A convergence theorem of an implicit iterative method with multiple mappings in geodesic spaces
測地空間における複数の写像を用いた陰的な点列の収束定理

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Abstract

In this paper, we define a quadrilateral inequality and an iterative of Xu-Ori type in CAT(1) spaces. Furthermore, we prove a convergence theorem of the Xu-Ori type method in the same space by using the quadrilateral inequality.

1 Introduction

Fixed point theory has been investigated by many mathematicians in recent years. In particular, approximating a common fixed point of a nonlinear mapping is one of the major topics in this theory. We have been researching some types of approximating iteration for finding a fixed point in Banach spaces, Hilbert spaces and geodesic spaces. For example, there are some implicit type methods such as Browder type [7] and Xu-Ori type [8].

Recently, Kimura [3] proved the following convergence theorem with multiple anchor points in a complete CAT(0) space. It was inspired by the idea of the sequence of Browder's theorem.

Theorem 1.1 (Kimura [3]). *Let X be a Hadamard space, $T : X \to X$ be a nonexpansive mapping such that $F(T) \neq \varnothing$ and $u_1, u_2, \dots, u_r \in X$. Suppose $\{\alpha_n\} \subset]0, 1[$ is a real sequence such that $\alpha_n \to 0$. For $k = 1, 2, \dots, r$, let $\{\beta_n^k\} \subset [0, 1]$ such that*

$\sum_{k=1}^{r} \beta_n^k = 1$ *and* $\beta_n^k \to \beta^k \in [0,1]$. *Define* $\{x_n\} \subset X$ *by*

$$x_n = \operatorname*{argmin}_{y \in X} \left(\alpha_n \sum_{k=1}^{r} \beta_n^k d(y, u_k)^2 + (1 - \alpha_n) d(y, Tx_n)^2 \right)$$

for $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges to a unique minimizer of a function* g *on* $F(T)$, *where* $g : X \to \mathbb{R}$ *is defined by*

$$g(y) = \sum_{k=1}^{r} \beta^k d(y, u_k)^2,$$

for $y \in X$.

On the other hand, in the following Δ-convergence theorem with an implicit iterative scheme for a finite family of nonexpansive mappings proved by Kimura [4] took in the idea of Xu and Ori's iterative method.

Theorem 1.2 (Kimura [4])**.** *Let* X *be a Hadamard space and* $T_k : X \to X$ *a nonexpansive mapping for* $k = 1, 2, \ldots, N$ *such that* $\bigcap_{k=1}^{N} F(T_k) \neq \varnothing$. *Suppose* $\{\alpha_n^k\} \subset]0,1[$ *is a real sequence for* $k = 0, 1, \ldots, N$ *such that* $\sum_{k=0}^{N} \alpha_n^k = 1$. *For given* $x_1 \in X$, *generate a sequence* $\{x_n\} \subset X$ *satisfying that*

$$x_{n+1} = \operatorname*{argmin}_{y \in X} \left(\alpha_n^0 d(x_n, y)^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x_{n+1}, y)^2 \right)$$

for $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *is well-defined and* Δ*-convergent to some* $x_0 \in \bigcap_{k=1}^{N} F(T_k)$.

In this paper, we propose a quadrilateral inequality in a CAT(1) space and prove an implicit iterative method for some nonexpansive mappings in this space.

2 Preliminaries

Let X be a metric space. For $x, y \in X$, a mapping $c : [0,1] \to X$ is called a geodesic with endpoints $x, y \in X$ if it satisfies $c(0) = x, c(1) = y$, and $d(c(t), c(s)) = |t - s|$ for every $t, s \in [0,1]$. If a geodesic with endpoints y and z exists for any $y, z \in X$, we call X is a geodesic space. In this work, we suppose X has a unique geodesic for any $y, z \in X$. Then, we denote the image of the geodesic with $y, z \in X$ by $[y, z]$, which is well defined.

