Generically supercompact cardinals by forcing with chain conditions

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Abstract

A ccc-generically supercompact cardinal κ can be smaller than or equal to the continuum. On the other hand, such a cardinal κ still satisfies diverse largeness properties, like that it is a stationary limit of ccc-generically measurable cardinals (Theorem 4.1). This is in a strong contrast to $\mathcal{P}$-generically supercompact cardinals for the class $\mathcal{P}$ of all σ-closed posets, which can be $\aleph_n$ for any $n > 1$.

1 Introduction and preliminaries.

For a class $\mathcal{P}$ of posets, we say that a cardinal κ is *$\mathcal{P}$-generically measurable* (*$\mathcal{P}$-g. measurable*, for short) if there is $\mathbb{P} \in \mathcal{P}$ such that, for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that $\mathsf{V}[\mathbb{G}] \models j : \mathsf{V} \prec\!\!\to_\kappa M$ holds[1)]. If κ is $\{\mathbb{P}\}$-generically measurable, we shall also say that κ is $\mathbb{P}$-generically measurable.

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[1)] When we write $j : N \prec\!\!\to_\kappa M$, we mean N is a transitive model (possibly a class model) of ZFC^-, j is an elementary embedding of V into M, M is transitive, and κ is the critical point of j.

A cardinal κ is *$\mathcal{P}$-generically λ-supercompact* (*$\mathcal{P}$-g. λ-supercompact*, for short) for a given cardinal $\lambda \geq \kappa$, for short) if there is $\mathbb{P} \in \mathcal{P}$ with a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ and j, $M \subseteq \mathsf{V}[\mathbb{G}]$ with

$$\mathsf{V}[\mathbb{G}] \models j : \mathsf{V} \overset{\prec}{\to}_\kappa M,\ j(\kappa) > \lambda,\ j''\lambda \in M. \tag{1.1}$$

A cardinal κ is *$\mathcal{P}$-generically supercompact* (*$\mathcal{P}$-g. supercompact*, for short) if it is $\mathcal{P}$-g. λ-supercompact for all $\lambda \geq \kappa$.

Clearly, for $\kappa < \lambda < \lambda'$, $\mathcal{P}$-g. λ'-supercompactness of κ implies $\mathcal{P}$-g. λ-supercompactness of κ and $\mathcal{P}$-g. κ-supercompactness of κ is equivalent to $\mathcal{P}$-g. measurability of κ.

In the following we mainly consider the cases in which $\mathcal{P}$ is the class of all ν-cc posets for some uncountable ν. In this case, we shall say *ν-cc-generically measurable* (*ν-cc-g. measurable*, for short), or *ν-cc-generically λ-supercompact* (*ν-cc-g. λ-supercompact*, for short), in place of $\mathcal{P}$-generically measurable or $\mathcal{P}$-generically λ-supercompact, respectively.

Starting from a measurable (supercompact, resp.) cardinal κ, it is easy to obtain a model in which a ccc-g. measurable (supercompact, resp.) cardinal is less than or equal to the continuum. Actually, forcing with $\mathrm{Fn}(\lambda, 2)$ for any $\lambda \geq \kappa$ will create such a model.

We can also consider the generic versions of weak compactness: A cardinal κ is said to be *$\mathcal{P}$-generically weakly compact* for a class $\mathcal{P}$ of posets (or *$\mathcal{P}$-g. weakly compact*, for short), if, for any $A \subseteq \kappa$ ($A \in \mathsf{V}$), there is a transitive set model M of ZFC^- with κ, $A \in M$ such that, for some $\mathbb{P} \in \mathcal{P}$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have $j : M \overset{\prec}{\to}_\kappa N$ for some j, $N \in \mathsf{V}[\mathbb{G}]$.

We shall also say *ν-cc-g. weakly compact* etc. similarly to above.

Lemma 1.1 *For a class $\mathcal{P}$ of posets, if κ is $\mathcal{P}$-generically measurable then κ is $\mathcal{P}$-generically weakly compact.*

Proof. Suppose that $\mathbb{P} \in \mathcal{P}$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ are such that there are j^*, $M^* \subseteq \mathsf{V}[\mathbb{G}]$ with $j^* : \mathsf{V} \overset{\prec}{\to}_\kappa M^*$. Let $M := \mathcal{H}(\kappa^+)$, $N := \mathcal{H}(j^*(\kappa)^{+^{M^*}})^{M^*}$ and $j := j^* \restriction M$. Then, these M, $\mathbb{P}$, $\mathbb{G}$, j, N are witnesses of the property in the definition of the $\mathcal{P}$-g. weakly compactness for all $A \subseteq \kappa$. □ (Lemma 1.1)

We refer mainly [14] for results in connection with precipitousness and generic ultrapower while our notation tend to be more compatible with that of [15]. Names in forcing are denoted by alphabets with undertilde adopting the notation of [17].

2 Generically weakly compact cardinals

Lemma 2.1 *Suppose that κ is ν-cc-g. weakly compact for a $\nu < \kappa$. Then*

(1) *κ is weakly Mahlo.*

(2) *κ has the tree property.*

Proof. Assume that κ is ν-cc-g. weakly compact.

(1): (a) First, we prove that κ is not a successor cardinal[2]. Suppose, toward a contradiction, that κ is a successor cardinal, say $\kappa = \mu^+$. Note that $\nu \leq \mu$.

Let $A \subseteq \kappa$ be a set which codes $\langle s_\xi : \xi < \kappa \rangle$ where each s_ξ for $0 < \xi < \kappa$ is a surjection from μ to ξ.

Let M be a transitive model of ZFC^- such that κ, $A \in M$ and there is a ν-cc poset $\mathbb{P}$ with $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ such that there are j, $N \in \mathsf{V}[\mathbb{G}]$ such that

$$\mathsf{V}[\mathbb{G}] \models j : M \prec\!\!\!\rightarrow_\kappa N. \tag{2.1}$$

Now, since $A \in M$, we have $M \models \kappa = \mu^+$ and, since $j(\mu) = \mu$ by $\mu < \kappa$, we have $N \models$ "$j(\kappa) = \mu^+$" by elementarity. Thus

$$j(\kappa) = (\mu^+)^N \leq (\mu^+)^{V[\mathbb{G}]} \overbrace{=}^{\substack{\text{since } \mathbb{P} \text{ preserves cardinals} > \nu \\ \text{by the } \nu\text{-cc of } \mathbb{P}}} (\mu^+)^V = \kappa.$$

This is a contradiction to $\kappa = \mathit{crit}(j)$.

(b) Next, we prove that κ is regular. Suppose, again toward a contradiction, that κ is singular and let $\langle \kappa_\xi : \xi < \delta \rangle$ be a strictly increasing sequence of cardinals $< \kappa$ cofinal in κ and such that $\delta < \kappa$. Let $A \subseteq \kappa$ be a set which codes the sequence $\langle \kappa_\xi : \xi < \delta \rangle$.

Let M be a transitive model of ZFC^- such that κ, $A \in M$ and there is a ν-cc poset $\mathbb{P}$ with $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ such that there are j, $N \in \mathsf{V}[\mathbb{G}]$ such that (2.1) holds.

