FILTERS IN ZFC

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ABSTRACT. This is a survey of several filter constructions in ZFC.

1. Introduction.

There are many filter constructions by means of additional hypothesis to ZFC, such as the Martin's Axiom or the Continuum Hypothesis. Such hypothesis are used in order to construct filters with some predetermined combinatorial properties. The situation is somewhat different when dealing with constructions of filters by only making use of the ZFC-framework. Obviously, each Borel ideal has its own combinatorial properties, and the definition of any Borel ideal provides a construction in ZFC of such ideal. However, we mean constructions of a different flavor, and maybe the main interest of this paper is the techniques involved.

This paper present several classical constructions of filters in ZFC. Some of these techniques have been expanded by some authors. For example, the construction of an ultrafilter with character 2^ω has been expanded in [12] to construct an $F_{\sigma\delta\sigma}$ ideal for which $\mathscr{I}$-ultrafilters exist. In [15], the construction of OK points has been generalized by showing that any meager filter can be extended to an OK point.

The only cardinal invariants we need are the dominating and the unbounding numbers. Given $f, g \in \omega^\omega$, define $f \leq^* g$ if and only if $\{n \in \omega : f(n) \leq g(n)\}$ is cofinite. An family $\mathcal{B} \subseteq \omega^\omega$ is unbounded if for any $f \in \omega^\omega$ there is $g \in \mathcal{B}$ such that $g \not\leq^* f$. A family $\mathcal{D} \subseteq \omega^\omega$ is dominating if for any $f \in \omega^\omega$ there is $g \in \mathcal{D}$ such that $f \leq^* g$. The unbounding number is defined as follows:

$$\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \text{ is an unbounded family}\}$$

and the dominating number is defined as

$$\mathfrak{d} = \min\{|\mathcal{D}| : \mathcal{D} \text{ is an dominating family}\}$$

Let us recall that a family $\mathcal{I} \subseteq [\omega]^\omega$ is an independent family if for any finite partial function $f; \mathcal{I} \to 2$, the set

$$\bigcap_{X \in dom(f)} X^{f(X)}$$

is infinite, where $X^0 = X$ and $X^1 = \omega \setminus X$.

As usual, the natural numbers are denoted by ω, and when needed, the set of positive natural numbers are denoted by $\mathbb{N}$. The cofinite filter is the filter of cofinite sets on ω. The following theorem is widely known:

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Theorem 1.1 (Jalali-Naini, Talagrand, see [13] and [16]). *Let $\mathcal{F}$ be a filter on ω. The following are equivalent:*

(1) *The filter $\mathcal{F}$ is meager.*
(2) *$\mathcal{F}$ has the Baire property.*
(3) *$\mathcal{F}$ is bounded.*
(4) *There is a partition $\mathcal{I} = \langle I_n : n \in \omega \rangle$ of ω into intervals such that any $A \in \mathcal{F}$ has empty intersection with finitely many elements from $\mathcal{I}$.*
(5) *There is a finite to one function $f : \omega \to \omega$ such that $f(\mathcal{F})$ is the cofinite filter.*

Finally, recall that given an filter $\mathcal{F}$, as base for $\mathcal{F}$ is any family $\mathcal{B} \subseteq \mathcal{F}$ such that for any $A \in \mathcal{F}$ there is $B \in \mathcal{B}$ such that $B \subseteq A$; in this case we say that B generates the filter $\mathcal{F}$. The character of a filter is the minimal cardinality of a base for the filter.

2. A Tukey top ultrafilter.

Given two partial orderings $(\mathbf{P}, \leq_{\mathbf{P}})$ and $(\mathbf{Q}, \leq_{\mathbf{Q}})$, a function $f : \mathbf{P} \to \mathbf{Q}$ is cofinal if the image of any cofinal subset of $\mathbf{P}$ is a cofinal subset of $\mathbf{Q}$. If there exists such a map, we say that $\mathbf{Q}$ is Tukey reducible to $\mathbf{P}$, which is written as $\mathbf{Q} \leq_T \mathbf{P}$. In the case that both $\mathbf{Q} \leq_T \mathbf{P}$ and $\mathbf{P} \leq_T \mathbf{Q}$ hold, we say that $\mathbf{P}$ and $\mathbf{Q}$ are Tukey equivalent, which is written as $\mathbf{P} \equiv_T \mathbf{Q}$. N. Dobrinen and S. Todorcevic have extensively studied the Tukey types of ultrafilters on countable sets, that is, the Tukey types of the orderings $(\mathcal{U}, \supseteq)$, where $\mathcal{U}$ is an ultrafilter. They focus mainly on p-points and selective ultrafilters, but also obtain several general results. It is straightforward to see that $\mathcal{V} \leq_{RK} \mathcal{U}$ implies $\mathcal{V} \leq_T \mathcal{U}$. Also, the following relations are pointed out in [8] to be true for any ultrafilters $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$:

(1) $\mathcal{U} \times \mathcal{U} \equiv_T \mathcal{U}$.
(2) $\mathcal{U} \leq_T \mathcal{U} \times \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U} \times \mathcal{V}$.
(3) If $\mathcal{V} \leq_T \mathcal{W}$ and $\mathcal{U} \leq_T \mathcal{W}$, then $\mathcal{U} \times \mathcal{V} \leq_T \mathcal{W}$.

It is also proved that for any directed partial ordering $(P, \leq)$ with cardinality at most $\mathfrak{c}$, it holds $(P, \leq) \leq_T ([\mathfrak{c}]^{<\omega}, \subseteq)$. So in particular, the Tukey type of any ultrafilter on ω is at most the Tukey type of $([\mathfrak{c}]^{<\omega}, \subseteq)$. They actually prove that the maximum is attainable:

Theorem 2.1 (N. Dobrinen, S. Todorcevic, see [8]). *There is an ultrafilter $\mathcal{U}$ such that $(\mathcal{U}, \supseteq) \equiv_T ([\mathfrak{c}]^{<\omega}, \subseteq)$.*

The following lemma is the key in the proof of the previous theorem:

Lemma 2.2 (N. Dobrinen, S. Todorcevic, see [8]). *Let $\mathcal{U}$ be an ultrafilter. Then $(\mathcal{U}, \supseteq) \equiv_T ([\mathfrak{c}]^{<\omega}, \subseteq)$ if and only if there is $X \subseteq \mathcal{U}$ with cardinality $\mathfrak{c}$ such that for any infinite $Y \subseteq X$, $\bigcap Y \notin \mathcal{U}$.*

So everything reduces to construct an ultrafilter having the property stated in the previous lemma. The construction of such ultrafilter is quite well known, and it is in fact the same construction of an ultrafilter having character $\mathfrak{c}$. We need the existence an independent family of cardinality $\mathfrak{c}$:

Lemma 2.3. *There exists an independent family of cardinality $\mathfrak{c}$.*

Proof. There are several constructions of independent families with cardinality $\mathfrak{c}$. We choose the following one. Denote by Z the family of all pairs (a, F) such that $a \in [\omega]^{<\omega}$ and $F \in [[\omega]^{<\omega}]^{<\omega}$. Note that Z has cardinality ω. For each $X \subseteq \omega$. define the set

$$A_X = \{(a, \mathcal{B}) : X \cap a \in \mathcal{B}\}$$

Define $\mathcal{I} = \{A_X : X \subseteq \omega\}$. We claim that $\mathcal{I}$ is an independent family and has cardinality $\mathfrak{c}$. To see the second affirmation, let $X, Y \subseteq \omega$ be two different sets. Assume there is $n \in X \setminus Y$. Then $(\{n\}, \{\{n\}\}) \in A_X \setminus A_Y$. This implies that the function $X \to A_X$ is an inyective function.