For $x, y, z \in X$, a geodesic triangle $\triangle(x, y, z)$ is defined as the union of three segments $[y, z], [z, x]$, and $[x, y]$. Its comparison triangle $\bar{\triangle}(\bar{x}, \bar{y}, \bar{z})$ with $d(\bar{y}, \bar{z}) + d(\bar{z}, \bar{x}) + d(\bar{x}, \bar{y}) < 2\pi$ is defined as the triangle in the 2-dimensional unit sphere $\mathbb{S}^2$ whose length of each corresponding edge is identical with that of the original triangle;

$$d(y, z) = d_{\mathbb{S}^2}(\bar{y}, \bar{z}), \quad d(z, x) = d_{\mathbb{S}^2}(\bar{z}, \bar{x}), \quad d(x, y) = d_{\mathbb{S}^2}(\bar{x}, \bar{y}).$$

If for any geodesic triangle $\triangle(x,y,z)$ with $d(y,z)+d(z,x)+d(x,y) < 2\pi$ and $p,q \in \triangle(x,y,z)$ and their comparison points $\bar{p},\bar{q} \in \bar{\triangle}(\bar{x},\bar{y},\bar{z})$, the inequality

$$d(p,q) \leq d_{\mathbb{S}^2}(\bar{p},\bar{q})$$

holds, then we call X a CAT(1) space. Moreover, X is called admissible if for any $u,v \in X$, $d(u,v) < \frac{\pi}{2}$.

For $x,y \in X$ with $d(x,y) < \pi$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that $d(x,z) = (1-t)d(x,y)$ and $d(z,y) = td(x,y)$. We denote it by $tx \oplus (1-t)y$. In an admissible CAT(1) space X, the following inequality holds;

$$\cos d(z, tx \oplus (1-t)y) \sin d(x,y) \geq \cos d(z,x) \sin td(x,y) + \cos d(z,y) \sin(1-t)d(x,y),$$

for every $x,y,z \in X$ and $t \in [0,1]$.

A mapping $U : X \to X$ is a contraction if there exists $\beta \in [0,1[$ such that

$$d(Ux,Uy) \leq \beta d(x,y)$$

for every $x,y \in X$. The famous Banach contraction principle guarantees the existence and uniqueness of a fixed point of U. Furthermore, a mapping $T : X \to X$ is called nonexpansive if

$$d(Tx,Ty) \leq d(x,y)$$

for every $x,y \in X$. We know the set $F(T) = \{z \in X : z = Tz\}$ of all fixed points of nonexpansive T is closed and convex.

Let $\{x_n\} \subset X$ be a bounded sequence. Put $r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n)$ for $x \in X$. Then, the asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is defined by

$$r(\{x_n\}) = \inf_{x\in X} r(x,\{x_n\}).$$

The asymptotic center $\mathrm{AC}(\{x_n\})$ of $\{x_n\}$ is a set of point $p \in X$ satisfying

$$r(p,\{x_n\}) = r(\{x_n\}).$$

We say $\{x_n\}$ is Δ-convergent to $q \in X$ if q is the unique asymptotic center of any subsequence of $\{x_n\}$. In an admissible CAT(1) space, we know that an asymptotic center of $\{x_n\}$ is always singleton.

The following theorem is helpful to show the well-defineness of some mappings in this paper.

Theorem 2.1 (Sasaki [6]). *Let X be a complete admissible* CAT(1) *space and C a nonempty closed convex subset of X. For $u_1,u_2,\dots,u_N \in X$ and $\beta_1,\beta_2,\dots,\beta_N \in [0,1]$ with $\sum_{k=1}^N \beta_k = 1$, define a function $g : X \to]0,1]$ by*

$$g(x) = \sum_{k=1}^N \beta_k \cos d(u_k,x)$$

for all $x \in X$. Then, g has a unique maximizer on C.

3 Main results

To prove our main result, we first show the following lemmas. These lemmas are proved in Kimura and Torii [5]. For the sake of completeness, we show the proofs.