By $A \in M$, we have $\langle \kappa_\xi : \xi < \delta \rangle \in M$. By elementarity and $\mathit{crit}(j) = \kappa$, $j(\langle \kappa_\xi : \xi < \delta \rangle) = \langle \kappa_\xi : \xi < \delta \rangle$. Hence

$$N \models \text{"}j(\kappa) = \lim(j(\langle \kappa_\xi : \xi < \delta \rangle)) = \lim(\langle \kappa_\xi : \xi < \delta \rangle) = \kappa\text{"}.$$

This is contradiction to $\kappa = \mathit{crit}(j)$ [3].

(c) Finally, we prove that κ is weakly Mahlo. Suppose that $C \subseteq \kappa$ is a club. Let $A \subseteq \kappa$ be such that it codes C as well as witnesses of singularity of all singular cardinals and successorship of the successor cardinals $< \kappa$.

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[2] We can skip (a) since (b) implies (c) and this establishes (1) (see Lemma 3.1).

[3] Actually, we do not need the ν-cc or any other condition on $\mathcal{P}$ to prove (b).

Let M be a transitive model of ZFC^- such that κ, $A \in M$ and there is a ν-cc poset $\mathbb{P}$ with $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ such that there are j, $N \in \mathsf{V}[\mathbb{G}]$ such that (2.1) holds.

Since $C \in M$ by $A \in M$ and $M \models$ "C is a club subset of $j(\kappa)$", we have $N \models$ "$j(C)$ is a club subset of κ" by elementarity. Since $j(C) \cap \kappa = C$ by $\mathit{crit}(j) = \kappa$, it follows that $\kappa \in j(C)$. κ is regular by (b). Since $\mathbb{P}$ preserves cardinality and cofinality $\geq \nu$ by its ν-cc, $\mathsf{V}[\mathbb{G}] \models$ "κ is regular". It follows that $N \models$ "κ is regular". Thus $N \models$ "$j(C)$ contains a regular cardinal" and $M \models$ "C contains a regular cardinal" by elementarity. By the choice of A the weakly inaccessible cardinal in $C \cap M$ is really weakly inaccessible.

Since C was arbitrary, this shows that κ is a weakly Mahlo cardinal.

(2): Suppose that T is a κ-tree. We want to show that T has a κ-branch.

Since we have $|T| = \kappa$, we may assume without loss of generality that the underlying set of T is κ.

Let $A \subseteq \kappa$ code the tree ordering $\leq_T$ as well as the witnesses asserting that T is a κ-tree. Let M be a transitive model of ZFC^- such that κ, $A \in M$ and there is a ν-cc poset $\mathbb{P}$ with $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ such that there are j, $N \in \mathsf{V}[\mathbb{G}]$ such that (2.1) holds.

We have $T \in M$ and $M \models$ "T is a κ-tree" by $A \in M$. It follows that $N \models$ "$j(T)$ is a $j(\kappa)$-tree" by elementarity, and $j(T)_{<\kappa} = T$. Since $j(\kappa) > \kappa$, there is $t^* \in j(T)$ such that $N \models$ "$t^* \in j(T)_\kappa$".

Let $\underset{\sim}{\leq}$ and $\underset{\sim}{t}$ be $\mathbb{P}$-names of $j(\leq_T)$ and t^*.

Back in V, let

$$T_0 := \{t \in T : \Vdash_{\mathbb{P}} \text{"}\check{t} \underset{\sim}{\leq} \underset{\sim}{t}\text{"}\}.$$

T_0 is a tree of height κ and, by the ν-cc of $\mathbb{P}$, it is of width $\leq \nu$ and $\nu^+ < \kappa$ (in V). By a theorem of Kurepa (Proposition 7.90 in [15]), it follows that there is a κ-branch b in T_0. Clearly b_0 is also a κ-branch of T. □ (Lemma 2.1)

3 Generically measurable cardinals

Let us call a cardinal κ *greatly weakly Mahlo* if κ is weakly inaccessible and there exists a non-trivial $<\kappa$-complete normal filter $\mathcal{F}$ over κ such that $\{\mu < \kappa : \mu \text{ is a regular cardinal}\} \in \mathcal{F}$, and $\mathcal{F}$ is closed with respect to the Mahlo operation [4]:

$$(3.1) \qquad S \mapsto \mathrm{M}\ell(S) := \{\alpha \in S : \alpha \text{ has uncountable cofinality and } S \cap \alpha \text{ is stationary in } \alpha\}.$$

[4] Closedness here means that for any $S \in \mathcal{F}$, we have $\mathrm{M}\ell(S) \in \mathcal{F}$.

This definition of the Mahlo operation is slightly different from the one given in [2].

For $\alpha \in \mathrm{On}$, we define the notion of α-weakly Mahloness for all cardinals κ by induction on α.

(3.2) κ is 0-*weakly Mahlo* if κ is weakly Mahlo;

(3.3) κ is 1-*weakly Mahlo* if κ is weakly Mahlo[5] and $\{\mu < \kappa : \mu \text{ is weakly Mahlo}\}$ is stationary;

(3.4) for $1 < \alpha \leq \kappa$, κ is α-*weakly Mahlo* if $\{\mu < \kappa : \mu \text{ is } \beta\text{-weakly Mahlo}\}$ is stationary in κ for all $\beta < \alpha$.

(3.5) κ is *hyper-weakly Mahlo* if $\triangle_{\alpha<\kappa}\{\mu < \kappa : \mu \text{ is } \alpha\text{-weakly Mahlo}\}$ is stationary[6].

Lemma 3.1 *For an ordinal κ, if $S \subseteq \kappa$ is a stationary set consisting of regular cardinals, then κ is also regular and hence κ is weakly Mahlo.*

Proof. Suppose that S is as above but κ is not regular.

We have $cf\,\kappa > \omega$, since if $cf(\kappa) = \omega$, then any increasing ω-sequence of successor ordinals cofinal in κ is a club in κ disjoint from S.

Say, $cf(\kappa) = \mu < \kappa$. Let $\langle \xi_\alpha : \alpha < \mu \rangle$ be a continuously increasing sequence of ordinals cofinal in κ such that $\xi_0 > \mu$. By the assumption on S, there is $\lambda \in S \cap \{\xi_\alpha : \alpha < \mu\}$. Say, $\lambda = \xi_{\alpha^*}$. Then $cf(\lambda) \leq \alpha^* < \mu < \lambda$. This is a contradiction since λ as an element of S must be regular. □ (Lemma 3.1)

Lemma 3.2 *Suppose $\alpha \leq \beta \leq \kappa$. If κ is β-weakly Mahlo, then κ is α-weakly Mahlo.*

Proof. By induction on β. □ (Lemma 3.2)

For $S \subseteq \kappa$ and $\alpha < \kappa$, let $\mathrm{M}\ell^{\,\alpha}(S)$ be defined inductively by

(3.6) $\mathrm{M}\ell^{\,0}(S) := S$;

(3.7) $\mathrm{M}\ell^{\,\alpha+1}(S) := \mathrm{M}\ell\,(\mathrm{M}\ell^{\,\alpha}(S))$;

(3.8) $\mathrm{M}\ell^{\,\gamma}(S) := \bigcap_{\alpha<\gamma} \mathrm{M}\ell^{\,\alpha}(S)$ for a limit $\gamma < \kappa$.

