ON THE CENTEREDNESS OF KUNEN'S SATURATED IDEAL

KENTA TSUKUURA

Abstract. We show that Kunen's saturated ideal over $\aleph_1$ is not centered.

1. Introduction

In [5], Kunen established

Theorem 1.1 (Kunen [5]). *Suppose that j is a huge embedding with critical point κ. Then there is a poset P such that $P * \dot{S}(\kappa, j(\kappa))$ forces that $\aleph_1$ carries a saturated ideal.*

This theorem has been improved by some ways. One is due to Foreman and Laver [3]. They establised

Theorem 1.2 (Foreman–Laver [3]). *Suppose that j is a huge embedding with critical point κ. Then there is a poset P such that $P * \dot{R}(\kappa, j(\kappa))$ forces that $\aleph_1$ carries a centered ideal.*

Centered ideal is one of the strengthenings of saturated ideal. See Section 2 for the definition of centered ideal. Foreman and Laver introduced the poset $R(\kappa, \lambda)$, to obtain the centeredness, while Kunen used the Silver collapse $S(\kappa, \lambda)$. In their paper, it is mentioned that the ideal in Theorem 1.1 is not centered without proof. We give a proof of this. That is,

Theorem 1.3. *The saturated ideal on $\aleph_1$ in Theorem 1.1 is not centered.*

The structure of this paper is as follows: In Section 2, we recall basic properties of forcing and Silver collapses. In Section 3, we show Theorem 1.3. To show Theorem 1.3, we also recall the model of Theorem 1.1. In Section 4, we will give some observations about collapsing posets. The contents in Section 4 are not directly related to Theorem 1.3 but these were arisen by the context of centered ideal.

2. Preliminaries

In this section, we recall some definitions. We use [4] as a reference for set theory in general.

Our notation is standard. We use κ, λ, μ to denote a regular cardinal unless otherwise stated. We write $[\kappa, \lambda)$ for the set of all ordinals between κ and λ. By Reg, we mean the set of all regular cardinals. For $\kappa < \lambda$, $E^{\lambda}_{\geq \kappa}$ and $E^{\lambda}_{< \kappa}$ denote the set of all ordinals below λ of cofinality $\geq \kappa$ and $< \kappa$, respectively.

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Throughout this paper, we identify a poset P with its separative quotient. Thus, $p \leq q \leftrightarrow \forall r \leq p(r||q) \leftrightarrow p \Vdash q \in \dot{G}$, where $\dot{G}$ is the canonical name of (V, P)-generic filter.

We say that P is κ-centered if there is a sequence of centered subsets $\langle C_\alpha \mid \alpha < \kappa \rangle$ with $P = \bigcup_{\alpha<\kappa} C_\alpha$. A centered subset is $C \subseteq P$ such that every $X \in [C]^{<\omega}$ has a lower bound). We call such sequence a centering of P. It is easy to see that the κ-centeredness implies the κ^+-c.c.

We say that P is well-met if $\prod X \in P$ for all $X \subseteq P$ with X has a lower bound. If P is well-met, the κ-centeredness of P is equivalent to the existence of κ-many filters that cover P. Note that every poset that we will deal with in this paper is well-met.

For a filter $F \subseteq Q$, by Q/F, we mean the subset $\{q \in Q \mid \forall p \in F(p \parallel q)\}$ ordered by $\leq_Q$. For a given complete embedding $\tau : P \to Q$, $P * (Q/\tau``\dot{G})$ is forcing equivalent with $Q$. We also write $Q/\dot{G}$ for $Q/\tau``\dot{G}$ if τ is obvious from context. If the inclusion mapping $P \to Q$ is complete, we say that P is a complete suborder of Q, denoted by $P \lessdot Q$.

In this paper, by ideal, we mean normal and fine ideal. For an ideal I over κ^+, $\mathcal{P}(\kappa^+)/I$ is $\mathcal{P}(\kappa^+) \setminus I$ ordered by $A \leq B \leftrightarrow A \setminus B \in I$. We say that I is saturated and centered if $\mathcal{P}(\kappa^+)/I$ has the κ^{++}-c.c. and $\mathcal{P}(\kappa^+)/I$ is κ^+-centered, respectively.