On the other hand, let $X_0, \ldots, X_n, Y_0, \ldots, Y_m$ be different subsets of ω. We prove that $\bigcap_{i \leq n} A_{X_i} \cap \bigcap_{j \leq m} Z \setminus A_{Y_j}$ is infinite. For $i \leq n$ and $j \leq m$, pick some $k_{i,j} \in (X_i \setminus Y_j) \cup (Y_j \setminus X_i)$. Now let $B \in [\omega]^{<\omega}$ be such that $k_{i,j} \in B$ for $i \leq n$ and $j \leq m$. Then we have that $B \cap X_i \neq B \cap Y_j$. Define $\mathcal{B} = \{B \cap X_i : i \leq n\}$. Then we have that $(B, \mathcal{B}) \in A_{X_i}$ for all $i \leq n$, and $(B, \mathcal{B}) \notin A_{Y_j}$ for all $j \leq m$. □

Proof of Theorem 2.1. Let $\mathcal{I}$ be an independent family with cardinality $\mathfrak{c}$. Define $\mathcal{F}$ to be the following family:

$$\mathcal{I} \cup \{\omega \setminus \bigcap B : B \subseteq \mathcal{I} \text{ is infinite}\}$$

It is easy to see that $\mathcal{F}$ has the finite intersection property. Let $\mathcal{U}$ be any ultrafilter extending $\mathcal{F}$. Then $\mathcal{I} \subseteq \mathcal{U}$, and by the definition of $\mathcal{F}$, there is no infinite $B \subseteq \mathcal{I}$ such that $\bigcap B \in \mathcal{U}$(since the complement of any infinite intersection of elements of $\mathcal{I}$ belongs to the family $\mathcal{F}$). It follows by Lemma 2.2 that $\mathcal{U}$ has maximal Tukey type.

3. OK-points.

OK-points were introduced by K. Kunen in [14], in order to prove that $\beta\omega^*$ is not homogeneous. Previously to him, it was established under additional hypothesis, such as the Continuum Hypothesis, the non-homogeneity of $\beta\omega^*$. It turns out that the ultrafilter he constructed is a weak p-point: it does not belong to the topological closure of any countable subset of $\beta\omega^*$. His construction makes use of a clever combinatorial device called *independent linked families*. In this section we deal with his construction of OK-points

Definition 3.1. *Let X be a topological space, $p \in X$ and $\langle U_n : n \in \omega \rangle$ a sequence of neighborhoods of p. A κ-refinement system for $\langle U_n : n \in \omega \rangle$ is a sequece $\langle V_\alpha : \alpha \in \kappa \rangle$ of neighborhoods of p such that for all $n \geq 1$, for any $\alpha_0 < \alpha_1 < \ldots < \alpha_{n-1}$, it holds that $\bigcap_{i<n} V_{\alpha_i} \subseteq U_n$.*

Definition 3.2. *Let X be a topological space. A point $p \in X$ is a κ-OK point if for any sequence $\langle U_n : n \in \omega \rangle$ of neighborhoods of p there is a κ-refinement system.*

So we now start with the construction of a $\mathfrak{c}$-OK point in $\beta\omega^*$ by introducing Kunen's independent linked families.

Definition 3.3 (K. Kunen, see [14])**.** *Let $\mathcal{F}$ be a filter on ω. Then:*

(1) *Let n be a positive natural number. A family $\{A_i : i \in I\}$ is precisely n-linked with respect to(w. r. t. from now on) $\mathcal{F}$, if for any $a \in [I]^n$, $\bigcap_{i \in a} A_i \in \mathcal{F}^+$, but for any $a \in [I]^{n+1}$, $\bigcap_{i \in a} A_i$ is finite.*

(2) *An indexed family* $\{A_{i,n} : i \in I, n \in \omega, n > 0\}$ *is a linked system w. r. t.* $\mathcal{F}$*, if for any positive* $n \in \omega$*,* $\{A_{i,n} : i \in I\}$ *is precisely* n*-linked w. r. t.* $\mathcal{F}$*, and for any positive* $n \in \omega$ *and* $i \in I$*,* $A_{i,n} \subseteq A_{i,n+1}$*.*

(3) *An indexed family* $\{A^j_{i,n} : i \in I, j \in J, n \in \omega, n > 0\}$ *is an* $I \times J$ *independent linked family w. r. t.* $\mathcal{F}$*, if for any* $j \in J$*,* $\{A^j_{i,n} : i \in I, n \in \omega, n > 0\}$ *is a linked system w. r. t.* $\mathcal{F}$*, and for any* $\tau \in [J]^{<\omega}$ *and* $\{(n_j, \sigma_j) : j \in \tau\}$ *such that* $\sigma_j \in [I]^{n_j}$*, where* $n_j \in \omega$ *is positive, we have that:*

$$\bigcap_{j\in\tau}\left(\bigcap_{i\in\sigma_j} A^j_{i,n_j}\right) \in \mathcal{F}^+$$

The following lemma shows that independent linked families actually exist.

Lemma 3.4 (K. Kunen, see [14])**.** *There is a* $\mathfrak{c} \times \mathfrak{c}$ *independent linked family with respect to* fin^**.*

Proof. Define the set $S = \{(k, f) : k \in \omega \wedge f \in (\mathcal{P}(\mathcal{P}(k)))^{\mathcal{P}(k)}\}$. Clearly S is a countable set. The independent linked family will be defined over S. For $X, Y \subseteq \omega$ and a positive $n \in \omega$, define $A^Y_{X,n}$ as follows:

$$A^Y_{X,n} = \{(k, f) \in S : |f(Y \cap k)| \le n \wedge X \cap k \in f(Y \cap k)\}$$

Fix $Y \subseteq \omega$. Let us see that for any positive $n \in \omega$, the family $\{A^Y_{X,n} : X \subseteq \omega\}$ is a precisely n-linked system w. r. t. fin. Let $X_0, \dots, X_{n-1}$ be different subsets of ω. Let $k \in \omega$ be big enough so the sets $X_0 \cap k, \dots, X_{n-1} \cap k$ are all different. Then define $f : \mathcal{P}(k) \to \mathcal{P}(\mathcal{P}(k))$ such that $f(Y \cap k) = \{X_0 \cap k, \dots, X_{n-1} \cap k\}$. It follows that $(k, f) \in \bigcap_{i<n} A^Y_{X_i,n}$. On the other hand, for $X_0, \dots, X_n$ all different, there are finitely many k such that $\{X_0 \cap k, \dots, X_n \cap k\}$ contains at most n elements, so the clause $X \cap k \in f(Y \cap k)$ in the definition of $A^Y_{X,n}$ can be true only for finitely many $k \in \omega$. Since $(k, f) \in \bigcap_{i\le n} A^Y_{X_i,n}$ implies $X_i \cap k \in f(Y \cap k)$ for all $i \le n$, and $|f(Y \cap k)| \le n$, there are only finitely many pairs (k, f) which belong to $\bigcap_{i\le n} A^Y_{X_i,n}$. Now fix $X \subseteq \omega$. It follows directly from the definition that if $(k, f) \in A^Y_{X,n}$, then $(k, f) \in A^Y_{X,n+1}$.

