Asymptotic behavior of the resolvents of equilibrium problems in complete geodesic spaces
完備測地距離空間における均衡問題のリゾルベントの漸近的挙動

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Abstract

Equilibrium problems are one of the important problems which are applied in a variety of fields such as natural science, economics, statistics and so on. The idea of the resolvent are considerable notion to solve these problems and has been studied by many researchers. In this paper, we consider the problem of asymptotic behavior of resolvents defined for equilibrium problems.

1 Introduction

The concept of resolvent has been deeply studied and applied for solving convex minimization problems and fixed point problems in various settings of spaces. A geodesic space is a metric space having a convex structure and a Hadamard space is one of complete geodesic spaces in which resolvents are considered. For a convex function on a Hadamard space, its resolvent is often defined as follows. Let X be a Hadamard space and g a convex lower semicontinuous function on X. We define a resolvent $R_g \colon X \to X$ by

$$R_g(a) = \operatorname*{argmin}_{y \in X} \{f(y) + d(y,a)^2\}$$

for $a \in X$. This R_g is well-defined as a single valued mapping; see [4]. Therefore, we can consider a resolvent $R_{\lambda g}$ with positive parameter λ. That is,

$$R_{\lambda g}(a) = \operatorname*{argmin}_{y \in X}\{\lambda f(y) + d(y,a)^2\} = \operatorname*{argmin}_{y \in X}\left\{f(y) + \frac{1}{\lambda}d(y,a)^2\right\}.$$

The asymptotic behavior of resolvent at infinity is a problem to consider the limit of $R_{\lambda g}x$ at $\lambda \to \infty$. As for this problem, the following result is known.

Theorem 1.1 (See [1]). *Let X be a Hadamard space, g a convex lower semicontinuous function on X, $x \in X$, and $\lambda > 0$. Define $R_{\lambda g}\colon X \to X$ by*

$$R_{\lambda g}(a) = \operatorname*{argmin}_{y \in X}\left\{f(y) + \frac{1}{\lambda}d(y,a)^2\right\}$$

for each $a \in X$. If there exists a sequence $\{\mu_n\}$ such that $\mu_n \to \infty$ and $\{J_{\mu_n f}x\}$ is bounded, then $\operatorname{argmin} f \neq \emptyset$ *and,*

$$\lim_{\lambda \to \infty} J_{\lambda f}x = P_{\mathrm{Equil}\, f}x.$$

Equilibrium problems are important problems containing minimization problems, fixed point problems, saddle point problems, and Nash equilibria. Let K be a nonempty set and $f\colon K \times K \to \mathbb{R}$ a bifunction on K. An equilibrium problem for f is a problem of finding

$$z \in K \text{ such that } f(z,y) \geq 0 \text{ for all } y \in K.$$

These problems are studied by many researchers. For example, see [2]. We consider a resolvent for such a bifunction f of equilibrium problems. A resolvent of equilibrium problems on Hadamard spaces is defined in [3].

In this paper, we study the property of a resolvent of equilibrium problems with a positive parameter λ and consider the asymptotic behavior of its resolvent at λ to infinity.

2 Preliminaries

Let X be a metric space. For $x, y \in X$, a geodesic $c_{xy}\colon [0, d(x,y)] \to X$ is a mapping which satisfies $c_{xy}(0) = x$, $c_{xy}(d(x,y)) = y$, and $d(c_{xy}(u), c_{xy}(v)) = |u - v|$ for $u, v \in [0, d(x,y)]$. If for any two points, there exists a unique geodesic, X is called a uniquely geodesic space. We define convex combination $tx \oplus (1-t)y$ between x and y in a unique geodesic by

$$tx \oplus (1-t)y = c_{xy}((1-t)d(x,y)),$$

for $t \in [0,1]$. In particular, we denote $\frac{1}{2}x \oplus \frac{1}{2}y$ by $\frac{x \oplus y}{2}$. A complete uniquely geodesic space X is called a Hadamard space if it holds that

$$d(tx \oplus (1-t)y, z)^2 \leq td(x,z)^2 + (1-t)d(y,z)^2 - t(1-t)d(y,z)^2$$

for any three point $x, y, z \in X$, and $t \in [0,1]$. Let S be a subset of X. The convex hull $\operatorname{co} S$ of S is defined by

$$\operatorname{co} S = \bigcup_{j=1}^{\infty} S_j,$$

where $S_1 = S$ and $S_{j+1} = \{tx \oplus (1-t)y \in X \mid x, y \in S_j, t \in [0,1]\}$. We say X has the convex hull finite property if for every subset S and continuous mapping $T\colon \overline{\operatorname{co} S} \to \overline{\operatorname{co} S}$, T has a fixed point in $\overline{\operatorname{co} S}$.

