\tau_{i+2}^{(k+\eta_i,t)}\tau_i^{(k+\eta_i+1,t)}}{\tau_{i+1}^{(k+\eta_i+1,t)}\tau_{i+1}^{(k+\eta_i,t)}}. \tag{2.18}$$

Proof. First, we will prove (2.13). The assertion is clear when $i = 0$; thus, we assume $i > 0$. We can write the polynomial $(x + s^{(t)})p_i^{(k,t+1)}(x)$ using a linear combination of polynomials $p_0^{(k,t)}(x), p_1^{(k,t)}(x), ..., p_{i+1}^{(k,t)}(x)$ as

$$(x + s^{(t)})p_i^{(k,t+1)}(x) = \sum_{l=0}^{i+1} c_l p_l^{(k,t)}(x), \quad c_l \in \mathbb{C}. \tag{2.19}$$

Multiplying both sides of (2.19) by $r_j^{(k,t)}(x)$ and using (2.9), we obtain

$$\mathcal{L}^{(k,t+1)}[p_i^{(k,t+1)}(x)r_j^{(k,t)}(x)] = \sum_{l=0}^{i+1} c_l \mathcal{L}^{(k,t)}[p_l^{(k,t)}(x)r_j^{(k,t)}(x)]. \tag{2.20}$$

When $j = 0$, the only remaining term on the right-hand side of (2.20) is c_0, while the left-hand side is 0. Thus, $c_0 = 0$. Continuing the argument successively for $j = 1, 2, ..., i-1$, we obtain $c_0 = c_1 = \cdots = c_{i-1} = 0$. We define $\widetilde{q}_i^{(k,t)} = c_i$ and conclude the proof of (2.13). The proof of (2.15) is performed similarly (2.13).

Next, we will prove (2.16). Suppose $\epsilon_i = 0$. We write $p_{i+1}^{(k,t)}$ as

$$p_{i+1}^{(k,t)}(x) = \sum_{l=0}^{i+1} d_l p_l^{(k+1,t)}(x), \quad d_l \in \mathbb{C}. \tag{2.21}$$

Multiplying both sides of (2.21) by $r_j^{(k,t)}(x)$ and using (2.8), we obtain

$$\mathcal{L}^{(k,t)}[xp_{i+1}^{(k,t)}(x)r_j^{(k,t)}(x)] = \sum_{l=0}^{i+1} d_l \mathcal{L}^{(k+1,t)}[p_l^{(k+1,t)}(x)r_j^{(k,t)}(x)].$$

For $j = 0, ..., i-1$, we can write $xr_j^{(k,t)}(x)$ as a linear combination of $x^{\nu_0}, x^{\nu_1}, ..., x^{\nu_{j+1}}$ since $\epsilon_i = 0$. Therefore, from the same argument as before, it follows that $d_0 = d_1 = \cdots = d_{i-1} = 0$. The case of $\epsilon_i = 1$ can be shown in the same way.

Finally, we will prove (2.14). When $\epsilon_i = 0$, we apply the same proof as before. Thus we suppose $\epsilon_i = 1$. We can write $p_{i+1}^{(k,t+1)}(x)$ as

$$p_{i+1}^{(k,t+1)}(x) = p_{i+1}^{(k,t)}(x) + \sum_{l=0}^{i} \gamma_l p_l^{(k-1,t+1)}(x), \quad \gamma_l \in \mathbb{C}. \qquad (2.22)$$

Multiplying both side of (2.22) by $r_j^{(k-1,t+1)}(x)$ and taking functional $\mathcal{L}^{(k-1,t+1)}$, we have

$$\mathcal{L}^{(k,t+1)}[x^{-1}p_{i+1}^{(k,t+1)}(x)r_j^{(k-1,t+1)}(x)] = \mathcal{L}^{(k,t)}[(1+s^{(t)}x^{-1})p_{i+1}^{(k,t)}(x)r_j^{(k-1,t+1)}]$$

$$+ \sum_{l=0}^{i} \gamma_l \mathcal{L}^{(k-1,t+1)}[p_l^{(k-1,t+1)}(x)r_j^{(k-1,t+1)}(x)].$$

Here we used relations (2.8) and (2.9). Since $\epsilon_i = 1$, the polynomial $x^{-1}r_j^{(k-1,t+1)}(x)$ can be written as a linear combination of $x^{\nu_0}, x^{\nu_1}, ..., x^{\nu_{j+1}}$ when $j = 0, 1, ..., i-1$. Therefore we have $\gamma_0 = \gamma_1 = \cdots = \gamma_{i-1} = 0$.

For the determinantal expression of $\widetilde{q}_i^{(k,t)}$, we first set $j = i$ in (2.20). Then, we have $h_i^{(k,t+1)} = \widetilde{q}_i^{(k,t)} h_i^{(k,t)}$, and we obtain the determinantal expression for $\widetilde{q}_i^{(k,t)}$ using (2.10). Determinantal expressions for other variables can be obtained similarly way using bilinear relations (2.11) and (2.12). This concludes the proof. □

Proposition 2.1.4 is a key result of Chapter 2 as it enables us to derive nd-eToda orbits in the suitable form for an application to box-ball systems. Combining (2.15) and (2.16), we can express the coefficients of the three-term recurrence relation (2.3) in terms of the variables $q_i^{(k,t)}$ and $e_i^{(k,t)}$.

Let us rewrite (2.13)–(2.16) in matrix-vector form. We introduce matrices

$$R^{(k,t)} := \sum_{i=1}^{N} q_{i-1}^{(k,t)} E_{i,i} + \sum_{i=1}^{N-1} E_{i,i+1},$$

$$\widetilde{R}^{(k,t)} := \sum_{i=1}^{N} \widetilde{q}_{i-1}^{(k,t)} E_{i,i} + \sum_{i=1}^{N-1} E_{i,i+1},$$

$$(L_1^{(k,t)})^{-1} := I_N + \sum_{i=1}^{N-1} -\epsilon_{i-1} e_{i-1}^{(k,t)} E_{i+1,i},$$

$$L_2^{(k,t)} := I_N + \sum_{i=1}^{N-1} (1-\epsilon_{i-1}) e_{i-1}^{(k,t)} E_{i+1,i},$$

$$(\widetilde{L}_1^{(k,t)})^{-1} := I_N + \sum_{i=1}^{N-1} -\epsilon_{i-1} \widetilde{e}_{i-1}^{(k,t)} E_{i+1,i},$$

$$\widetilde{L}_2^{(k,t)} := I_N + \sum_{i=1}^{N-1} (1-\epsilon_{i-1}) \widetilde{e}_{i-1}^{(k,t)} E_{i+1,i},$$

where $E_{i,j} = (\delta_{ik}\delta_{jl})_{k,l=1}^N$ denotes a matrix element and I_N is an $N \times N$ identity matrix. Then the relations (2.13)–(2.16) can be written in matrix form as

$$(x + s^{(t)})\boldsymbol{p}^{(k,t+1)} = \widetilde{R}^{(k,t)}\boldsymbol{p}^{(k,t)} + \boldsymbol{p}_N^{(k,t)}, \tag{2.23}$$

$$\widetilde{L}_2^{(k,t)}\boldsymbol{p}^{(k,t+1)} = \boldsymbol{p}^{(k,t)} + ((\widetilde{L}_1^{(k,t)})^{-1} - I_N)\boldsymbol{p}^{(k-1,t+1)}, \tag{2.24}$$

$$x\boldsymbol{p}^{(k+1,t)} = R^{(k,t)}\boldsymbol{p}^{(k,t)} + \boldsymbol{p}_N^{(k,t)}, \tag{2.25}$$

$$L_2^{(k,t)}\boldsymbol{p}^{(k+1,t)} = (L_1^{(k,t)})^{-1}\boldsymbol{p}^{(k,t)}, \tag{2.26}$$

where $\boldsymbol{p}^{(k,t)} = (p_0^{(k,t)}, p_1^{(k,t)}, ..., p_{N-1}^{(k,t)})^T$ and $\boldsymbol{p}_N^{(k,t)} = (0, ..., 0, p_N^{(k,t)})^T$. Let $\widetilde{L}^{(k,t)} = \widetilde{L}_1^{(k,t)}\widetilde{L}_2^{(k,t)}$, $L^{(k,t)} = L_1^{(k,t)}L_2^{(k,t)}$ and $\overline{L}^{(k,t)} = \widetilde{L}_2^{(k,t)} + (I_N - (\widetilde{L}_1^{(k,t)})^{-1})L^{(k-1,t+1)}$. Then, we have

$$(x + s^{(t)})\boldsymbol{p}^{(k,t+1)} = (\overline{L}^{(k,t+1)}\widetilde{R}^{(k,t+1)} + (s^{(t)} - s^{(t+1)})I_N)\boldsymbol{p}^{(k,t+1)} + \overline{L}^{(k,t+1)}\boldsymbol{p}_N^{(k,t+1)}, \tag{2.27}$$

$$(x + s^{(t)})\boldsymbol{p}^{(k,t+1)} = \widetilde{R}^{(k,t)}\overline{L}^{(k,t)}\boldsymbol{p}^{(k,t+1)} + \boldsymbol{p}_N^{(k,t)}, \tag{2.28}$$

$$x\boldsymbol{p}^{(k+1,t)} = R^{(k,t)}L^{(k,t)}\boldsymbol{p}^{(k+1,t)} + \boldsymbol{p}_N^{(k,t)}, \tag{2.29}$$

$$x\boldsymbol{p}^{(k+1,t)} = L^{(k+1,t)}R^{(k+1,t)}\boldsymbol{p}^{(k+1,t)} + L^{(k+1,t)}\boldsymbol{p}_N^{(k+1,t)}. \tag{2.30}$$

By Proposition 2.1.3, the polynomial $p_N^{(k,t)}(x)$ does not depend on parameters k and t as it only depends on the support of a linear functional $\mathcal{L}^{(k,t)}$. Thus, we see that $\overline{L}^{(k,t+1)}\boldsymbol{p}_N^{(k,t+1)} = \boldsymbol{p}_N^{(k,t)}$ and $L^{(k+1,t)}\boldsymbol{p}_N^{(k+1,t)} = \boldsymbol{p}_N^{(k,t)}$.

Therefore, the compatibility conditions of (2.27)–(2.30) are written in the form

$$R^{(k,t)}L^{(k,t)} = \overline{L}^{(k+1,t)}\widetilde{R}^{(k+1,t)} - s^{(t)}I_N, \tag{2.31}$$
$$\widetilde{R}^{(k+1,t)}\overline{L}^{(k+1,t)} = R^{(k,t+1)}L^{(k,t+1)} + s^{(t)}I_N. \tag{2.32}$$

From (2.31) and (2.32), we have

$$\begin{aligned} R^{(k,t)}L^{(k,t)}\overline{L}^{(k+1,t)} &= \overline{L}^{(k+1,t)}(\widetilde{R}^{(k+1,t)}\overline{L}^{(k+1,t)} - s^{(t)}I_N) \\ &= \overline{L}^{(k+1,t)}R^{(k,t+1)}L^{(k,t+1)}. \end{aligned}$$

Since $\overline{L}^{(k+1,t)}$ is a regular matrix, the two matrices $R^{(k,t)}L^{(k,t)}$ and $R^{(k,t+1)}L^{(k,t+1)}$ are similar. From (2.31) and (2.32), we have

$$\widetilde{q}_i^{(k+1,t)} = q_i^{(k,t)} + e_i^{(k,t)} - \widetilde{e}_{i-1}^{(k+1,t)} + s^{(t)}, \tag{2.33}$$
$$\widetilde{e}_i^{(k+1,t)}(\widetilde{q}_i^{(k+1,t)} + \epsilon_i e_{i-1}^{(k,t+1)}) = e_i^{(k,t)}(q_{i+1}^{(k,t)} + \epsilon_{i+1}e_{i+1}^{(k,t)}) \tag{2.34}$$
$$q_i^{(k,t+1)} = \widetilde{q}_i^{(k+1,t)} + \widetilde{e}_i^{(k+1,t)} - e_i^{(k,t+1)} - s^{(t)}, \tag{2.35}$$
$$e_{i-1}^{(k,t+1)}(\epsilon_i(e_i^{(k,t+1)} - \widetilde{e}_i^{(k+1,t)}) + q_i^{(k,t+1)}) = \widetilde{q}_i^{(k+1,t)}\widetilde{e}_{i-1}^{(k+1,t)}. \tag{2.36}$$

We set

$$\begin{aligned} q_i^{(t)} &:= q_i^{(0,t+1)}, \quad e_i^{(t)} := e_i^{(0,t+1)}, \\ \widetilde{q}_i^{(t)} &:= \widetilde{q}_i^{(1,t)}, \quad \widetilde{e}_i^{(t)} := \widetilde{e}_i^{(1,t)}. \end{aligned}$$

Then, from (2.33)–(2.36), these variables satisfy the following equations:

$$\widetilde{q}_i^{(t+1)} + \widetilde{e}_{i-1}^{(t+1)} = q_i^{(t)} + e_i^{(t)} + s^{(t+1)} \tag{2.37}$$
$$\widetilde{e}_i^{(t+1)}(\widetilde{q}_i^{(t+1)} + \epsilon_i e_{i-1}^{(t+1)}) = e_i^{(t)}(q_{i+1}^{(t)} + \epsilon_{i+1}e_{i+1}^{(t)}) \tag{2.38}$$
$$q_i^{(t+1)} + e_i^{(t+1)} = \widetilde{q}_i^{(t+1)} + \widetilde{e}_i^{(t+1)} - s^{(t+1)}, \tag{2.39}$$
$$e_{i-1}^{(t+1)}(\epsilon_i(e_i^{(t+1)} - \widetilde{e}_i^{(t+1)}) + q_i^{(t+1)}) = \widetilde{q}_i^{(t+1)}\widetilde{e}_{i-1}^{(t+1)} \tag{2.40}$$

We call the system (2.37)–(2.40) the *nonautonomous discrete elementary Toda orbits* (nd-eToda orbits). Note that recurrences (2.37)–(2.40) cannot be written in explicit form without restricting parameters ϵ_i to 0 or 1.

In order to write the evolutions of (2.37)–(2.40) in subtraction-free form, we introduce auxiliary variables $d_i^{(t)}$ and $f_i^{(t)}$. First, we set $d_0^{(t+1)}$ and $f_i^{(t)}$ as

$$\begin{aligned} d_0^{(t+1)} &= q_0^{(t)} + s^{(t+1)}, \\ f_i^{(t)} &= q_i^{(t)} + e_i^{(t)}. \end{aligned}$$

Then, we compute $d_1^{(t+1)}, d_2^{(t+1)}, ..., d_{N-1}^{(t+1)}$ and the rest of dependent variables as

$$\widetilde{q}_i^{(t+1)} = \begin{cases} d_i^{(t+1)} + e_i^{(t)} & \epsilon_i = 0, \\ \dfrac{f_i^{(t)} d_{i-1}^{(t+1)}}{\widetilde{q}_{i-1}^{(t+1)}} + s^{(t+1)} & \epsilon_i = 1, \end{cases} \tag{2.41}$$

$$d_i^{(t+1)} = \frac{q_i^{(t)}}{\widetilde{q}_{i-1}^{(t+1)}} d_{i-1}^{(t+1)} + s^{(t+1)}, \tag{2.42}$$

$$\widetilde{e}_i^{(t+1)} = \left(\frac{d_{i-1}^{(t+1)}}{\widetilde{q}_{i-1}^{(t+1)}} \right)^{\epsilon_i} \frac{(q_{i+1}^{(t)} + \epsilon_{i+1} e_{i+1}^{(t)})}{\widetilde{q}_i^{(t+1)}} e_i^{(t)}, \tag{2.43}$$

$$q_i^{(t+1)} = \frac{d_{i-1}^{(t+1)} \widetilde{q}_i^{(t+1)}}{\widetilde{q}_{i-1}^{(t+1)} d_i^{(t+1)}} q_i^{(t)}, \tag{2.44}$$

$$e_i^{(t+1)} = \begin{cases} \dfrac{\widetilde{q}_i^{(t+1)} d_{i+1}^{(t+1)}}{d_i^{(t+1)} q_{i+1}^{(t)}} \widetilde{e}_i^{(t+1)} & \epsilon_{i+1} = 0, \\ \dfrac{\widetilde{q}_i^{(t+1)} \widetilde{q}_{i+1}^{(t+1)}}{d_i^{(t+1)} f_{i+1}^{(t)}} \widetilde{e}_i^{(t+1)} & \epsilon_{i+1} = 1, \end{cases} \tag{2.45}$$

for $i = 0, 1, ..., N-1$, where $d_{-1}^{(t)}, \widetilde{q}_{-1}^{(t)} \equiv 1$ and $e_{N-1}^{(t)} \equiv 0$. We see that equations (2.41)–(2.45) are in subtraction-free form, so we can perform ultradiscretization. Various integrable systems can be obtained as a special case from the system (2.41)–(2.45) by restricting parameters ϵ or $s^{(t)}$. For example, when $\epsilon = (0, 0, ..., 0)$, the system reduces to the *modified nonautonomous discrete Toda lattice* introduced in [48], whose ultradiscretization is a time evolution equation of the BBS with a carrier capacity [70]. In this case, the nonautonomous parameter in the ultradiscretized system is interpreted as the capacity of the carrier. One can also obtain the *discrete relativistic Toda lattice* [67, 75] from (2.41)–(2.45) by restricting $\epsilon = (1, 1, ..., 1, 1)$. In Section 3, we give the rule of a BBS obtained from ultradiscretization of equations (2.41)–(2.45) for general $\epsilon \in \{0, 1\}^N$.

