Generalized dual expression for set relations by means of sublinear scalarization functions

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Abstract

In the literature, characterization for set relations are of various application to optimality conditions of set optimization problems, variational principles for set-valued maps, theorems of the alternative, certain robustness of vector optimization problems, and so on. In this paper, the author presents properties of scalarization functions as dual expression of set relations. Comparing to existing results, one can confirm their uniqueness in their relaxed conditions. Also, we show our results suggest some robust positive semi-definite optimization problems.

1 Introduction

For set optimization, binary relations between two sets (usually called set relations) are used to find an optimal set. However, their relations are abstract for the most part and not easily calculable so that many papers scalarize given sets to figure out their relations hold or not. Scalarization is basically a fundamental tool to estimate objects that their magnitude can be mathematically quantified, for which size, volume, or probability are familiar examples. This approach is done by scalarization functions which are real-valued set functions based on a convex cone. These functions representing their corresponding set relations are utilized in the literature for optimizing algorithm, variational principles, or optimality conditions for set optimization problems [3, 8–10, 12]. Recently, oriented distance functions ([4–6]) playing similar roles to scalarization functions are vigorously researched as a parallel approach to estimate set relations.

To prove equivalence between set relations and scalarized values, given sets are required to have topological conditions: some paper describe compactness, cone-compactness, closedness, properness. This paper is a review of [10, 11] and studies six kinds of expression with cone-compactness and cone-closedness as a generalization and propose tiny application for positive semi-definite optimization problems.

2 Basic notations

This thesis is involved in a topological vector space X with a convex solid cone C where $\mathrm{int}C \neq \emptyset$. The relation $\leq_C$ denotes the vector ordering where $x \leq_C y$ is defined to be $x \in y - C$.

We use convex cone properties with respect to C: A is C-closed if $A+C$ is closed, A is C-bounded if it holds that $A \subset U + C$ for any open neighborhood U of the zero, A is C-compact if any cover of S being like $\{U_\lambda + C \mid U_\lambda \text{ is open}\}$ admits a finite subcover. We clearly see C-compactness leads to $C-$closedness and C-boundedness. For sets $A, B \subset X \setminus \{\emptyset\}$, the algebraic sum and the scalar multiplication are denoted by $A + B = \{a + b \mid a \in A, b \in B\}$, $\alpha A = \{\alpha a \mid a \in A\}$

At first, we introduce the six types of set relations originally proposed in [7]: for nonempty sets $A, B \subset X \setminus \{\emptyset\}$, the relations $\preceq_C^{(\cdot)}$ are defined by

$$A \preceq_C^{(\mathrm{i})} B \iff A \subset \bigcap_{b \in B} (b - C);$$

$$A \preceq_C^{(\mathrm{ii})} B \iff A \cap \bigcap_{b \in B} (b - C) \neq \emptyset;$$

$$A \preceq_C^{(\mathrm{iii})} B \iff B \subset A + C;$$

$$A \preceq_C^{(\mathrm{iv})} B \iff A \cap \bigcap_{a \in A} (a + C) \neq \emptyset;$$

$$A \preceq_C^{(\mathrm{v})} B \iff A \subset B - C;$$

$$A \preceq_C^{(\mathrm{vi})} B \iff B \cap (A + C) \neq \emptyset.$$

By the above definition, we see that $\preceq_C^{(\mathrm{i})}$ implies $\preceq_C^{(\mathrm{ii})}$ and $\preceq_C^{(\mathrm{iv})}$, which lead to $\preceq_C^{(\mathrm{iii})}$ and $\preceq_C^{(\mathrm{v})}$ respectively. The last relation $\preceq_C^{(\mathrm{vi})}$ follows from any others. Moreover, these relations $\preceq_C^{(\cdot)}$ coincide with $\leq_C$ when two compared set A, B are both singleton.

3 Scalarization functions

For quantification of set relations, we use the following Minkovski functional [2] proposed by Garstewitz: for a given vector $x \in X$ and a fixed direction $d \in X$, the scalarization functions $\varphi_{C,d} : X \to \mathbb{R} \cup \{\infty\}$ is defined by

$$\varphi_{C,d}(x) := \inf\{\gamma \in \mathbb{R} \mid x \leq_C \gamma d\}.$$

Note that this function $\varphi_{C,d}$ coincides with the linear functional $f \in X^*$ where $C := \{x \in X \mid f(x) \geq 0\}$ is a half space. Nishizawa et.al. studied theorems of the alternative for set-valued maps with the function in [9]. In addition, [3, 8] proposed generalized types of scalarization by using set relations: for a given set A, a fixed reference set B, and a fixed direction d, the characterization functions $\Phi_{C,B,d}^{(\cdot)} : 2^X \to \mathbb{R} \cup \{\infty\}$ are defined by

$$\Phi_{C,B,d}^{(\cdot)}(A) := \inf\{\gamma \in \mathbb{R} \mid A \preceq_C^{(\cdot)} B + \gamma d\}.$$