Silver collapse $S(\kappa, \lambda)$ is the set of all p with the following properties:

- $p \in \prod_{\gamma \in [\kappa^+, \lambda) \cap \mathrm{Reg}} {}^{<\kappa}\gamma$.
- $|p| \leq \kappa$.
- There is a $\xi < \kappa$ with $\forall \gamma \in \mathrm{dom}(p)(\mathrm{dom}(p(\gamma)) \subseteq \xi)$.

$S(\kappa, \lambda)$ is ordered by reverse inclusion. The following properties are well known.

Lemma 2.1. (1) *$S(\kappa, \lambda)$ is κ-closed.*
(2) *If λ is inaccessible, then $S(\kappa, \lambda)$ has the λ-c.c. Therefore, $S(\kappa, \lambda) \Vdash \kappa^+ = \lambda$.*
(3) *If $\mu < \lambda$ then $S(\kappa, \mu) \lessdot S(\kappa, \lambda)$.*

3. Proof of Theorem 1.3

First, we recall about P and the ideal in Theorem 1.1. Let j be a huge embedding $j : V \to M$ with critical point κ. Fix $\mu < \kappa$. Define P by the $< \mu$-support iteration of $\langle P_\alpha \mid \alpha < \kappa \rangle$ such that

- $P_0 = S(\mu, \kappa)$.
- $P_{\alpha+1} = \begin{cases} P_\alpha * S^{P_\alpha \cap V_\alpha}(\alpha, \kappa) & \alpha \text{ is good} \\ P_\alpha & \text{otherwise} \end{cases}$.

Here, we say that α is good if $P_\alpha \cap V_\alpha \lessdot P_\alpha$ has the α-c.c. and α is inaccessible. The set $P_\alpha * S^{P_\alpha \cap V_\alpha}(\alpha, \kappa)$ is the set of all $\langle p, \dot{q} \rangle$ such that $p \in P_\alpha$ and $\dot{q}$ is a $P_\alpha \cap V_\alpha$-name for an element of $S^{P_\alpha \cap V_\alpha}(\alpha, \kappa)$. Note that, for every $p \in P$ and good α, $p(\alpha)$ is $P_\alpha \cap V_\alpha$-name. This P is called a universal collapse. P has the following properties:

Lemma 3.1. (1) *P is μ-directed closed and has the κ-c.c.*
(2) *κ is good for $j(P)$. In particular, $j(P)_\kappa \cap V_\kappa = P \lessdot j(P)_\kappa$.*
(3) *There is a complete embedding $\tau : P * \dot{S}(\kappa, j(\kappa)) \to j(P)$ such that $\tau(p, -) = p$ for all $p \in P$.*

Theorem 3.2 (Kunen [5] for $\mu = \omega$, Laver)**.** *$P * \dot{S}(\kappa, j(\kappa))$ forces that $\mu^+ = \kappa$ carries a saturated ideal $\dot{I}$.*

Proof. See Section 7.7 in [2]. □

The ideal which we call "Kunen's saturated ideal" is this $\dot{I}$. Studying the saturation of $\dot{I}$ will be reduced to that of some quotient forcing. Indeed, Theorem 1.3 follows by Lemmas 3.3 and 3.4. We give the proof of Lemma 3.4.

Lemma 3.3 (Foreman–Magidor–Shelah [1])**.** *$P * \dot{S}(\kappa, j(\kappa))$ forces $\mathcal{P}(\mu^+)/\dot{I} \simeq j(P)/\dot{G} * \dot{H}$. Here, $\dot{G} * \dot{H}$ is the canonical name for generic filter.*

Proof. See Claim 7 in [1]. □

Lemma 3.4. *$P * S(\kappa, j(\kappa))$ forces $j(P)/\dot{G} * \dot{H}$ is not κ-centered.*

Proof. Note that $\{\alpha < j(\kappa) \mid \alpha \text{ is good}\}$ is stationary in $j(\kappa)$. We fix a good $\alpha > \kappa$. It is enough to prove that $j(P)_{\alpha+1}/\dot{G} * \dot{H}$ is not κ-centered in the extension.