2.1.3 Solutions of nonautonomous discrete elementary Toda orbits

Assume there exists a pair of monic ϵ-BLP sequences $\{p_i^{(k,t)}(x)\}_{i=0}^N$ and $\{r_i^{(k,t)}(x)\}_{i=0}^{N-1}$ for all $k, t \in \mathbb{Z}_{\geq 0}$ and $p_N^{(0,0)}(x)$ has N distinct zeros $x_0, x_1, ..., x_{N-1} \in \mathbb{R}$. Suppose $0 < |x_0| < |x_1| < \cdots < |x_{N-1}|$ and $0 < |x_0 + s^{(t)}| < |x_1 + s^{(t)}| < \cdots < |x_{N-1} + s^{(t)}|$ for all $t \in \mathbb{Z}_{\geq 0}$. Then from Proposition 2.1.3, $\mathcal{L}^{(0,0)}$ has the form

$$\mathcal{L}^{(0,0)}[\pi(x)] = \sum_{i=0}^{N-1} w_i \pi(x_i)$$

for some constants $w_i \in \mathbb{C}$. Then the moments $\mu_n^{(k,t)}$ can be expressed in terms of x_i and w_i as

$$\mu_n^{(k,t)} = \sum_{i=0}^{N-1} w_i x_i^{k+n} \prod_{l=0}^{t-1} (x_i + s^{(l)}). \tag{2.46}$$

We use the expression (2.46) to write moment determinants $\tau_i^{(k,t)}$ in terms of parameters $w_i, x_i, s^{(t)}$. We set $\alpha_i^{(t)} = w_i \prod_{l=0}^{t-1}(x_i + s^{(l)})$ and, by using the

Cauchy-Binet formula, we expand $\tau_i^{(k,t)}$ as

$$\tau_i^{(k,t)} = \left| \begin{pmatrix} \alpha_0^{(t)} & \alpha_1^{(t)} & \cdots & \alpha_{N-1}^{(t)} \\ x_0\alpha_0^{(t)} & x_1\alpha_1^{(t)} & \cdots & x_{N-1}\alpha_{N-1}^{(t)} \\ \vdots & \vdots & & \vdots \\ x_0^{i-1}\alpha_0^{(t)} & x_1^{i-1}\alpha_1^{(t)} & \cdots & x_{N-1}^{i-1}\alpha_{N-1}^{(t)} \end{pmatrix} \begin{pmatrix} x_0^k & x_0^{k+1} & \cdots & x_0^{k+i-1} \\ x_1^k & x_1^{k+1} & \cdots & x_1^{k+i-1} \\ \vdots & \vdots & & \vdots \\ x_{N-1}^k & x_{N-1}^{k+1} & \cdots & x_{N-1}^{k+i-1} \end{pmatrix} \right|$$

$$= \sum_{0\le r_0<r_1<\cdots<r_{i-1}\le N-1} \begin{vmatrix} \alpha_{r_0}^{(t)} & \alpha_{r_1}^{(t)} & \cdots & \alpha_{r_{i-1}}^{(t)} \\ x_{r_0}\alpha_{r_0}^{(t)} & x_{r_1}\alpha_{r_1}^{(t)} & \cdots & x_{r_{i-1}}\alpha_{r_{i-1}}^{(t)} \\ \vdots & \vdots & & \vdots \\ x_{r_0}^{i-1}\alpha_{r_0}^{(t)} & x_{r_1}^{i-1}\alpha_{r_1}^{(t)} & \cdots & x_{r_{i-1}}^{i-1}\alpha_{r_{i-1}}^{(t)} \end{vmatrix} \begin{vmatrix} x_{r_0}^k & x_{r_0}^{k+1} & \cdots & x_{r_0}^{k+i-1} \\ x_{r_1}^k & x_{r_1}^{k+1} & \cdots & x_{r_1}^{k+i-1} \\ \vdots & \vdots & & \vdots \\ x_{r_{i-1}}^k & x_{r_{i-1}}^{k+1} & \cdots & x_{r_{i-1}}^{k+i-1} \end{vmatrix}$$

$$= \sum_{0\le r_0<r_1<\cdots<r_{i-1}\le N-1} (x_{r_0}x_{r_1}\cdots x_{r_{i-1}})^k$$
$$\begin{vmatrix} \alpha_{r_0}^{(t)} & \alpha_{r_1}^{(t)} & \cdots & \alpha_{r_{i-1}}^{(t)} \\ x_{r_0}\alpha_{r_0}^{(t)} & x_{r_1}\alpha_{r_1}^{(t)} & \cdots & x_{r_{i-1}}\alpha_{r_{i-1}}^{(t)} \\ \vdots & \vdots & & \vdots \\ x_{r_0}^{i-1}\alpha_{r_0}^{(t)} & x_{r_1}^{i-1}\alpha_{r_1}^{(t)} & \cdots & x_{r_{i-1}}^{i-1}\alpha_{r_{i-1}}^{(t)} \end{vmatrix} \begin{vmatrix} 1 & x_{r_0} & \cdots & x_{r_0}^{i-1} \\ 1 & x_{r_1} & \cdots & x_{r_1}^{i-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_{r_{i-1}} & \cdots & x_{r_{i-1}}^{i-1} \end{vmatrix}$$

$$= \sum_{0\le r_0<r_1<\cdots<r_{i-1}\le N-1} (x_{r_0}x_{r_1}\cdots x_{r_{i-1}})^k$$
$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_{r_0} & x_{r_1} & \cdots & x_{r_{n-1}} \\ \vdots & \vdots & & \vdots \\ x_{r_0}^{i-1} & x_{r_1}^{i-1} & & x_{r_{i-1}}^{i-1} \end{vmatrix} \begin{vmatrix} \alpha_{r_0}^{(t)} & & & \\ & \alpha_{r_1}^{(t)} & & \\ & & \ddots & \\ & & & \alpha_{r_{i-1}}^{(t)} \end{vmatrix} \begin{vmatrix} 1 & x_{r_0} & \cdots & x_{r_0}^{i-1} \\ 1 & x_{r_1} & \cdots & x_{r_1}^{i-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_{r_{i-1}} & \cdots & x_{r_{i-1}}^{i-1} \end{vmatrix}$$

$$= \sum_{0\le r_0<r_1<\cdots<r_{i-1}\le N-1} \prod_{j=0}^{i-1}\left(x_{r_j}^k w_{r_j} \prod_{l=0}^{t-1}(x_{r_j}+s^{(l)}) \right) \prod_{0\le j_1<j_2\le i-1} (x_{r_{j_1}}-x_{r_{j_2}})^2. \tag{2.47}$$

By substituting (2.47) into (2.17) and (2.18), we obtain a family of particular solutions to the system (2.37)–(2.40). From (2.47) and the assumption $0 < |x_0| < |x_1| < \cdots < |x_{N-1}|$ and $0 < |x_0 + s^{(t)}| < |x_1 + s^{(t)}| < \cdots < |x_{N-1} + s^{(t)}|$, we can compute the limits of dependent variables $q_i^{(k,t)}, e_i^{(k,t)}, \widetilde{q}_i^{(k,t)}, \widetilde{e}_i^{(k,t)}$ at $t \to +\infty$ as

$$\lim_{t\to+\infty} q_i^{(k,t)} = x_{N-1-i}, \qquad \lim_{t\to+\infty} e_i^{(k,t)} = 0, \tag{2.48}$$

$$\lim_{t\to+\infty} (\widetilde{q}_i^{(k,t)} - s^{(t+1)}) = x_{N-1-i}, \qquad \lim_{t\to+\infty} \widetilde{e}_i^{(k,t)} = 0. \tag{2.49}$$

This enables us to compute the eigenvalues of the matrix $R^{(k,t)}L^{(k,t)}$ by the nd-eToda orbits since each step of a time evolution $R^{(k,t)}L^{(k,t)} \to R^{(k,t+1)}L^{(k,t+1)}$ of the nd-eToda orbits is a similar transformation and $R^{(k,t)}L^{(k,t)}$ tends to an upper triangular matrix as $t \to +\infty$ by (2.48).

2.1.4 Conserved quantities of nonautonomous discrete elementary Toda orbits

Proposition 2.1.3 implies that the polynomial $p_N^{(k,t)}(x)$ does not depend on parameters k and t since $p_N^{(k,t)}(x)$ only depends on the support of a linear functional $\mathcal{L}^{(k,t)}$. Therefore, coefficients of the polynomial $p_N^{(k,t)}(x)$ give conserved quantities of the nd-eToda orbits. We write coefficients of $p_N^{(k,t)}(\mathrm{x})$ as

$$p_N^{(k,t)}(x) = x^N - C_{N,1}x^{N-1} + \cdots + (-1)^{N-1}C_{N,N-1}x + (-1)^N C_{N,N}.$$

We can express conserved quantities $C_{N,1}, ..., C_{N,N}$ in terms of dependent variables $q_i^{(k,t)}$ and $e_i^{(k,t)}$ of the nd-eToda orbits.

Proposition 2.1.5. Suppose $\epsilon = (\epsilon_0, \epsilon_1, ..., \epsilon_{N-1})$. Then $C_{N,i}$ are constructed as follows: First, arrange $q_n^{(k,t)}, e_n^{(k,t)}$ as

$$q_0 \quad e_0 \quad q_1 \quad \cdots \quad q_{N-2} \quad e_{N-2} \quad q_{N-1}. \tag{2.50}$$

Here, we omit the superscripts of $q_i^{(k,t)}, e_i^{(k,t)}$. We consider a collection $\mathcal{V}$ of dependent variables q_n, e_n satisfying the following conditions:

- $|\mathcal{V}| = i$ and variables in a collection $\mathcal{V}$ are not adjacent in the sequence (2.50), and
- For any i for which $\epsilon_i = 1$, a collection $\mathcal{V}$ does not include both e_i and e_{i-1}.

Then, $C_{N,i} = \sum_{\mathcal{V}} \prod_{v \in \mathcal{V}} v$, where the sum is taken over all $\mathcal{V}$'s satisfying the above conditions.

Proposition 2.1.5 can be proved by induction on the size of the system N using a three-term recurrence formula of Proposition 2.1.1.

2.2 Ultradiscrete elementary Toda orbits and ϵ-BBS

In this section, we ultradiscretize the nd-eToda orbits (2.41)–(2.45) and propose a family of BBSs that we call ϵ-BBS.

2.2.1 Ultradiscretization

Let $\varepsilon > 0$. We consider the transformations of variables $q_i^{(t)} = e^{-Q_n^{(t)}/\varepsilon}, \widetilde{q}_i^{(t)} = e^{-\widetilde{Q}_n^{(t)}/\varepsilon}, e_i^{(t)} = e^{-\widetilde{E}_n^{(t)}/\varepsilon}, f_i^{(t)} = e^{-F_n^{(t)}/\varepsilon}, d_i^{(t)} = e^{-D_n^{(t)}/\varepsilon}, s^{(t)} = e^{-S^{(t)}/\varepsilon}$. By applying these to (2.41)–(2.45) and using

$$\lim_{\varepsilon \to +0} -\varepsilon \log(e^{-A/\varepsilon} + e^{-B/\varepsilon}) = \min(A, B),$$

we obtain the following piecewise-linear system:

$$F_i^{(t)} = \min(Q_i^{(t)}, E_i^{(t)}), \tag{2.51}$$

$$D_i^{(t+1)} = \min(D_{i-1}^{(t+1)} + Q_i^{(t)} - \widetilde{Q}_{i-1}^{(t+1)}, S^{(t+1)}), \tag{2.52}$$

$$\widetilde{Q}_i^{(t+1)} = \begin{cases} \min(D_i^{(t+1)}, E_i^{(t)}) & \epsilon_i = 0, \\ \min(F_i^{(t)} + D_{i-1}^{(t+1)} - \widetilde{Q}_{i-1}^{(t+1)}, S^{(t+1)}) & \epsilon_i = 1, \end{cases} \tag{2.53}$$

$$\widetilde{E}_i^{(t+1)} = E_i^{(t)} + \min(Q_{i+1}^{(t)}, \mathcal{E}_{i+1} + E_{i+1}^{(t)}) + \epsilon_i(D_{i-1}^{(t+1)} - \widetilde{Q}_{i-1}^{(t+1)}) - \widetilde{Q}_i^{(t+1)} \tag{2.54}$$

$$Q_i^{(t+1)} = Q_i^{(t)} + D_{i-1}^{(t+1)} + \widetilde{Q}_i^{(t+1)} - D_i^{(t+1)} - \widetilde{Q}_{i-1}^{(t+1)}, \tag{2.55}$$

$$E_i^{(t+1)} = \begin{cases} \widetilde{E}_i^{(t+1)} + \widetilde{Q}_i^{(t+1)} - D_i^{(t+1)} + D_{i+1}^{(t+1)} - Q_{i+1}^{(t)} & \epsilon_{i+1} = 0, \\ \widetilde{E}_i^{(t+1)} + \widetilde{Q}_i^{(t+1)} - D_i^{(t+1)} + \widetilde{Q}_{i+1}^{(t+1)} - F_{i+1}^{(t)} & \epsilon_{i+1} = 1 \end{cases} \tag{2.56}$$

for $i = 0, 1, ..., N-1$, where $\mathcal{E}_i$ is defined as

$$\mathcal{E}_i = \begin{cases} +\infty & \epsilon_i = 0, \\ 0 & \epsilon_i = 1 \end{cases}$$

and $\widetilde{Q}_{-1}^{(t+1)}, D_{-1}^{(t+1)} \equiv 0$ and $E_{N-1}^{(t+1)} \equiv +\infty$. We call (2.51)–(2.56) the *nonautonomous ultradiscrete elementary Toda orbits* (nu-eToda orbits). To interpret the nu-eToda orbits as a cellular automaton, we need the following proposition.

Proposition 2.2.1. Suppose $S^{(t+1)} > 0$. Then, the nu-eToda orbits (2.51)–(2.56) define the map

$$\begin{array}{ccc} (\mathbf{Z}_{>\mathbf{0}})^{2N-1} & \longrightarrow & (\mathbf{Z}_{>\mathbf{0}})^{2N-1} \\ \cup & & \cup \\ (Q_0^{(t)}, ..., Q_{N-1}^{(t)}, E_0^{(t)}, ..., E_{N-2}^{(t)}) & \longmapsto & (Q_0^{(t+1)}, ..., Q_{N-1}^{(t+1)}, E_0^{(t+1)}, ..., E_{N-2}^{(t+1)}) \end{array}$$

where $\mathbf{Z}_{>0}$ denotes the set of positive integers.

Proof. Let $A_i^{(t+1)} = D_i^{(t+1)} - \widetilde{Q}_i^{(t+1)}$ for $i = 0, 1, ..., N-1$. First, we prove $A_i^{(t+1)} \geq 0$ for $i = 0, 1, ..., N-1$ by induction on the index i. We immediately see that $A_0^{(t+1)} = D_0^{(t+1)} - \widetilde{Q}_0^{(t+1)} \geq 0$. Suppose we have proved $A_{i-1}^{(t+1)} \geq 0$. When $\epsilon_i = 0$, we have $A_i^{(t+1)} = D_i^{(t+1)} - \widetilde{Q}_i^{(t+1)} = D_i^{(t+1)} - \min(D_i^{(t+1)}, E_i^{(t)}) \geq 0$. When $\epsilon_i = 1$, we have

$$D_i^{(t+1)} - \widetilde{Q}_i^{(t+1)} = \min(A_{i-1}^{(t+1)} + Q_i^{(t)}, S^{(t+1)}) - \min(A_{i-1}^{(t+1)} + F_i^{(t)}, S^{(t+1)}).$$

Thus, we obtain $A_i^{(t+1)} \geq 0$, because $Q_i^{(t)} - F_i^{(t)} = Q_i^{(t)} - \min(Q_i^{(t)}, E_i^{(t)}) \geq 0$. Combining $A_{i-1}^{(t+1)} \geq 0$ and $S^{(t+1)} > 0$, we also see that $D_i^{(t+1)} > 0$ and $\widetilde{Q}_i^{(t+1)} > 0$.

Next, we prove $Q_i^{(t+1)} > 0$. From (2.55), we have

$$Q_i^{(t+1)} = Q_i^{(t)} + A_{i-1}^{(t+1)} - \min(Q_i^{(t)} + A_{i-1}^{(t+1)}, S^{(t+1)}) + \widetilde{Q}_i^{(t+1)}.$$

Thus $Q_i^{(t+1)} > 0$.