Let X be a Hadamard space and g a function from X to $\mathbb{R}$. g is said to be lower semicontinuous if it satisfies

$$g(x) \leq \liminf_{n\to\infty} g(x_n)$$

for all $x \in X$ and sequences $\{x_n\}$ of X with $x_n \to x$. Further, g is said to be convex if it satisfies

$$g(tx \oplus (1-t)y) \leq tg(x) + (1-t)g(y)$$

for $x, y \in X$ and $t \in [0,1]$. Moreover, g is said to be upper hemicontinuous if it satisfies

$$\limsup_{t\to 0} g(tx \oplus (1-t)y) \leq g(y).$$

for $x, y \in X$. Let C be a closed convex subset of X. Then, for each $x \in X$, there exists a unique point $x_0 \in C$ such that

$$d(x_0, x) = \inf_{y\in C} d(y, x).$$

We define the metric projection $P_C\colon X \to C$ by

$$P_C(a) = \operatorname*{argmin}_{y\in C} d(y, a)$$

for $a \in X$.

Let X be a Hadamard space, K a closed convex subset of X, and f a bifunction from $K \times K$ to $\mathbb{R}$. For this f, we denote the set of solution of equilibrium problems by $\operatorname{Equil} f$. That is,

$$\operatorname{Equil} f = \left\{ z \in K \;\middle|\; \inf_{y\in K} f(z, y) \geq 0 \right\}.$$

In what follows, we assume f satisfies the following four conditions.

(E1) $f(x, x) = 0$ for all $x \in K$;
(E2) $f(x, y) + f(y, x) \leq 0$ for all $x, y \in K$;
(E3) $f(x, \cdot)\colon K \to \mathbb{R}$ is convex and lower semicontinuous for all $x \in K$;
(E4) $f(\cdot, x)\colon K \to \mathbb{R}$ is upper hemicontinuous for all $x \in K$.

If f satisfies these conditions, for $x \in X$, the function $f_x\colon K \times K \to \mathbb{R}$ which is defined by

$$f_x(z, y) = f(z, y) + d(y, x)^2 - d(z, x)^2$$

is also satisfies (E1)–(E4). We denote the set of solutions of equilibrium problems for f_x by $J_f x$. That is

$$J_f x = \operatorname{Equil} f_x = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left(f(z, y) + d(y, x)^2 - d(z, x)^2 \right) \geq 0 \right\}.$$

If X has the convex hull finite property, then $J_f x$ is a singleton; see [3]. Therefore, we can consider J_f as a mapping from X to K. We call such a mapping J_f a resolvent of equilibrium problems for f.

3 Main result

Let X be a Hadamard space which has the convex hull finite property and f a bi-function on K satisfying (E1)–(E4). For a positive parameter λ, we define a resolvent $J_{\lambda f}$ as follows;

$$J_{\lambda f}(a) = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left(f(z, y) + \frac{1}{\lambda} \left(d(y, a)^2 - d(z, a)^2 \right) \right) \geq 0 \right\}.$$

We consider the asymptotic behavior of a resolvent $J_{\lambda f}$ at λ to infinity.

We first show the following lemma.

Lemma 3.1. *Let $\xi\colon \mathbb{R} \to \mathbb{R}$ be an increasing function. If for some sequence $\{\mu_n\} \subset \mathbb{R}$ diverging to ∞, $\{\xi(\mu_n)\}$ is bounded, then $\{\xi(\lambda_n)\}$ is bounded for any sequence $\{\lambda_n\} \subset \mathbb{R}$ diverging to ∞.*

Proof. Let $\{\lambda_n\}$, $\{\mu_n\}$ be real sequences such that $\lambda_n, \mu_n \to \infty$ and $\xi\colon \mathbb{R} \to \mathbb{R}$ an increasing function. Suppose $\{\xi(\mu_n)\}$ is bounded and $\{\xi(\lambda_n)\}$ not. Then there exists $M > 0$,

$$\xi(\mu_n) \leq M.$$

for all $n \in \mathbb{N}$ and we can find a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ such that

$$\xi(\lambda_{n_i}) > M.$$

for all $i \in \mathbb{N}$. Then there exists $k \in \mathbb{N}$ such that $\lambda_{n_1} \leq \mu_k$. Since ξ is increasing, we have

$$M < \xi(\lambda_{n_1}) \leq \xi(\mu_l) \leq M.$$

This is a contradiction and it completes the proof. □

By using this lemma, we get a result about asymptotic behavior of an equilibrium problems as follows.