We show by contradiction. Suppose $j(P)_{\alpha+1}/\dot{G} * \dot{H}$ has a centering $\langle \dot{C}_\xi \mid \xi < \kappa \rangle$ in some extension. We may assume that each $\dot{C}_\xi$ is forced to be a filter. For simplification of notation, we assume $P * \dot{S}(\kappa, j(\kappa))$ forces the existence of such centering.

By the κ-c.c. of P, for every $\langle p, \dot{q} \rangle \in P * \dot{S}(\kappa, j(\kappa))$, $P \Vdash \dot{q} \in \dot{S}(\kappa, \beta)$ for some $\beta < j(\kappa)$. For each $q \in j(P)_{\alpha+1}$, let $\rho(q)$ be defined by the following way:

For $\xi < \delta$, let $\mathcal{A}^\xi_q \subseteq P * \dot{S}(\kappa, j(\kappa))$ be a maximal anti-chain such that, for every $r \in \mathcal{A}^\xi_q$, r decides $q \in \dot{C}_\xi$. Let $\rho(q)$ be the least ordinal $\beta < j(\kappa)$ such that $P \Vdash \dot{q} \in \dot{S}(\kappa, \beta)$ for every $\langle p, \dot{q} \rangle \in \bigcup_\xi \mathcal{A}^\xi_q$.

Let $C \subseteq j(\kappa)$ be a club generated by $\beta \mapsto \sup\{\rho(q) \mid q \in Q * S^Q(\alpha, \beta)\}$. Here, $Q = j(P)_\alpha \cap V_\alpha$. Since $j(\kappa)$ is inaccessible, we can find a strong limit cardinal $\delta \in C \cap E^{j(\kappa)}_{\geq\kappa} \cap E^{j(\kappa)}_{<\alpha} \setminus (\alpha + 1)$.

Claim 3.5. *$P * \dot{S}(\kappa, \delta)$ forces that $Q * S^Q(\alpha, \delta)/\dot{G} * \dot{H}_\delta$ is κ-centered. Here, $\dot{H}_\delta$ is the canonical name for $(V[\dot{G}], S(\kappa, \delta))$-generic.*

Proof of Claim. Note that $P * \dot{S}(\kappa, \delta)$ forces $Q * S^Q(\alpha, \delta)/\dot{G} * \dot{H}_\delta = Q * S^Q(\alpha, \delta)/\dot{G} * \dot{H}_\alpha$. For every $q \in \bigcup_{\beta<\delta} Q * S^Q(\alpha, \beta)$, by $\rho(q) < \delta$, the statement $q \in \dot{C}_\xi$ has been decided by $P * \dot{S}(\kappa, \delta)$ for all $\xi < \kappa$. Let $G * H$ be an arbitrary $(V, P * \dot{S}(\kappa, j(\kappa)))$-generic filter. Note that $G * H_\delta = G * H \cap (P * \dot{S}(\kappa, \delta))$ is $(V, P * \dot{S}(\kappa, \delta))$-generic. Let D_ξ be defined by

$$\langle p, \dot{q} \rangle \in D_\xi \text{ if and only if } \langle p, \dot{q} \restriction \beta \rangle \in \dot{C}_\xi \text{ forced by } G * H_\delta \text{ for every } \beta < \delta.$$

It is easy to see that D_ξ is a filter over $Q * S^Q(\alpha, \delta)/G * H_\delta$. We claim that $\{D_\xi \mid \xi < \kappa\}$ covers $Q * S^Q(\alpha, \delta)/G * H_\delta$. For each $\langle p, \dot{q} \rangle \in Q * \dot{S}(\alpha, \delta)/G * H_\delta$, in $V[G][H]$, there is a ξ such that $\langle p, \dot{q} \rangle \in \dot{C}^{G*H}_\xi$. Then $\langle p, \dot{q} \restriction \beta \rangle \in \dot{C}^{G*H}_\xi$ for every $\beta < \delta$, since $\dot{C}^{G*H}_\xi$ is a filter. In particular, $\langle p, \dot{q} \restriction \beta \rangle \in \dot{C}_\xi$ is forced by $G * H_\delta$ for every $\beta < \delta$. By the definition of D_ξ, $\langle p, \dot{q} \rangle \in D_\xi$ in $V[G][H_\delta]$, as desired. □