Finally, we prove $E_{i-1}^{(t+1)} > 0$. When $\epsilon_i = 0$, we have

$$E_{i-1}^{(t+1)} = E_{i-1}^{(t)} + \epsilon_{i-1} A_{i-2}^{(t+1)} + D_i^{(t+1)} - \min(A_{i-2}^{(t+1)} + Q_{i-1}^{(t)}, S^{(t+1)}).$$

If $D_{i-1}^{(t+1)} + Q_i^{(t)} - \widetilde{Q}_{i-1}^{(t+1)} \geq S^{(t+1)}$, then $D_i^{(t+1)} = S^{(t+1)}$. Hence, we have $E_{i-1}^{(t+1)} > 0$. If $D_{i-1}^{(t+1)} + Q_i^{(t)} - \widetilde{Q}_{i-1}^{(t+1)} \leq S^{(t+1)}$, then

$$\begin{aligned} E_{i-1}^{(t+1)} &= E_{i-1}^{(t)} + \epsilon_{i-1} A_{i-2}^{(t+1)} + Q_i^{(t)} - \widetilde{Q}_{i-1}^{(t+1)}, \\ &= \begin{cases} E_{i-1}^{(t)} - \min(D_{i-1}^{(t+1)}, E_{i-1}^{(t)}) + Q_i^{(t)} & \epsilon_{i-1} = 0, \\ E_{i-1}^{(t)} + A_{i-2}^{(t+1)} - \min(F_{i-1}^{(t)} + A_{i-2}^{(t+1)}, S^{(t+1)}) + Q_i^{(t)} & \epsilon_{i-1} = 1. \end{cases} \end{aligned}$$

As $E_{i-1}^{(t)} + A_{i-2}^{(t+1)} \geq F_{i-1}^{(t)} + A_{i-2}^{(t+1)} \geq \min(F_{i-1}^{(t)} + A_{i-2}^{(t+1)}, S^{(t+1)})$, we obtain $E_{i-1}^{(t+1)} > 0$. The case $\epsilon_i = 1$ can be shown in the same way. □

2.2.2 ϵ-BBS

We can interpret the nu-eToda orbits (2.51)–(2.56) as a time evolution of a cellular automaton, which we call the ϵ-BBS. We define $E_{-1}^{(t)}$ as

$$E_{-1}^{(t+1)} = E_{-1}^{(t)} + \min(Q_0^{(t)}, \mathcal{E}_0 + E_0^{(t)}, S^{(t)}), \quad E_{-1}^{(0)} = 0.$$

Let us associate the dependent variables of the nu-eToda orbits $Q_i^{(t)} > 0, E_i^{(t)} > 0$ with a 01-sequence $u^{(t)} = (u_0^{(t)}, u_1^{(t)}, ...) \in \{0,1\}^{\mathbb{N}}$. We call '0' an empty box and '1' a ball. Let $k_i^{(t)},\ i = -1, 0, ..., 2N$, be

$$k_{2i}^{(t)} = \sum_{l=-1}^{i-1} E_l^{(t)} + \sum_{l=0}^{i-1} Q_l^{(t)}, \quad i = 0, ..., N-1,$$

$$k_{2i-1}^{(t)} = \sum_{l=-1}^{i-2} E_l^{(t)} + \sum_{l=0}^{i-1} Q_l^{(t)}, \quad i = 1, ..., N,$$

$k_{-1}^{(t)} = 0$ and $k_{2N}^{(t)} = +\infty$. Then we define $u_n^{(t)}$ as

$$u_n^{(t)} = \begin{cases} 0 & n \in [k_{2i-1}^{(t)}, k_{2i}^{(t)}) \text{ for some } i = 0, 1, ..., N, \\ 1 & \text{otherwise.} \end{cases}$$

The integers $k_i^{(t)}$ can be rephrased as follows:

- $k_{2i}^{(t)}$: the position of the first ball in the $(i+1)$-st block of consecutive balls at time t and
- $k_{2i+1}^{(t)}$: the position of the first empty box in the $(i+1)$-st block of empty boxes at time t.

The above identification is the same as the following:

- $Q_i^{(t)}$: the length of the $(i+1)$-st block of consecutive balls at time t and
- $E_i^{(t)}$: the number of empty boxes between the $(i+1)$-st and the $(i+2)$-nd blocks of consecutive balls at time t.

Example 2.2.1. For $\epsilon = (1, 0, 1, 0, 1, 0)$, $S^{(t)} \equiv 3$ and initial values

$$\begin{aligned} &Q_0^{(0)} = 3, \quad Q_1^{(0)} = 2, \quad Q_2^{(0)} = 2, \quad Q_3^{(0)} = 3, \quad Q_4^{(0)} = 3, \quad Q_5^{(0)} = 2, \\ &E_{-1}^{(0)} = 1, \quad E_0^{(0)} = 1, \quad E_1^{(0)} = 2, \quad E_2^{(0)} = 2, \quad E_3^{(0)} = 2, \quad E_4^{(0)} = 1, \end{aligned}$$

the corresponding sequence $u^{(0)}$ and the first few time evolutions of $u^{(0)}$ by the ϵ-BBS can be written:

$$\begin{aligned} u^{(0)} &: 0111011001100111001110110000000000000\cdots \\ u^{(1)} &: 0010111001110001101110011100000000000\cdots \\ u^{(2)} &: 0001000110011110001001111000111000000\cdots \\ u^{(3)} &: 0000100001100011110100001111000111000\cdots \\ u^{(4)} &: 0000010000011001101110000111100011100\cdots \end{aligned}$$

□

Before explaining the rule of the time evolution $T : u^{(0)} \to u^{(1)}$ of the ϵ-BBS, we prepare some notations. In what follows, we omit the superscript indicating a time variable as $u := u^{(0)}$, $S := S^{(0)}$, $Q_i := Q_i^{(0)}$ and $E_i := E_i^{(0)}$. Let $I = \{i \in \{0, ..., N-1\} \mid \epsilon_i = 1\}$ and denote elements of I as $I = \{i_0, i_1, ..., i_{K-1} \mid i_0 < i_1 < \cdots < i_{K-1}\}$, where $K = |I|$. We set $i_{-1} = 0$. Let $m_j = k_{2i_j}$ for $j = 0, 1, ..., K-1$. We decompose the sequence u into subsequences $v^{(j)} = (u_{m_{j-1}}, u_{m_{j-1}+1}, ..., u_{m_j-1})$ for $j = 0, 1, ..., K$, where

$m_{-1} = 0$ and $m_K = +\infty$. In the 01-sequence $u^{(0)}$ of Example 2.2.1, we have $i_0 = 0, i_1 = 2$ and $i_2 = 4$. Hence, $m_0 = 1, m_1 = 9$ and $m_2 = 18$ and the decomposition of a sequence $u^{(0)}$ is $v^{(0)} = (0)$, $v^{(1)} = (1,1,1,0,1,1,0,0), v^{(2)} = (1,1,0,0,1,1,1,0,0)$, and $v^{(3)} = (1,1,1,0,1,1,0,0,0,0,...)$.

We explain the rule of the ϵ-BBS in terms of a carrier that moves from left to right. Let $c^{(-1)} = 0$. We construct a map that takes $v^{(j)}$ and $c^{(j-1)}$ as inputs and outputs a 01-sequence $\widetilde{v}^{(j)}$ and a nonnegative integer $c^{(j)}$ for $j = 0, 1, ..., K$.

First, starting with a carrier containing $c^{(j-1)}$ balls, move the carrier from left to right until it reaches the right end of $v^{(j)}$. As the carrier passes each position, perform one of the following:

- When the carrier comes across a ball, load it onto the carrier as long as the number of the balls on the carrier does not exceed $S^{(0)}$.
- When the carrier comes across an empty box and contains no ball, do nothing.
- When the carrier comes across an empty box and contains at least one ball, unload a ball. However, when unloading a ball for the first time in step j, remove $c^{(j-1)}$ balls from the carrier (This procedure is indicated by the double-lined arrow in diagrams (2.57) and (2.58) in Example 2.2.2 below).

Then, we obtain the finite 10-sequence $(v')^{(j)}$ and the carrier contents $c^{(j)}$. Next, we add $c^{(j-1)}$ balls into the first (leftmost) block of balls of $(v')^{(j)}$ and, if $j > 0$, delete $\max(\min(Q_{i_{j-1}}, S - c^{(j-1)}) - E_{i_{j-1}}, 0)$ boxes from the first (leftmost) block of empty boxes of $(v')^{(j)}$. We define $\widetilde{v}^{(j)}$ to be the resulting sequence.

After executing the above procedures for $j = 0, 1, ..., K$, we concatenate sequences $\widetilde{v}^{(0)}, \widetilde{v}^{(1)}, ..., \widetilde{v}^{(K)}$ to obtain $u^{(1)} = \widetilde{v}^{(0)}\widetilde{v}^{(1)} \cdots \widetilde{v}^{(K)}$.

Example 2.2.2. Let us give an example of the above rule for the 01-sequence $u^{(0)}$ in Example 2.2.1. First, we explain the above procedure for $j = 1$ (in the case of $j = 0$, we trivially obtain $\widetilde{v}^{(0)} = (0)$). Let $c = (c_j)_{j=1}^S$ and $\widetilde{c} = (\widetilde{c}_j)_{j=1}^S$ be respectively the states of the carrier before and after it passes through position $v_l^{(1)}$. The following diagram illustrates the changes in the state of the carrier and $v_l^{(1)}$ of the 01-sequence before and after the carrier passes:

$$\begin{array}{ccc} & v_l^{(1)} & \\ (c_0, c_1, \cdots, c_S) & \longrightarrow\!\!\!\!\!\!\!\!\downarrow\;\;\; & (\widetilde{c}_0, \widetilde{c}_1, \cdots, \widetilde{c}_S) \\ & v_l'^{(1)} & \end{array}$$

The diagram below shows how the state of the carrier changes as it moves from u_{m_0} to u_{m_1-1}:

$$\begin{array}{ccccccccccccccccc}
 & 1 & & 1 & & 1 & & 0 & & 1 & & 1 & & 0 & & 0 & \\
000 & \to & 001 & \to & 011 & \to & 111 & \to & 011 & \to & 111 & \to & 111 & \to & 011 & \to & 001 \\
 & 0 & & 0 & & 0 & & 1 & & 0 & & 1 & & 1 & & 1 &
\end{array}$$

Here, we abbreviate the notation of the state of the carrier $(c_0, c_1, ..., c_{S^{(0)}})$ as $c_0c_1 \cdots c_{S^{(0)}}$. After the carrier passes u_{m_1-1}, we obtain the sequence $v'^{(1)} = (0, 0, 0, 1, 0, 1, 1, 1)$ and the carrier contents $c^{(1)} = 1$. Then we delete $\max(\min(3, 3) - 1, 0) = 2$ empty boxes from $v'^{(1)}$ and obtain the resulting sequence $\widetilde{v}^{(1)} = (0, 1, 0, 1, 1, 1)$.

Next, let us consider the cases $j = 2$ and $j = 3$. When $j = 2$, we have following diagram:

$$\begin{array}{ccccccccccccccccccc}
 & 1 & & 1 & & 0 & & 0 & & 1 & & 1 & & 1 & & 0 & & 0 & \\
001 & \to & 011 & \to & 111 & \Rightarrow & 001 & \to & 000 & \to & 001 & \to & 011 & \to & 111 & \to & 011 & \to & 001 \\
 & 0 & & 0 & & 1 & & 1 & & 0 & & 0 & & 0 & & 1 & & 1 &
\end{array} \tag{2.57}$$

Therefore we obtain $(v')^{(2)} = (0, 0, 1, 1, 0, 0, 0, 1, 1)$ and $c^{(2)} = 1$. Because $c^{(1)} = 1$ and $\max(\min(2, 3-1)-2, 0) = 0$, we have $\widetilde{v}^{(2)} = (0, 0, 1, 1, 1, 0, 0, 0, 1, 1)$. When $j = 3$, we have the following diagram:

$$\begin{array}{ccccccccccccccccccc}
 & 1 & & 1 & & 1 & & 0 & & 1 & & 1 & & 0 & & 0 & & 0 & \\
001 & \to & 011 & \to & 111 & \to & 111 & \Rightarrow & 001 & \to & 011 & \to & 111 & \to & 011 & \to & 001 & \to & 000 \\
 & 0 & & 0 & & 1 & & 1 & & 0 & & 0 & & 1 & & 1 & & 1 &
\end{array} \tag{2.58}$$

Therefore we obtain $(v')^{(3)} = (0, 0, 1, 1, 0, 0, 1, 1, 1, 0, ...)$ and $c^{(3)} = 0$. Because $c^{(2)} = 1$ and $\max(\min(3, 3-1)-1, 0) = 1$, we have $\widetilde{v}^{(3)} = (0, 1, 1, 1, 0, 0, 1, 1, 1, 0, ...)$. Finally, concatenating sequences

$$\widetilde{v}^{(0)} = (0), \quad \widetilde{v}^{(1)} = (0, 1, 0, 1, 1, 1), \quad \widetilde{v}^{(2)} = (0, 0, 1, 1, 1, 0, 0, 0, 1, 1),$$
$$\widetilde{v}^{(3)} = (0, 1, 1, 1, 0, 0, 1, 1, 1, 0, ...),$$

we obtain $u^{(1)} =$ "0010111001110001101110011100..". □

When $\epsilon = (1, 1, ..., 1, 0)$ and $S^{(t)} = +\infty$ for all t, we obtain a BBS associated with the ultradiscrete relativistic Toda lattice as a special case of the ϵ-BBS. The ultradiscretization of the discrete relativistic Toda lattice and its interpretation in terms of cellular automata is also given in [34], but here we give an alternative interpretation using balls and boxes.

Let us explain a rule of the ϵ-BBS $T\colon u^{(0)} \to u^{(1)}$ with an example when $\epsilon = (1, 1, ..., 1, 0)$ and $S^{(t)} = +\infty$ for all t. Let $Q_i, E_{i-1},\ i = 0, 1, ..., N-1$ be positive integers and u be a 01-sequence constructed from Q_i and E_i by the above identification. We set $E_{N-1} = +\infty$. We use the sequence $u^{(0)} =$ "01111000111001110000.." for illustration.

First, move $\min(Q_i, E_i)$ balls in the $(i+1)$-st block of balls to the nearest empty boxes on the right for $i = 0, 1, ..., N-1$:

$$01111000111001110000... \to 00001\overline{111}001\overline{11}0001\overline{11}0...$$

Here, the overlines indicate the balls that were moved in this step. Then, insert the remaining balls in the $(i+1)$-st block of balls in front of the $(i+2)$-nd block of balls, and delete the empty boxes created by moving the balls in this step.

$$00001\overline{111}001\overline{11}0001\overline{11}0... \to 0000\overline{111}00\overline{11}\underline{1}000\overline{111}\underline{1}0...$$

We underline the balls that were inserted in this step. The resulting sequence "00001110011100011110..." is $u^{(1)}$.

Example 2.2.3. We give an example of the first few time evolution of the 01-sequence $u^{(0)} =$ "0111011001100111001110000000000000000...".

$$
\begin{aligned}
u^{(0)} &: \ 0111011001100111001110000000000000000\cdots \\
u^{(1)} &: \ 0010011110011001100011110000000000000\cdots \\
u^{(2)} &: \ 0001000110011110011000001111000000000\cdots \\
u^{(3)} &: \ 0000100001100110011110000000111100000\cdots \\
u^{(4)} &: \ 0000010000011001100001111000000011110\cdots
\end{aligned}
$$

□

Chapter 3

Generalization of the ϵ-BBS and the Schensted insertion algorithm

In the previous chapter, we define the ϵ-BBS which generalizes Takahashi-Satsuma's BBS and the BBS associated with the relativistic Toda lattice. In this chapter, we consider a multi-color extension of the ϵ-BBS, i.e., an extension of the ϵ-BBS to many kinds of balls (we call it the hungry ϵ-BBS). Then, we show that some of the conserved quantities of the hungry ϵ-BBS can be obtained by a combinatorial algorithm called the Schensted insertion.

3.1 Preliminaries

3.1.1 Notations and basic properties of Young tableaux

In this section, we provide some basic facts about Young tableaux and notations used throughout the paper. Let $[m] = \{1, 2, ..., m\}$ be a set of m letters equipped with the usual ordering on integers. A finite sequence $v = v_1 v_2 \cdots v_l$ using the letters $[m]$ is called a *word*. A word v is *non-decreasing* if $v_k \leq v_{k+1}$ for each $1 \leq k < l$. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$, $(k \leq m)$ be a Young diagram. A *semi-standard tableau* (SST) of shape λ is obtained by assigning a letter in $[m]$ to each box of λ so as to satisfy the followings:

- In each row, the letters are non-decreasing from left to right, and
- In each column, the letters are strictly increasing from top to bottom.

The *Schensted insertion* of a letter $i \in [m]$ into an SST T is defined as follows:

1. Set $k := 1$ and $x := i$.

2. Find the leftmost letter in the k-th row of T that is greater than x. If no such letter is found, then append x to the right end of the k-th row of T, and then terminate. If such a letter j is found, then replace it with x and set $x := j,\ k := k + 1$, then go back to the start of Step 2.

The SST obtained by inserting a letter i into an SST T is denoted by $T \leftarrow i$. The following is an example of the Schensted insertion:

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 & 4 \\ \hline 3 & 4 & 6 \\ \cline{1-3} \end{array} \longleftarrow 1 \quad = \quad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 3 & 4 \\ \hline 3 & 3 & 6 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}.$$

Let $w = w_1 w_2 \cdots w_l$ be a word. We define $T \leftarrow w$ as an SST obtained by $((((T \leftarrow x_1) \leftarrow x_2) \leftarrow \cdots) \leftarrow x_l)$. Let T be an SST. Denote each row of T by $r_1, r_2, ..., r_k$. The *row word* $w_{\mathrm{row}}(T)$ of T is defined as $w_{\mathrm{row}}(T) = r_k r_{k-1} \cdots r_2 r_1$. A product of two SSTs T and T' is defined as $T \cdot T' := T \leftarrow w_{\mathrm{row}}(T')$. We will use the following property of this product later.

Proposition 3.1.1. The product defined above is associative, that is, for any SST T_1, T_2 and T_3, we have $(T_1 \cdot T_2) \cdot T_3 = T_1 \cdot (T_2 \cdot T_3)$.

See for example [17] for a proof of Proposition 3.1.1. Two different words can give rise to the same SST by the Schensted insertion:

$$\varnothing \leftarrow 132 \;=\; \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}, \qquad \varnothing \leftarrow 312 \;=\; \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}.$$

Definition 3.1.1. The following two transformations (and their inverse) for three consecutive letters in a word are called *elementary Knuth transformation*:

$$yzx \mapsto yxz, \quad x < y \leq z, \tag{3.1}$$

$$xzy \mapsto zxy, \quad x \leq y < z. \tag{3.2}$$

When two words w and w' are transformed into each other by a finite sequence of elementary Knuth transformations, we say that w and w' are *Knuth equivalent*.

Proposition 3.1.2. The two words w and w' are Knuth equivalent if and only if $\varnothing \leftarrow w$ and $\varnothing \leftarrow w'$ give the same SST.

Let $u_1, u_2, ..., u_l \in [m]$ and $v_1, v_2, ..., v_l \in [n]$. An array consisting of two rows

$$\omega = \begin{pmatrix} u_1 & u_2 & \cdots & u_l \\ v_1 & v_2 & \cdots & v_l \end{pmatrix}$$

is called a *biword* if the following conditions are satisfied:

- $u_1 \le u_2 \le \cdots \le u_l$, and
- For all $1 \le i < l$, $v_i \le v_{i+1}$ if $u_i = u_{i+1}$.