Now let us see that for any $Y_0, \dots, Y_l \subseteq \omega$, any positive $n_0, \dots, n_l \in \omega$ and any different $X^i_0, \dots, X^i_{n_i-1} \subseteq \omega$ for $i \le l$, the set

$$\bigcap_{i\le l}\left(\bigcap_{j<n_i} A^{Y_i}_{X_{j,n_i}}\right)$$

is infinite. Let $k \in \omega$ be big enough so $Y_0 \cap k, \dots, Y_l \cap k$ are all different, and for each $i \le l$, the sets $X^i_0 \cap k, \dots, X^i_{n_i-1} \cap k$ are all different. Then define $f : \mathcal{P}(k) \to \mathcal{P}(\mathcal{P}(k))$ as $f(Y_i \cap k) = \{X^i_j \cap k : j < n_i\}$. Then $(k, f) \in \bigcap_{i\le l}\left(\bigcap_{j<n_i} A^{Y_i}_{X_{j,n_i}}\right)$. □

Now we are ready to prove Kunen's theorem.

Theorem 3.5 (K. Kunen, see [14])**.** *There is* $\mathcal{U} \in \beta\omega^*$ *which is a* $\mathfrak{c}$*-OK point.*

Proof. The proof goes over a recursion of length $\mathfrak{c}$. Let $\langle X_\alpha : \alpha < \mathfrak{c} \wedge \alpha \text{ is even}\rangle$ be the enumeration of all subsets of ω. Also, let $\langle \vec{C}_\alpha : \alpha < \mathfrak{c} \wedge \alpha \text{ is odd}\rangle$ the enumeration of all $\subseteq$-decreasing sequences $\vec{C}_\alpha = \langle C^\alpha_n : n \in \omega\rangle$ of infinite subsets of ω, and such that each such sequence appears cofinally often. Let $\{A^{\beta}_{\alpha,n} : \alpha, \beta < \mathfrak{c}, n \in \omega, n > 0\}$ be a $\mathfrak{c} \times \mathfrak{c}$ independent linked family w. r. t. fin^*.

We construct an $\subseteq$-increasing sequence $\langle \mathcal{F}_\alpha : \alpha < \mathfrak{c}\rangle$ of filters and a $\subseteq$-decreasing sequence $\langle K_\alpha : \alpha < \mathfrak{c}\rangle$ of subsets of $\mathfrak{c}$, such that $\mathcal{F}_0 = \mathsf{fin}^*$ and $K_0 = \mathfrak{c}$, and at step γ of the construction the following holds:

(1) $\{A^{\beta}_{\alpha,n} : \alpha < \mathfrak{c}, n \in \omega, n > 0, \beta \in K_\gamma\}$ is an independent linked family w. r. t. $\mathcal{F}_\gamma$.
(2) If γ is a limit ordinal, $\mathcal{F}_\gamma = \bigcup_{\alpha<\gamma} F_\alpha$ and $K_\gamma = \bigcup_{\alpha<\gamma} K_\alpha$.
(3) $K_\gamma \setminus K_{\gamma+1}$ is finite.
(4) If γ is even, then $X_\gamma \in \mathcal{F}_{\gamma+1}$ or $\omega \setminus X_\gamma \in \mathcal{F}_{\gamma+1}$.
(5) If γ is odd and $C^\gamma_n \in \mathcal{F}_\gamma$ for all $n \in \omega$, then there is $\{D_{\gamma,\alpha} : \alpha < \mathfrak{c}\} \subseteq \mathcal{F}_{\gamma+1}$ such that for all positive $n \in \omega$ and $\alpha_0 < \ldots, < \alpha_{n-1} < \mathfrak{c}$, the set $\bigcap_{i<n} D_{\gamma,\alpha_i} \setminus C^\gamma_n$ is finite.

Suppose the construction has been done and define $\mathcal{U} = \bigcup_\gamma \mathcal{F}_\gamma$. Condition (4) makes sure that $\mathcal{U}$ is an ultrafilter, while condition (5) makes sure that $\mathcal{U}$ is a $\mathfrak{c}$-OK point.

So let us see know how to achieve the construction. Assume $\mathcal{F}_\gamma$ and K_γ have been constructed. We have two cases:

(1) γ is even. Let $\mathcal{V}$ be the filter generated by $\mathcal{F}_\gamma \cup \{X_\gamma\}$. If $\mathcal{V}$ is a proper filter and $\{A^{\beta}_{\alpha,n} : \alpha < \mathfrak{c}, n \in \omega, n > 0, \beta \in K_\gamma\}$ is independent w. r. t. $\mathcal{V}$, define $\mathcal{F}_{\gamma+1} = \mathcal{V}$, and $K_{\gamma+1} = K_\gamma$. Otherwise, there are $A \in \mathcal{F}_\gamma$, $\tau \in [K_\gamma]^{<\omega}$ and $\{(n_\beta, \sigma_\beta) : \beta \in \tau\}$ such that $\sigma_\beta \in [\mathfrak{c}]^{n_\beta}$ such that:

$$X_\gamma \cap A \cap \bigcap_{\beta\in\tau} \left(\bigcap_{\alpha\in\sigma_\beta} A^{\beta}_{\alpha,n_\beta} \right) = \emptyset$$

Define $K_{\gamma+1} = K_\gamma \setminus \tau$ and let $\mathcal{F}_{\gamma+1}$ be the filter generated by $\mathcal{F}_\gamma$ and $\bigcap_{\beta\in\tau}\left(\bigcap_{\alpha\in\sigma_\beta} A^{\beta}_{\alpha,n_\beta}\right)$. Then we have that $\omega \setminus X_\gamma \in \mathcal{F}_{\gamma+1}$.

(2) γ is odd. If there is $n \in \omega$ such that $C^\gamma_n \notin \mathcal{F}_\gamma$, define $F_{\gamma+1} = \mathcal{F}_\gamma$ and $K_{\gamma+1} = K_\gamma$. Otherwise, pick some $\beta_0 \in K_\gamma$ and define $K_{\gamma+1} = K_\gamma \setminus \{\beta_0\}$. For $\alpha < \mathfrak{c}$ define $D_{\gamma,\alpha}$ as follows:

$$D_{\gamma,\alpha} = \left(\bigcap_{n\in\omega} C^\gamma_n \right) \cup \left(\bigcup_{n\in\omega, n>0} A^{\beta_0}_{\alpha,n} \cap (C^\gamma_n \setminus C^\gamma_{n+1}) \right)$$

Then define $\mathcal{F}_{\gamma+1}$ as the filter generated by $\mathcal{F}_\gamma$ and $\{D_{\gamma,\alpha} : \alpha < \mathfrak{c}\}$. Let us check that condition (5) is satisfied. Let $n \in \omega$ be any positive number, and choose $\alpha_0 < \ldots, < \alpha_{n-1} < \mathfrak{c}$. Note that if $n = 1$, then $D_{\gamma,\alpha_0} \setminus C_1 = \emptyset$. If $n > 1$ we have the following:

$$\bigcap_{k=1,\ldots,n} D_{\gamma,\alpha_k} \setminus C^\gamma_n \subseteq \bigcap_{k=1,\ldots,n} \left(\bigcup_{m=1,\ldots,n-1} A^{\beta_0}_{\alpha_k,m} \cap (C^\gamma_m \setminus C^\gamma_{m+1}) \right) \subseteq \bigcap_{k=1,\ldots,n} A^{\beta_0}_{\alpha_k,n-1}$$

Where the last inclusion follows from the fact that $A^{\beta_0}_{\alpha,k} \subseteq A^{\beta_0}_{\alpha,k+1}$ for any positive $k \in \omega$. Since the family $\{A^{\beta_0}_{\alpha,n-1} : \alpha < \mathfrak{c}\}$ is $(n-1)$-linked, it follows that $\bigcap_{k=1,\dots,n} A^{\beta_0}_{\alpha_k,n-1}$ is finite.