Lemma 3.1. *Let X be an admissible* CAT(1) *space and $x, y, z, w \in X$. Then, the following inequality holds:*

$$\cos d(x,y) + \cos d(y,z) + \cos d(z,w) + \cos d(w,x) \leq 4\cos\frac{d(x,z)}{2}\cos\frac{d(y,w)}{2}.$$

Proof. Let $m_1 = \frac{1}{2}y \oplus \frac{1}{2}w, m_2 = \frac{1}{2}x \oplus \frac{1}{2}z$. Then we have

$$\begin{aligned}
&\cos d(m_1, m_2)\cos\frac{d(x,z)}{2}\cos\frac{d(y,w)}{2} \\
&\geq \frac{1}{2}\cos d(m_1, x)\cos\frac{d(y,w)}{2} + \frac{1}{2}\cos d(m_1, z)\cos\frac{d(y,w)}{2} \\
&= \frac{1}{2}\cos d\left(\frac{1}{2}y \oplus \frac{1}{2}w, x\right)\cos\frac{d(y,w)}{2} + \frac{1}{2}\cos d\left(\frac{1}{2}y \oplus \frac{1}{2}w, z\right)\cos\frac{d(y,w)}{2} \\
&\geq \frac{1}{2}\left(\frac{1}{2}\cos d(y,x) + \frac{1}{2}\cos d(w,x)\right) + \frac{1}{2}\left(\frac{1}{2}\cos d(y,z) + \frac{1}{2}\cos d(w,z)\right) \\
&= \frac{1}{4}\left(\cos d(x,y) + \cos d(y,z) + \cos d(z,w) + \cos d(w,x)\right).
\end{aligned}$$

Since $1 \geq \cos d(m_1, m_2)$, we get

$$\cos d(x,y) + \cos d(y,z) + \cos d(z,w) + \cos d(w,x) \leq 4\cos\frac{d(x,z)}{2}\cos\frac{d(y,w)}{2},$$

the desired result. □

Lemma 3.2 (The quadrilateral inequality in CAT(1) spaces)**.** *Let X be an admissible* CAT(1) *space and $x, y, z, w \in X$. Then, the following inequality holds:*

$$\begin{aligned}
&(1 - \cos d(x,z)) + (1 - \cos d(y,w)) \\
&\leq (1 - \cos d(x,y)) + (1 - \cos d(y,z)) + (1 - \cos d(z,w)) + (1 - \cos d(w,x)).
\end{aligned}$$

Proof. From Lemma 3.1, we get

$$\begin{aligned}
&\frac{1}{4}(\cos d(x,y) + \cos d(y,z) + \cos d(z,w) + \cos d(w,x)) \\
&\leq \cos\frac{d(x,z)}{2}\cos\frac{d(y,w)}{2} = \sqrt{\cos^2\frac{d(x,z)}{2}\cos^2\frac{d(y,w)}{2}} \\
&\leq \frac{1}{2}\left(\cos^2\frac{d(x,z)}{2} + \cos^2\frac{d(y,w)}{2}\right)
\end{aligned}$$

$$= \frac{1}{2}\left(\frac{1+\cos d(x,z)}{2} + \frac{1+\cos d(y,w)}{2}\right)$$
$$= \frac{1}{4}(\cos d(x,z) + \cos d(y,w) + 2).$$

Therefore, we get

$$(1-\cos d(x,z)) + (1-\cos d(y,w))$$
$$\leq (1-\cos d(x,y)) + (1-\cos d(y,z)) + (1-\cos d(z,w)) + (1-\cos d(w,x)).$$

This is the desired result. □

Lemma 3.3. *Let X be a metric space with $d(u,v) \leq \frac{\pi}{2}$ for any $u,v \in X$ and $U: X \to X$ a mapping. Suppose $\beta \in [0,1[$. If for any $x,y \in X$,*

$$1-\cos d(Ux,Uy) \leq \beta(1-\cos d(x,y)),$$

then U is a contraction.