These functions $\Phi^{(\cdot)}_{C,B,d}$ vary themselves by types (i)–(vi), and coincide with $\varphi_{C,d}$ when $A = \{x\}$, $B = \{\mathbf{0}_X\}$. Also, it is easy to show $A \preceq^{(\cdot)}_C B$ implies $\Phi^{(\cdot)}_{C,B,d}(A) \le 0$.

In recent years, Ogata et.al. proposed the scalarization functions can characterize the corresponding set relations based on an open or closed convex cone [10, 12] under some compactness. This fact is a similar situation to that where $a \le_C b$ is equivalent to $\langle c, b-a\rangle \le 0$ for all $c \in C^*$.

Proposition 3.1. Let $A, B \in 2^X \setminus \{\emptyset\}$.

- If A is compact, then $A \preceq^{(ii)}_{\mathrm{cl}C} B$ and $A \preceq^{(iii)}_{\mathrm{cl}C}$ follow from $\Phi^{(ii)}_{\mathrm{cl}C,B,d}(A) \le 0$ and $\Phi^{(iii)}_{\mathrm{cl}C,B,d}(A) \le 0$ for some $d \in X$, respectively.
- If B is compact, then $A \preceq^{(iv)}_{\mathrm{cl}C} B$ and $A \preceq^{(iv)}_{\mathrm{cl}C}$ follow from $\Phi^{(v)}_{\mathrm{cl}C,B,d}(A) \le 0$ and $\Phi^{(v)}_{\mathrm{cl}C,B,d}(A) \le 0$ for some $d \in X$, respectively.
- If both A, B are compact, then $A \preceq^{(vi)}_{\mathrm{cl}C} B$ follows from $\Phi^{(vi)}_{\mathrm{cl}C,B,d}(A) \le 0$ for some $d \in X$.

Note that the case (i) is clearly true without assuming any compactness. The other cases for an open cone are also proved as follows.

Proposition 3.2. Let $A, B \in 2^X \setminus \{\emptyset\}$.

- If both A, B are compact, then $A \preceq^{(i)}_{\mathrm{int}C} B$ follows from $\Phi^{(i)}_{\mathrm{int}C,B,d}(A) \le 0$ for some $d \in X$.
- If B is compact, then $A \preceq^{(ii)}_{\mathrm{int}C} B$ and $A \preceq^{(iii)}_{\mathrm{int}C}$ follow from $\Phi^{(ii)}_{\mathrm{int}C,B,d}(A) \le 0$ and $\Phi^{(iii)}_{\mathrm{int}C,B,d}(A) \le 0$ for some $d \in X$, respectively.
- If A is compact, then $A \preceq^{(iv)}_{\mathrm{int}C} B$ and $A \preceq^{(iv)}_{\mathrm{int}C}$ follow from $\Phi^{(v)}_{\mathrm{int}C,B,d}(A) \le 0$ and $\Phi^{(v)}_{\mathrm{int}C,B,d}(A) \le 0$ for some $d \in X$, respectively.

Note that the case (vi) is clearly true without assuming any compactness.

Theorem 3.1. Let S be a nonempty set, $A, B \in 2^X \setminus \{\emptyset\}$. If

- A is C-compact for case (ii);
- A is C-closed for case (iii);
- B is $(-C)$-compact for case (iv);
- B is $(-C)$-closed for case (v);
- A is C-closed and B is $(-C)$-compact,
  or A is C-compact and B is $(-C)$-closed for case (vi),

then

$$A \preceq^{(\cdot)}_{\mathrm{cl}C} B \iff \exists k \in \mathrm{int}C \text{ s.t. } \Phi^{(\cdot)}_{C,k}(A, B) \le 0$$

for the cases (i)–(vi).

We illustrate Theorem 3.1 with Figures 1–3 to indicate how it fails under broken conditions in each case.