The fact that $\{A^{\beta}_{\alpha,n} : \alpha < \mathfrak{c}, n \in \omega, n > 0, \beta \in K_{\gamma+1}\}$ is an independent linked family w. r. t. $\mathcal{F}_{\gamma+1}$ follows from $A^{\beta_0}_{\alpha,n} \cap C^{\gamma}_{n} \subseteq D_{\gamma,\alpha}$, which is easy to see it's true.

□

Kunen's construction has been extended in [15] by showing that any meager filter can be extended to a $\mathfrak{c}$-OK point, besides some other results.

4. Adding a real always destroys an ultrafilter.

Preservation of ultrafilters have turned out to be a quite useful tool. The preservation theorem for p-points along countable support iterations has been useful in proving the consistency of several classical cardinal invariant inequalities. However, it is natural to ask if it is possible to preserve all ultrafilters after adding a real to a model V of ZFC. It was proved in [2] the existence of an ultrafilter which is destroyed whenever a real is added to the model V.

Theorem 4.1. *Let* V *and* W *be two models of* ZFC *such that* $\mathsf{V} \subseteq \mathsf{W}$*, and let* $r \in \mathsf{W}$ *be a real which does not belong to* V*. Then there is an ultrafilter* $\mathcal{U}$ *in* V *which does not generate an ultrafilter in* W*.*

Proof. Fix an increasing sequence of natural numbers $\langle k_n : n \in \omega\rangle$. Define a tree $T \subseteq 2^{<\omega}$ such that the following holds:

1. For any $s \in T$, s is an splitting node if and only if $|s| = k_n$ for some $n \in \omega$.
2. For any $n \in \omega$, let $\{s_1, \dots, s_{2^n}\}$ be the lexicographical ordering of 2^{k_n}. Then for any $w \subseteq \mathcal{P}(2^n) \setminus \{\emptyset, 2^n\}$, there is $k \in (k_{n-1}, k_n)$ such that $s_l(m) = 0$ if and only if $l \in w$.
3. There is no $m \in \omega$ such that for all $s \in T \cap 2^{m+1}$, $s(0) = 0$ or for all $n \in T \cap 2^{m+1}$ $s(m) = 1$.

Now, for any subtree $S \subseteq T$, define the following sets:

$$A^0_S = \{m \in \omega : (\forall s \in S \cap 2^{m+1})(s(m) = 0)\}$$

$$A^1_S = \{m \in \omega : (\forall s \in S \cap 2^{m+1})(s(m) = 1)\}$$

Let $\mathscr{I}$ be the ideal generated by the family $\{A^0_S, A^S_1 : S \subseteq T \text{ is a perfect tree}\}$.

Claim 1. The ideal $\mathscr{I}$ is a proper ideal. Let $S_0, \dots S_n$ be perfect subtrees of T. We prove that $\bigcup_{i\leq n} A^0_{S_i} \cup A^1_{S_i}$ does not almost cover ω. Let $n \in \omega$ be such that for all $i \leq m$, $|S_i \cap 2^{k_m}| > n$, and let $\{s_i : i \leq 2^{k_m}\}$ be the lexicographical enumeration of $T \cap 2^{k_m}$. Define $w_i \subseteq 2^{k_m}$ such that $S_i \cap 2^{k_m} = \{s_i : i \in w_i\}$, and $w = \{\min(w_i) : i \leq n\}$. Then $w_i \nsubseteq w$, since w_i has cardinality bigger than n. Also, $w \cap w_i \neq \emptyset$ for all $i \leq n$. By (2) in the construction of T, there is $k \in (k_m, k_{m+1})$ such that $w = \{j : s_j(k) = 0\}$. Now, since $w_j \nsubseteq w$ and $w_j \cap w \neq \emptyset$, there are $s^0, s^1 \in S_j \cap 2^{k_m}$ such that $s^i(k) = i$, for $i \in 2$. It follows that $k \notin \bigcup_{i\leq n} A^0_{S_i} \cup A^1_{S_i}$. Since this happens for infinitely many k_m, the claim follows.

Claim 2. For any ultrafilter $\mathcal{U}$ extending $\mathscr{I}^*$, $\mathcal{U}$ does not generate an ultrafilter in W. Assume towards a contradiction that $\mathcal{U}$ generates an ultrafilter. Let $r \in (2^\omega)^{\mathsf{W}}$ be a real not in V, and define $X_r = \{n \in \omega : r(n) = 1\}$. Then there is $X \in \mathcal{U}$

such that $X \subseteq X_r$. Let us define $S = \{s \in T : (\forall k \in X)(|s| > k \Rightarrow s(k) = 1)\}$. Note that by definition S belongs to V. Also note that r is a branch of S in W, so S contains a perfect subtree in V, say $R \subseteq S$. Then $X \subseteq A^1_R \in \mathscr{I}$, which is a contradiction.

□

More recently the previous theorem has been generalized in [7] by proving that any ultrafilter which is disjoint from the density zero ideal $\mathcal{Z}$ is not preserved as an ultrafilter after adding a new real to the ground model. More generally, any ultrafilter which is not a $\mathcal{Z}$-ultrafilter is destroyed as an ultrafilter after adding a real to the ground model.

5. There is an $\mathscr{I}$-ultrafilter for some $F_{\sigma\delta\sigma}$ ideal.

The notion of $\mathscr{I}$-ultrafilter was introduced by J. Baumgartner in 1992, in his article *Ultrafilters on* ω (see [3]). This notion has proved to be very useful in the classification of combinatorial properties of ultrafilters, and has been extensively studied by several authors, among them we can find J. Baumgartner, J. Brendle, O. Guzmán-González, M. Hrušák and many more. The precise definition is as follows:

Definition 5.1 (J. Baumgartner, see [3]). *Let $\mathscr{I}$ be an ideal on ω. An ultrafilter $\mathcal{U}$ is an $\mathscr{I}$-ultrafilter if for any function $f : \omega \to \omega$, there is $A \in \mathcal{U}$ such that $f[A] \in \mathscr{I}$.*

Several combinatorial properties of ultrafilters can be stated as being an $\mathscr{I}$-ultrafilter for a suitable ideal $\mathscr{I}$, tipically with low Borel complexity. However, it was an open question the existence in ZFC of a Borel ideal $\mathscr{I}$ for which there is an $\mathscr{I}$-ultrafilters. This was answered in the positive by O. Guzmán González and M. Hrušák, and below we reproduce their proof. For the proof it will be essential the concept of independent family, which was defined in section 1. The proof makes use of the cardinal invariant $\mathfrak{ge}(\mathscr{I})$ introduced by J. Brendle and J. Flašková in [5]. The relevance of this cardinal invariant is that the cardinal equation $\mathfrak{ge}(\mathscr{I}) = 2^\omega$ is equivalent to the generic existence of $\mathscr{I}$-ultrafilters, so the strategy is to find a Borel ideal $\mathscr{I}$ for which $\mathfrak{ge}(\mathscr{I}) = 2^\omega$ is a theorem of ZFC. Let us recall that generic existence of $\mathscr{I}$-ultrafilters means that any filter $\mathcal{F}$ which is generated by strictly less than continuum many sets, there is an $\mathscr{I}$-ultrafilter extending $\mathcal{F}$.