Theorem 3.1. *Let X be a Hadamard space having the convex hull finite property, K a closed convex subset of X, $x \in X$, and $\lambda > 0$. Suppose $f\colon K \times K \to \mathbb{R}$ satisfies* (E1)–(E4)*. Define $J_{\lambda f}\colon X \to K$ by*

$$J_{\lambda f} a = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left(f(z,y) + \frac{1}{\lambda} \left(d(y,a)^2 - d(z,a)^2 \right) \right) \geq 0 \right\}$$

for each $a \in X$. If there exists a sequence $\{\mu_n\}$ such that $\mu_n \to \infty$ and $\{J_{\mu_n f} x\}$ is bounded, then $\operatorname{Equil} f \neq \emptyset$ *and,*

$$\lim_{\lambda \to \infty} J_{\lambda f} x = P_{\operatorname{Equil} f} x$$

.

Proof. Let $x \in X$ and $\{\lambda_n\}$ a positive increasing sequence diverging to ∞. We put $x_n = J_{\lambda_n f} x$ for each $n \in \mathbb{N}$ and suppose $n, m \in \mathbb{N}$ satisfy $n \leq m$. Assume that there exists a sequence $\{\mu_n\}$ such that $\mu_n \to \infty$ and $\{J_{\mu_n f} x\}$ is bounded. Then $\{x_n\}$ is bounded from Lemma 3.1. First, we show $d(x_n, x) \leq d(x_m, x)$, $f(x_n, x_m) \leq 0$ and $f(x_m, x_n) \geq 0$. From the definition of the resolvent, it holds that

$$0 \leq f(x_n, x_m) + \frac{1}{\lambda_n} \left\{ d(x_m, x)^2 - d(x_n, x)^2 \right\}$$

and

$$0 \leq f(x_m, x_n) + \frac{1}{\lambda_m} \left\{ d(x_n, x)^2 - d(x_m, x)^2 \right\}.$$

From these inequalities and (E2), we have

$$\begin{aligned} 0 &\leq f(x_n, x_m) + f(x_n, x_m) + \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_m} \right) \left\{ d(x_m, x)^2 - d(x_n, x)^2 \right\} \\ &\leq \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_m} \right) \left\{ d(x_m, x)^2 - d(x_n, x)^2 \right\}. \end{aligned}$$

Since $\frac{1}{\lambda_n} - \frac{1}{\lambda_m} > 0$ from the monotonicity of $\{\lambda_n\}$, we get

$$0 \leq d(x_m, x)^2 - d(x_n, x)^2,$$

which is equivalent to $d(x_n, x) \leq d(x_m, x)$. By the monotonicity of $\{d(x_n, x)\}$ and (E2), we have

$$\begin{aligned} 0 &\leq f(x_m, x_n) + \frac{1}{\lambda_m} \left\{ d(x_n, x)^2 - d(x_m, x)^2 \right\} \\ &\leq f(x_m, x_n) \\ &\leq -f(x_n, x_m). \end{aligned}$$

Hence, we get $f(x_m, x_n) \leq 0$ and $f(x_n, x_m) \geq 0$. Next, we show

$$d(x_n, x) \leq d\left(\frac{x_n \oplus x_m}{2}, x\right).$$

From (E3) and (E1), we have

$$\begin{aligned}
0 &\leq f\left(x_n, \frac{x_n \oplus x_m}{2}\right) + \frac{1}{\lambda_n}\left\{d\left(\frac{x_n \oplus x_m}{2}, x\right)^2 - d(x_n, x)^2\right\} \\
&\leq \frac{f(x_n, x_n) + f(x_n, x_m)}{2} + \frac{1}{\lambda_n}\left\{d\left(\frac{x_n \oplus x_m}{2}, x\right)^2 - d(x_n, x)^2\right\} \\
&= \frac{1}{2}f(x_n, x_m) + \frac{1}{\lambda_n}\left\{d\left(\frac{x_n \oplus x_m}{2}, x\right)^2 - d(x_n, x)^2\right\}.
\end{aligned}$$

Similarly, it holds that

$$0 \leq \frac{1}{2}f(x_m, x_n) + \frac{1}{\lambda_m}\left\{d\left(\frac{x_n \oplus x_m}{2}, x\right)^2 - d(x_m, x)^2\right\}.$$