Proof. Since

$$\frac{1-\cos d(Ux,Uy)}{2} \leq \frac{\beta(1-\cos d(x,y))}{2},$$

we have

$$\sin^2 \frac{d(Ux,Uy)}{2} \leq \beta \sin^2 \frac{d(x,y)}{2}.$$

Since $\sin(\cdot)$ is a concave function on $[0,\frac{\pi}{2}]$, we get

$$\sin\frac{d(Ux,Uy)}{2} \leq \sqrt{\beta}\sin\frac{d(x,y)}{2} \leq \sin\frac{\sqrt{\beta}d(x,y)}{2}.$$

Since $\sin(\cdot)$ is increasing on $[0,\frac{\pi}{2}]$, we get

$$\frac{d(Ux,Uy)}{2} \leq \frac{\sqrt{\beta}d(x,y)}{2}.$$

Therefore, we have

$$d(Ux,Uy) \leq \sqrt{\beta}d(x,y)$$

Since $\beta \in [0,1[$, this implies U is a contraction. □

By using above lemmas, we can show the following theorem.

Theorem 3.4. *Let X be a complete* CAT(1) *space with $d(u,v) < \frac{\pi}{3}$ for any $u,v \in X$ and $T_i: X \to X$ nonexpansive mappings for $i = 1,2,\ldots,k$. Let $u \in X$ and $\alpha \in]\frac{1}{2},1[$. For $i = 1,2,\ldots,k$, let $\beta_i \in]0,1[$ such that $\sum_{i=1}^k \beta^i = 1$. Define $U: X \to X$ by*

$$Ux = \operatorname*{argmax}_{z\in X}\left\{\alpha\cos d(z,u) + (1-\alpha)\sum_{i=1}^k \beta^i \cos d(z,T_i x)\right\}$$

for every $x \in X$. Then U is well-defined and a contraction.

Proof. U is well-defined as a single-valued mapping on X by Theorem 2.1. Let $x, y \in X$. If $d(Ux, Uy) = 0$, then it is obvious that $d(Ux, Uy) \leq \beta d(x, y)$ for any $\beta \in [0, 1[$. Thus, we consider the case of $d(Ux, Uy) \neq 0$. For $t \in]0, 1[$, we have

$$\begin{aligned}
&\left(\alpha \cos d(Ux, u) + (1-\alpha) \sum_{i=1}^{k} \beta^i \cos d(Ux, T_i x)\right) \sin d(Ux, Uy) \\
&\geq \alpha \cos d(tUx \oplus (1-t)Uy, u) \sin d(Ux, Uy) \\
&\quad + (1-\alpha) \sum_{i=1}^{k} \beta^i \cos d(tUx \oplus (1-t)Uy, T_i x) \sin d(Ux, Uy) \\
&\geq \alpha(\cos d(Ux, u) \sin td(Ux, Uy) + \cos d(Uy, u) \sin(1-t)d(Ux, Uy)) \\
&\quad + (1-\alpha) \sum_{i=1}^{k} \beta^i (\cos d(Ux, T_i x) \sin td(Ux, Uy) + \cos d(Uy, T_i x) \sin(1-t)d(Ux, Uy)) \\
&= \left(\alpha \cos d(Ux, u) + (1-\alpha) \sum_{i=1}^{k} \beta^i \cos d(Ux, T_i x)\right) \sin td(Ux, Uy) \\
&\quad + \left(\alpha \cos d(Uy, u) + (1-\alpha) \sum_{i=1}^{k} \beta^i \cos d(Uy, T_i x)\right) \sin(1-t)d(Ux, Uy).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\left(\alpha \cos d(Ux, u) + (1-\alpha) \sum_{i=1}^{k} \beta^i \cos d(Ux, T_i x)\right) \frac{\sin d(Ux, Uy) - \sin td(Ux, Uy)}{\sin(1-t)d(Ux, Uy)} \\
&\geq \alpha \cos d(Uy, u) + (1-\alpha) \sum_{i=1}^{k} \beta^i \cos d(Uy, T_i x).
\end{aligned}$$