Definition 5.2 (J. Brendle, J. Flašková, see [5]). *Let $\mathscr{I}$ be an ideal on ω, The cardinal invariant $\mathfrak{ge}(\mathscr{I})$ is defined as follows:*

$$\mathfrak{ge}(\mathscr{I}) = \min\{\mathsf{cof}(\mathscr{J}) : \mathscr{J} \text{ is an ideal and } \mathscr{I} \subseteq \mathscr{J}\}$$

Lemma 5.3 (J. Brendle, J. Flašková, see [5]). *Let $\mathscr{I}$ be an ideal over ω. Then $\mathscr{I}$-ultrafilters exist generically if and only if $\mathfrak{ge}(\mathscr{I}) = 2^\omega$.*

Definition 5.4 (O. Guzmán-González, M. Hrušák, see [12]). *A tree $T \subseteq 2^{<\omega}$ is said to be independent if the family of all brances of T, denoted by $[T]$, is an independent family. For an independet tree $[T]$, there is an associated ideal denoted by $Pos(T)$, which is defined as the ideal generated by the family $\{\omega \setminus A : A \in [T]\} \cup \{\bigcap \mathcal{C} : \mathcal{C} \subseteq [T] \wedge \mathcal{C}$ is countable$\} \cup [\omega]^{<\omega}$.*

Lemma 5.5 (O. Guzmán-González, M. Hrušák, see [12]). *The ideal $Pos(T)$ is a proper ideal.*

Proof. This follows easily from the fact that $Pos(T)$ is an independent family. □

Lemma 5.6 (O. Guzmán-González, M. Hrušák, see [12]). *For any independent tree $T \subseteq 2^{<\omega}$, $\mathfrak{ge}(Pos(T)) = 2^\omega$.*

Proof. This is essentially the same proof as Theorem 2.1. □

So the previous two lemmas show the existence of an analytic ideal $\mathscr{I}$ for which it holds $\mathfrak{ge}(\mathscr{I}) = 2^\omega$. However, it is not clear that the complexity of such ideal can be Borel by only choosing an appropriated tree T. The solution given in [12] consisted in finding a Borel ideal which contains $Pos(T)$. Note that for any two ideals $\mathscr{I} \subseteq \mathscr{J}$, it follows that $\mathfrak{ge}(\mathscr{I}) \leq \mathfrak{ge}(\mathscr{J})$, so for any Borel ideal $\mathscr{I}$ such that $Pos(T) \subseteq \mathscr{I}$ we have that $\mathfrak{ge}(\mathscr{I}) = 2^\omega$, and then $\mathscr{I}$-ultrafilters exist generically. So, aiming to find a Borel ideal extending $Pos(T)$:

Definition 5.7 (O. Guzmán-González, M. Hrušák, see [12]). *For a tree T and $m \in \omega$, let $Z_m(T)$ be the family of all inyective sequences having length m of elements of $[T]$. Let T be an independent tree:*

(1) *For $x \in [[T]]^n$, define $C(x) = \bigcup_{c\in x} \omega \setminus c$ and $D(x) = \bigcap_{c\in x} c$.*
(2) *For $x \in [[T]]^n$ and $y_1, \dots, y_k \in Z_m(T)$, define $H(x, y_1, \dots, y_k) = C(x) \cup \bigcup_{i=1,\dots k} D(y_i)$.*
(3) *For positive $n \in \omega$, $\mathcal{H}_n$ is defined as the family of all sets $A \subseteq \omega$ such that for any $m > n$ there are $k \geq 1$, $x \in [T]^n$ and $y_1, \dots, y_k \in Z_m(T)$ such that $A \subseteq H(x, y_1, \dots, y_k)$.*

Finally define $Pos_B(T) = \bigcup_{k\in\omega} \mathcal{H}_k$.

Theorem 5.8 (O. Guzmán-González, M. Hrušák, see [12]). *$Pos_B(T)$ is a proper $F_{\sigma\delta\sigma}$ ideal whenever T is an independent tree. Moreover, $Pos(T) \subseteq Pos_B(T)$, so it follows that $Pos_B(T)$-ultrafilters exist generically.*

Proof. Fix $A, B \in Pos_B(T)$ and let $k \in \omega$ be such that $A, B \in \mathcal{H}_k$(clearly such k exist since $\langle \mathcal{H}_k : k \in \omega \rangle$ is $\subseteq$-increasing). We will see that $A \cup B \in \mathcal{H}_{2k}$. Fix $m > 2k$. By definition of $\mathcal{H}_k$, there are $a, b \in [[T]]^k$ and $\bar{a}_1, \dots, \bar{a}_{k_1} \in Z_m(T)$ and $\bar{b}_1, \dots, \bar{b}_{k_2} \in Z_m(T)$ such that $A \subseteq H(a, \bar{a}_1, \dots, \bar{a}_{k_1})$ and $B \subseteq H(b, \bar{b}_1, \dots, \bar{b}_{k_2})$. It follows that $A \cup B \subseteq H(a \cup b, a, \bar{a}_1, \dots, \bar{a}_{k_1}, \bar{b}_1, \dots, \bar{b}_{k_2}) \in \mathcal{H}_{2k}$. This implies that $Pos_{\mathcal{B}}(T)$ is closed under finite unions. The fact that $Pos_{\mathcal{B}}(T)$ is a proper ideal follows from $[T]$ being an independent family and thus $[T]$ generates a proper ideal.

To see that $Pos(T) \subseteq Pos_B(T)$, pick any $x \in Pos(T)$ and let $F \subseteq [T]$ be finite and a countable $C \subseteq [T]$ such that $x \subseteq \left(\bigcup_{A\in F} \omega \setminus A\right) \cap \bigcap C$. Thus, for any $m > 1$, fix $A \in F$ and choose $B \in [C]^m$, $\omega \setminus x \subseteq A \cup \bigcap B$, so $x \in \mathcal{H}_1$.

Let us see that $Pos_{\mathcal{B}}(T)$ has Borel complexity $F_{\sigma\delta\sigma}$. For positive $n, m \in \omega$, $n < m$, define the following:

(1) $a_m(T) = \{(s_1, \dots, s_m) \in T^m : (\forall i, j \leq m)(i \neq j \longrightarrow s_i \nsubseteq s_j \wedge s_j \nsubseteq s_i)\}$.
(2) For $\vec{s} \in a_m(T)$ define $\langle \vec{s} \rangle = \{(x_1, \dots, x_m) : x_i \in [T \restriction s_i]\}$.
(3) For positive $k \in \omega$ and $\vec{s}_1, \dots, \vec{s}_k \in a_m(T)$:

$$\mathcal{G}(n, m, \vec{s}_1, \dots, \vec{s}_k) = \{H(x, \vec{y}_1, \dots, \vec{y}_k) : x \in [T]^n \wedge \forall i \leq n, \vec{y}_i \in \langle \vec{s}_i \rangle\}$$

(4) $\mathcal{G}^d(n, m, \vec{s}_1, \dots, \vec{s}_k) = \{A \subseteq \omega : (\exists B \in G(n, m, \vec{s}_1, \dots, \vec{s}_k)(A \subseteq B))\}$
(5) $\mathcal{B}(n, m) = \bigcup_{k>0} \{\mathcal{G}^d(n, m, \vec{s}_1, \dots, \vec{s}_k) : \vec{s}_1, \dots, \vec{s}_k \in a_m(T)\}$.