We write the first row of ω as u and the second row of ω as v. The *P-symbol* $P(\omega)$ is the SST obtained by $\varnothing \leftarrow v$. The *Q-symbol* $Q(\omega)$ is the SST of the same shape as $P(\omega)$ constructed as follows: The tableau $Q(\omega)$ is obtained by adding the box with letter u_k to the place where v_k is inserted in $P(\omega)$. The correspondence $\omega \mapsto (P(\omega), Q(\omega))$ is called the *RSK correspondence.*

Proposition 3.1.3. The RSK correspondence

$$\omega \mapsto (P(\omega), Q(\omega))$$

gives a bijection between biwords and tuples of SSTs of the same shape.

In this paper we consider only the P-symbol.

3.1.2 Piecewise-linear formula for Schensted insertion

The Schensted insertion can be written in the form of a piecewise-linear equation. It first appeared in [42] and further investigated in [56] where its relation to the discrete Toda lattice was pointed out. Let v and w be non-decreasing words consisting of letters in $[m]$. Let $x = (x_1, x_2, ..., x_m)$ and $a = (a_1, a_2, ..., a_m)$ be *coordinate representations* of v and w, respectively, i.e., x_i (resp. a_i) is the number of i's in the word v (resp. w). The SST obtained by a Schensted insertion $w \leftarrow v$ consists of two rows, with its first row denoted by w' and the second row by v'. Let $y = (y_1, y_2, ..., y_m)$ and $b = (b_1, b_2, ..., b_m)$ be coordinate representations of w' and v', respectively. There exists a piecewise-linear formula to compute y and b from x and a. First, we define $\eta_i,\ i = 1, 2, ..., m$, as

$$\eta_1 = y_1, \quad \eta_j = \eta_{j-1} + y_j,\ j = 2, 3, ..., m.$$

Then, $\eta_j,\ j = 1, 2, ..., m$, is expressed in terms of x and a as

$$\eta_j = \max_{1 \le k \le j} \{x_1 + x_2 + \cdots + x_k + a_k + a_{k+1} + \cdots + a_j\}, \tag{3.3}$$

by which we can recover y_i's and x_i's because $x_i + a_i = y_i + b_i$ holds for all $i = 1, 2, ..., m$. The proof of formula (3.3) is given in [56].

3.1.3 Box-ball system

In this section, we review the work by Fukuda [15], in which the P-symbol of the RSK correspondence was shown to be a conserved quantity of the generalized BBS. First, we present the definition of the generalized BBS. Let $u = (u_i)_{i=0}^{\infty}$ be a semi-infinite sequence of letters in $[m] \cup \{e\}$ and $u_i = e$

for all but finitely many $i \in \mathbb{Z}_{\geq 0}$. We regard the letter e to be greater than any element of $[m]$. Let Ω be the set of all such sequences. The letter $i \in [m]$ represents a 'ball of color i' and e represents an 'empty box'. We define the map $T\colon \Omega \to \Omega$ as follows:

1. Set $i := 1$.
2. Move the leftmost ball of color i to the nearest empty box on the right. Repeat this procedure for the other balls of color i until all of them have been moved once.
3. If $i = m$, then terminate. Otherwise, set $i := i + 1$ and go to Step 2.

For an initial sequence $u^{(0)} \in \Omega$, the time evolution of the generalized BBS is defined as $u^{(t+1)} = T(u^{(t)})$. The figure below shows an example of the time evolutions of the generalized BBS (here, the letter e is replaced by an underscore symbol '_'.).

```
t = 0 : _132__12_413________________
t = 1 : ___312__1_2413______________
t = 2 : ____3_12_1__2413____________
t = 3 : _____3__121___2413__________
t = 4 : ______3___211___2413________
t = 5 : _______3___2_11___2413______
t = 6 : ________3___2__11___2413____
```

For $u \in \Omega$, let $f(u)$ denote a finite subsequence of u obtained by removing all e's. For $u^{(0)}$ in the above example, we have $f(u^{(0)}) = 13212413$.

Proposition 3.1.4 ([15])**.** For any $u \in \Omega$, the following two SST coincide:

$$\varnothing \leftarrow f(u), \quad \varnothing \leftarrow f(T(u)).$$

That is, the P-symbol of the RSK correspondence gives a conserved quantity of the generalized BBS.

Although the proof of Proposition 3.1.4 can be found in [15], to make this paper self-contained, we write the proof here. The idea is to realize the time evolution of the generalized BBS by successive applications of Knuth transformations. First, let us rewrite the above time evolution rule into a *carrier rule*. We use above sequences $u^{(0)} = e132ee12e413ee\cdots$ and $u^{(1)} = T(u^{(0)}) = eee312ee1e2413ee\cdots$ as examples. First, let N be the number of indices i such that $u_i \neq 0$. Let $C^{(0)} = \underbrace{ee\cdots e}_{N}$ be a finite sequence consisting of N copies of e's, which is called a *carrier*. We consider a sequence

$v^{(0)} = (v_i^{(0)})_{i=0}^{\infty}$ obtained by concatenating the carrier $C^{(0)}$ to the left end of $u^{(0)}$:

$$v^{(0)} = \underline{eeeeeeee}e132ee12e413ee\cdots$$

Here, the part of the sequence $v^{(0)}$ corresponding to the carrier is underlined. We also use the notation $v^{(0)} = C^{(0)}e132ee12e413ee\cdots$ for the same sequence. We define $C^{(1)}$ and the sequence $v^{(1)} = v_0^{(1)}C^{(1)}v_{N+1}^{(1)}v_{N+2}^{(1)}\cdots$ from $v^{(0)}$ as follows: Let x be the letter $v_N^{(0)}$. In other words, x is the letter to the right of the carrier $C^{(0)}$. Then, we perform procedure (A) or (B) depending on whether there is a letter in $C^{(0)}$ greater than x.

(A) If there is a letter in $C^{(0)}$ that is greater than x, we name the leftmost one y. Then, we

  (i) replace y with x, then

  (ii) remove y from the carrier and concatenate it to the left of the carrier.

(B) If there is no letter in $C^{(0)}$ that is greater than x, we first append x to the rightmost position of the carrier. Then, we remove the leftmost letter y of the carrier and concatenate it to the left of the carrier.

This results in a new carrier $C^{(1)}$ and a sequence $v^{(1)} = v_0^{(1)}C^{(1)}v_{N+1}^{(1)}v_{N+2}^{(1)}\cdots$. We continue the above procedure until there are no balls to the right of the carrier and the carrier consists of N copies of e. An example is given below.

$v^{(0)} = \underline{eeeeeeee}\ e132ee12e413eeeeeee\cdots$

$v^{(1)} = e\ \underline{eeeeeeee}\ 132ee12e413eeeeeee\cdots$

$v^{(2)} = ee\ \underline{1eeeeeee}\ 32ee12e413eeeeeee\cdots$

$v^{(3)} = eee\ \underline{13eeeeee}\ 2ee12e413eeeeeee\cdots$

$v^{(4)} = eee3\ \underline{12eeeeee}\ ee12e413eeeeeee\cdots$

$v^{(5)} = eee31\ \underline{2eeeeeee}\ e12e413eeeeeee\cdots$

$v^{(6)} = eee312\ \underline{eeeeeeee}\ 12e413eeeeeee\cdots$

$v^{(7)} = eee312e\ \underline{1eeeeeee}\ 2e413eeeeeee\cdots$

$v^{(8)} = eee312ee\ \underline{12eeeeee}\ e413eeeeeee\cdots$

$v^{(9)} = eee312ee1\ \underline{2eeeeeee}\ 413eeeeeee\cdots$

$v^{(10)} = eee312ee1e\ \underline{24eeeeee}\ 13eeeeeee\cdots$

$v^{(11)} = eee312ee1e2\ \underline{14eeeeee}\ 3eeeeeee\cdots$

$v^{(12)} = eee312ee1e24\ \underline{13eeeeee}\ eeeeeee\cdots$

$v^{(13)} = eee312ee1e241\ \underline{3eeeeeee}\ eeeeee\cdots$

$v^{(14)} = eee312ee1e2413\ \underline{eeeeeeee}\ eeeee\cdots$

Finally, we delete the carrier $\underline{eeeeeeee}$ from $v^{(0)}$ and $v^{(14)}$ to obtain $u^{(0)} \mapsto u^{(1)}$. The equivalence between this procedure and the time evolution rule defined above is shown in [15]. Procedures (A) and (B) above can be realized by a sequence of elementary Knuth transformations. For (B) the assertion is obvious as it does not change sequence itself. Let us consider the case of (A). Let $x_1x_2\cdots x_l y z_1 z_2 \ldots z_{k-1}z_k$ be the state of the carrier. Step (i) of the procedure (A) can be achieved via a sequence of elementary Knuth transformations (3.1) ($bca \mapsto bac$ for $a < b \leq c$) as follows:

$$\begin{aligned}
&x_1\ x_2\ \cdots\ x_{l-1}\ x_l\ y\ z_1\ z_2\ \cdots\ z_{k-2}\ z_{k-1}\ \overline{z_k\ x}\\
&x_1\ x_2\ \cdots\ x_{l-1}\ x_l\ y\ z_1\ z_2\ \cdots\ z_{k-2}\ \overline{z_{k-1}\ x}\ z_k\\
&\qquad\vdots\\
&x_1\ x_2\ \cdots\ x_{l-1}\ x_l\ y\ \overline{z_1\ x}\ z_2\ \cdots\ z_{k-2}\ z_{k-1}\ z_k\\
&x_1\ x_2\ \cdots\ x_{l-1}\ x_l\ y\ x\ z_1\ z_2\ \cdots\ z_{k-2}\ z_{k-1}\ z_k
\end{aligned}$$

Here, the letters whose positions are to be exchanged are marked as $\overline{a\ b}$. Next, we evacuate y from the carrier (Step (ii) of the procedure (A)) by a sequence of elementary Knuth transformations (3.2) ($acb \mapsto cab$ for $a \leq b < c$) as follows:

$$\begin{aligned}
&x_1\ x_2\ \cdots\ x_{l-1}\ \overline{x_l\ y}\ x\ z_1\ z_2\ \cdots\ z_{k-1}\ z_k\\
&x_1\ x_2\ \cdots\ \overline{x_{l-1}\ y}\ x_l\ x\ z_1\ z_2\ \cdots\ z_{k-1}\ z_k\\
&\qquad\vdots\\
&\overline{x_1\ y}\ x_2\ \cdots\ x_{l-1}\ x_l\ x\ z_1\ z_2\ \cdots\ z_{k-1}\ z_k\\
&y\ x_1\ x_2\ \cdots\ x_{l-1}\ x_l\ x\ z_1\ z_2\ \cdots\ z_{k-1}\ z_k
\end{aligned}$$

Thus, two words $w =$ "$eeeeeeeee132ee12e413$" and $w' =$ "$eee312ee1e2413eeeeeeee$" result in the same SST according to Proposition 3.1.2. Since removing the letter e from a word does not affect the position of letters in $[m]$ after the Schensted insertion, we have $\varnothing \leftarrow f(w) = \varnothing \leftarrow f(w')$. This concludes the proof.

3.2 Hungry ϵ-BBS

In Chapter 2, a family of box-ball systems called the ϵ-BBS was introduced. The ϵ-BBS contains Takahashi-Satsuma's BBS [68] as a special case. In this section, we first derive the discrete hungry elementary Toda orbits (d-heToda orbits) and then obtain the hungry ϵ-BBS by ultradiscretizing them. The d-heToda orbits contains a positive integer parameter M. When M is set to 1, then the d-heToda orbits specializes to the discrete elementary Toda orbits. Thus, the hungry ϵ-BBS is a multi-color extension of the ϵ-BBS in the sense that the parameter M corresponds to the number of colors of balls. Furthermore, we present birational transformations between different orbits of the discrete hungry elementary Toda orbits.

3.2.1 (ϵ, M)-biorthogonal Laurent polynomials and discrete hungry elementary Toda orbits

Let us first generalize the ϵ-BLP defined in Section 2.1. Let N and M be positive integers and define $\epsilon = (\epsilon_0, \epsilon_1, ..., \epsilon_{N-1}) \in \{0,1\}^N$. Define $\{p_i(x)\}_{i=0}^{N}$

and $\{r_i(x)\}_{i=0}^{N-1}$ as in Section 2.1.1. Then we define a biorthogonal relation as follows:

Definition 3.2.1. We call the (Laurent) polynomial sequences $\{p_i(x)\}_{i=0}^{N}$ and $\{r_i(x)\}_{i=0}^{N-1}$ the pair of (finite) monic (ϵ, M)-*biorthogonal Laurent polynomial sequences* ((ϵ, M)-BLP) with respect to $\mathcal{L}$ if

$$\begin{aligned}
&\mathcal{L}[p_i(x)r_j(x^M)] = h_i\delta_{ij}, \quad h_i \neq 0, \quad i,j = 0,1,...,N-1, \\
&\mathcal{L}[p_N(x)\pi(x^M)] = 0, \quad {}^\forall\pi(x) \in \mathbb{C}[x, x^{-1}]
\end{aligned}$$

holds.

We introduce a discrete parameter $t \in \mathbb{Z}_{\geq 0}$ to the functional $\mathcal{L}^{(0)} := \mathcal{L}$ as

$$\mathcal{L}^{(t+1)}[\cdot] = \mathcal{L}^{(t)}[x\cdot].$$

Assume there exists a pair of monic (ϵ, M)-BPL $\{p_i^{(t)}(x)\}_{i=0}^{N}$ and $\{r_i^{(t)}(x)\}_{i=0}^{N-1}$ for all $t \in \mathbb{Z}_{\geq 0}$. Then we obtain the following contiguous relations of the (ϵ, M)-BLPs.

Proposition 3.2.1. There exist constants $q_i^{(t)}, e_i^{(t)} \in \mathbb{C}$ such that

$$xp_i^{(t+1)}(x) = p_{i+1}^{(t)}(x) + q_i^{(t)}p_i^{(t)}(x), \tag{3.4}$$

$$p_{i+1}^{(t)}(x) - \epsilon_i e_i^{(t)}p_i^{(t)}(x) = p_{i+1}^{(t+M)}(x) + (1-\epsilon_i)e_i^{(t)}p_i^{(t+M)}(x). \tag{3.5}$$

The proof of Proposition 3.2.1 goes the same as Proposition 2.1.4, thus we omit it.

3.2.2 Discrete hungry elementary Toda orbits

Let N be a positive integer and $\epsilon = (\epsilon_0, \epsilon_1, ..., \epsilon_{N-1}) \in \{0,1\}^N$. We define $R^{(t)}, L_1^{(t)}$ and $L_2^{(t)}$ as

$$\begin{aligned}
R^{(t)} &= \sum_{i=1}^{N} q_{i-1}^{(t)}E_{i,i} + \sum_{i=1}^{N-1} E_{i,i+1}, \\
(L_1^{(t)})^{-1} &= I_N + \sum_{i=1}^{N-1} -\epsilon_{i-1}e_{i-1}^{(t)}E_{i+1,i}, \\
L_2^{(t)} &= I_N + \sum_{i=1}^{N-1}(1-\epsilon_{i-1})e_{i-1}^{(t)}E_{i+1,i}.
\end{aligned}$$

Then the relations (3.4) and (3.5) can be written in the matrix form as

$$\begin{aligned}
x\boldsymbol{p}^{(t+1)} &= R^{(t)}\boldsymbol{p}^{(t)} + \boldsymbol{p}_N^{(t)}, \\
L_2^{(t)}\boldsymbol{p}^{(t+M)} &= (L_1^{(t)})^{-1}\boldsymbol{p}^{(t)}
\end{aligned}$$

where $\boldsymbol{p}^{(t)} = (p_0^{(t)}, p_1^{(t)}, ..., p_{N-1}^{(t)})^T$ and $\boldsymbol{p}_N^{(t)} = (0, ..., 0, p_N^{(t)})^T$. We can prove an assertion similar to Proposition 2.1.3, by which we can see that $P_N^{(t)}$ only depends on the support of the functional $\mathcal{L}^{(t)}$. Therefore, the compatibility conditions of relations (3.4) and (3.5) are written as

$$L_1^{(t+1)} L_2^{(t+1)} R^{(t+M)} = R^{(t)} L_1^{(t)} L_2^{(t)}. \tag{3.6}$$

Equation (3.6) is equivalent to the following system of equations:

$$q_i^{(t+M)} = q_i^{(t)} + e_i^{(t)} - e_{i-1}^{(t+1)}, \quad i = 0, 1, ..., N-1, \tag{3.7}$$

$$e_i^{(t+1)} = \frac{q_{i+1}^{(t)} + \epsilon_{i+1} e_{i+1}^{(t)}}{q_i^{(t+M)} + \epsilon_i e_{i-1}^{(t+1)}}\, e_i^{(t)}, \quad i = 0, 1, ..., N-2, \tag{3.8}$$

where $e_{-1}^{(t)} = 0$ for all t. We regard the system of equations (3.7), (3.8) as a time evolution

$$\begin{aligned}(q_i^{(t)}, q_i^{(t+1)}, ..., q_i^{(t+M-1)})_{i=0}^{N-1}, (e_i^{(t)})_{i=0}^{N-2} \\ \mapsto (q_i^{(t+M)}, q_i^{(t+M+1)}, ..., q_i^{(t+2M-1)})_{i=0}^{N-1}, (e_i^{(t+M)})_{i=0}^{N-2}.\end{aligned} \tag{3.9}$$

We call an orbit of (3.9) through any given initial value an *ϵ-orbit*. Define $X^{(t)}$ as

$$X^{(t)} = L_1^{(t)} L_2^{(t)} R^{(t+M-1)} R^{(t+M-2)} \cdots R^{(t)}.$$

Then we have

$$X^{(t+M)} = (L_1^{(t)} L_2^{(t)})^{-1} X^{(t)} L_1^{(t)} L_2^{(t)}.$$

Thus the characteristic polynomial $\phi(x) = \det(X^{(t)} - xI_N)$ is a conserved quantity of the d-heToda orbits. The system (3.7) and (3.8) can be rewritten in the following *subtraction-free* form:

$$\begin{cases} d_i^{(t+1)} = \dfrac{q_i^{(t)}}{q_{i-1}^{(t+M)}} d_{i-1}^{(t+1)} \\ q_i^{(t+M)} = d_i^{(t+1)} + e_i^{(t)} \end{cases}, \quad (\epsilon_{i-1}, \epsilon_i) = (0, 0) \tag{3.10}$$

$$\begin{cases} d_i^{(t+1)} = q_i^{(t)} + e_i^{(t)} \\ q_i^{(t+M)} = \dfrac{d_i^{(t+1)}}{d_{i-1}^{(t+1)}} q_{i-1}^{(t)} \end{cases}, \quad (\epsilon_{i-1}, \epsilon_i) = (1, 1) \tag{3.11}$$

$$\begin{cases} d_i^{(t+1)} = q_i^{(t)} + e_i^{(t)} \\ q_i^{(t+M)} = \dfrac{d_i^{(t+1)}}{q_{i-1}^{(t+M)}} d_{i-1}^{(t+1)} \end{cases}, \quad (\epsilon_{i-1}, \epsilon_i) = (0, 1) \tag{3.12}$$

$$\begin{cases} d_i^{(t+1)} = \dfrac{q_i^{(t)}}{d_{i-1}^{(t+1)}} q_{i-1}^{(t)} \\ q_i^{(t+M)} = d_i^{(t+1)} + e_i^{(t)} \end{cases}, \quad (\epsilon_{i-1}, \epsilon_i) = (1, 0). \tag{3.13}$$

$$e_i^{(t+1)} = \frac{q_{i+1}^{(t)} + \epsilon_{i+1} e_{i+1}^{(t)}}{q_i^{(t+M)} + \epsilon_i e_{i-1}^{(t+1)}} \, e_i^{(t)}, \tag{3.14}$$

where $q_{-1}^{(t)}, d_{-1}^{(t)} \equiv 1$ and $e_{N-1}^{(t)} \equiv 0$. We note that, since the right-hand sides of equations (3.10)–(3.14) do not contain any subtraction, they can be ultradiscretized. The BBS obtained from the ultradiscretization of (3.10)–(3.14) will be discussed later.