Letting $t \to 1$, we have

$$\begin{aligned}
&\left(\alpha \cos d(Ux, u) + (1-\alpha) \sum_{i=1}^{k} \beta^i \cos d(Ux, T_i x)\right) \cos d(Ux, Uy) \\
&\geq \alpha \cos d(Uy, u) + (1-\alpha) \sum_{i=1}^{k} \beta^i \cos d(Uy, T_i x).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\left(\alpha \cos d(Uy, u) + (1-\alpha) \sum_{i=1}^{k} \beta^i \cos d(Uy, T_i y)\right) \cos d(Ux, Uy) \\
&\geq \alpha \cos d(Ux, u) + (1-\alpha) \sum_{i=1}^{k} \beta^i \cos d(Ux, T_i y).
\end{aligned}$$

Therefore, we get

$$\left(\alpha \cos d(Ux,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Ux,T_ix)\right.$$
$$\left. + \alpha \cos d(Uy,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Uy,T_iy)\right) \cos d(Ux,Uy)$$
$$\geq \alpha \cos d(Uy,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Uy,T_ix)$$
$$+ \alpha \cos d(Ux,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Ux,T_iy),$$

Therefore, we get

$$\left(\alpha \cos d(Ux,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Uy,T_ix)\right.$$
$$\left. + \alpha \cos d(Uy,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Ux,T_iy)\right) (1-\cos d(Ux,Uy))$$
$$\leq (1-\alpha)\cos d(Ux,Uy)\sum_{i=1}^{k}\beta^i \left((1-\cos d(Uy,T_ix)) + (1-\cos d(Ux,T_iy)) \right.$$
$$\left. -(1-\cos d(Ux,T_ix)) - (1-\cos d(Uy,T_iy))\right).$$

By Lemma 3.2, we obtain

$$\left(\alpha \cos d(Ux,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Uy,T_ix)\right.$$
$$\left. +\alpha \cos d(Uy,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Ux,T_iy)\right) (1-\cos d(Ux,Uy))$$
$$\leq (1-\alpha)\cos d(Ux,Uy)\sum_{i=1}^{k}\beta^i((1-\cos d(Ux,Uy)) + (1-\cos d(T_ix,T_iy)))$$
$$\leq (1-\alpha)((1-\cos d(Ux,Uy)) + (1-\cos d(x,y))).$$

Thus, we get

$$\Bigg(\alpha \cos d(Ux,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Uy,T_ix)$$
$$+\alpha \cos d(Uy,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Uy,T_iy) - (1-\alpha)\Bigg)(1-\cos d(Ux,Uy))$$
$$\leq (1-\alpha)(1-\cos d(x,y)).$$

Since $\frac{1}{2} < \cos d(u,v) \leq 1$ for any $u, v \in X$, we get

$$\alpha \cos d(Ux,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Ux,Tx)$$
$$+ \alpha \cos d(Uy,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Uy,Ty) - (1-\alpha)$$
$$> 2\left(\frac{1}{2}\alpha + \frac{1}{2}(1-\alpha)\right) - 1 + \alpha = \alpha,$$

that is, we get

$$\alpha(1-\cos d(Ux,Uy)) \leq (1-\alpha)(1-\cos d(x,y)).$$

and thus

$$1-\cos d(Ux,Uy) \leq \frac{1-\alpha}{\alpha}(1-\cos d(x,y)).$$

Since $\frac{1}{2} < \alpha < 1$, we have $0 < \frac{1-\alpha}{\alpha} < 1$. Hence U is a contraction by Lemma 3.3. □

In Theorem 3.4, U has a unique fixed point $x \in X$. That is, it satisfies that

$$x = Ux = \operatorname*{argmax}_{z\in X}\left\{\alpha \cos d(z,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(z,T_ix)\right\}.$$

Thus we can define a sequence $\{x_n\}$ by

$$x_n = \operatorname*{argmax}_{z\in X}\left\{\alpha_n \cos d(z,u) + (1-\alpha_n)\sum_{i=1}^{k}\beta_n^i \cos d(z,T_ix_n)\right\}$$

where $\alpha_n \in]\frac{1}{2},1[$ and $\beta_n^i \in]0,1[$ for $i = 1,2,\ldots,k$ in a CAT(1) space. However, since $\{\alpha_n\}$ cannot tend to 0, we cannot show the Browder's type convergence theorem in CAT(1) spaces for this mapping. On the other hand, we can show the following convergence theorem.