On the other hand, this poset does not satisfy the κ^+-c.c. as follows:

Claim 3.6. (1) *$P * \dot{S}(\kappa, \delta)$ forces $(\delta^+)^V \geq \kappa^+$.*

(2) *$P * \dot{S}(\kappa, \delta)$ forces that $Q * S^Q(\alpha, \delta)/\dot{G} * \dot{H}_\delta$ has an anti-chain of size κ^+.*

Proof of Claim. (1) follows by P has the κ-c.c. and $P \Vdash |\dot{S}(\kappa, \delta)| = |\delta|$.

For (2), by (1), we have $(\delta^{\mathrm{cf}(\delta)})^V \geq (\delta^+)^V \geq \kappa^+$ in the extension by $P * \dot{S}(\kappa, \delta)$. It is enough to find an anti-chain of size $(\delta^{\mathrm{cf}(\delta)})^V$. We fix an increasing sequence of regular cardinals $\langle \delta_i \mid i < \mathrm{cf}(\delta) \rangle$ which converges to δ. For each i, we let $p^i_\xi = \{\langle 0, \xi \rangle\}$. For every $f \in \prod_{i<\mathrm{cf}(\delta)} \delta_i$, define p_f by $\mathrm{dom}(p_f) = \{\delta_i \mid i < \mathrm{cf}(\delta)\}$ and $p_f(\delta_i) = p^i_{f(i)}$. Then $\{p_f \mid f \in \prod_{i<\mathrm{cf}(\delta)} \delta_i\}$ is an anti-chain of $S(\alpha, \delta)$. It is easy to see that $P * \dot{S}(\kappa, \delta) \Vdash \{\langle \emptyset, \check{p_f} \rangle \mid f \in \prod_{i<\mathrm{cf}(\delta)} \delta_i\} \subseteq Q * S^Q(\alpha, \delta)/\dot{G} * \dot{H}_\delta$ witnesses. □

Claim 3.6 (2) shows that $Q * S^Q(\alpha, \delta)/\dot{G} * \dot{H}_\delta$ does not have the κ^+-c.c. in the extension. But this poset has the κ^+-c.c. by Claim 3.5. This is a contradiction. The proof is completed. □

Lastly, we list other saturation properties of $\dot{I}$.

Theorem 3.7. *$P * \dot{S}(\kappa, j(\kappa))$ forces that*

(1) *$\dot{I}$ is $(j(\kappa), j(\kappa), < \mu)$-saturated.*
(2) *$\dot{I}$ is not $(j(\kappa), \mu, \mu)$-saturated. In particular, $\dot{I}$ is not strongly saturated.*
(3) *$\dot{I}$ is layered.*
(4) *$\dot{I}$ is not centered. In particular, $\dot{I}$ is not strongly layered.*

Proof. For (1) and (2), we refer to [10]. (3) has been proven in [1]. (4) follows by Lemma 3.4. Note that Shelah showed that every strongly layered ideal is centered in [7]. Thus $\dot{I}$ is not strongly layered. □

Here, we say that ideal I over κ^+ is $(\alpha, \beta, < \gamma)$-saturated if $\mathcal{P}(\kappa^+)/I$ has the $(\alpha, \beta, < \gamma)$-c.c. Whenever I is $(\kappa^{++}, \kappa^{++}, \kappa)$-saturated, we say that I is strongly saturated. A poset P has the $(\alpha, \beta, < \gamma)$-c.c. if and only if $\forall X \in [P]^\alpha \exists Y \in [X]^\beta \forall Z \in [Y]^{<\gamma}(Z$ has a lower bound). We say that I is layered and strongly layered if $\mathcal{P}(\kappa^+)/I$ is S-layered for some stationary $S \subseteq E^{\kappa^{++}}_{\kappa^+}$ and $E^{\kappa^{++}}_{\kappa^+}$-layered, respectively. For a stationary subset $S \subseteq \lambda$, P is S-layered if there is a sequence $\langle P_\alpha \mid \alpha < \lambda \rangle$ of complete suborders of P such that $|P_\alpha| < \lambda$ and there is a club $C \subseteq \lambda$ such that $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$ for all $\alpha \in S \cap C$.