Note that $\mathcal{G}(n, m, \vec{s}_1, \ldots, \vec{s}_k)$ is a closed set, so $\mathcal{G}^d(n, m, \vec{s}_1, \ldots, \vec{s}_k)$ is closed as well. Then $\mathcal{B}(n, m)$ is an F_σ set, and $\mathcal{H}_n = \bigcap_{m>n} \mathcal{B}(n, m)$, which is an $F_{\sigma\delta}$ set. It follows that $Pos_{\mathcal{B}}(T) = \bigcup_{n\in\omega} \mathcal{H}_n$, which is an $F_{\sigma\delta\sigma}$ set.

□

It is worth mentioning that, regarding generic existence of $\mathscr{I}$-ultrafilters and Borel complexity, the previous theorem is optimal: it has been proved in [12] that consistently there is no $F_{\sigma\delta}$ ideal for which generic existence of $\mathscr{I}$-ultrafilters holds.

6. Friendly $\mathscr{I}_{1/n}$-ultrafilter.

The previous section established the existence of an $F_{\sigma\delta\sigma}$ ideal $\mathscr{I}$ for which $\mathscr{I}$-ultrafilters exist generically. It has been proved in [6] that consistently there is no F_σ ideal $\mathscr{I}$ for which $\mathscr{I}$-ultrafilters exist, and moreover, consistently there is no weak $\mathscr{I}$-ultrafilter. As a counterpart, J. Flašková has proved in ZFC the existence of friendly $\mathscr{I}_{1/n}$-ultrafilters, where $\mathscr{I}_{1/n}$ is the summable ideal defined on $\mathbb{N}$ by the function $f(n) = \frac{1}{n}$, and friendly means that for any inyective function $f : \mathbb{N} \to \mathbb{N}$, there is $A \in \mathscr{I}_{1/n}$ such that $f^{-1}[A]$ is an element of the ultrafilter (see [9]). This is an improvement of a Gryzlov's theorem, which states the existence of frendly $\mathcal{Z}$-ultrafilters (see [10, 11]). So, Flašková and Gryzlov's results are optimal when passing from inyective functions to finite to one functions. In this section we present Flašková's construction of a friendly $\mathscr{I}_{1/n}$-ultrafilter.

In this section $\mathbb{N}$ denotes the positive natural numbers, *summable* set $A \subseteq \mathbb{N}$ is a set such that $\sum_{n\in A} \frac{1}{n}$ is finite, and a family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is summable if for any inyective $f : \mathbb{N} \to \mathbb{N}$ there is $A \in \mathcal{F}$ such that $f[A]$ is a summable set.

Lemma 6.1. *For every positive natural number k, it holds $\sum_{m=1}^{k} \frac{1}{m} \leq 1 + ln(k) \leq 1 + log_2(k)$.*

Lemma 6.2 (J. Flašková, see [9])**.** *Let $\langle \mathcal{F}_k : k \in \mathbb{N}\rangle$ be such that for every $k \in \mathbb{N}$, $\mathcal{F}_k$ a k-linked family of infinite subsets of $\mathbb{N}$. Then $\mathcal{F} = \{F \subseteq \mathbb{N} : (\forall k \in \mathbb{N})(\exists A \in \mathcal{F})(A \subseteq^* F)\}$ is a centered family. If moreover, $\mathscr{I}$ is a p-ideal and $f : \mathbb{N} \to \mathbb{N}$ is an inyective function such that for all $k \in \mathbb{N}$ there is $U_k \in \mathcal{F}_k$ such that $f[U_k] \in \mathscr{I}$, then there is $U \in \mathcal{F}$ such that $f[U] \in \mathscr{I}$.*

Proof. Pick $F_1, \ldots, F_n \in \mathcal{F}$. Then for every $k \geq n$ and $i \leq n$, there is $A_i^k \in \mathcal{F}_i$ such that $A_i^k \subseteq^* F_k$. Since $\mathcal{F}_k$ is k-linked, $\bigcap_{j=1,\ldots,n} A_i^k$ is infinite and it is almost contained in $\bigcap_{j=1,\ldots,n} F_j$. Thus, $\mathcal{F}$ is a centered family.

Now, if $\{U_k : k \in \omega\} \subseteq \mathcal{F}$ and inyective $f : \mathbb{N} \to \mathbb{N}$ are such that $f[U_k] \in \mathscr{I}$ for all k, with $\mathscr{I}$ being a p-ideal, then there is $Z \in \mathscr{I}$ such that $f[U_k] \subseteq^* \subseteq Z$ for all k, and since f is inyective, it follows that $U_k \subseteq^* f^{-1}[Z]$, and therefore $f^{-1}[Z] \in \mathcal{F}$. □

Lemma 6.3 (J. Flašková, see [9])**.** *Let A be an infinite subset of ω. For every $k \in \mathbb{N}$ there is $\mathcal{F}_k \subset \mathcal{P}(A)$ which is a summable k-linked family.*

Proof. Following [9], in this proof $\prod$ denotes the product of sets and $\odot$ denotes the product of numbers. Fix a positive $k \in \omega$. For any $n \in \mathbb{N}$ and $j \leq k$, define $q_j^n = 2^{2^j n}$; note that for $i \leq k$, $2^n \odot_{j=0}^{k} q_j^n = 2^{n+\sum_{j=0}^{i} q_j^n} = 2^{n+(2^i-1)n} = 2^{2^i n}$. Now, let $\langle B_n : n \in \mathbb{N}\rangle$ be a partition of A into finite sets such that $B_n = \{b_\phi : \phi \in \prod_{j=0}^{k} q_j^n\}$.

For each $i \leq k$, $m \in q_i^n$ and $s \in \prod_{j=i+1}^{k} q_j^n$, define $B_n(i, m, s) = \{b_{\eta^\frown m^\frown s} : \eta \in \prod_{j=0}^{i-1} q_j^n\}$. Given an inyective $f : \mathbb{N} \to \mathbb{N}$, define $l_m^f = \min(f[B_n(i, m, s)])$ and

$m(f,s) \in q_i^n$ such that $l^f_{m(f,s)} = \max\{l^f_m : m \in q_i^n\}$. Now, given $f : \mathbb{N} \to \mathbb{N}$ and $i \leq k$, define the following:

(1) $B_n^f(i,s) = B_n(i, m(f,s), s)$.
(2) $B_n^f(i) = \bigcup\{B_n^f(i,s) : s \in \prod_{j=i+1}^k q_j^n\}$.
(3) $B_n^f = \bigcup_{i=0,\ldots,k} B_n^f(i)$.
(4) $A_f = \bigcup_{n\in\mathbb{N}} B_n^f$.