From these inequalities and (E2), we obtain

$$\begin{aligned}
0 &\leq \frac{1}{2}f(x_n, x_m) + \frac{1}{2}f(x_m, x_n) \\
&\quad + \frac{1}{\lambda_n}\left\{d\left(\frac{x_n \oplus x_m}{2}, x\right)^2 - d(x_n, x)^2\right\} + \frac{1}{\lambda_m}\left\{d\left(\frac{x_n \oplus x_m}{2}, x\right)^2 - d(x_m, x)^2\right\} \\
&= \left(\frac{1}{\lambda_n} + \frac{1}{\lambda_m}\right) d\left(\frac{x_n \oplus x_m}{2}, x\right)^2 - \frac{1}{\lambda_n}d(x_n, x)^2 - \frac{1}{\lambda_m}d(x_m, x)^2 \\
&\leq \left(\frac{1}{\lambda_n} + \frac{1}{\lambda_m}\right) d\left(\frac{x_n \oplus x_m}{2}, x\right)^2 - \frac{1}{\lambda_n}d(x_n, x)^2 - \frac{1}{\lambda_m}d(x_n, x)^2 \\
&= \left(\frac{1}{\lambda_n} + \frac{1}{\lambda_m}\right) \left\{d\left(\frac{x_n \oplus x_m}{2}, x\right)^2 - d(x_n, x)^2\right\}
\end{aligned}$$

since $d(x_n, x) \leq d(x_m, x)$. Thus, we get

$$d(x_n, x) \leq d\left(\frac{x_n \oplus x_m}{2}, x\right).$$

Then, we show the sequence $\{x_n\}$ is convergent. Using the parallelogram law, we have

$$d(x_n, x)^2 \leq d\left(\frac{x_n \oplus x_m}{2}, x\right)^2$$

$$\leq \frac{1}{2}d(x_n,x)^2+\frac{1}{2}d(x_m,x)^2-\frac{1}{4}d(x_n,x_m)^2,$$

and this implies $d(x_n,x_m)^2 \leq 2\{d(x_m,x)^2-d(x_n,x)^2\}$. Therefore, since $\{d(x_n,x)\}$ is bounded and increasing, $\{x_n\}$ is a Cauchy sequence on K. Since K is a closed subset of complete metric space X, $\{x_n\}$ converges to some point $q \in K$. Finally, we show $q = P_{\mathrm{Equil}\, f}$. From the lower semicontinuity of f for the second argument,

$$\limsup_{n\to\infty}(-f(y,x_n)) \leq -\liminf_{n\to\infty} f(y,x_n) \leq -f(y,q)$$

for all $y \in K$. From (E2), we have

$$\begin{aligned}0 &\leq f(x_n,y)+\frac{1}{\lambda_n}\left\{d\left(y,x\right)^2-d(x_n,x)^2\right\}\\ &\leq -f(y,x_n)+\frac{1}{\lambda_n}\left\{d\left(y,x\right)^2-d(x_n,x)^2\right\}.\end{aligned}$$

Since $\{d(x_n,x)\}$ is bounded, letting $n\to\infty$, we obtain

$$0 \leq \limsup_{n\to\infty}(-f(y,x_n)) \leq -f(y,q)$$

for all $y \in K$. Let $w \in K$ and $t \in \,]0,1[$. Since f is lower semicontinuous for the second argument and K is convex, we have

$$\begin{aligned}0 &= f(tw\oplus(1-t)q, tw\oplus(1-t)q)\\ &\leq tf(tw\oplus(1-t)q,w)+(1-t)f(tw\oplus(1-t)q,q)\\ &\leq tf(tw\oplus(1-t)q,w).\end{aligned}$$

Dividing by t and letting $t\to 0$, we obtain

$$0 \leq \limsup_{t\to 0} f(x_t,w) \leq f(q,w)$$

for all $w \in K$ from (E4). Therefore, we know

$$q \in \mathrm{Equil}\, f \neq \emptyset.$$

For all $z \in \mathrm{Equil}\, f$, we have

$$\begin{aligned}0 &\leq f(x_n,z)+\frac{1}{\lambda_n}\left\{d(z,x)^2-d(x_n,x)^2\right\}\\ &\leq \frac{1}{\lambda_n}\left\{d(z,x)^2-d(x_n,x)^2\right\}\end{aligned}$$

from the definition of the resolvent. This implies $d(x_n,x) \leq d(z,x)$. By the lower semicontinuity of the distance function, we get

$$d(q,x) \leq \liminf_{n\to\infty} d(x_n,x) \leq d(z,x)$$

for all $z \in \operatorname{Equil} f$, which is equivalent to $q = P_{\operatorname{Equil} f} x$.

Since $J_{\lambda_n f} x \to P_{\operatorname{Equil} f} x$ for every positive increasing sequence $\{\lambda_n\}$ which diverges to infinity, we conclude

$$\lim_{\lambda \to \infty} J_{\lambda f} x = P_{\operatorname{Equil} f} x.$$

This is the desired result. □

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