3.2.3 Conserved quantities of discrete hungry elementary Toda orbits

We give an explicit formula of coefficients C_l of the characteristic polynomial $\phi(x) = x^N - C_1 x^{N-1} + \cdots + (-1)^l C_l x^{N-l} + \cdots + (-1)^N C_N$, which are conserved quantities of the d-heToda orbits. For the remainder of this subsection, we omit superscript of $e_i^{(0)}$ as e_i for $i = 0, 1, ..., N-2$. First, we define

$$\widetilde{q}_i := q_i^{(0)} q_i^{(1)} \cdots q_i^{(M-1)},$$

for $i = 0, 1, ..., N-1$. Let $\mathcal{I}$ be the set of all non-empty subsets I of $\{0, 1, ..., N-2\}$ satisfying the following two conditions:

- The set I consists of consecutive integers as $I = \{i, i+1, ..., j\}$ and satisfies $|I| \le M$.
- $\epsilon_k = 1$ for $i + 1 \le k \le j$.

Let $n = M - |I|$. Denote by $\mathcal{J}_{n,i}$ a set of all n-tuple of integers $J = (j_1, j_2, ..., j_n)$ satisfying $i \le j_1 \le j_2 \le \cdots \le j_n \le M - n + i$. For each $I \in \mathcal{I}$ and $J \in \mathcal{J}_{n,i}$, we define $\widetilde{e}_I^{(J)}$ to be

$$\widetilde{e}_I^{(J)} := e_i e_{i+1} \cdots e_j q_{j_1}^{(M-1-j_1+i)} q_{j_2}^{(M-2-j_2+i)} \cdots q_{j_n}^{(M-n-j_n+i)}.$$

where $I = \{i, i+1, ..., j\}$. Let k and l be integers satisfying $0 \le k \le l \le N$. We say that an l-tuple $(i_1, ..., i_k, I_1, ..., I_{l-k})$ for $0 \le i_j \le N-1, I_j \in \mathcal{I}$ is *admissible* if it satisfies the followings:

- If $i < j$, then $x < y$ for all $x \in I_i$ and $y \in I_j$.
- For all $1 \le i < l - k$, there exists an integer a such that $\max I_i < a \le \min I_{i+1}$ and $\epsilon_a = 0$.
- $\bigcup_{1 \le i \le l-k} \{\min I_i, \min I_i + 1, ..., \max I_i + 1\} \cap \{i_1, i_2, ..., i_k\} = \emptyset$.

Denote by $T_{k,l}$ the set of all admissible l-tuples with k integers. Suppose that an admissible l-tuples $(i_1, ..., i_k, I_1, ..., I_{l-k}) \in T_{k,l}$ is given. Let $n_i = |I_i|$ and $m_i = \min I_i$ for $1 \le i \le l-k$ and define $\mathcal{J}_{n_1,n_2,...,n_{l-k}}^{m_1,m_2,...,m_{l-k}} = \mathcal{J}_{n_1,m_1} \times \mathcal{J}_{n_2,m_2} \times \cdots \times \mathcal{J}_{n_{l-k},m_{l-k}}$. We denote the following condition on $(J_1, J_2, ..., J_{l-k}) \in \mathcal{J}_{n_1,n_2,...,n_{l-k}}^{m_1,m_2,...,m_{l-k}}$ by $(*)$:

- For all $1 \le i < l-k$, if $\max I_i = \min I_{i+1} - 1$, then the integer $\min I_{i+1}$ does not appear in $(J_1, J_2, ..., J_{l-k})$ more than or equal to M times — $(*)$.

We conjecture that the coefficients C_l of the characteristic polynomial $\phi(x)$ is expressed by the following formula:

$$C_l = \sum_{k=0}^{l} \sum_{(i_1,...,i_k,I_1,...,I_{l-k}) \in T_{k,l}} \sum_{(J_1,J_2,...,J_{l-k})} \widetilde{q}_{i_1} \cdots \widetilde{q}_{i_k} e_{I_1}^{(J_1)} \cdots e_{I_{l-k}}^{(J_{l-k})},$$

where the sum $\sum_{(J_1,J_2,...,J_{l-k})}$ is taken over all $(J_1, J_2, ..., J_{l-k}) \in \mathcal{J}_{n_1,n_2,...,n_{l-k}}^{m_1,m_2,...,m_{l-k}}$ with $n_i = |I_i|$ and $m_i = \min I_i$ satisfying the condition $(*)$.

3.2.4 Birational transformations of discrete hungry elementary Toda orbits

There is a birational transformation from the ϵ-orbit for given $\epsilon \in \{0,1\}^N$ to the ϵ'-orbit for another parameter $\epsilon' \in \{0,1\}^N$. Such a birational transformation is considered in [9] for the continuous case. In this section, we extend it to the discrete and hungry case. We define $E_i^{(t)} = I_N + e_i^{(t)} E_{i+2,i+1}$ for $i = 0, 1, ..., N-2$. Let $I = \{i_0 < i_1 < \cdots < i_{k-1} \mid \epsilon_{i_j+1} = 0\}$. We denote $E_{i_l}^{(t)} E_{i_l-1}^{(t)} \cdots E_{i_{l-1}+1}^{(t)}$ by $E_{[i_l, i_{l-1}+1]}^{(t)}$ where $i_{-1} = -1$. Then,

$$L_1^{(t)} L_2^{(t)} = E_{[i_0,0]}^{(t)} E_{[i_1,i_0+1]}^{(t)} \cdots E_{[i_{k-1},i_{k-2}+1]}^{(t)}.$$

Example 3.2.1. When $N = 6, \epsilon = (0,1,1,0,1,0)$, we have $i_0 = 2, i_1 = 4$ and

$$L_1^{(t)} L_2^{(t)} = E_2^{(t)} E_1^{(t)} E_0^{(t)} E_4^{(t)} E_3^{(t)}.$$

Suppose there is an index i such that $\epsilon_i = 0, \epsilon_{i+1} = 1$. Define $\epsilon' = (\epsilon'_0, \epsilon'_1, ..., \epsilon'_{N-1})$ as

$$\epsilon'_j = \begin{cases} 1 & j = i, \\ 0 & j = i+1, \\ \epsilon_j & \text{otherwise.} \end{cases}$$

Then consider the following transformations of matrices:

$$\begin{aligned} E_i^{(t)} R^{(t+M-1)} R^{(t+M-2)} \cdots R^{(t)} &= \widetilde{R}^{(t+M-1)} E_i^{(t,1)} R^{(t+M-2)} \cdots R^{(t)} \\ &= \widetilde{R}^{(t+M-1)} \widetilde{R}^{(t+M-2)} E_i^{(t,2)} \cdots R^{(t)} \\ &\vdots \\ &= \widetilde{R}^{(t+M-1)} \widetilde{R}^{(t+M-2)} \cdots \widetilde{R}^{(t)} E_i^{(t,M)}, \end{aligned}$$

where $\widetilde{R}^{(t+M-j)}$ and $E_i^{(t,j)}$ are matrices of the form

$$\begin{aligned} \widetilde{R}^{(t+M-j)} &= \sum_{l=1}^{N} \widetilde{q}_{l-1}^{(t+M-j)} E_{l,l} + \sum_{l=1}^{N-1} E_{l,l+1}, \\ E_i^{(t,j)} &= I_N + e_i^{(t,j)} E_{i+2,i+1}, \quad j = 1,2,...,M. \end{aligned}$$

We denote $\widetilde{E}_j^{(t)} = E_j^{(t,M)}$ and $\widetilde{e}_j^{(t)} = e_j^{(t,M)}$. Let us express $\widetilde{q}_i^{(t+M-j)}, \widetilde{q}_{i+1}^{(t+M-j)}$ and $\widetilde{e}_i^{(t)}$ by $q_i^{(t+M-j)}, q_{i+1}^{(t+M-j)}$ and $e_i^{(t)}$. Let $e_i^{(t,0)} := e_i^{(t)}$. Then

$$e_i^{(t,j)} = \frac{e_i^{(t,j-1)} q_i^{(t+M-j)}}{e_i^{(t,j-1)} + q_{i+1}^{(t+M-j)}}, \tag{3.15}$$

$$\widetilde{q}_i^{(t+M-j)} = \frac{q_{i+1}^{(t+M-j)} q_i^{(t+M-j)}}{e_i^{(t,j-1)} + q_{i+1}^{(t+M-j)}}, \tag{3.16}$$

$$\widetilde{q}_{i+1}^{(t+M-j)} = e_i^{(t,j-1)} + q_{i+1}^{(t+M-j)}, \tag{3.17}$$

$$\widetilde{q}_l^{(t)} = q_l^{(t)}, \quad l \neq i, i+1, \tag{3.18}$$

$$\widetilde{e}_l^{(t)} = e_l^{(t)}, \quad l \neq i. \tag{3.19}$$

We denote the rational transformation (3.15)–(3.19) by $\varphi_i,\ i = 0,1,...,N-2$.

Proposition 3.2.2. The transformation φ_i commutes with the time evolution of the d-hetoda orbits.

Proof. Suppose $\epsilon_0 = 0$ and $I = \{i_0 < i_1 < \cdots < i_{k-1} \mid \epsilon_{i_j+1} = 0\}$. Then $L_1^{(t)} L_2^{(t)}$ has the form

$$L_1^{(t)} L_2^{(t)} = E_{[i_0,0]} E_{[i_1,i_0+1]} \cdots E_{[i_{k-1},i_{k-2}+1]}.$$

We consider the case $i = 0$ and $\epsilon_1 = 1$. In this case, $E_{[i_0,0]}$ is the product of two or more matrices since $i_0 \geq 1$. The general case can be shown in the same way. From the definition of the ϵ-orbits (3.6), we have

$$\begin{aligned} &R^{(t+l)} E^{(t+l)}_{[i_0,0]} E^{(t+l)}_{[i_1,i_0+1]} \cdots E^{(t+l)}_{[i_{k-1},i_{k-2}+1]} \\ &\qquad = E^{(t+l+1)}_{[i_0,0]} E^{(t+l+1)}_{[i_1,i_0+1]} \cdots E^{(t+l+1)}_{[i_{k-1},i_{k-2}+1]} R^{(t+l+M)}, \end{aligned} \tag{3.20}$$

for $l = 0, 1, ..., M-1$. From the definition of the birational transformation (3.15)–(3.19), we also have

$$E_0^{(t,l)} R^{(t+M-1-l)} = \widetilde{R}^{(t+M-1-l)} E_0^{(t,l+1)}, \quad l = 0, 1, ..., M-1, \tag{3.21}$$

where $E_0^{(t,0)} := E_0^{(t)}$. We define $\widetilde{E}_0^{(t)} := E_0^{(t,M)}$ and $\widetilde{E}_l^{(t)} := E_l^{(t)}$ for $l = 1, 2, ..., N-2$. The time evolution of the ϵ'-orbits for $\epsilon' = (1, 0, \epsilon_2, \epsilon_3, ..., \epsilon_{N-1})$ is

$$\begin{aligned} &\widetilde{R}^{(t+l)} \widetilde{E}_0^{(t+l)} \widetilde{E}^{(t+l)}_{[i_0,1]} \widetilde{E}^{(t+l)}_{[i_1,i_0+1]} \cdots \widetilde{E}^{(t+l)}_{[i_{k-1},i_{k-2}+1]} \\ &\qquad = \widetilde{E}_0^{(t+l+1)} \widetilde{E}^{(t+l+1)}_{[i_0,1]} \widetilde{E}^{(t+l+1)}_{[i_1,i_0+1]} \cdots \widetilde{E}^{(t+l+1)}_{[i_{k-1},i_{k-2}+1]} \widetilde{R}^{(t+l+M)}, \end{aligned} \tag{3.22}$$

for $l = 0, 1, ..., M-1$. We define matrices $\overline{R}^{(t+M+l)}$ for $l = 0, 1, ..., M-1$ and $\overline{E}_0^{(t+M)}$ by

$$E_0^{(t+M,l)} R^{(t+2M-1-l)} = \overline{R}^{(t+2M-1-l)} E_0^{(t+M,l+1)}, \quad l = 0, 1, ..., M-1 \tag{3.23}$$

where $E_0^{(t+M,0)} := E_0^{(t+M)}$ and $\overline{E}_0^{(t+M)} = E_0^{(t+M,M)}$. We also define $\overline{E}_l^{(t+M)} = E_l^{(t+M)}$ for $l = 1, 2, ..., N-2$. We must show that $\widetilde{R}^{(t+M+l)} = \overline{R}^{(t+M+l)}$ for $l = 0, 1, ..., M-1$ and $\widetilde{E}_l^{(t+M)} = \overline{E}_l^{(t+M)}$ for $l = 0, 1, ..., N-2$. From (3.21) and (3.22), we have

$$\begin{aligned} &\widetilde{R}^{(t)} \widetilde{E}_0^{(t)} E^{(t)}_{[i_0,1]} E^{(t)}_{[i_1,i_0+1]} \cdots E^{(t)}_{[i_{k-1},i_{k-2}+1]} \\ &\qquad = \widetilde{R}^{(t)} \widetilde{E}_0^{(t)} \widetilde{E}^{(t)}_{[i_0,1]} \widetilde{E}^{(t)}_{[i_1,i_0+1]} \cdots \widetilde{E}^{(t)}_{[i_{k-1},i_{k-2}+1]} \\ &\qquad = E_0^{(t,M-1)} R^{(t)} \widetilde{E}^{(t)}_{[i_0,1]} \widetilde{E}^{(t)}_{[i_1,i_0+1]} \cdots \widetilde{E}^{(t)}_{[i_{k-1},i_{k-2}+1]} \\ &\qquad = E_0^{(t,M-1)} \widetilde{E}^{(t+1)}_{[i_0,1]} \widetilde{E}^{(t+1)}_{[i_1,i_0+1]} \cdots \widetilde{E}^{(t+1)}_{[i_{k-1},i_{k-2}+1]} \widetilde{R}^{(t+M)}. \end{aligned} \tag{3.24}$$

Comparing (3.24) with (3.22), we obtain $\widetilde{E}_0^{(t+1)} = E_0^{(t,M-1)}$. From (3.20), we have

$$
\begin{aligned}
R^{(t)}E^{(t)}_{[i_0,0]}E^{(t)}_{[i_1,i_0+1]}\cdots E^{(t)}_{[i_{k-1},i_{k-2}+1]} &= R^{(t)}E^{(t)}_{[i_0,1]}E^{(t)}_{[i_1,i_0+1]}\cdots E^{(t)}_{[i_{k-1},i_{k-2}+1]}E_0^{(t)} \\
&= E^{(t+1)}_{[i_0,1]}E^{(t+1)}_{[i_1,i_0+1]}\cdots E^{(t+1)}_{[i_{k-1},i_{k-2}+1]}R'^{(t+M)}E_0^{(t)} \\
&= E^{(t+1)}_{[i_0,1]}E^{(t+1)}_{[i_1,i_0+1]}\cdots E^{(t+1)}_{[i_{k-1},i_{k-2}+1]}E_0^{(t+1)}R^{(t+M)},
\end{aligned}
$$

for an upper bidiagonal matrix $R'^{(t+M)}$, owing to the relation $E^{(t)}_\alpha E^{(t)}_\beta = E^{(t)}_\beta E^{(t)}_\alpha$ for $|\alpha-\beta|>1$ and $i_0 \geq 1$. Thus we obtain

$$
\begin{aligned}
&\widetilde{E}_l^{(t+1)} = E_l^{(t+1)}, \quad l=1,2,...,N-2, \\
&R'^{(t+M)} = \widetilde{R}^{(t+M)}, \\
&\widetilde{R}^{(t+M)}E_0^{(t)} = E_0^{(t+1)}R^{(t+M)},
\end{aligned}
$$

owing to the uniqueness of the LU-decomposition. By repeating this argument inductively, we obtain

$$
\begin{aligned}
&\widetilde{E}_0^{(t+k+1)} = E_0^{(t,M-k-1)}, && (3.25)\\
&\widetilde{E}_l^{(t+k+1)} = E_l^{(t+k+1)}, \quad l=1,2,...,N-2, \\
&\widetilde{R}^{(t+M+k)}E_0^{(t+k)} = E_0^{(t+k+1)}R^{(t+M+k)}, && (3.26)
\end{aligned}
$$

for $k=0,1,...,M-1$. From (3.23) and (3.26), we have

$$
\overline{R}^{(t+2M-1)}E_0^{(t+M,1)} = \widetilde{R}^{(t+2M-1)}E_0^{(t+M-1)}.
$$

Thus, we have $\overline{R}^{(t+2M-1)} = \widetilde{R}^{(t+2M-1)}$ and $E_0^{(t+M,1)} = E_0^{(t+M-1)}$. By repeating this argument inductively, we obtain

$$
\begin{aligned}
&\overline{R}^{(t+2M-l)} = \widetilde{R}^{(t+2M-l)}, \\
&E_0^{(t+M,l)} = E_0^{(t+M-l)}, && (3.27)
\end{aligned}
$$

for $l=1,2,...,M$. From (3.25) and (3.27), we have

$$
\overline{E}_0^{(t+M)} = E_0^{(t+M,M)} = E_0^{(t)} = E_0^{(t,0)} = \widetilde{E}_0^{(t+M)}.
$$

This concludes the proof. □

The inverse of the transformation (3.15)–(3.19) is also rational:

$$e_i^{(t,j-1)} = \frac{e_i^{(t,j)} \widetilde{q}_{i+1}^{(t+M-j)}}{e_i^{(t,j)} + \widetilde{q}_i^{(t+M-j)}}, \tag{3.28}$$

$$q_i^{(t+M-j)} = e_i^{(t,j)} + \widetilde{q}_i^{(t+M-j)}, \tag{3.29}$$

$$q_{i+1}^{(t+M-j)} = \frac{\widetilde{q}_i^{(t+M-j)} \widetilde{q}_{i+1}^{(t+M-j)}}{e_i^{(t,j)} + \widetilde{q}_i^{(t+M-j)}}. \tag{3.30}$$

Note that the right-hand sides of the rational transformation (3.15)–(3.19) and (3.28)–(3.30) have no subtractions; this is important in the proof of the main result. We also remark that successive applications of the transformations (3.28)–(3.30) yield a time evolution of the discrete hungry elementary Toda orbits (3.6).