Finally, let

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[6] The definition of "hyper Mahloness" (i.e. the strongly Mahlo version of the hyper-weakly Mahloness defined here) has several deviations: in some cases κ-Mahloness (which is apparently slightly weaker than the hyper Mahloness parallel to the hyper-weakly Mahloness as defined here) is called hyper Mahlo.

[6] By Lemma 3.1, the weak Mahloness of κ follows from the second condition.

(3.9) $\quad \mathrm{M}\ell^{\kappa}(S) := \triangle_{\alpha<\kappa}\mathrm{M}\ell^{\alpha}(S).$

Note that stationary sets are not necessarily closed with respect to intersection of decreasing sequence of short length: Let κ be an uncountable cardinal with $\kappa \geq \omega_\omega$. For $n \in \omega$, let $S_n := \{\alpha < \kappa : \omega_n \leq cf(\alpha) < \omega_\omega\}$. Then each S_n, $n \in \omega$ is stationary. But $\bigcap_{n\in\omega} S_n = \emptyset$.

Lemma 3.3 (1) *For a regular κ, a filter $\mathcal{F}$ over κ is uniform (i.e. every end-segment of κ is in $\mathcal{F}$) and normal, if and only if $\mathcal{F}$ is non-principal, $<\kappa$-complete and normal.*

(2) *If $\mathcal{F}$ is a uniform normal filter over a regular κ, then $C \in \mathcal{F}$ for all club $C \subseteq \kappa$. It follows that all $S \in \mathcal{F}$ are stationary in κ.*

(3) *If κ is greatly weakly Mahlo and $\mathcal{F}$ is as in the definition of the greatly weak Mahloness of κ, then for all $\alpha < \kappa$ $\{\xi < \kappa : \xi$ is α-Mahlo$\} \in \mathcal{F}$.*

Proof. (1): "$\Leftarrow$" is trivial. For "$\Rightarrow$", suppose that $\delta < \kappa$ and $S_\alpha \in \mathcal{F}$ for all $\alpha < \delta$.

For $\alpha < \kappa$, let

$$S^*_\alpha = \begin{cases} S_\alpha \setminus \delta, & \text{if } \alpha < \delta; \\ \kappa & \text{otherwise.} \end{cases}$$

We have $S^*_\alpha \in \mathcal{F}$ for $\alpha < \delta$ as $\kappa \setminus \delta \in \mathcal{F}$ since $\mathcal{F}$ is uniform.

Then $\mathcal{F} \ni \triangle_{\alpha<\kappa} S^*_\alpha = \bigcap_{\alpha<\kappa} S_\alpha \setminus \delta \subseteq \bigcap_{\alpha<\kappa} S_\alpha$. Thus $\cap_{\alpha<\kappa} S_\alpha \in \mathcal{G}$.

(2): We show first that $Lim(\kappa)$ $(= \{\alpha < \kappa : \alpha$ is a limit ordinal$\})$ is an element of $\mathcal{F}$. This follows from $Lim(\kappa) = \triangle_{\alpha<\kappa}\kappa \setminus (\alpha + 1) \in \mathcal{F}$.

For a club $C \subseteq \kappa$, let $\langle c_\alpha : \alpha < \kappa\rangle$ be an increasing enumeration of C. Then we have $C \supseteq Lim(\kappa) \cap \triangle_{\alpha<\kappa}\kappa \setminus c_\alpha \in \mathcal{F}$.

For an $S \in \mathcal{F}$, $S \cap C \in \mathcal{F}$ and hence $S \cap C \neq \emptyset$ for all club $C \subseteq \kappa$. Thus S is stationary in κ.

(3): By induction on α. (Lemma A 3.0)

The following Proposition is a variant of Proposition 16.8 in [15].

Proposition 3.4 *Suppose that κ is greatly weakly Mahlo, and let $\mathcal{F}$ be a non-trivial $<\kappa$-complete normal filter over κ such that*

(3.10) $\quad Reg(\kappa) := \{\mu < \kappa : \mu$ *is regular*$\} \in \mathcal{F}$, *and*

(3.11) $\quad \mathcal{F}$ *is closed with respect to the Mahlo operation (as defined in* (3.1)*).*

Then, for any $1 \leq \alpha < \kappa$,

(1) $\mathrm{M}\ell^{\,\alpha}(Reg(\kappa)) \in \mathcal{F}$,

(2) $\mathrm{M}\ell^{\,\alpha}(Reg(\kappa)) = \{\mu < \kappa : \mu$ *is* β*-weakly Mahlo for all* $\beta < \alpha\}$
$= \{\mu < \kappa : \mu$ *is* α_0*-weakly Mahlo*$\}$ *for all* $1 \leq \alpha < \omega$ *where* α_0 *is such that* $\alpha = \alpha_0 + 1$*;*
$\mathrm{M}\ell^{\,\alpha}(Reg(\kappa)) = \{\mu < \kappa : \mu$ *is* β*-weakly Mahlo for all* $\beta \leq \alpha\}$
$= \{\mu < \kappa : \mu$ *is* α*-weakly Mahlo*$\}$ *for all* $\omega \leq \alpha < \kappa$,

(3) κ *is hyper-weakly Mahlo.*

Proof. We first prove (1) and (2) simultaneously by induction on $1 \leq \alpha < \kappa$. Note that the last equality in both of the cases in (2) follows from Lemma 3.2.

For $\alpha = 1$, we have

$$\begin{aligned} \underbrace{\mathcal{F} \ni \mathrm{M}\ell\,(Reg(\kappa))}_{\text{by (3.10) and (3.11)}} &= \mathrm{M}\ell^{\,1}(Reg(\kappa)) \\ &= \{\mu \in Reg(\kappa) : \mu \cap Reg(\kappa) \text{ is stationary in } \mu\} \\ &= \{\mu \in Reg(\kappa) : \mu \text{ is weakly Mahlo}\}. \\ &= \{\mu \in Reg(\kappa) : \mu \text{ is 0-weakly Mahlo}\}. \end{aligned}$$

Suppose that $\gamma < \kappa$ is a limit ordinal, and (1), (2) hold for all $\alpha < \gamma$. Then

$$\mathrm{M}\ell^{\,\gamma}(Reg(\kappa)) \underbrace{=}_{\text{by (3.8)}} \bigcap\nolimits_{\alpha<\gamma} \mathrm{M}\ell^{\,\alpha}(Reg(\kappa)) \in \mathcal{F}$$

by the induction hypothesis about (1) and $< \kappa$-completeness of $\mathcal{F}$.

Suppose that $\mu \in \mathrm{M}\ell^{\,\gamma}(Reg(\kappa))$. Then, by the induction hypothesis about (2), μ is $(\beta + 1)$-weakly Mahlo for all $\beta < \gamma$. By (3.4), it follows that $\{\xi < \mu : \xi$ is β-weakly Mahlo$\}$ is stationary in μ. Thus, again by (3.4), μ is γ-weakly Mahlo.

Conversely, if $\mu < \kappa$ is γ-weakly Mahlo, then, by Lemma 3.2, μ is α-weakly Mahlo for all $\alpha < \gamma$. Thus, by the induction hypothesis about (2),
$\mu \in \bigcap_{\alpha<\gamma} \mathrm{M}\ell^{\,\alpha}(Reg(\kappa)) = \mathrm{M}\ell^{\,\gamma}(Reg(\kappa))$. This shows that (2) holds for γ.