The family $\{A_f : f : \mathbb{N} \to \mathbb{N} \text{ is inyective}\}$ is k-linked. Let $f_1, \ldots, f_k : \mathbb{N} \to \mathbb{N}$ be inyective functions. For each $n \in \mathbb{N}$ we will find $\phi \in \prod_{j=0}^k q_j^n$ such that $b_\phi \in \bigcap_{j=0}^k B_n^{f_j}$. The sequence ϕ is constructed recursively backwards: start with $s_0 = \emptyset$ and $s_1 = \langle m(f_1, s_0)\rangle$. Suppose s_i is defined, then let $s_{i+1} = m(f_{i+1}, s_i)^\frown s_i$ be the next element of the sequence. Define $\phi = s_k$. Note that by construction $b_\phi \in \bigcap_{j=0,\ldots,k} B_n^{f_j}(k-j, s_j) \subseteq \bigcap_{j=0,\ldots,k} B_n^{f_j}$.

What remains is to prove that $f[A_f]$ is summable. Note first that by definition of $B_n^f(i,s)$ we have $|B_n^f(i,s)| = \odot_{j=0}^{i-1} q_j^n$, so

$$\sum_{a\in B_n^f(i)} \frac{1}{f(a)} \leq \frac{\odot_{j=0}^{i-1} q_j^n}{\min(f[B_n^f(i,s)])} = \frac{2^{n(2^i-1)}}{l^f_{m(f,s)}}$$

Define $r_i^n = \odot_{j=i+1}^k q_j^n$ and let $\{m_l : l = 1, \ldots, r_i^n\}$ be the increasing enumeration of $\{l^f_{m(f,s)} : s \in \prod_{j=i+1}^k q_j^n\}$. Note that $m_l \geq l \cdot q_i^n$, so we have, by Lemma 6.1,

$$\sum_{l=1}^{r_i^n} \frac{1}{m_l} \leq \frac{1}{q_i^n} \sum_{l=1}^{r_i^n} \frac{1}{l} \leq$$

$$\frac{1 + log_2(r_i^n)}{q_i^n} \leq \frac{1 + \sum_{j=i+1}^k log_2(q_j^n)}{q_i^n} \leq$$

$$\frac{q + n\sum_{j=0}^k 2^j}{q_i^n} = \frac{1 + (2^{k+1}-1)n}{q_i^n} \leq \frac{2^{k+1}n}{q_i^n}$$

Thus, we have

$$\sum_{a\in B_n^f} \frac{1}{f(a)} \leq \sum_{j=0}^k \sum_{a\in B_n^f(j)} \frac{1}{f(a)} \leq \frac{2^{k+1}n(k+1)}{q_i^n}$$

And since $q_i^n = 2^{2^i n} \geq 2^n$, it also holds that

$$\sum_{a\in B_n^f} \frac{1}{f(a)} \leq \frac{2^{k+1}n(k+1)}{2^n}$$

So taking the sum overall A_f,

$$\sum_{a\in A_f} \frac{1}{f(a)} \leq 2^{k+2}(k+1)$$

which concludes the proof. □

Theorem 6.4 (J. Flašková, see [9]). *There is a friendly $\mathscr{I}_{1/n}$-ultrafilter.*

Proof. Apply the previous lemma to each $k \in \mathbb{N}$ and $\mathbb{N}$ to obtain a summable $k-$linked family. Then apply Lemma 6.2 to the family $\{\mathcal{F}_k : k \in \mathbb{N}\}$ to obtain a summable centered family $\mathcal{F}$. Then extend $\mathcal{F}$ to an ultrafilter $\mathcal{U}$. Since $\mathcal{F}$ provides witnesses of summability for any inyective f, it follows that U is a friendly $\mathscr{I}_{1/n}$-ultrafilter. □

7. A measurable filter without the Baire property.

Null filters have a similar characterization to the Theorem 1.1, given by T. Bartoszyński in 1992, which is needed for the proof of Theorem 7.2. It is the following theorem:

Theorem 7.1 (T. Bartoszyński, see [1]). *Let $\mathcal{F}$ be a filter on ω. Then $\mathcal{F}$ is a measurable filter if and only if there is a family $\langle F_n : n \in \omega \rangle$ such that the following holds:*

(1) *For each $n \in \omega$, F_n is a finite collection of finite subsets of ω.*
(2) *For different $n, m \in \omega$, $\bigcup F_n \cap \bigcup F_m = \emptyset$.*
(3) $\sum_{n=0}^{\infty} \mu(\{X \subseteq \omega : (\exists a \in F_n)(a \subseteq X)\}) < \infty$.
(4) $\mathcal{F} \subseteq \bigcap_{n \in \omega} \bigcup_{m \geq n} \{X \subseteq \omega : (\exists a \in F_m)(a \subseteq X)\}$.

Theorem 7.2 (T. Bartoszyński, see [1]). *Every measurable filter can be extended to a measurable filter without the Baire property.*

Proof. Let $\mathcal{F}$ be a measurable filter on ω and $\langle F_n : n \in \omega \rangle$ the family given by the previous theorem. Given $X \subseteq \omega$, define

$$supp(X) = \{y \in \bigcup_{n \in \omega} F_n : y \subseteq X\}$$

Define $\mathcal{H} = \{supp(X) : X \in \mathcal{F}\}$, which is a centered family since $\mathcal{F}$ is a filter and the choice of the family $\langle F_n : n \in \omega \rangle$. Let $\mathcal{U}$ be an ultrafilter extending $\mathcal{H}$ and define $\tilde{\mathcal{F}} = \{X \in \mathcal{P}(\omega) : supp(X) \in \mathcal{U}\}$. Then $\tilde{\mathcal{F}}$ is a measurable filter, since $\mathcal{U}$ is an ultrafilter and $\langle F_n : n \in \omega \rangle$ still witnesses that $\tilde{\mathcal{F}}$ has measure zero. Let us see that $\tilde{\mathcal{F}}$ is not meager. Let $\{P_n : n \in \omega\}$ be a partition of ω such that for all $n \in \omega$ $F_n \subseteq \mathcal{P}(P_n)$. Pick any interval partition $\langle I_n : n \in \omega \rangle$ and partition ω into finite sets $\langle B_n : n \in \omega \rangle$ such that for each $n \in \omega$ there is $k \in \omega$ such that $I_k \subseteq \bigcup_{i \in B_n} P_i$. Since $\mathcal{U}$ is an ultrafilter, there is $Z \subseteq \omega$ coinfinite such that $\bigcup_{n \in Z} \bigcup_{i \in B_n} F_i \in \mathcal{U}$. Define $X = \bigcup_{n \in Z} \bigcup_{i \in B_n} F_i$. Then $X \in \tilde{F}$ and X has empty intersection with infinitely many intervals I_n, namely, those contained in $\bigcup_{i \in B_n} P_i$ for some $n \notin Z$. Since $\langle I_n : n \in \omega \rangle$ was arbitrary, it follows that $\tilde{\mathcal{F}}$ is not meager.

□

One may be tempted to ask about a meager filter which is not measurable. However, it has been proved in [2] that every filter with the Baire property is measurable, so the existence of a nonmeasurable meager filter is not provable in ZFC.