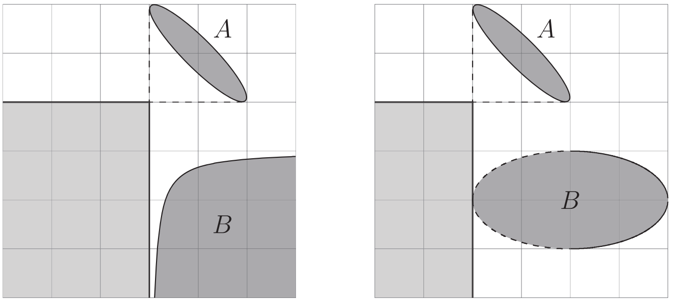

Figure 1: Counter examples for case (ii) of Theorem 3.1

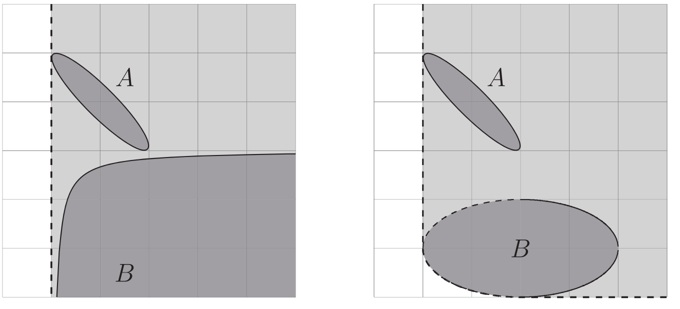

Figure 2: Counter examples for case (iii) of Theorem 3.1

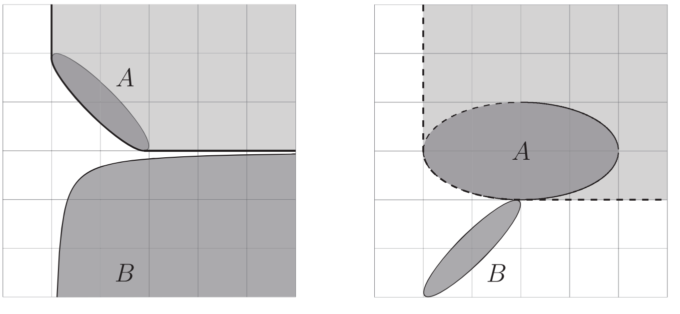

Figure 3: Counter examples for case (vi) of Theorem 3.1

4 Example

Let S^n be the set of $n \times n$ symmetric matrices. The set of non-negative (positive) semidefinite matrices in S^n are denoted by S^n_+ (S^n_{++}), $f : \mathbb{R}^m \to S^n \setminus \{\emptyset\}$, $V \subset S^n$ a nonempty set. In this part, the trace of $A \subset S^n$ is denoted by $\operatorname{tr} A := \sum_{i=1}^n a_{ii}$. Note that S^n_+ is a closed cone and $S^n_{++} = \operatorname{int} S^n_+$. For detail, see [1].

Proposition 4.1. If a nonempty set $V \subset S^n$ is S^n_+-closed, it holds that

$$\exists x \in \mathbb{R}^m \text{ s.t. } \{f(x)\} \preceq^{(\cdot)}_{S^n_+} V \Longleftrightarrow \exists M \subset S^n_{++} \text{ s.t. } \Phi^{(\cdot)}_{S^n_+,M}(\{f(x)\}, V) \leq 0.$$

The above proposition follows directly from Theorem 3.1 by setting A as a singleton and note that the cases (i)–(iii), and (iv)–(vi) respectively coincide in this assertion.

Let us consider the following semidefinite optimization problem and its dual problem for $M \in S^n_{++}$ and $x \in \mathbb{R}^m$:

$$\begin{aligned} \text{(SDP) minimize} \quad & \operatorname{tr}(-(f(x) - V)X) \\ \text{subject to} \quad & \operatorname{tr}(MX) = -1, \\ & X \in S^n_+; \end{aligned}$$

$$\begin{aligned} \text{(SDD) maximize} \quad & -t \\ \text{subject to} \quad & -tM + S = -f(x) + V, \\ & S \in S^n_+ \end{aligned}$$

togeter with the following robust problems with a perturbation $V \in W$ for (SDD):

$$\begin{aligned} \text{(RDP1) minimize} \quad & t \\ \text{subject to} \quad & f(x) \in \bigcap_{V \in W} \left(tM + V - S^n_+\right); \end{aligned}$$

$$\begin{aligned} \text{(RDP2) minimize} \quad & t \\ \text{subject to} \quad & f(x) \in tM + W - S^n_+. \end{aligned}$$

By Proposition 4.1 the optimal value of (RDP1) and (RDP2) are attained by calculating $\Phi^{(\cdot)}_{S^n_+,M}(\{f(x)\}, W)$ for cases (i)–(iii) and (iv)–(vi), respectively since the infimum value $\inf\{t \mid f(x) \in tM + V - S^n_+\}$ is equal to the optimal value of (SDD).

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