Theorem 3.5. *Let X be a complete* CAT(1) *space with $d(u,v) < \frac{\pi}{3}$ for any $u, v \in X$. For $i = 1, 2, \dots, k$, let $T_i\colon X \to X$ be nonexpansive mappings with $\bigcap_{i=1}^k F(T_i) \neq \varnothing$. Suppose $\{\alpha_n\} \subset \mathbb{R}$ and $a \in \mathbb{R}$ such that $\frac{1}{2} < \alpha_n \leq a < 1$ for $n \in \mathbb{N}$. Let $\{\beta_n^i\} \subset]0,1[$ for $i = 1, 2, \dots, k$ such that $\sum_{i=1}^k \beta_n^i = 1$. Let $x_1 \in X$ and generate $\{x_n\}$ as follows: For $n \in \mathbb{N}$ and given $x_n \in X$, let x_{n+1} be a unique point in X satisfying that*

$$x_{n+1} = \operatorname*{argmax}_{z \in X} \left\{ \alpha_n \cos d(z, x_n) + (1 - \alpha_n) \sum_{i=1}^k \beta_n^i \cos d(z, T_i x_{n+1}) \right\}.$$

Then, $\{x_n\}$ is well-defined and Δ-convergent to some $x_0 \in \bigcap_{i=1}^k F(T_i)$.

Proof. Define a mapping $V_n\colon X \to X$ by

$$V_n x = \operatorname*{argmax}_{z \in X} \left\{ \alpha_n \cos d(z, x_n) + (1 - \alpha_n) \sum_{i=1}^k \beta_n^i \cos d(z, T_i x) \right\}$$

for every $x \in X$. Then, V_n is well-defined as a single-valued mapping on X by Theorem 2.1. We can show that V_n is a contraction in the same way as the proof of Theorem 3.4 and thus it has a unique fixed point $x_{n+1} \in X$. That is, it satisfies that

$$x_{n+1} = V_n x_{n+1} = \operatorname*{argmax}_{z \in X} \left\{ \alpha_n \cos d(z, x_n) + (1 - \alpha_n) \sum_{i=1}^k \beta_n^i \cos d(z, T_i x_{n+1}) \right\}.$$

This implies that x_{n+1} satisfying this equation exists uniquely, and hence $\{x_n\}$ is well-defined. Next, we show $\{x_n\}$ is Δ-convergent to some $x_0 \in \bigcap_{i=1}^k F(T_i)$. Let $p \in \bigcap_{i=1}^k F(T_i)$ and $t \in]0,1[$. Then, we have

$$\begin{aligned}
&\left(\alpha_n \cos d(x_n, x_{n+1}) + (1 - \alpha_n) \sum_{i=1}^k \beta_n^i \cos d(T_i x_{n+1}, x_{n+1}) \right) \sin d(x_{n+1}, p) \\
&= \left(\alpha_n \cos d(x_n, V_n x_{n+1}) + (1 - \alpha_n) \sum_{i=1}^k \beta_n^i \cos d(T_i x_{n+1}, V_n x_{n+1}) \right) \sin d(x_{n+1}, p) \\
&\geq \alpha_n \cos d(x_n, t x_{n+1} \oplus (1-t) p) \sin d(x_{n+1}, p) \\
&\quad + (1 - \alpha_n) \sum_{i=1}^k \beta_n^i \cos d(T_i x_{n+1}, t x_{n+1} \oplus (1-t) p) \sin d(x_{n+1}, p) \\
&\geq \alpha_n (\cos d(x_n, x_{n+1}) \sin t d(x_{n+1}, p) + \cos d(x_n, p) \sin (1-t) d(x_{n+1}, p)) \\
&\quad + (1 - \alpha_n) \sum_{i=1}^k \beta_n^i (\cos d(T_i x_{n+1}, x_{n+1}) \sin t d(x_{n+1}, p) \\
&\qquad + \cos d(T_i x_{n+1}, p) \sin d(1-t)(x_{n+1}, p))
\end{aligned}$$

$$= \left(\alpha_n \cos d(x_n, x_{n+1}) + (1-\alpha_n)\sum_{i=1}^{k}\beta_n^i \cos d(T_i x_{n+1}, x_{n+1})\right)\sin t d(x_{n+1}, p)$$
$$+ \left(\alpha_n \cos d(x_n, p) + (1-\alpha_n)\sum_{i=1}^{k}\beta_n^i \cos d(T_i x_{n+1}, p)\right)\sin(1-t) d(x_{n+1}, p).$$