8. A non-feeble ultrafilter with character $\mathfrak{b}$.

A feeble filter is a filter $\mathcal{F}$ for which there is a finite to one function such that $f(\mathcal{F})$ is the cofinite filter. It follows from Jalali-Naini-Talagrand theorem that any meager $\mathcal{F}$ filter is feeble, and ultrafilters are easily seen to be not feeble, and more generally, any unbounded[1] filter is not feeble, by Jalali-Naini-Talagrand's thereom again. In this section we present a construction of a non-feeble filter having character $\mathfrak{b}$. This construction appears in [4].

Definition 8.1. *A filter $\mathcal{F}$ on ω is a feeble filter if there is a finite to one function $f : \omega \to \omega$ such that $f(\mathcal{F})$ is the frechet filter.*

We need the following lemma. Given a function $f : \omega \to \omega$, f^{it} is defined recursively as $f^{it}(0) = 0$ and $f^{it}(n+1) = f(f^{it}(n))$.

Lemma 8.2. *Let $\mathcal{D}$ be a dominating family of strictly increasing functions and such that $f(0) > 0$ for all $f \in \mathcal{D}$. Let $g \in \omega^\omega$ be a strictly increasing function. There is $f \in \mathcal{D}$ such that for all but finitely many $n \in \omega$, there exist $m \in \omega$ such that $f^{it}(n) < g(m) < g(m+1) < f^{it}(n+1)$.*

Proof. Let $g \in \omega^\omega$ be an strictly increasing function. Define $h_g : \omega \to \omega$ as follows:

$$h_g(n) = \min\{k \in \omega : k \geq n \wedge (\exists m \in \omega)(n < g(m) < g(m+1) < k)\}$$

Let $f \in \mathcal{D}$ dominating h_g. Let $k_0 \in \omega$ be such that for all $k \geq k_0$, $h_g(k) \leq f(k)$. Note that $f(k) \leq f^{it}(k)$. Then we have that for any $k \geq k_0$, $h_g(f^{it}(k)) \leq f(f^{it}(k)) = f^{it}(k+1)$, so by the definition of h_g, there is $m \in \omega$ such that $f^{it}(k) < g(m) < g(m+1) < f^{it}(k+1)$. □

Theorem 8.3. *There is a non-feeble filter with character $\mathfrak{b}$.*

Proof. We have two cases:

Case 1: $\mathfrak{b} < \mathfrak{d}$. Let $\mathcal{B} \subseteq \omega^\omega$ be an unbounded family with cardinality $\mathfrak{b}$ which is closed under $\max\{f_0, \ldots, f_n\}$, and each element from $\mathcal{B}$ is strictly increasing. Since $\mathfrak{b} < \mathfrak{d}$, there is $h \in \omega^\omega$ increasing which is not bounded by $\mathcal{B}$. For any $f \in \mathcal{B}$, define $A_f = \{n \in \omega : f(n) < h(n)\}$, and let $\mathcal{F}$ be the filter generated by $\{A_f : f \in \mathcal{B}\}$(since $\mathcal{B}$ is closed under taking maximum of finitely many elements of it, the family $\{A_f : f \in \mathcal{B}\}$ is closed under finite intersections). We claim that $\mathcal{F}$ is not bounded. Assume otherwise and let $\langle I_n : n \in \omega\rangle$ be such that any $A \in \mathcal{F}$ avoids finitely many intervals I_n. Define $g(n) = h(\max(I_{k+1}))$ where k is such that $n \in I_k$. Then g dominates each element in $\mathcal{B}$. Indeed, let $f \in \mathcal{B}$ and let $k_0 \in \omega$ be such that for all $m \geq k_0$, $A_f \cap I_m \neq \emptyset$, pick $n \geq min(I_{k_0})$, let $k \in \omega$ be such that $n \in I_k$. Then there is $i \in A_f \cap I_{k+1}$ such that $f(i) < h(i)$, and since f and h are a strictly increasing functions it follows that $f(n) < f(i) < h(i) < h(\max(I_{k+1})) = g(n)$.

Case 2: $\mathfrak{b} = \mathfrak{d}$. Let $\mathcal{D}$ be a dominating family of strictly increasing functions and well ordered by $\leq^*$ (this is possible since $\mathfrak{b} = \mathfrak{d}$). Given two interval partitions $\mathcal{I}$ and $\mathcal{J}$, define $\mathcal{J} \sqsubseteq \mathcal{I}$ if all but finitely many intervals from $\mathcal{I}$ contain an interval from $\mathcal{J}$. Note that Lemma 8.2 implies that from $\mathcal{D}$ we can construct a family $\tilde{\mathcal{D}}$ of interval partitions such that given any other interval partition $\mathcal{J}$, there is $\mathcal{I} \in \tilde{\mathcal{D}}$ such that $\mathcal{J} \sqsubseteq \mathcal{I}$, and moreover, $\tilde{\mathcal{D}}$ is well ordered by $\sqsubseteq$. A straightforward modification of the argument for Lemma 8.2 shows that for any family $\mathcal{A} \subseteq [\omega]^\omega$ with cardinality smaller than $\mathfrak{b}$, there is a partition $\mathcal{I}$ such that for any $A \in \mathcal{A}$, all but finitely many intervals from $\mathcal{I}$ have non-empty intersection with $\mathcal{A}$. We construct a sequence of centered families $\langle \mathcal{F}_\alpha : \alpha < \mathfrak{d}\rangle$ recursively as follows:

(1) $\mathcal{F}_0$ is the cofinite filter.

[1] A filter $\mathcal{F}$ is unbounded if the family $\{e_A : A \in \mathcal{F}\}$ is an unbounded family, where e_A is the increasing enumeration of the set A.

(2) For limit ordinal α, $\mathcal{F}_\alpha = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$.
(3) If $\mathcal{F}_\alpha$ is defined, then there is $\gamma_\alpha \geq \alpha$ such that almost every interval from $\mathcal{I}_{\gamma_\alpha}$ intersects all the elements from the filter generated by $\mathcal{F}_\alpha$(this is possible since $\mathcal{F}_\alpha$ has cardinality smaller than $\mathfrak{b}$ and the remark following the definition of $\tilde{D}$ in the previous paragraph). Let X_α be the union of the even intervals from $\mathcal{I}_{\gamma_\alpha}$ and define $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{X_\alpha\}$.

Let $\mathcal{F}$ be the filter generated by $\bigcup_{\beta<\mathfrak{d}} \mathcal{F}_\beta$. Note that by construction, for any interval $\mathcal{I}$ partition there is $A \in \mathcal{F}$ such that A has empty intersection with infinitely many intervals from $\mathcal{I}$. By the Jalali-Naini-Talagrand Theorem 1.1 $\mathcal{F}$ is not a feeble filter. □

9. An ultrafilter which is not a p-point neither a q-point.

Finally, there is the following well known result:

Proposition 9.1. *There is an ultrafilter which is not a p-point neither a q-point.*

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter and define:

$$\mathcal{U} \times \mathcal{U} = \{A \subseteq \omega \times \omega : \{m \in \omega : (A)_m \in \mathcal{U}\} \in \mathcal{U}\}$$

Where $(A)_m = \{n \in \omega : (m, n) \in A\}$. It is easy to see that $\mathcal{U} \times \mathcal{U}$ is an ultrafilter. It is not a p-point since $\{[n, \infty) \times \omega : n \in \omega\}$ has no pseudointersection in $\mathcal{U} \times \mathcal{U}$. It is not a q-point since the projection on the second coordinate has a restriction which is finite to one, but no restriction is one to one. □

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