4. Absorption property of Collapsing posets

The proof of Lemma 3.4 is based on that of the following lemma.

Lemma 4.1 (Tsukuura [10])**.** *If λ is inaccessible then* $\mathrm{Coll}(\kappa, < \lambda) \Vdash \mathrm{Coll}^V(\alpha, < \lambda)$ *is not κ-centered for every $\alpha \in [\kappa^+, \lambda)$.*

Here, $\mathrm{Coll}(\kappa, < \lambda)$ is the usual Levy collapse $\prod^{<\kappa}_{\alpha \in [\kappa^+, \lambda)} {}^{<\kappa}\alpha$. We cannot improve $[\kappa^+, \lambda)$ in Lemma 4.1 as we see in Proposition 4.3.

Lemma 4.2. *Suppose that κ is regular, $\kappa^{<\kappa} = \kappa$, and $\{P_\alpha \mid \alpha \in K\}$ is κ-centered posets. If $|K| \leq 2^\kappa$ then $\prod^{<\kappa}_{\alpha \in K} P_\alpha$ is κ-centered.*

Proof. The proof is due to Foreman–Laver [3]. For $\alpha \in K$, let $F_\alpha : P_\alpha \to \kappa$ be a centering function, that is, $F_\alpha^{-1}\{\xi\}$ is a centered subset of P_α for all $\xi < \kappa$. Let $D : K \to {}^\kappa 2$ be an injection.

For each $p \in \prod^{<\kappa}_{\alpha \in K} P_\alpha$, there is a $\delta < \kappa$ such that $D(\alpha) \restriction \delta \neq D(\beta) \restriction \delta$ for all $\alpha \neq \beta$ in $\mathrm{dom}(p)$. For $D(\alpha) \restriction \delta$ with $\alpha \in \mathrm{dom}(p)$, define a function J_p by

$J_p(D(\alpha) \upharpoonright \delta) = F_\alpha(p(\alpha))$. Note that $J_p \in \bigcup_{\delta<\kappa}\{{}^d\kappa \mid d \in [2^\delta]^{<\kappa}\}$. By $\kappa^{<\kappa} = \kappa$, $X = \bigcup_{\delta<\kappa}\{{}^d\kappa \mid d \in [2^\delta]^{<\kappa}\}$ is of size κ. For each $J \in X$, let $C_J = \{q \in \prod^{<\kappa}_{\alpha\in K} \mid J_q = J\}$. It is easy to see that each C_J is a centered subset and $\bigcup_J C_J = \prod^{<\kappa}_{\alpha\in K} P_\alpha$. □

Proposition 4.3. *If λ is inaccessible then* $\mathrm{Coll}(\kappa, <\lambda)$ *forces that* $\mathrm{Coll}(\kappa, <\lambda)^V$ *is κ-centered.*

Proof. Note that $\mathrm{Coll}(\kappa, <\lambda)$ forces

- $\mathrm{Coll}(\kappa, <\lambda)^V = (\prod_{\alpha\in[\kappa^+,\lambda)\cap\mathrm{Reg}} {}^{<\kappa}\alpha)^V = \prod^{<\kappa}_{\alpha\in[\kappa^+,\lambda)\cap\mathrm{Reg}^V} ({}^{<\kappa}\alpha)^V$ and
- $|{}^{<\kappa}\alpha| = \kappa$ for all $\alpha \in [\kappa^+, \lambda) \cap \mathrm{Reg}^V$.

Thus, $\mathrm{Coll}(\kappa, <\lambda)^V$ is forced to be a $<\kappa$-support product of $\lambda = 2^\kappa$-many κ-centered posets. By Lemma 4.2, $\mathrm{Coll}(\kappa, <\lambda)^V$ is κ-centered in the extension. □

Note that the same proof shows that $\mathrm{Coll}(\kappa, <\lambda)$ forces $\mathrm{Coll}(\alpha, <\lambda)^V$ is κ-centered for all $\alpha < \kappa$. This proposition shows the negation of absorption property as follows. We say that Q absorbs P if there is a complete embedding from P to Q.