Suppose now that (1) and (2) hold for $1 \leq \alpha < \kappa$.

If $\alpha < \omega$, this means in particular that for α_0 such that $\alpha = \alpha_0 + 1$,

$$\mathrm{M}\ell^{\,\alpha}(Reg(\kappa)) = \{\mu < \kappa : \mu \text{ is } \alpha_0\text{-weakly Mahlo}\} \in \mathcal{F}.$$

By the definition (3.7) of the iteration of Mahlo operation and (3.11), we have

$$\mathrm{M}\ell^{\,\alpha+1}(Reg(\kappa)) = \{\mu < \kappa : \mu \text{ is } \underbrace{(\alpha_0 + 1)}_{=\alpha}\text{-weakly Mahlo}\} \in \mathcal{F}.$$

If $\omega \leq \alpha < \omega$, our assumption is

$$\mathrm{M}\ell^{\,\alpha}(Reg(\kappa)) = \{\mu < \kappa \,:\, \mu \text{ is } \alpha\text{-weakly Mahlo}\} \in \mathcal{F}.$$

Thus, similarly to above, we obtain

$$\mathrm{M}\ell^{\,\alpha+1}(Reg(\kappa)) = \{\mu < \kappa \,:\, \mu \text{ is } (\alpha+1)\text{-weakly Mahlo}\} \in \mathcal{F}.$$

(1) and (2) imply (3):

$$\triangle_{\alpha<\kappa}\{\mu < \kappa \,:\, \mu \text{ is } \alpha\text{-weakly Mahlo}\} \underbrace{=}_{\text{by (2)}} \triangle_{\alpha<\kappa}\mathrm{M}\ell^{\,\alpha}(Reg(\kappa)) \underbrace{\in}_{\text{by (1) and normality of } \mathcal{F}} \mathcal{F}.$$

In particular $\triangle_{\alpha<\kappa}\{\mu < \kappa \,:\, \mu \text{ is } \alpha\text{-weakly Mahlo}\}$ is stationary, and this proves that κ is hyper-weakly Mahlo. □ (Proposition 3.4)

The following theorem actually holds already for a ν-cc-g. weakly compact κ for some $\nu < \kappa$. This will be addressed in the forthcoming [13].

Theorem 3.5 *If κ is a ν-cc-g. measurable cardinal for a $\nu < \kappa$, then κ is greatly weakly Mahlo.*

Proof. Let $\mathbb{P}$ be a ccc poset with $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ such that there are classes j, $M \subseteq \mathsf{V}[\mathbb{G}]$ with $j : \mathsf{V} \overset{\prec}{\to}_{\kappa} M$.

Note that, since generically large cardinals are definable (see [12]), we may apply forcing theorems in the arguments which involve j and M. In particular, we may assume that

(3.12) $\quad \Vdash_{\mathbb{P}} \text{“}j : \mathsf{V} \overset{\prec}{\to}_{\kappa} M\text{”}.$

In $\mathsf{V}[\mathbb{G}]$, let $\tilde{\mathcal{F}} := \{S \subseteq \kappa \,:\, S \in \mathsf{V}, j(S) \ni \kappa\}$ and let $\underset{\sim}{\mathcal{F}}$ be a $\mathbb{P}$-name of $\tilde{\mathcal{F}}$.
In V, let $\mathcal{F} := \{S \subseteq \kappa \,:\, \Vdash_{\mathbb{P}} \text{“}\check{S} \,\varepsilon\, \underset{\sim}{\mathcal{F}}\text{”}\} = \{S \subseteq \kappa \,:\, \Vdash_{\mathbb{P}} \text{“}j(\check{S}) \ni \kappa\text{”}\}$. Then

Claim 3.5.1 (1) *$\mathcal{F}$ is a non-trivial $<\kappa$-complete normal filter.*

(2) *$Reg(\kappa) \in \mathcal{F}$.*

(3) *$\mathcal{F}$ is closed with respect to Mahlo operation.*

$\vdash$ (1): It is clear that $\mathcal{F}$ is a non-trivial filter.

Suppose that $\vec{S} := \langle S_\alpha \,:\, \alpha < \mu\rangle \in \mathsf{V}$ for some $\mu < \kappa$ is a sequence of length μ of elements of $\mathcal{F}$. Then $\Vdash_{\mathbb{P}} \text{“}\vec{S}$ is a sequence of elements of $\underset{\sim}{\mathcal{F}}$ of length $\mu\text{”}$. Since $\Vdash_{\mathbb{P}} \text{“}j(\vec{S}) = \langle j(S_\alpha) \,:\, \alpha < \mu\rangle\text{”}$ by (3.12), we have

$$\Vdash_{\mathbb{P}} \text{“}j(\bigcap \vec{S}) = \bigcap j(\vec{S}) = \bigcap\{j(S_\alpha) \,:\, \alpha < \mu\} \ni \kappa\text{”}.$$

Thus $\Vdash_{\mathbb{P}} \text{“}\bigcap \vec{S} \in \underset{\sim}{\mathcal{F}}\text{”}$, and hence $\bigcap \vec{S} \in \mathcal{F}$.

If $\vec{S} := \langle S_\alpha \,:\, \alpha < \kappa\rangle$ is a sequence in V of elements of $\mathcal{F}$. then

$$\Vdash_{\mathbb{P}} \text{“}\kappa \in \bigcap\{j(S_\alpha) : \alpha < \kappa\} = \bigcap\big((j(\vec{S})) \restriction \kappa\text{”}\big).$$

Since $\Vdash_{\mathbb{P}} \text{“}\kappa \in \bigcap\big((j(\vec{S})) \restriction \kappa\big) \Leftrightarrow \kappa \in \triangle j(\vec{S})^\bullet = j(\triangle\vec{S})\text{”}$, it follows that $\Vdash_{\mathbb{P}} \text{“}\triangle\vec{S} \in \underset{\sim}{\mathcal{F}}\text{”}$ and thus $\triangle\vec{S} \in \mathcal{F}$.

(2): Let $R := Reg(\kappa) \setminus \nu$. Then we have $\Vdash_{\mathbb{P}} \text{“}j(R) = Reg(j(\kappa))^M \setminus \nu\text{”}$ by (3.12). By the ν-cc of $\mathbb{P}$, it follows that $\Vdash_{\mathbb{P}} \text{“}\kappa \text{ is regular and } \kappa > \nu\text{”}$. Thus $\Vdash_{\mathbb{P}} \text{“}\kappa \in j(R)\text{”}$, and hence $R \in \mathcal{F}$.

(3): If $S \in \mathcal{F}$, then S is stationary by (1) and Lemma 3.3, (2).

Since $\mathbb{P}$ is ν-cc, $\Vdash_{\mathbb{P}} \text{“}V^{\mathbb{P}} \models S \text{ is stationary in } \kappa\text{”}$.