3.2.5 Hungry ϵ-BBS

Let $\varepsilon > 0$. We consider the transformations of variables $q_i^{(t)} = e^{-Q_i^{(t)}/\varepsilon}, e_i^{(t)} = e^{-E_i^{(t)}/\varepsilon}, d_i^{(t)} = e^{-D_i^{(t)}/\varepsilon}$. By applying them to (3.10)–(3.14) and using

$$\lim_{\varepsilon \to +0} -\varepsilon \log(e^{-A/\varepsilon} + e^{-B/\varepsilon}) = \min(A, B),$$

we obtain the following piecewise-linear system:

$$\begin{cases} D_i^{(t+1)} = Q_i^{(t)} + D_{i-1}^{(t+1)} - Q_{i-1}^{(t+M)} \\ Q_i^{(t+M)} = \min(D_i^{(t+1)}, E_i^{(t)}) \end{cases}, \quad (\epsilon_{i-1}, \epsilon_i) = (0, 0) \tag{3.31}$$

$$\begin{cases} D_i^{(t+1)} = \min(Q_i^{(t)}, E_i^{(t)}) \\ Q_i^{(t+M)} = D_i^{(t+1)} + Q_{i-1}^{(t)} - D_{i-1}^{(t+1)} \end{cases}, \quad (\epsilon_{i-1}, \epsilon_i) = (1, 1) \tag{3.32}$$

$$\begin{cases} D_i^{(t+1)} = \min(Q_i^{(t)}, E_i^{(t)}) \\ Q_i^{(t+M)} = D_i^{(t+1)} + D_{i-1}^{(t+1)} - Q_{i-1}^{(t+M)} \end{cases}, \quad (\epsilon_{i-1}, \epsilon_i) = (0, 1) \tag{3.33}$$

$$\begin{cases} D_i^{(t+1)} = Q_i^{(t)} + Q_{i-1}^{(t)} - D_{i-1}^{(t+1)} \\ Q_i^{(t+M)} = \min(D_i^{(t+1)}, E_i^{(t)}) \end{cases}, \quad (\epsilon_{i-1}, \epsilon_i) = (1, 0) \tag{3.34}$$

$$E_i^{(t+1)} = \min(Q_{i+1}^{(t)}, \mathcal{E}_{i+1} + E_{i+1}^{(t)}) - \min(Q_i^{(t+M)}, \mathcal{E}_i + E_{i-1}^{(t+1)}) + E_i^{(t)}, \tag{3.35}$$

for $i = 0, 1, ..., N-1$. Here, $\mathcal{E}_i$ is defined as

$$\mathcal{E}_i = \begin{cases} +\infty & \epsilon_i = 0, \\ 0 & \epsilon_i = 1, \end{cases}$$

and $D_{-1}^{(t+1)} \equiv 0$ and $E_{N-1}^{(t+1)} \equiv +\infty$. We call the system (3.31)–(3.35) the *ultradiscrete hungry elementary Toda orbits* (u-hToda). We consider an auxiliary variable $E_{-1}^{(t)}$ and its time evolution

$$E_{-1}^{(t+1)} = \min(Q_0^{(t)}, \mathcal{E}_0 + E_0^{(t)}) + E_{-1}^{(t)}.$$

From $Q = (Q_i^{(0)}, Q_i^{(1)}, ..., Q_i^{(M-1)})_{i=0}^{N-1} \in \mathbb{Z}_{>0}^{MN}$ and $E = (E_i^{(0)})_{i=-1}^{N-2} \in \mathbb{Z}_{\geq 0}^{N}$, we construct a sequence $u := \Phi_N(Q, E) \in \Omega$ by the following rule:

- $Q_i^{(j)}$ denotes the number of balls of color $j+1$ in the $(i+1)$-st block of balls (the balls in each block are arranged in increasing order), and
- $E_i^{(0)}$ denotes the number of empty boxes between the $(i+1)$-st and the $(i+2)$-nd blocks of balls.

The map Φ_N is a bijection between $\mathbb{Z}_{>0}^{NM} \times \mathbb{Z}_{\geq 0}^{N}$ and the set $\Omega_N (\subset \Omega)$ of sequences satisfying

$$\begin{cases} \mathrm{des}(u) = N-1 & u_0 \neq e \\ \mathrm{des}(u) = N & u_0 = e \end{cases}$$

where $\mathrm{des}(u)$ is the number of descents in u, i.e., the number of indexes i such that $u_i > u_{i+1}$. The time evolution of the u-hToda orbits

$$\begin{aligned} &Q_i^{(j)} \mapsto Q_i^{(j+M)}, \quad 0 \leq i \leq N-1,\ 0 \leq j \leq M-1, \\ &E_i^{(0)} \mapsto E_i^{(M)}, \quad 0 \leq i \leq N-2, \end{aligned}$$

together with

$$E_{-1}^{(t+1)} = E_{-1}^{(t)} + \min(Q_0^{(t)}, \mathcal{E}_0 + E_0^{(t)}),$$

coincides with the rule of the hungry ϵ-BBS which will be explained below. When $M = 1$ it coincides with the ϵ-BBS introduced in Chapter 2 with the nonautonomous parameter $S^{(t)}$ set to $+\infty$ for all $t \in \mathbb{Z}_{\geq 0}$. The hungry ϵ-BBS is a discrete dynamical system on Ω with the time evolution $T_\epsilon \colon \Omega \to \Omega$ defined by the following:

1. Set $i := 1$.
2. For balls of color i, compute a time evolution of the ϵ-BBS as if there are no balls other than the balls of color i.
3. If $i = M$, then terminate. Otherwise set $i := i+1$ and go back to Step 2.

Example 3.2.2. The following is an example of time evolutions of the hungry ϵ-BBS for $\epsilon = (0, 1, 0, 0)$:

$t = 0:$ _1111222___112233____133_____11223________________________________

$t = 1:$ ________11122___1112223__1333______11223__________________________

$t = 2:$ _____________11122_112___11223333___11223_________________________

$t = 3:$ __________________122_11112________1122333_112233_________________

$t = 4:$ _____________________122___11112__________12233__11122333_________

$t = 5:$ ________________________122_____11112__________12233_____11122333_

The following is an example of time evolutions of the same initial sequence, but for $\epsilon = (0, 1, 1, 0)$.

$t = 0:$ _1111222___112233____133_____11223________________________________

$t = 1:$ ________11122___1112223__1333_______11223_________________________

$t = 2:$ _____________11122_112___11223333________11223____________________

$t = 3:$ __________________122_11112________11223333_____11223_____________

$t = 4:$ _____________________122___11112___________11223333__11223________

$t = 5:$ ________________________122_____11112____________112233_1122333___

3.3 P-symbol as a conserved quantity of the hungry ϵ-BBS

For $u \in \Omega$, let $f(u)$ denotes a finite subsequence of u obtained by removing all e's. For $u^{(0)}$ in the above example, we have $f(u^{(0)}) = 111122211223313311223$. The purpose of this section is to prove the following proposition.

Proposition 3.3.1. For any $u \in \Omega$ and $\epsilon \in \{0, 1\}^N$, two SSTs, $\emptyset \leftarrow f(u)$ and $\emptyset \leftarrow f(T_\epsilon(u))$, coincide.

Proposition 3.3.1 gives conserved quantities of the hungry ϵ-BBS. To prove Proposition 3.3.1, we use the birational transformation described in Section 3.2.4 We ultradiscretize (3.15)–(3.17) to obtain

$$E_i^{(t,j)} = E_i^{(t,j-1)} + Q_i^{(t+M-j)} - \min(E_i^{(t,j-1)}, Q_{i+1}^{(t+M-j)}), \tag{3.36}$$

$$\widetilde{Q}_i^{(t+M-j)} = Q_{i+1}^{(t+M-j)} + Q_i^{(t+M-j)} - \min(E_i^{(t,j-1)}, Q_{i+1}^{(t+M-j)}), \tag{3.37}$$

$$\widetilde{Q}_{i+1}^{(t+M-j)} = \min(E_i^{(t,j-1)}, Q_{i+1}^{(t+M-j)}). \tag{3.38}$$

We also consider transformation from $(1, \epsilon_1, \epsilon_2, ...)$-orbit to $(0, \epsilon_1, \epsilon_2, ...)$-orbit as

$$E_{-1}^{(t,j)} = E_{-1}^{(t,j-1)} - Q_0^{(t+M-j)}, \quad j = 1, 2, ..., M, \tag{3.39}$$

where $E_{-1}^{(t,0)} := E_{-1}^{(t)}$ and define $\widetilde{E}_{-1}^{(t)} = E_{-1}^{(t,M)}$. The transformation (3.39) also commute with the h-uToda orbits. We denote the (tropical) birational transformation (3.36)–(3.38) act on a pair $(\epsilon_i, \epsilon_{i+1})$ by the same symbol φ_i for $i = 0, ..., N-2$, and define φ_{-1} as (3.39). The transformation φ_i acting on $u \in \Omega_N$ is given by $\widetilde{\varphi}_i(u) := \Phi_N \circ \varphi_i \circ \Phi_N^{-1}(u)$. To prove Proposition 3.3.1, it is sufficient to show the following:

Proposition 3.3.2. For all $u \in \Omega$ and $i = -1, 0, ..., N-2$, two SSTs $\emptyset \leftarrow f(u)$ and $\emptyset \leftarrow f(\widetilde{\varphi}_i(u))$ coincide.

It is easy to see that for any $\epsilon \in \{0,1\}^N$, there is a sequence $i_1, i_2, ..., i_k \in \{-1, 0, ..., N-2\}$ such that $\widetilde{\varphi} := \widetilde{\varphi_{i_k}} \circ \cdots \circ \widetilde{\varphi}_{i_2} \circ \widetilde{\varphi}_{i_1}$ is the transformation from the ϵ-orbit to the ϵ_0-orbit where $\epsilon_0 = (0, 0, ..., 0)$. Thus, by combining Proposition 3.2.1 with Proposition 3.3.2, we obtain Proposition 3.3.1. Let us prove Proposition 3.3.2. As the product for SSTs is associative (Proposition 3.1.1), it is sufficient to show that Proposition 3.3.2 holds for a sequence of the following form:

$$u = \underbrace{11...1}_{Q_0^{(0)}} \underbrace{22...2}_{Q_0^{(1)}} \ ... \ \underbrace{MM...M}_{Q_0^{(M-1)}} \underbrace{ee...e}_{E_0} \underbrace{11...1}_{Q_1^{(0)}} \underbrace{22...2}_{Q_1^{(1)}} \ ... \ \underbrace{MM...M}_{Q_1^{(M-1)}} \ ee \ ...$$

Let $(Q, E) = \Phi_2^{-1}(u)$. We will prove that

$$\eta_n = \max_{1 \le k \le n} \left\{ Q_0^{(0)} + Q_0^{(1)} + \cdots + Q_0^{(k-1)} + Q_1^{(k-1)} + \cdots + Q_1^{(n-1)} \right\} \tag{3.40}$$

for $n = 1, 2, ..., M$, are conserved under $\widetilde{\varphi}_0$. First, we introduce a notion of the *inverse-ultradiscretization*. The inverse-ultradiscretization is an operation of replacing $(\min, +)$ into $(+, \times)$ as

$$\min(A, B) \mapsto a + b, \tag{3.41}$$

$$A + B \mapsto ab. \tag{3.42}$$

New variables obtained by this operation are called the *geometric liftings* of the original variables. For example, in (3.41) and (3.42), variables a and b are geometric liftings of A and B, respectively. We denote by $\mathrm{trop}^{-1}(A)$ the geometric lifting of the variable A. We perform the inverse-ultradiscretization of (3.40) to obtain

$$\mathrm{trop}^{-1}(\eta_n) = \frac{\prod_{i=1}^n a_i}{\sum_{j=1}^n \prod_{i=1, i \neq j}^n a_i}, \tag{3.43}$$

where a_k is

$$a_k = \prod_{i=0}^{k-1} q_0^{(i)} \prod_{j=k-1}^{n-1} q_1^{(j)},$$

and $q_i^{(j)}$ and e_0 denote geometric liftings of $Q_i^{(j)}$ and E_0, respectively. We call (3.43) the geometric Schensted insertion. We define $\beta_k,\ k=0,1,...,M-1$ as

$$\beta_i = \beta_{i-1} q_0^{(i)} + \prod_{j=0}^{i-1} q_1^{(j)}, \tag{3.44}$$
$$\beta_0 = 1.$$

Then (3.43) is written as

$$\mathrm{trop}^{-1}(\eta_n) = \frac{q_0^{(0)} q_0^{(1)} \cdots q_0^{(n-1)} q_1^{(0)} q_1^{(1)} \cdots q_1^{(n-1)}}{\beta_{n-1}}.$$

As $q_0^{(j)} q_1^{(j)},\ j=0,1,...,N-1$, are conserved by the transformation φ_0, it suffice to show that β_n is unchanged by the transformation φ_0. We also define $\alpha_i,\ i=0,1,...,M-1$, by

$$\alpha_i = \alpha_{i-1} q_1^{(M-i-1)} + e_0 \prod_{j=0}^{i-1} q_0^{(M-j-1)}, \tag{3.45}$$
$$\alpha_0 = e_0 + q_1^{(M-1)}.$$

With $\alpha_i,\ i=0,1,...,M-1$, variables $\widetilde{q}_0^{(i)}, \widetilde{q}_1^{(i)},\ i=0,1,...,M-1$, are written as

$$\widetilde{q}_0^{(i)} = \frac{q_1^{(i)} q_0^{(i)} \alpha_{M-2-i}}{\alpha_{M-1-i}}, \quad \widetilde{q}_1^{(i)} = \frac{\alpha_{M-1-i}}{\alpha_{M-2-i}}. \tag{3.46}$$

We use the following relation between α_i's and β_i's.

Lemma 3.3.1. For $i=0,1,...,M-1$, we have

$$\alpha_{M-1} = \alpha_{M-1-i} \prod_{j=0}^{i-1} q_1^{(j)} + \beta_{i-1} e_0 \prod_{j=i}^{M-1} q_0^{(j)}, \tag{3.47}$$

where $\beta_{-1}=0$.

Proof. We prove (3.47) by induction on i. For $i=0$, (3.47) trivially holds. Suppose (3.47) holds for some $i \geq 0$. From (3.45), we have $\alpha_{M-1-i} = \alpha_{M-2-i} q_1^{(i)} + e_0 \prod_{j=i+1}^{M-1} q_0^{(j)}$, thus

$$\begin{aligned}
\alpha_{M-1} &= \alpha_{M-1-i} \prod_{j=0}^{i-1} q_1^{(j)} + \beta_{i-1} e_0 \prod_{j=i}^{M-1} q_0^{(j)} \\
&= \alpha_{M-2-i} \prod_{j=0}^{i} q_1^{(j)} + (\beta_{i-1} q_0^{(i)} + \prod_{j=0}^{i-1} q_1^{(j)}) e_0 \prod_{j=i+1}^{M-1} q_0^{(j)} \\
&= \alpha_{M-2-i} \prod_{j=0}^{i} q_1^{(j)} + \beta_i e_0 \prod_{j=i+1}^{M-1} q_0^{(j)}.
\end{aligned}$$

Therefore (3.47) holds for $i+1$. $\square$

Proposition 3.3.3. $\beta_i,\ i=0,1,2,...,M-1$, are conserved by the transformation $\widetilde{\varphi}_0$.