Thus, we get

$$\left(\alpha_n \cos d(x_n, x_{n+1}) + (1-\alpha_n)\sum_{i=1}^{k}\beta_n^i \cos d(T_i x_{n+1}, x_{n+1})\right)$$
$$\times \frac{\sin d(x_{n+1}, p) - \sin t d(x_{n+1}, p)}{\sin(1-t) d(x_{n+1}, p)}$$
$$\geq \alpha_n \cos d(x_n, p) + (1-\alpha_n)\cos d(T x_{n+1}, p).$$

Letting $t \to 1$, we have

$$\cos d(x_{n+1}, p)$$
$$\geq \left(\alpha_n \cos d(x_n, x_{n+1}) + (1-\alpha_n)\sum_{i=1}^{k}\beta_n^i \cos d(T_i x_{n+1}, x_{n+1})\right)\cos d(x_{n+1}, p)$$
$$\geq \alpha_n \cos d(x_n, p) + (1-\alpha_n)\cos d(x_{n+1}, p).$$

Therefore, since $\{\alpha_n\} \subset]\frac{1}{2}, a]$, we obtain

$$\cos d(x_{n+1}, p) \geq \cos d(x_n, p).$$

This implies $d(x_n, p) \geq d(x_{n+1}, p)$. Since the real sequence $\{d(x_n, p)\}$ is nonincreasing, there exists

$$\lim_{n\to\infty} d(x_n, p) = c_p \in \left[0, \frac{\pi}{2}\right[.$$

Then, we have

$$1 \geq \alpha_n \cos d(x_n, x_{n+1}) + (1-\alpha_n)\sum_{i=1}^{k}\beta_n^i \cos d(T_i x_{n+1}, x_{n+1})$$
$$\geq (\alpha_n \cos d(x_n, p) + (1-\alpha_n)\cos d(x_{n+1}, p))\frac{1}{\cos d(x_{n+1}, p)}$$
$$\geq \frac{\alpha_n(\cos d(x_n, p) - \cos d(x_{n+1}, p))}{\cos d(x_{n+1}, p)} + 1$$
$$\to 1$$

as $n \to \infty$. This implies

$$\lim_{n\to\infty}\left(\alpha_n \cos d(x_n, x_{n+1}) + (1-\alpha_n)\sum_{i=1}^{k}\beta_n^i \cos d(T_i x_{n+1}, x_{n+1})\right) = 1.$$

Then we obtain

$$\lim_{n\to\infty} \cos d(x_n, x_{n+1}) = \lim_{n\to\infty} \cos d(T_i x_{n+1}, x_{n+1}) = 1$$

for $i = 1, 2, \ldots, k$. In fact, assume $\{\cos d(x_n, x_{n+1})\}$ does not converges to 1t. Then, there exist $\varepsilon > 0$ and a subsequence $\{\cos d(x_{n_j}, x_{n_j+1})\}$ of $\{\cos d(x_n, x_{n+1})\}$ such that $\cos d(x_{n_j}, x_{n_j+1}) \leq 1-\varepsilon$ for $j \in \mathbb{N}$. Furthermore, since $\{\alpha_{n_j}\} \subset]\frac{1}{2}, a]$, we may assume $\alpha_{n_j} \to \alpha_0 \in [\frac{1}{2}, a]$ without loss of generality. Then we have

$$\begin{aligned}
1 &= \limsup_{j\to\infty} \left(\alpha_{n_j} \cos d(x_{n_j}, x_{n_j+1}) + (1-\alpha_{n_j}) \sum_{j=1}^{k} \beta^i_{n_j} \cos d(T_i x_{n_j}, x_{n_j+1}) \right) \\
&\leq \alpha_0 \limsup_{j\to\infty} \cos d(x_{n_j}, x_{n_j+1}) + (1-\alpha_0) \sum_{i=1}^{k} \beta^i_{n_j} \limsup_{j\to\infty} \cos d(T_i x_{n_j+1}, x_{n_j+1}) \\
&\leq \alpha_0(1-\varepsilon) + (1-\alpha_0) = 1 - \alpha_0\varepsilon < 1.
\end{aligned}$$