Since $\Vdash_{\mathbb{P}} \text{“}S = j(S) \cap \kappa\text{”}$, it follows that

$$\Vdash_{\mathbb{P}} \text{“}\kappa \in \mathrm{M}\ell^{M}(j(S)) = j(\mathrm{M}\ell^{\mathsf{V}}(S))\text{”}.$$

Thus, $\Vdash_{\mathbb{P}} \text{“}\mathrm{M}\ell^{\mathsf{V}}(S) \in \underset{\sim}{\mathcal{F}}\text{”}$ and hence $\mathrm{M}\ell(S) \in \mathcal{F}$. ⊣ (Claim 3.5.1)

□ (Theorem 3.5)

Proposition 3.6 *For a regular cardinals κ, ν with $\nu < \kappa$, the following are equivalent:*

(a) *κ is ν-cc-g. measurable.*

(b) *There is a non-trivial, non-principal and ν-saturated $< \kappa$-complete ideal over κ.*

(c) *there are a ν-cc poset $\mathbb{P}$, a $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$, and j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that $\mathsf{V}[\mathbb{G}] \models \text{“}j : \mathsf{V} \overset{\prec}{\to}_\kappa M\text{”}$ and $({}^\kappa M)^{\mathsf{V}[\mathbb{G}]} \subseteq M$.*

Proof. “(c) ⇒ (a)”: is clear. So we shall prove “(a) ⇒ (b)” and “(b) ⇒ (c)”.

“(a) ⇒ (b)”: Let $\mathbb{P}$ be a ν-cc poset such that, for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ and j, $M \subseteq \mathsf{V}[\mathbb{G}]$, we have $\mathsf{V}[\mathbb{G}] \models j : V \overset{\prec}{\to}_\kappa M$.

In V, let $\mathcal{I} := \{A \subseteq \kappa : \Vdash_{\mathbb{P}} \text{“}\kappa \notin j(\check{A})\text{”}\}$. Note that $\mathcal{I}$ is the dual ideal of the filter of $\mathcal{F}$ in the proof of Proposition 3.4.

Claim 3.6.1 *$\mathcal{I}$ is $< \kappa$-complete and ν-saturated ideal (in V).*

Proof. $< \kappa$-completeness follows from Claim 3.5.1, (1).

In the following, we argue in V. To prove that $\mathcal{I}$ is ν-saturated, assume, toward a contradiction, that $\langle A_\xi : \xi < \nu\rangle$ is a pairwise incompatible sequence of elements in $\mathbb{P}(\kappa) \setminus \mathcal{I}$. By the $< \kappa$-completeness of $\mathcal{I}$, we may choose the sequence such that A_ξ, $\xi < \nu$ are pairwise disjoint. For each $\xi < \nu$, since $A_\xi \notin \mathcal{I}$, there is $\mathbb{p}_\xi \in \mathbb{P}$, such that $\mathbb{p}_\xi \Vdash_{\mathbb{P}} \text{“}\kappa \in j(\check{A}_\xi)\text{”}$. By ν-cc of $\mathbb{P}$, there are $\xi < \eta < \nu$ such that $\mathbb{p}_\xi$ and $\mathbb{p}_\eta$ are compatible, say $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}_\xi, \mathbb{p}_\eta$. But then $\mathbb{r} \Vdash_{\mathbb{P}} \text{“}\kappa \in j(\check{A}_\xi) \cap j(\check{A}_\eta)\text{”}$ and hence

$\mathbb{r} \Vdash_{\mathbb{P}}$ "$\check{A}_\xi \cap \check{A}_\eta \neq \emptyset$". It follows that $A_\xi \cap A_\eta \neq \emptyset$. This is a contradiction to the choice of A_ξ, $\xi < \nu$. $\dashv$ (Claim 3.6.1)

"(b) $\Rightarrow$ (c)": Let $\mathcal{I}$ be a ν-saturated κ-complete ideal over κ. $\mathbb{P}_\mathcal{I} := \mathbb{P}(\kappa) \setminus \mathcal{I}$ satisfies then the ν-cc.

$\mathcal{I}$ is precipitous (see e.g. Lemma 22.22 in [14]). Let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P}_\mathcal{I})$-generic filter, and let $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$ be the canonical elementary embedding of V into the Mostowski collapse of the generic ultrapower by $\mathbb{G}$. By Lemma 22.31 in [14], we have $({}^\kappa M)^{\mathsf{V}[\mathbb{G}]} \subseteq M$. $\square$ (Proposition 3.6)

Theorem 3.7 *Suppose that κ is a ν-cc-g. measurable cardinal for some regular $\nu < \kappa$. Then κ is the stationary limit of ν-cc-g. weakly compact cardinals.*

Proof. Suppose that κ is ν-cc-g. measurable and let $C \subseteq \kappa$ be an arbitrary club subset of κ. We have to show that C contains a ν-cc-g. weakly compact cardinal. Let $\mathbb{P}$ be a ν-cc poset with a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ and j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$ and

$$({}^\kappa M)^{\mathsf{V}[\mathbb{G}]} \subseteq M \tag{3.13}$$

(see Proposition 3.6).

Since $M \models$ "$j(C)$ is a club subset of $j(\kappa)$" and $\kappa \in j(C)$ by the closedness, the following claim completes the proof.

Claim 3.7.1 $M \models$ "*κ is ν-cc-g. weakly compact*".

$\vdash$ In M, suppose $A \subseteq \kappa$. We have to show in M that there is a transitive model M_0 of ZFC^- with κ, $A \in M_0$ and $j_0 : M_0 \overset{\prec}{\to}_\kappa N_0$ for some j_0, N_0 in some ν-cc generic extension.

By Proposition 3.6, there is a ν-saturated $< \kappa$-complete ideal $\mathcal{I}$ on κ in V.

In $\mathsf{V}[\mathbb{G}]$, let $\mathcal{J}$ be the ideal over κ generated by $\mathcal{I}$. By ν-cc of $\mathbb{P}$, it is easy to see that $\mathcal{J}$ is $< \kappa$-complete (in $\mathsf{V}[\mathbb{G}]$). $\mathcal{J}$ is ν-saturated (in $\mathsf{V}[\mathbb{G}]$) by Prikry's Theorem (see e.g. Theorem 17.1 in [15]). Let $\mathbb{P}_\mathcal{J} := (\mathcal{P}(\kappa) \setminus \mathcal{J})^{\mathsf{V}[\mathbb{G}]}$. Then $\mathsf{V}[\mathbb{G}] \models$ "$\mathbb{P}_\mathcal{J}$ has the ν-cc".

Working further in $\mathsf{V}[\mathbb{G}]$, let θ be sufficiently large and let M_1 be such that

$$M_1 \prec \mathcal{H}(\theta), \tag{3.14}$$

$$\kappa + 1 \cup \{A, \mathcal{J}\} \subseteq M_1, \text{ and} \tag{3.15}$$

$$|\, M_1 \,| = \kappa. \tag{3.16}$$

Let $m : M_1 \to M_0$ be the Mostowski collapse. Note that

(3.17) $m \restriction \kappa + 1 = id_{\kappa+1}$

by (3.15). $M_0 \in M$ by (3.13).

By $A \subseteq \kappa$ and (3.17), we have $A = m(A) \in M_0$.

Let $\mathcal{J}_0 := m(\mathcal{J})$. $\mathcal{J}_0 \in M_0$ by this definition and $\mathcal{J}_0 = \mathcal{J} \cap M_0$ by (3.17). By the elementarity (3.14) (and since θ is taken sufficiently large), we have $M_1 \models$ "$\mathcal{J}$ is a ν-saturated, $< \kappa$-complete ideal over κ". It follows that

$$M_0 \models \text{"}\mathcal{J}_0 \text{ is a } \nu\text{-saturated, } < \kappa\text{-complete ideal over } \kappa\text{"}.$$

In particular, $\mathcal{J}_0$ is precipitous in connection with M_0 by Lemma 22.22 in [14].