Proposition 4.4. *If λ is inaccessible then* $\mathrm{Coll}(\kappa, <\lambda)$ *does not absorb* $\mathrm{Coll}(\alpha, <\lambda)$ *for every $\alpha \in [\kappa^+, \lambda) \cap \mathrm{Reg}$.*

Proof. Note that $\mathrm{Coll}(\alpha, <\lambda)$ forces that $\mathrm{Coll}(\alpha, <\lambda)^V$ is α-centered. If $\mathrm{Coll}(\kappa, <\lambda)$ absorbs $\mathrm{Coll}(\alpha, <\lambda)$ then $\mathrm{Coll}(\kappa, <\lambda)$ forces $\mathrm{Coll}(\alpha, <\lambda)^V$ is $|\alpha| = \kappa$-centered. This contradicts to Lemma 4.1. □

Note that we can replace Levy collapse by Silver collapse in the above observation.

Let us consider two collapses. One is the Foreman–Laver collapse $R(\kappa, \lambda)$, which was introduced in [3]. $R(\kappa, \lambda)$ is the full support product $\prod_n R^n(\kappa, \lambda)$. Here, $R^{n+1}(\kappa, \lambda) = \prod^{<\kappa}_{\alpha\in[\kappa,\lambda)\cap\mathrm{Reg}} R^n(\alpha, \lambda)$ and $R^0(\kappa, \lambda) = S(\kappa, \lambda)$. If λ is Mahlo then $R(\kappa, \lambda)$ has the λ-c.c. This forces $\kappa^+ = \lambda$. $R(\kappa, \lambda)$ has the following properties:

Proposition 4.5. *If λ is Mahlo, then for every $\alpha \in [\kappa, \lambda) \cap \mathrm{Reg}$,*

(1) *If $\kappa \geq \omega_1$ then $R(\kappa, \lambda)$ forces $R(\alpha, \lambda)^V$ is κ-centered.*
(2) *$R(\kappa, \lambda)$ absorbs $R(\alpha, \lambda)$.*

Proof. First, we show (2). By the definition of $R^n(\kappa, \lambda)$, $R^{n+1}(\kappa, \lambda)$ absorbs $R^n(\alpha, \lambda)$ for each $n < \omega$. Thus $R(\kappa, \lambda)$ absorbs $R(\alpha, \lambda)$.

For (1), by Lemma 4.2 and induction on $n < \omega$, we have $R^{n+1}(\kappa, \lambda)$ forces that $R^n(\alpha, \lambda)$ is κ-centered for all $n < \omega$. By (2), $R(\kappa, \lambda)$ forces $R^n(\alpha, \lambda)$ is κ-centered for all $n < \omega$. Then Lemma 4.2 shows $R(\kappa, \lambda)$ forces $R(\alpha, \lambda)$ is κ-centered by $\kappa \geq \omega_1$. □

Another one is the Easton collapse $E(\kappa, \lambda)$, which was introduced by Shioya [8]. $E(\kappa, \lambda)$ is the Easton support product $\prod^E_{\alpha\in[\kappa^+,\lambda)\cap\mathrm{Reg}} {}^{<\kappa}\alpha$. Note that (2) in Proposition 4.5 holds if we replaced Foreman–Laver collapse by Easton collapse.

Proposition 4.6. *For every $\alpha \in [\kappa^+, \lambda) \cap \mathrm{Reg}$, $E(\kappa, \lambda)$ absorbs $E(\alpha, \lambda)$.*

Proof. See Section 3 in [8]. □

But we don't know for (1).

Question 4.7. *For $\alpha \in [\kappa^+, \lambda)$, does $E(\kappa, \lambda)$ force that $E(\alpha, \lambda)^V$ is not κ-centered?*

This question is related to

Question 4.8. *Is Laver's saturated ideal over* $\aleph_1$ *in* [6] *centered?*

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University of Tsukuba, Tsukuba, 305-8571, Japan
Email address: tukuura@math.tsukuba.ac.jp