This is a contradiction. Thus we have $\lim_{n\to\infty} \cos d(x_n, x_{n+1}) = 1$, and similarly we get $\lim_{n\to\infty} \cos d(T_i x_{n+1}, x_{n+1}) = 1$ for $i = 1, 2, \ldots, k$. Hence we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(T_i x_{n+1}, x_{n+1}) = 0$$

for $i = 1, 2, \ldots, k$. Let $x_0 \in X$ be a unique asymptotic center of a sequence $\{x_n\}$ and let $u \in X$ be an asymptotic center of any subsequence $\{x_{n_j}\}$ of $\{x_n\}$. We will show $u = x_0$. From the definition of asymptotic center, we have

$$\begin{aligned}
r(\{x_{n_j}\}) &= \limsup_{j\to\infty} d(x_{n_j}, u) \\
&\leq \limsup_{j\to\infty} d(x_{n_j}, T_i u) \\
&\leq \limsup_{j\to\infty} (d(x_{n_j}, T_i x_{n_j}) + d(T_i x_{n_j}, T_i u)) \\
&= \limsup_{j\to\infty} d(T_i x_{n_j}, T_i u) \\
&\leq \limsup_{j\to\infty} d(x_{n_j}, u) = r(\{x_{n_j}\}).
\end{aligned}$$

for $i = 1, 2, \ldots, k$. This implies $T_i u \in \mathrm{AC}(\{x_{n_j}\})$ for $i = 1, 2, \ldots, k$. From the uniqueness of an asymptotic center, we get $u = T_i u$ for $i = 1, 2, \ldots, k$, that is, $u \in \bigcap_{i=1}^{k} F(T_i)$. It follows that $\{d(x_n, u)\}$ is convergent to c_u. Therefore, we obtain

$$\begin{aligned}
r(\{x_n\}) &= \limsup_{n\to\infty} d(x_n, x_0) \\
&\leq \limsup_{n\to\infty} d(x_n, u) = c_u = \lim_{j\to\infty} d(x_{n_j}, u) \\
&\leq \limsup_{j\to\infty} d(x_{n_j}, x_0)
\end{aligned}$$

$$\leq \limsup_{n\to\infty} d(x_n, x_0) = r(\{x_n\}).$$

Thus $u \in \mathrm{AC}(\{x_n\})$. From the uniqueness of an asymptotic center, we get $u = x_0$. Hence, $\{x_n\}$ is Δ-convergent to $x_0 \in \bigcap_{i=1}^k F(T_i)$. This is the desired result. $\square$

We get the next proposition from Theorem 3.5. This is proved in [5].

Corollary 3.1. *Let X be a complete $\mathrm{CAT}(1)$ space with $d(u,v) < \frac{\pi}{3}$ for any $u, v \in X$ and $T\colon X \to X$ nonexpansive with $F(T) \neq \varnothing$. Suppose $\{\alpha_n\} \subset \mathbb{R}$ and $a \in \mathbb{R}$ such that $\frac{1}{2} < \alpha_n \leq a < 1$ for $n \in \mathbb{N}$. Let $x_1 \in X$ and generate $\{x_n\}$ as follows:*
For $n \in \mathbb{N}$ and given $x_n \in X$, let x_{n+1} be a unique point in X satisfying that

$$x_{n+1} = \operatorname*{argmax}_{z\in X}\{\alpha_n \cos d(z, x_n) + (1-\alpha_n)\cos d(z, Tx_{n+1})\}$$

Then, $\{x_n\}$ is well-defined and Δ-convergent to some $x_0 \in F(T)$.

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