Let $\mathbb{Q} := (\mathcal{P}(\kappa) \setminus \mathcal{J}_0)^{M_0}$ (note that $\mathbb{Q} \in M$ since $M_0 \in M$). $m^{-1} \restriction \mathbb{Q} = id_{\mathbb{Q}}$ is then an order-preserving and incompatibility preserving embedding of $\mathbb{Q}$ into $\mathbb{P}_{\mathcal{J}}$. Since $\mathbb{P}_{\mathcal{J}}$ is ν-cc in $\mathsf{V}[\mathbb{G}]$, $\mathbb{Q}$ is also ν-cc in $\mathsf{V}[\mathbb{G}]$. It follows that $\mathbb{Q}$ is also ν-cc in M (note that $(\nu^+)^M = (\nu^+)^{\mathsf{V}[\mathbb{G}]}$ by (3.13)).

Let $\mathbb{H}$ be a $(M, \mathbb{Q})$-generic filter and let $j_0 : M_0 \prec\!\!\to_\kappa N_0$ where N_0 is the Mostowski collapse of the generic ultrapower of M_0 by $\mathbb{H}$ (in $M[\mathbb{H}]$).

Clearly, M_0 together with these j_0 and N_0 is as desired. ⊣ (Claim 3.7.1)

□ (Theorem 3.7)

4 Generically supercompact cardinals

Theorem 4.1 *Suppose that κ is ν-cc-g. 2^κ-supercompact for some uncountable cardinal $\nu < \kappa$. Then κ is the stationary limit of ν-cc-g. measurable cardinals.*

Proof. Let $\mathbb{P}$ be a ν-cc poset with a $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$ and j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that, in $\mathsf{V}[\mathbb{G}]$, $j : \mathsf{V} \prec\!\!\to_\kappa M$, $j(\kappa) > (2^\kappa)^{\mathsf{V}}$ and

(4.1) $j''(2^\kappa)^{\mathsf{V}} \in M$.

As in the proof of Theorem 3.7, it is enough to show that $M \models$ "κ is ν-cc-g. measurable". Thus, by Proposition 3.6, we are done by showing that $M \models$ "there is a ν-saturated, $< \kappa$-complete ideal over κ".

Since κ is ν-cc g. measurable, there is a ν-saturated, $< \kappa$-complete ideal $\mathcal{I}$ over κ in V by Proposition 3.6. In $\mathsf{V}[\mathbb{G}]$, let $\mathcal{J}$ be the ideal over κ generated by $\mathcal{I}$. By Prikry's theorem, we have

(4.2) $\mathsf{V}[\mathbb{G}] \models$ "$\mathcal{J}$ is ν-saturated, $< \kappa$-complete ideal".

Note that

$$\begin{aligned}\mathcal{J} &= \{A \in \mathcal{P}(\kappa)^{\mathsf{V}[\mathbb{G}]} \,:\, A \subseteq B \text{ for some } B \in \mathcal{I}\}\\ &= \{A \in \mathcal{P}(\kappa)^{\mathsf{V}[\mathbb{G}]} \,:\, A \subseteq j(B) \text{ for some } B \in \mathcal{I}\}\\ &= \{A \in \mathcal{P}(\kappa)^{\mathsf{V}[\mathbb{G}]} \,:\, A \subseteq C \text{ for some } C \in j''\mathcal{I}\}.\end{aligned}$$

Since $j''\mathcal{I} \in M$ by (4.1), it follows that

$$\mathcal{J} \cap M = \{A \in \mathcal{P}(\kappa)^M \,:\, A \subseteq C \text{ for some } C \in j''\mathcal{I}\}$$

is an element of M.

Thus, we have $M \models$ "$\mathcal{J} \cap M$ is ν-saturated, $<\kappa$-complete ideal" by (4.2).

□ (Theorem 4.1)

5 Reflection properties down to $<$ a generically supercompact cardinal

An interesting fact about the notion of generic large cardinals is that the continuum can be generic large, or in some cases the continuum can be strictly larger than many generic large cardinals (cf. Theorem 3.7, Theorem 4.1). This is in particular the case with ccc-generically supercompact cardinals (e.g. obtained by starting from a model with a supercompact κ and then by forcing by $\mathrm{Fn}(\kappa, 2)$). The largeness properties of generic supercompact cardinals for forcing with chain condition discussed in the previous sections can thus also be situations with the continuum.

Since generic large cardinals are reflection points of diverse reflection statements (as discussed below), the continuum can be also the reflection point of the same reflection statements.

Let $\mathcal{S}$ be a class of (not necessarily first-order) structures with a notion $\sqsubseteq_{\mathcal{S}}$ of substructure relation where $(\mathcal{S}, \sqsubseteq_{\mathcal{S}})$ should satisfy certain reasonable properties like that

(5.1) $\mathfrak{A} \in \mathcal{S}$ and $\mathfrak{A} \cong \mathfrak{B}$ imply $\mathfrak{B} \in \mathcal{S}$,

$\sqsubseteq_{\mathcal{S}}$ is transitive,

$\mathfrak{A} \sqsubseteq_{\mathcal{S}} \mathfrak{B}$ and $\langle \mathfrak{B}, \mathfrak{A}\rangle \cong \langle \mathfrak{B}', \mathfrak{A}'\rangle$ imply $\mathfrak{A}' \sqsubseteq_{\mathcal{S}} \mathfrak{B}'$,

etc.

We also assume that $\mathcal{S}$ is absolute and $\sqsubseteq_{\mathcal{S}}$ is upward absolute, meaning that if M, N are transitive (class or set) models of ZFC^- and M is an inner model of N, then, for any $\mathfrak{A}$, $\mathfrak{B} \in M$, $M \models \mathfrak{A} \in \mathcal{S} \Leftrightarrow N \models \mathfrak{A} \in \mathcal{S}$ and $M \models \mathfrak{A} \sqsubseteq_{\mathcal{S}} \mathfrak{B} \Rightarrow N \models \mathfrak{A} \sqsubseteq_{\mathcal{S}} \mathfrak{B}$. For a structure $\mathfrak{A}$, we denote by $|\mathfrak{A}|$ the underlying set of $\mathfrak{A}$ and by $\|\mathfrak{A}\|$ the cardinality of the underlying set of the structure $\mathfrak{A}$.

Given such a class $\mathcal{S} = (\mathcal{S}, \sqsubseteq_{\mathcal{S}})$ of structures, and a property P, the reflection number of P (in connection with $\mathcal{S}$) is defined as

$$\begin{aligned}\mathfrak{refl}_{stat}(\mathcal{S}, P) := \min\{\kappa \in Reg \ : \ &\text{for any } \mathfrak{A} \in \mathcal{S} \text{ with } \mathfrak{A} \models P \text{ and } \|\mathfrak{A}\| \geq \kappa,\\ &\text{the set } \{\, |\mathfrak{B}| \ : \ \mathfrak{B} \sqsubseteq_{\mathcal{S}} \mathfrak{A}, \mathfrak{B} \models P,\ \|\mathfrak{B}\| < \kappa\}\\ &\text{is stationary subset of } [\ |\mathfrak{A}|\ ]^{<\kappa} \quad \}\end{aligned}$$

where we define $\min \emptyset := \infty$.