Proof. We prove the assertion by induction on i. For $i=0$ it is trivial. Suppose the assertion holds for $i-1$ (that is, we have $\widetilde{\beta}_{i-1}=\beta_{i-1}$.). From (3.44) and (3.46), we have

$$\begin{aligned}\widetilde{\beta}_i &= \beta_{i-1}\widetilde{q}_0^{(i)} + \prod_{j=0}^{i-1}\widetilde{q}_1^{(j)} \\ &= \frac{\beta_{i-1}q_0^{(i)}q_1^{(i)}\alpha_{M-2-i}+\alpha_{M-1}}{\alpha_{M-i-1}}. \end{aligned} \tag{3.48}$$

We have $\alpha_{M-1}=\alpha_{M-1-i}\prod_{j=0}^{i-1}q_1^{(j)}+\beta_{i-1}e_0\prod_{j=i}^{M-1}q_0^{(j)}$ and $\alpha_{M-2-i}q_1^{(i)}=\alpha_{M-1-i}-e_0\prod_{j=i+1}^{M-1}q_0^{(j)}$ because of Lemma 3.3.1 and (3.45), respectively. Thus, the numerator of the right-hand side of (3.48) is transformed as

$$\begin{aligned}\beta_{i-1}q_0^{(i)}q_1^{(i)}\alpha_{M-2-i}+\alpha_{M-1} &= \beta_{i-1}q_0^{(i)}(\alpha_{M-1-i}-e_0\prod_{j=i+1}^{M-1}q_0^{(j)})+\alpha_{M-1-i}\prod_{j=0}^{i-1}q_1^{(j)} \\ &\quad + \beta_{i-1}e_0\prod_{j=i}^{M-1}q_0^{(j)} \\ &= \alpha_{M-1-i}(\beta_{i-1}q_0^{(i)}+\prod_{j=0}^{i-1}q_1^{(j)}) \\ &= \alpha_{M-1-i}\beta_i.\end{aligned}$$

Therefore we have $\widetilde{\beta}_i=\beta_i$. $\square$

Obviously, the inverses of transformations (3.36)–(3.38) also preserve the P-symbol. Thus, together with the remark stated in the last sentence of Section 3.2.3, we also see that Proposition 3.3.1 follows without going through Proposition 3.1.4.

Chapter 4

The ultradiscrete Toda lattice and the Smith normal form of bidiagonal matrices

In this chapter, we give a method to compute invariant factors of a certain matrix over a principal ideal domain as an application of the ϵ-BBS. First, we explain how to compute invariant factors of a bidiagonal matrix by the ultradiscrete Toda lattice. Next, we show that this method can be naturally extended to the ϵ-BBS.

4.1 Preliminaries

First, we derive the ultradiscrete Toda lattice from the discrete Toda lattice. This is a special case of the discussion in Chapter 2, but we give the derivation for the convenience of the reader. First, we start with the discrete Toda lattice:

$$\begin{cases} q_n^{(t+1)} = q_n^{(t)} + e_n^{(t)} - e_{n-1}^{(t+1)}, \\ e_n^{(t+1)} = q_{n+1}^{(t)} e_n^{(t)} / q_n^{(t+1)}, \\ e_{-1}^{(t)} = e_{N-1}^{(t)} = 0. \end{cases} \tag{4.1}$$

We rewrite (4.1) as

$$\begin{cases} q_n^{(t+1)} = e_n^{(t)} + \dfrac{\prod_{j=0}^{n} q_j^{(t)}}{\prod_{j=0}^{n-1} q_j^{(t+1)}}, \\ e_n^{(t+1)} = e_n^{(t)} q_{n+1}^{(t)} / q_n^{(t+1)}, \\ e_{-1}^{(t)} = e_{N-1}^{(t)} = 0, \end{cases} \tag{4.2}$$

by which we can compute the discrete Toda lattice without subtractions. Then we ultradiscretize (4.2) to yield

$$\begin{cases} Q_n^{(t+1)} = \min\left(E_n^{(t)},\ \sum_{j=0}^{n} Q_j^{(t)} - \sum_{j=0}^{n-1} Q_j^{(t+1)}\right), \\ E_n^{(t+1)} = E_n^{(t)} + Q_{n+1}^{(t)} - Q_n^{(t+1)}, \\ E_{-1}^{(t)} = E_{N-1}^{(t)} = +\infty. \end{cases} \tag{4.3}$$

System (4.3) is known as the *ultradiscrete Toda lattice* (ud-Toda lattice). The following property is easy to prove, but important in our study.

Proposition 4.1.1. The ud-Toda lattice (4.3) defines the map

$$\begin{array}{ccc} (\mathbf{R}_{\geq \mathbf{0}})^{2N-1} & \longrightarrow & (\mathbf{R}_{\geq \mathbf{0}})^{2N-1} \\ \cup & & \cup \\ (Q_0^{(t)}, ..., Q_{N-1}^{(t)}, E_0^{(t)}, ..., E_{N-2}^{(t)}) & \longmapsto & (Q_0^{(t+1)}, ..., Q_{N-1}^{(t+1)}, E_0^{(t+1)}, ..., E_{N-2}^{(t+1)}) \end{array}$$

where $\mathbf{R}_{\geq 0}$ denotes the set of nonnegative real numbers.

Proof. Clearly, $Q_0^{(t+1)} = \min(E_0^{(t)}, Q_0^{(t)}) \geq 0$. Suppose we have proved that $Q_0^{(t+1)}, Q_1^{(t+1)}, ..., Q_n^{(t+1)} \geq 0$ and $E_0^{(t+1)}, Q_1^{(t+1)}, ..., E_{n-1}^{(t+1)} \geq 0$ for some $n \geq 1$. We will show that $E_n^{(t+1)}, Q_{n+1}^{(t+1)} \geq 0$. From (4.3), it follows that

$$\begin{aligned} E_n^{(t+1)} &= E_n^{(t)} + Q_{n+1}^{(t)} - Q_n^{(t+1)} \\ &= E_n^{(t)} - \min\left(E_n^{(t)},\ \sum_{j=0}^{n} Q_j^{(t)} - \sum_{j=0}^{n-1} Q_j^{(t+1)}\right) + Q_{n+1}^{(t)} \geq 0. \end{aligned}$$

Similarly, $E_n^{(t)} \geq 0$ and

$$\begin{aligned} \sum_{j=0}^{n+1} Q_j^{(t)} - \sum_{j=0}^{n} Q_j^{(t+1)} &= Q_{n+1}^{(t)} + \sum_{j=0}^{n} Q_j^{(t)} - \sum_{j=0}^{n-1} Q_j^{(t+1)} - Q_n^{(t+1)} \\ &= Q_{n+1}^{(t)} + \sum_{j=0}^{n} Q_j^{(t)} - \sum_{j=0}^{n-1} Q_j^{(t+1)} - \min\left(E_n^{(t)},\ \sum_{j=0}^{n} Q_j^{(t)} - \sum_{j=0}^{n-1} Q_j^{(t+1)}\right) \\ &\geq 0. \end{aligned}$$

Hence, $Q_{n+1}^{(t+1)} \geq 0$. □

In what follows, we assume $Q_n^{(0)}, E_n^{(0)} \in \mathbb{Z}_{\geq 0}$ for all n. In this case, we have $Q_n^{(t)}, E_n^{(t)} \in \mathbb{Z}_{\geq 0}$ for all t by Proposition 4.1.1. Let us consider the BBS with N solitons. System (4.3) is regarded as the time evolution of the BBS by the identification [55],

- $Q_n^{(t)}$: the length of the $(n+1)$-st block of consecutive balls at time t and
- $E_n^{(t)}$: the number of empty boxes between the $(n+1)$-st and the $(n+2)$-nd blocks of consecutive balls at time t.

The conserved quantities of the BBS can be expressed in terms of the dependent variables of the ud-Toda lattice as follows. First, we define

$$W_{2i+1}^{(t)} = Q_i^{(t)}, \quad i = 0, ..., N-1,$$
$$W_{2i}^{(t)} = E_i^{(t)}, \quad i = 0, ..., N-2,$$

and

$$\begin{aligned}
uC_1 &= \min_{1 \le j_1 \le 2N-1} W_{j_1}^{(t)}, \\
uC_2 &= \min_{1 \le j_1 < j_2 - 1 \le 2N-1} (W_{j_1}^{(t)} + W_{j_2}^{(t)}), \\
&\vdots \\
uC_{l-1} &= \min_{1 \le j_1 < j_2 - 1 < \cdots < j_l - l + 1 \le 2N-1} (W_{j_1}^{(t)} + W_{j_2}^{(t)} + \cdots + W_{j_l}^{(t)}), \\
&\vdots \\
uC_N &= \min_{1 \le j_1 < j_2 - 1 < \cdots < j_N - N + 1 \le 2N-1} (W_{j_1}^{(t)} + W_{j_2}^{(t)} + \cdots + W_{j_N}^{(t)}).
\end{aligned}$$

Then the following proposition holds.

Proposition 4.1.2 ([73])**.** The N independent conserved quantities for the ud-Toda lattice (4.3) are given by $uC_1, uC_2, ..., uC_N$.

The dependent variables $Q_0^{(t)}, \cdots Q_{N-1}^{(t)}, E_0^{(t)}, ..., E_{N-2}^{(t)}$ of the ud-Toda lattice satisfy the following lemma.

Lemma 4.1.1 ([73])**.** There exists a positive integer T such that for all $t > T$,

$$Q_0^{(t)} \le Q_1^{(t)} \le \cdots \le Q_{N-1}^{(t)}.$$

Lemma 4.1.1 is often called the *sorting property.* We also need the following lemma, which follows from the Lemma 4.1.1.

Lemma 4.1.2. There exists a positive integer T such that for all $t > T$, the dependent variables $E_0^{(t)}, E_1^{(t)}, \cdots, E_{N-2}^{(t)}$ satisfy

$$Q_i^{(t)} \le E_i^{(t)}, \quad i = 0, 1, ..., N-2.$$

4.2 Ultradiscrete Toda lattice and invariant factors

In this section, we give the main theorem (Theorem 4.2.1) of Chapter 4 and present an algorithm for computing the Smith normal form of a bidiagonal matrix. Our main result holds not only on the ring of rational integers $\mathbb{Z}$ but also on general principal ideal domains. Thus, for the convenience of the reader, we review the definition and some basic facts about principal ideal domains in Section 4.2.1.

4.2.1 Basic definitions

All rings we consider are commutative and having a multiplicative identity. An element u in a ring R is called a *unit* if u has a multiplicative inverse. A ring R is called an *integral domain* if for all non-zero $a, b \in R$, their product ab is non-zero. A non-zero and non-unit element a in an integral domain R is called *irreducible* if a is not a product of two non-unit elements. Given elements a, b of an integral domain R, we say that a *divides* b and write $a \mid b$ if there exists an element $c \in R$ such that $ac = b$. Two elements $a, b \in R$ are said to be *associates* if $a \mid b$ and $b \mid a$.

Definition 4.2.1. An integral domain R is called a *unique factorization domain* if any non-zero element $x \in R$ can be written as

$$x = up_1p_2 \cdots p_n,$$

where u is a unit and $p_1, p_2, ..., p_n$ are irreducible elements of R, and the decomposition is unique in the following sense: For another decomposition

$$x = vq_1q_2 \cdots q_m,$$

where v is a unit and $q_1, q_2, ..., q_m$ are irreducible elements of R, we have $m = n$ and there exists a permutation $\sigma \in S_n$ such that p_i and $q_{\sigma(i)}$ are associates.

For two elements $a, b \in R$, an element $d \in R$ is called a *common divisor* of a and b if d divides both a and b. A common divisor d of a and b is called a *greatest common divisor* if it is divided by any common divisor of a and b. In a general integral domain, a greatest common divisor does not necessarily exist. However, in a unique factorization domain, any two elements have a greatest common divisor.

A non-empty subset I of the ring R is called an *ideal* of R if for all $a \in R$ and for all $x \in I$, we have $ax \in I$. An integral domain R is called a *principal ideal domain* if any ideal I of R is generated by a single element of R, that is, there exists an element $x \in I$ such that $I = \{ax \mid a \in R\}$. For example, the ring of rational integers $\mathbb{Z}$ and the ring $K[x]$ of polynomials of one variable x over a field K are principal ideal domains. In this thesis, we will use the following properties of principal ideal domains:

- Any principal ideal domain is a unique factorization domain.
- The *Bézout identity* holds for any pair of elements of a principal ideal domain R, namely, for any pair of elements $a, b \in R$, there exist elements $x, y \in R$ such that

$$ax + by = d,$$

  where d is a greatest common divisor of a and b.

See, for example, Chapter 2 of [45] for proof.

Any matrix over a principal ideal domain R can be transformed into a particular form of diagonal matrix by unimodular transformations. That is, for any matrix $A \in R^{m\times n}$, there exist invertible matrices $P \in R^{m\times m}$ and $Q \in R^{n\times n}$ such that the matrix $S = PAQ$ vanishes off the main diagonal, and whose main diagonal has the form $(e_1, e_2, ..., e_r, 0, ..., 0)$, where e_i divides e_{i+1} for $1 \le i \le r-1$. The matrix S is called the *Smith normal form* of A and the quantities $e_1, e_2, ..., e_r$ are the *invariant factors* of A. Invariant factors are unique up to unit factors.

4.2.2 Ultradiscrete Toda lattice and invariant factors

Let R be a principal ideal domain. If $a \mid b$, then we define b/a to be the element $c \in R$ such that $b = ac$. The set of all units of R is denoted by R^*. Instead of using the ud-Toda lattice (4.3) directly, we replace $(+, -, \min)$ in (4.3) with $(\times, /, \gcd)$, where gcd denotes the greatest common divisor. That is:

$$\begin{cases} q_n^{(t+1)} = \gcd\left(e_n^{(t)},\ \prod_{j=0}^{n} q_j^{(t)} / \prod_{j=0}^{n-1} q_j^{(t+1)}\right) \\ e_n^{(t+1)} = e_n^{(t)} q_{n+1}^{(t)} / q_n^{(t+1)} \\ e_{-1}^{(t)} = e_{N-1}^{(t)} = 0 \end{cases}, \qquad (4.4)$$

where $e_n^{(t)}, q_n^{(t)} \in R$. We call the system (4.4) the *gcd-Toda lattice*. System (4.4) is considered as an extended expression of the ud-Toda lattice (4.3). When the dependent variables in (4.4) have only one irreducible factor $p \in R$, i.e., $q_n^{(t)} = p^{Q_n^{(t)}}, e_n^{(t)} = p^{E_n^{(t)}}$ for a single irreducible element $p \in R$ and $Q_n^{(t)}, E_n^{(t)} \in \mathbb{Z}_{\ge 0}$, then the exponents $Q_n^{(t)}, E_n^{(t)}$ satisfy the ud-Toda lattice (4.3), since $\gcd(q^a, q^b) = q^{\min(a,b)}$ for $a, b \in \mathbb{Z}_{\ge 0}$. Thus, when the dependent variables have more than one irreducible factors, the equations (4.4) is equivalent to running the ud-Toda lattice simultaneously on each irreducible factors without performing prime factorization. This also proves that the divisions in (4.4) can always be performed. The above observation is important for connecting ultradiscrete systems and computation of invariant factors. Let $X^{(0)} \in M(n, R)$ be a lower bidiagonal matrix, and denote elements of $X^{(0)}$ as

$$X^{(0)} = \begin{pmatrix} q_0^{(0)} & & & & \\ e_0^{(0)} & q_1^{(0)} & & & \\ & \ddots & \ddots & & \\ & & e_{N-3}^{(0)} & q_{N-2}^{(0)} & \\ & & & e_{N-2}^{(0)} & q_{N-1}^{(0)} \end{pmatrix}. \tag{4.5}$$

Suppose $q_0^{(0)}, q_1^{(0)}, \cdots, q_{N-1}^{(0)}$ and $e_0^{(0)}, e_1^{(0)}, \cdots, e_{N-2}^{(0)}$ are nonzero. We compute $q_n^{(t)}, e_n^{(t)}$ for $t = 1, 2, ...$ by (4.4). Then we obtain

$$X^{(t)} = \begin{pmatrix} q_0^{(t)} & & & & \\ e_0^{(t)} & q_1^{(t)} & & & \\ & \ddots & \ddots & & \\ & & e_{N-3}^{(t)} & q_{N-2}^{(t)} & \\ & & & e_{N-2}^{(t)} & q_{N-1}^{(t)} \end{pmatrix}.$$

The following theorem is the main result of Chapter 4.

Theorem 4.2.1. For sufficiently large $t > 0$, the diagonal part of the matrix $X^{(t)}$ coincides with the Smith normal form of the initial matrix $X^{(0)}$. In other words, the dependent variables $q_0^{(t)}, q_1^{(t)}, ..., q_{N-1}^{(t)}$ of the gcd-Toda lattice (4.4) converge to the invariant factors of $X^{(0)}$ in a finite time.

Before giving a proof of Theorem 4.2.1, we introduce *determinantal divisors*. The i-th determinantal divisor $d_i(A)$ of a matrix A is the gcd of all $i \times i$ minors of A. The i-th invariant factor $s_i(A)$ of A is expressed as $s_i(A) = d_i(A)/d_{i-1}(A)$, where $d_0(A) = 1$. The determinantal divisors of the bidiagonal matrix $X^{(t)}$ are expressed in a simpler form by means of the elements of the matrix $X^{(t)}$.

Lemma 4.2.1. Define the variables

$$w_{2i+1}^{(t)} = q_i^{(t)}, \quad i = 0, ..., N-1,$$
$$w_{2i}^{(t)} = e_i^{(t)}, \quad i = 0, ..., N-2.$$

Then, the determinantal divisors of the matrix $X^{(t)}$ are given by

$$\begin{aligned}
d_0^{(t)} &= \gcd_{1\le j_1\le 2N-1} w_{j_1}^{(t)},\\
d_1^{(t)} &= \gcd_{1\le j_1<j_2-1\le 2N-1} w_{j_1}^{(t)} w_{j_2}^{(t)},\\
&\vdots\\
d_{l-1}^{(t)} &= \gcd_{1\le j_1<j_2-1<\cdots<j_l-l+1\le 2N-1} w_{j_1}^{(t)} w_{j_2}^{(t)}\cdots w_{j_l}^{(t)},\\
&\vdots\\
d_{N-1}^{(t)} &= \gcd_{1\le j_1<j_2-1<\cdots<j_N-N+1\le 2N-1} w_{j_1}^{(t)} w_{j_2}^{(t)}\cdots w_{j_N}^{(t)},
\end{aligned}$$

where $\gcd_{1\le i\le n} a_i$ denotes the gcd of all $a_1, a_2, ..., a_n$.