The reflection spectrum of P is

$$\begin{aligned}\mathfrak{REFL}_{stat}(\mathcal{S}, P) := \{\kappa \in Reg \ : \ &\text{for any } \mathfrak{A} \in \mathcal{S} \text{ with } \mathfrak{A} \models P \text{ and } \|\mathfrak{A}\| \geq \kappa,\\ &\text{the set } \{\, |\mathfrak{B}| \ : \ \mathfrak{B} \sqsubseteq_{\mathcal{S}} \mathfrak{A}, \mathfrak{B} \models P,\ \|\mathfrak{B}\| < \kappa\}\\ &\text{is stationary subset of } [\ |\mathfrak{A}|\ ]^{<\kappa} \quad \}\end{aligned}$$

Example 5.1 *Let $\mathcal{S}$ be the class of all first countable topological spaces $X = \langle X, \tau\rangle$ where τ is an open basis for the space. $\sqsubseteq_{\mathcal{S}}$ is the subspace relation.*

For P = non-metrizability, the consistency of $\mathfrak{refl}_{stat}(\mathcal{S}, P) = \aleph_2$ is known as Hamburger's Problem which has been open for almost a half century. □

A property P is *downward absolute* if, for any transitive (class or set) models M, N of ZFC^- such that M is an inner model of N, and for any structure $\mathfrak{A}$, if $N \models$ "$\mathfrak{A} \models P$" implies $M \models$ "$\mathfrak{A} \models P$".

For a class $\mathcal{P}$ of posets, $\mathcal{P}$ *preserves* P, if $\mathfrak{A} \models P$ then for any $\mathbb{P} \in \mathcal{P}$, $\Vdash_{\mathbb{P}}$ "$\mathfrak{A} \models P$" holds.

Theorem 5.2 *Suppose that $\mathcal{S} = (\mathcal{S}, \sqsubseteq_{\mathcal{S}})$ is a class of structures, $\mathcal{P}$ a class of posets, and κ a $\mathcal{P}$-g. supercompact cardinal. If a property P satisfies:*

(5.2) *P is downward absolute and*

(5.3) *$\mathcal{P}$ preserves P,*

then $\kappa \in \mathfrak{REFL}_{stat}(\mathcal{S}, P)$ and hence $\mathfrak{refl}_{stat}(\mathcal{S}, P) \leq \kappa$.

Proof. Suppose that $\mathfrak{A} \in \mathcal{S}$ and $\|\mathfrak{A}\| \geq \kappa$. By replacing $\mathfrak{A}$ with an isomorphic structure, we may assume that $|\mathfrak{A}| = \lambda \in Card$ (see (5.1)).

Let $\mathbb{P} \in \mathcal{P}$ be such that for an $(\mathsf{V}, \mathcal{P})$-generic $\mathbb{G}$ and j, $M \subseteq \mathsf{V}[\mathbb{G}]$, we have $\mathsf{V}[\mathbb{G}] \models$ "$j : \mathsf{V} \overset{\prec}{\to}_{\kappa} M$",

(5.4) $j(\kappa) > \mu$ and $j''\mu \in M$ where $\mu = \beth_n(\lambda)$ for sufficiently large n.[7)]

By the last condition (5.4), we have $j(\mathfrak{A}) \restriction j''\lambda \in M$.

[7)] We need $\beth_n(\lambda)$ here since the elements of $\mathcal{S}$ may be $(n+1)$-th order structures for $n \geq 1$.

$\mathsf{V}[\mathbb{G}] \models$ "$\mathfrak{A} \models P$" by (5.3). By $\mathsf{V}[\mathbb{G}] \models$ "$\mathfrak{A} \cong j(\mathfrak{A}) \restriction j''\lambda$", it follows that $\mathsf{V}[\mathbb{G}] \models$ "$j(\mathfrak{A}) \restriction j''\lambda \models P$". Since $|j(\mathfrak{A}) \restriction j''\lambda| = j''\lambda \in M$ by (5.4), we have $j(\mathfrak{A}) \restriction j''\lambda \in M$ and hence $M \models$ "$\mathfrak{A} \restriction j''\lambda \models P$" by (5.2).

Suppose $\mathcal{D}$ is an arbitrary club subset of $[\lambda]^{<\kappa}$ in V. Since $M \models$ "$j''\mathcal{D}$ is cofinal in $[j''\lambda]^{<\kappa}$", we have $M \models$ "$j''\lambda = \bigcup(j''\mathcal{D}) \in j(\mathcal{D})$".

In V, let $\mathcal{S}_0 = \{\mathfrak{B} : \mathfrak{B} \sqsubseteq_{\mathcal{S}} \mathfrak{A}, \|\mathfrak{B}\| < \kappa\}$. Since $j(|\mathfrak{B}|) = j''|\mathfrak{B}|$ for all $\mathfrak{B} \in \mathcal{S}_0$, it follows that $\bigcup\{|\mathfrak{C}| : \mathfrak{C} \in j''\mathcal{S}_0\} = j''\lambda$. Thus, $M \models$ "$j(\mathfrak{A}) \restriction j''\lambda \sqsubseteq_{\mathcal{S}} j(\mathfrak{A})$" (for this, we have to assume that $\mathcal{S}$ satisfies the the property that the union of upward directed system of $\sqsubseteq_{\mathcal{S}}$-substructures is a $\sqsubseteq_{\mathcal{S}}$-substructure and certain Downward Löwenheim-Skolem theorem on $\sqsubseteq_{\mathcal{S}}$-substructures of a given structure in $\mathcal{S}$). with respect to $\sqsubseteq_{\mathcal{S}}$).

By elementarity, it follows that $\{|\mathfrak{B}| : \mathfrak{B} \sqsubseteq_{\mathcal{S}} \mathfrak{A}, \|\mathfrak{B}\| < \kappa\}$ is stationary in $[\lambda]^{<\kappa}$. ❑ (Theorem 5.2)

The following are some application of the theorem above.

Corollary 5.3 *Suppose that κ is a ccc-g. supercompact cardinal and $\mathcal{S} = (\mathcal{S}, \leq)$ is a variety and $\mathfrak{A} \in \mathcal{S}$ with $\|\mathfrak{A}\| \geq \kappa$ is not free. Then there are stationarily many non-free $\mathfrak{B} \leq \mathfrak{A}$ of cardinality $< \kappa$. In particular, with P being "being non-free", we have $\mathfrak{REFL}_{stat}(\mathcal{S}, P) \ni \kappa$ and $\mathfrak{refl}_{stat}(\mathcal{S}, P) \leq \kappa$.*

Proof. ccc posets preserve non-freeness of algebras in any variety (see [5]).

❑ (Corollary 5.3)

Note that the Corollary above applies e.g. to groups, abelian groups, Boolean algebras, etc.

We shall call posets of the form $\mathrm{Fn}(\lambda, 2)$ *generalized Cohen posets*. For $\mathcal{P} = \{\mathbb{P} : \mathbb{P}$ is forcing equivalent to some $\mathrm{Fn}(\lambda, 2)\}$, $\mathcal{P}$-g. supercompactness for this $\mathcal{P}$ will be also called *Cohen-g. supercompactness*.

In the following Corollaries, stationarity may be replaced with clubness since any topological space containing a non-metrizable subspace is non-metrizable and any tree containing a non-special subtree is non-special.

Corollary 5.4 *Suppose that κ is a Cohen-g. supercompact cardinal. Then any first countable non-metrizable topological space of cardinality $\geq \kappa$ have club many subspaces of size $< \kappa$ which are non-metrizable.*

Proof. Generalized Cohen posets preserve non-metrizability (see Dow, Tall and Weiss [4]). ❑ (Corollary 5.4)

Corollary 5.5 *Suppose that κ is a Cohen-g. supercompact. Then any non-special tree T has club many non-special subtrees of size $< \kappa$.*

Proof. Generalized Cohen posets preserve non-specialty of trees (Todorčević, see [6]). □ (Corollary 5.5)

The Cohen-g. supercompactness in Corollary 5.5 cannot be replaced by ccc-g. supercompactness:

We can prove (in ZFC) that there is a non-special tree of size $2^{\aleph_0}$ without branches of length ω_1 (e.g. $T := \{t : t : \alpha \to \omega, t \text{ is 1-1 for some } \alpha < \omega_1\}$ with the ordering $t/ \leq_T t' :\Leftrightarrow t \subseteq t'$ is such a tree).

By Baumgartner, Malitz, and Reinhardt [1], all trees of size $< 2^{\aleph_0}$ without branches of length ω_1 are special under Martin's Axiom.

The reflection number $\mathfrak{refl}_{stat}(\mathcal{C}, P)$ for $\mathcal{C} :=$ trees and $P :=$ non-special by $\mathfrak{refl}_{\mathsf{RC}}$ is related to Rado's Conjecture (which is the satement $\mathfrak{refl}_{stat}(\mathcal{C}, P) = \aleph_2$). If we denote this by $\mathfrak{refl}_{\mathsf{RC}}$, the results cited above can be put togetehr to show:

Proposition 5.6 (MA) $\quad \mathfrak{refl}_{\mathsf{RC}} > 2^{\aleph_0}$. □

Also, for $\mathcal{S}$ and P as in Example 5.1, we have $\mathfrak{refl}_{stat}(\mathcal{S}, P) > \mathfrak{b}$ (see [10]). Thus, denoting this reflection number connected to Hamburger's Problem (more presicely Hamburger's Problem is the question about the consistency of $\mathfrak{refl}_{stat}(\mathcal{S}, P) = \aleph_2$) by $\mathfrak{refl}_{\mathsf{HP}}$, we have

Proposition 5.7 (MA) $\quad \mathfrak{refl}_{\mathsf{HP}} > 2^{\aleph_0}$. □

If we start from a supercompact κ and force MA together with $2^{\aleph_0} = \kappa$ by the standard forcing construction, then in the resulting model κ $(= 2^{\aleph_0})$ is ccc-g. supercompact and MA holds. This shows (under the assumption of the consistency strength of a supercompact cardinal) that the assertion "the continuum is ccc-g. supercompact" is consistent with the non-reflection $\mathfrak{refl}_{\mathsf{RC}}$, $\mathfrak{refl}_{\mathsf{HP}} > 2^{\aleph_0}$.

On the other hand, it is consistent that $2^{\aleph_0}$ is ccc-g. supercompact, MA holds but a reasonably strong reflection principle with the reflection point $< \aleph_2$ still holds:

Start from a model of ZFC with two supercompact cardinals. Use the smaller supercompact to force Fodor-type Reflection Principle (FRP, which is a reflection principle with the reflection point $< \aleph_2$, and FRP follows from RC). Then force by the standard ccc forcing for MA to make the larger supercompact cardinal (which survives the first extension) to make it the continuum. In [7], it is shown that FRP is preserved by ccc generic extension. Thus in the resulting model, we still have FRP together with MA and that the continuum is ccc-g. supercompact.

Corollary 5.8 (König [16] see also [8]) *Suppose that κ is $\mathcal{P}$-g. supercompact where $\mathcal{P}$ is the class of all σ-closed posets. Then any non-special tree T has club many non-special subtrees of size $< \kappa$.*

Proof. σ-closed posets preserve non-specialty of trees (Todorčević, see [18]). □ (Corollary 5.8)

Corollary 5.9 (Diagonal Reflection Principle, see [3], [8]) *Let*

$$\mathcal{S} := \{\langle M, \langle S_a : a \in M\rangle\rangle : M \neq \emptyset,\ S_a \subseteq [M]^{\aleph_0} \text{ for all } a \in M\}.$$

For $\langle M, \langle S_a : a \in M\rangle\rangle$, $\langle N, \langle S_a : a \in N\rangle\rangle \in \mathcal{S}$, *let*

$$\langle M, \langle S^M_a : a \in M\rangle\rangle \sqsubseteq_{\mathcal{S}} \langle N, \langle S^N_a : a \in N\rangle\rangle \quad :\Leftrightarrow$$
$$M \subseteq N, \text{ and } S^M_a = S^N_a \cap [M]^{\aleph_0} \text{ for all } a \in M.$$

Let the property P be defined by stipulating that P holds in $\langle M, \langle S_a : a \in M\rangle\rangle \in \mathcal{S}$ if and only if S_a is a stationary subset of $[M]^{\aleph_0}$ for all $a \in M$.

Suppose that $\mathcal{P}$ is a class of posets such that all elements of $\mathcal{P}$ are proper. If κ is a $\mathcal{P}$-g. supercompact, then we have $\mathfrak{refl}_{stat}(\mathcal{S}, P) \leq \kappa$. □

The inequality $\mathfrak{refl}_{stat}(\mathcal{S}, P) \leq \kappa$ in Corollary 5.9 is optimal in the following sense: Suppose that κ is supercompact and $\mu < \kappa$ is such that there is a non-reflecting stationary set $S \subseteq [\mu]^{\aleph_0}$.[8] If $\mathbb{P} := \mathrm{Fn}(\kappa, 2)$ and $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter, then κ is *Cohen*-g. supercompact in $\mathsf{V}[\mathbb{G}]$. Since S remains a non-reflecting stationary subset of $[\mu]^{\aleph_0}$ in $\mathsf{V}[\mathbb{G}]$, we have $\mathsf{V}[\mathbb{G}] \models$ "$\mu < \mathfrak{refl}_{stat}(\mathcal{S}, P) \leq \kappa = 2^{\aleph_0}$".

On the other hand, it is also consistent (modulo large cardinals) that $\mathfrak{refl}_{stat}(\mathcal{S}, P) < \kappa = 2^{\aleph_0}$ holds for a ccc-g. supercompact cardinal κ: Suppose that $\kappa_0 < \kappa$ are two supercompact cardinals and $\mathbb{P}$ and $\mathbb{G}$ are as above. Then, in $\mathsf{V}[\mathbb{G}]$, κ_0 and κ are both Cohen-g. supercompact. Thus, by Corollary 5.9, we have $\mathsf{V}[\mathbb{G}] \models \mathfrak{refl}_{stat}(\mathcal{S}, P) \leq \kappa_0 < \kappa = 2^{\aleph_0}$.

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