The above lemma can be proved easily through direct calculation. We now return to the proof of Theorem 4.2.1.

Proof of Theorem 4.2.1. First, we show that the invariant factors of $X^{(t)}$ do not depend on the variable t. Let $p_1, p_2, ..., p_m$ be all irreducible elements that appear in the irreducible decomposition of $q_0^{(0)}, ..., q_{N-1}^{(0)}$ and $e_0^{(0)}, ..., e_{N-2}^{(0)}$, and suppose that none of them are associate to any of others. Then, no irreducible elements other than $p_1, p_2, ..., p_m$ appear in the decomposition of $q_i^{(t)}$ and $e_i^{(t)}$ for $t \ge 1$, because system (4.4) contains only multiplications, divisions, and gcd operations. The dependent variables $q_0^{(t)}, ..., e_{N-2}^{(t)}$ are expressed as

$$q_0^{(t)} = u_0^{(t)} p_0^{Q_{0,0}^{(t)}} p_1^{Q_{1,0}^{(t)}} \cdots p_m^{Q_{m,0}^{(t)}}, \quad \cdots, \quad q_{N-1}^{(t)} = u_{N-1}^{(t)} p_0^{Q_{0,N-1}^{(t)}} p_1^{Q_{1,N-1}^{(t)}} \cdots p_m^{Q_{m,N-1}^{(t)}},$$

$$e_0^{(t)} = v_0^{(t)} p_0^{E_{0,0}^{(t)}} p_1^{E_{1,0}^{(t)}} \cdots p_m^{E_{m,0}^{(t)}}, \quad \cdots, \quad e_{N-2}^{(t)} = v_{N-2}^{(t)} p_0^{E_{0,N-2}^{(t)}} p_1^{E_{1,N-2}^{(t)}} \cdots p_m^{E_{m,N-2}^{(t)}},$$

where $Q_{i,j}^{(t)},\ E_{i,j}^{(t)} \in \mathbf{Z}_{\ge 0},\ u_i^{(t)}, v_i^{(t)} \in R^*$. Because $q_0^{(t)}, ..., e_{N-2}^{(t)}$ satisfy the gcd-Toda lattice (4.4), exponents $Q_{i,0}^{(t)}, ...Q_{i,N-1}^{(t)}, E_{i,0}^{(t)}, ..., E_{i,N-2}^{(t)}$ of a single irreducible factor q_i satisfy the ud-Toda lattice (4.3). By Proposition 4.1.2, we have conserved quantities of the ud-Toda lattice $uC_{i,1}, ..., uC_{i,N}$ for each $i = 1, ..., m$. Therefore, we obtain conserved quantities $C_1, ..., C_N$ of the system (4.4) :

$$C_k = u_k p_0^{uC_{0,k}} p_1^{uC_{1,k}} \cdots p_m^{uC_{m,k}}, \quad k = 1, 2, ..., N,$$

where $u_k \in R^*$. By Lemma 4.2.1, we see that C_k and $d_k^{(t)}$ differ by a multiplicative factor of a unit; thus, invariant factors do not depend on the variable t. Next, we prove that the dependent variables $q_0^{(t)}, ..., q_{N-1}^{(t)}$ converge to the invariant factors. By Lemmas 4.1.1 and 4.1.2, there exists positive integer T such that for all $t > T$, we have

$$Q_{i,0}^{(t)} \leq Q_{i,1}^{(t)} \leq \cdots \leq Q_{i,N-1}^{(t)}$$

for $i = 0, ..., m$ and

$$Q_{i,j}^{(t)} \leq E_{i,j}^{(t)}$$

for $i = 0, ..., m$ and $j = 0, ..., N-2$. This means that $q_i^{(t)} \mid q_{i+1}^{(t)}$ and $q_i^{(t)} \mid e_i^{(t)}$ for all $i = 0, ..., N-2$ when $t > T$. Therefore, $X^{(t)}$ can be transformed into the Smith normal form by elementary row operations. This concludes the proof. □

Based on Theorem 4.2.1, we present a new method for computing invariant factors of matrices over a principal ideal domain. Let $A = (a_{ij})_{i,j=1}^n$ be an non-zero $n \times n$ matrix over a principal ideal domain R. We can transform A by unimodular transformations to a bidiagonal matrix $B = (b_{ij})_{i,j=1}^n$ with $b_{11}, b_{22}, ..., b_{kk} \neq 0, b_{21}, b_{32}, ..., b_{k,k-1} \neq 0$ for some k and $b_{jj} = b_{j+1,j} = 0$ for $j > k$. The element $b_{k+1,k}$ may or may not be zero. Suppose that the elements in the first row of matrix A are not all zero. Then we can set $a_{11} \neq 0$ by exchanging columns if necessary. If the elements in the first row of matrix A are all zero, then add a non-zero row to the first row of matrix A and exchange the columns so that $a_{11} \neq 0$.

Let $d = \gcd(a_{11}, a_{12})$. Then there exist $p, q, s, t \in R$ such that

$$\begin{aligned} &a_{11}p + a_{12}q = d, \\ &a_{11} = sd, \quad a_{12} = -td. \end{aligned}$$

Let the matrix $G(1,2)$ to be

$$G(1,2) = \begin{pmatrix} p & t & & & \\ q & s & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Then $\det G(1,2) = 1$ and the $(1,2)$-entry of the matrix $AG(1,2)$ is zero. Continuing this procedure, we can transform A by unimodular transformations to the matrix $A' = (a'_{ij})_{i,j=1}^n$ with $a'_{11} \neq 0$ and $a'_{12} = a'_{13} = \cdots = a'_{1n} = 0$. If the elements in the second row or below of the matrix A' are not all zero, we can set $a'_{21} \neq 0$ by elementary operations on A' as before with the first row of A unchanged. By applying the above procedure to the rows, we can transform A' by unimodular transformations to the matrix A'' with

$a''_{31} = a''_{41} = \cdots = a''_{n1} = 0$ and $a''_{12} = a''_{13} = \cdots = a''_{1n} = 0$. Denote A'' again as A. If the submatrix $\widetilde{A} = (a_{ij})_{i,j=2}^{n}$ of A is non-zero, we apply the above transformations inductively to A to obtain the bidiagonal matrix B satisfying the conditions stated in the beginning. Once the bidiagonalization is done, we can use the following algorithm by setting

$$X^{(0)} = \begin{pmatrix} b_{11} & & & & \\ b_{21} & b_{22} & & & \\ & b_{32} & \ddots & & \\ & & & b_{k-1,k-1} & \\ & & & b_{k,k-1} & b_{kk} \end{pmatrix}$$

if $b_{k+1,k} = 0$ or

$$X^{(0)} = \begin{pmatrix} b_{11} & & & & \\ b_{21} & b_{22} & & & \\ & b_{32} & \ddots & & \\ & & & b_{k,k} & \\ & & & b_{k+1,k} & 0 \end{pmatrix}$$

if $b_{k+1,k} \neq 0$. Note that the zero element at the bottom right of the latter matrix does not affect the correctness of the algorithm.

Algorithm 4.2.1.

(i) For a given lower bidiagonal matrix $X^{(0)}$, set the initial values of dependent variables of (4.4) as (4.5). Set $t = 0$.

(ii) Calculate $X^{(t+1)}$ using (4.4).

(iii) If terminating conditions $q_i^{(t+1)} \mid q_{i+1}^{(t+1)}$ and $q_i^{(t+1)} \mid e_i^{(t+1)}$ hold for all $i = 0, 1, ..., N-2$, then go to (iv), otherwise set $t := t+1$ and go to (ii).

(iv) Output $q_0^{(t+1)}, q_1^{(t+1)}, ..., q_{k-1}^{(t+1)}$.

Example 4.2.1. Let the initial matrix $X^{(0)}$ be

$$X^{(0)} = \begin{pmatrix} 2 & & \\ 4 & 6 & \\ & 3 & 9 \end{pmatrix}.$$

Then, Algorithm 4.2.1 proceeds as

$$X^{(0)} = \begin{pmatrix} 2 & & \\ 4 & 6 & \\ & 3 & 9 \end{pmatrix}, \quad X^{(1)} = \begin{pmatrix} 2 & & \\ 12 & 3 & \\ & 9 & 18 \end{pmatrix}, \quad X^{(2)} = \begin{pmatrix} 2 & & \\ 18 & 3 & \\ & 54 & 18 \end{pmatrix},$$

$$X^{(3)} = \begin{pmatrix} 2 & & \\ 27 & 3 & \\ & 324 & 18 \end{pmatrix}, \quad X^{(4)} = \begin{pmatrix} 1 & & \\ 81 & 6 & \\ & 972 & 18 \end{pmatrix}.$$

We see that the matrix $X^{(4)}$ satisfies the terminating conditions of Algorithm 4.2.1. Hence, the Smith normal form of the matrix $X^{(0)}$ is

$$\begin{pmatrix} 1 & & \\ & 6 & \\ & & 18 \end{pmatrix}.$$

4.3 Ultradiscrete elementary Toda orbits and invariant factors

In this section, we extend Theorem 4.2.1 to the ϵ-BBS. As in the previous section, we consider the gcd-version of the system (3.31)–(3.35) for $M = 1$:

$$\begin{cases} d_i^{(t+1)} = q_i^{(t)} d_{i-1}^{(t+1)} / q_{i-1}^{(t+1)} \\ q_i^{(t+1)} = \gcd(d_i^{(t+1)}, e_i^{(t)}) \end{cases}, \quad (\epsilon_{i-1}, \epsilon_i) = (0,0) \tag{4.6}$$

$$\begin{cases} d_i^{(t+1)} = \gcd(q_i^{(t)}, e_i^{(t)}) \\ q_i^{(t+1)} = d_i^{(t+1)} q_{i-1}^{(t)} / d_{i-1}^{(t+1)} \end{cases}, \quad (\epsilon_{i-1}, \epsilon_i) = (1,1) \tag{4.7}$$

$$\begin{cases} d_i^{(t+1)} = \gcd(q_i^{(t)}, e_i^{(t)}) \\ q_i^{(t+1)} = d_i^{(t+1)} d_{i-1}^{(t+1)} / q_{i-1}^{(t+1)} \end{cases}, \quad (\epsilon_{i-1}, \epsilon_i) = (0,1) \tag{4.8}$$

$$\begin{cases} d_i^{(t+1)} = q_i^{(t)} q_{i-1}^{(t)} / d_{i-1}^{(t+1)} \\ q_i^{(t+1)} = \gcd(d_i^{(t+1)}, e_i^{(t)}) \end{cases}, \quad (\epsilon_{i-1}, \epsilon_i) = (1,0). \tag{4.9}$$

$$e_i^{(t+1)} = \gcd(q_{i+1}^{(t)}, \epsilon_{i+1} e_{i+1}^{(t)}) e_i^{(t)} / \gcd(q_i^{(t+1)}, \epsilon_i e_{i-1}^{(t+1)}), \tag{4.10}$$

for $i = 0, 1, ..., N-1$, where $q_{-1}^{(t+1)}, d_{-1}^{(t+1)} \equiv 1$ and $e_{-1}^{(t+1)}, e_{N-1}^{(t+1)} \equiv 0$. We call (4.6)–(4.10) the *gcd-elementary Toda orbits*. We define $\widetilde{\epsilon}_i = \sum_{j=0}^{i-1} \epsilon_j \mod 2$. Let $X_\epsilon^{(t)}$ be a matrix

$$X_\epsilon^{(t)} = \sum_{i=1}^{N} q_{i-1}^{(t)} E_{i,i} + \sum_{i=1}^{N-1} \left(\widetilde{\epsilon}_{i-1} e_{i-1}^{(t)} E_{i,i+1} + (1 - \widetilde{\epsilon}_{i-1}) e_{i-1}^{(t)} E_{i+1,i} \right).$$

Example 4.3.1. If $\epsilon = (0,0,1,0,0)$, then $\widetilde{\epsilon} = (0,0,1,1,1)$. Thus $X_\epsilon^{(t)}$ is the form

$$X_\epsilon^{(t)} = \begin{pmatrix} q_0^{(t)} & & & & \\ e_0^{(t)} & q_1^{(t)} & & & \\ & e_1^{(t)} & q_2^{(t)} & e_2^{(t)} & \\ & & & q_3^{(t)} & e_3^{(t)} \\ & & & & q_4^{(t)} \end{pmatrix}$$

From Proposition 2.1.5 and the argument of the previous section, we have the following theorem:

Theorem 4.3.1. For sufficiently large $t > 0$, the diagonal part of the matrix $X_\epsilon^{(t)}$ coincides with the Smith normal form of the initial matrix $X_\epsilon^{(0)}$. In other words, the dependent variables $q_0^{(t)}, q_1^{(t)}, ..., q_{N-1}^{(t)}$ of the gcd-elementary Toda orbits (4.6)–(4.10) for a parameter ϵ converge to the invariant factors of $X_\epsilon^{(0)}$ in a finite time.

Chapter 5

Concluding remarks

In Chapter 2, we first introduced spectral transformations of the ϵ-BLP and obtained contiguous relations between them (Proposition 2.1.4). This is the key result of Chapter 2, because it allows us to derive the nonautonomous discrete elementary Toda orbit which agrees in a special case with the suitable form of the discrete Toda lattice for describing the BBS. We also gave its particular solutions and conserved quantities. Second, we obtained a subtraction-free form of the nd-eToda orbits which exhibits the positivity of the system. This property may also be useful in terms of numerical algorithms (see [12] for the usefulness of the positivity in numerical computations). As an application of the nd-eToda orbits, we proposed the ϵ-BBS. The ϵ-BBS is a generalization of Takahashi-Satsuma's BBS with a carrier capacity, and includes a box-ball system (BBS) associated with the ultradiscrete relativistic Toda lattice [34] as a special case. In the earlier work [34] on the cellular automaton associated with the ultradiscrete relativistic Toda lattice, authors realized ultradiscrete equations as a cellular automaton using "balls" and "kickers", which is rather different from the conventional "balls" and "boxes" description of the BBS. Our interpretation of nu-eToda orbits as the ϵ-BBS is consistent with box-ball description of the original BBS, thus it is considered to be useful in the unified study of the ϵ-BBS.

In Chapter 3, we first introduced the discrete hungry elementary Toda orbits and gave the hungry ϵ-BBS. The hungry ϵ-BBS contains Takahashi-Satsuma's BBS with several kind of balls as a special case. Next, we prove that the birational transformation of the elementary Toda orbits introduced in [9] commutes with the discrete hungry elementary Toda orbits. We remark that this transformation and its inverse is written without subtraction, which is important in the proof of the main theorem of Chapter 3. Finally, we show that the P-symbol of the RSK correspondence is a conserved quantity of the hungry ϵ-BBS, which generalizes the earlier work by Fukuda [15]. This follows from the fact that the birational transformation (3.15)–(3.19) preserves the image of geometric Schensted insertion by Noumi and Yamada [56]. The linearization of the hungry ϵ-BBS is possible in principle by combining the birational transformation (3.15)–(3.19) and the rigged configuration bijection of type $A_n^{(1)}$ (see [44] for the linearization of the $A_n^{(1)}$ automata). However we have not yet written the composition of those maps explicitly in combinatorial terms. We also have not yet investigated the crystal-theoretic interpretation of the transformation (3.15)–(3.19).

In Chapter 4, we introduced the gcd-Toda lattice and showed that its dependent variables converge to the invariant factors of a certain bidiagonal matrix over a principal ideal domain. Based on Theorem 4.2.1, we presented a new method for computing invariant factors of a given matrix. We also show that the ϵ-BBS can compute invariant factors of a certain tridiagonal matrix. This is the first instance of the usage of ultradiscrete integrable systems in the computation of invariant factors. It remains to be seen whether a practical algorithm can be obtained from the viewpoint of integrable systems based on the results in this thesis.

In summary, we investigated (ultra)discretization of the elementary Toda orbits and its applications. This work gives a connection between two important discrete integrable systems, discrete Toda lattice and discrete relativistic Toda lattice. We also presented a method to compute invariant factors of a matrix over a principal ideal domain by the ϵ-BBS. We hope these results gives a new perspective to the applications of integrable systems to informatics.

Acknowledgements

The author would like to express his deepest gratitude to his supervisor Associate Professor Satoshi Tsujimoto for his encouragement and instructions. With his generous support, the author was able to pursue his own interests without going off the road. He would also like to thank Professor Yoshimasa Nakamura for his valuable comments. He would also like to express his gratitude to Professor Hiroshi Miki, Hiroto Sekido, Kazuki Maeda, Kinji Kimura and Shuhei Kamioka for their fruitful comments.

He acknowledges the financial support by Research Fellowship for Young Scientists from Japan Society for the Promotion of Science.

Finally, he would like to thank all my colleagues in Applied Mathematical Analysis Laboratory at Graduate School of Informatics, Kyoto University, for fruitful discussions.

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List of author's papers and works related to the thesis

Original Papers

[1] K. Kobayashi and S. Tsujimoto, *The ultradiscrete Toda lattice and the Smith normal form of bidiagonal matrices*, J. Math. Phys. **62** (2021) 092701.
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[2] K. Kobayashi, *Nonautonomous discrete elementary Toda orbits and their ultradiscretization*, J. Phys. A: Math. Theor. **54** (2021) 455203. © IOP Publishing. Reproduced with permission. All rights reserved.

[3] K. Kobayashi and S. Tsujimoto, *Generalization of the ϵ-BBS and the Schensted insertion algorithm*, submitted. Preprint appears at arXiv:2202.09094.

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