Convex minimization problems on geodesic spaces and improvement of approximating sequences
測地距離空間における凸最小化問題と近似列の改良

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Abstract

In 2021, Hidaka and Kimura proved a Δ-convergence theorem in CAT(1) spaces. In this paper, we modified the coefficient condition and obtain another Δ-convergence theorem.

1 Introduction

In [1], the authors proved the following theorem.

Theorem 1 ([1]). *Let X be an admissible complete* CAT(1) *space. Let $f : X \to \,]-\infty, \infty]$ be a proper convex lower semicontinuous function and suppose that* $\operatorname{argmin} f \neq \emptyset$. *Let $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \beta_n = \infty$ and that both $\{\beta_n\}$ and $\{\gamma_n\}$ converge to 0. For an initial point $x_1 \in X$, generate a sequence $\{x_n\}$ as follows:*

$$
\begin{aligned}
y_n &= J_f x_n, \\
\alpha_n &\in [\min\{\beta_n, d(x_n, y_n) - \gamma_n\}, 1] \cap [0,1], \\
x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n y_n.
\end{aligned}
$$

Suppose that one of the following conditions holds:

- $\inf_{n\in\mathbb{N}} \alpha_n > 0$;
- $\sum_{n=1}^{\infty} \alpha_n < \infty$.

Then, $x_n \overset{\Delta}{\rightharpoonup} x_0 \in \operatorname{argmin} f$.

In this paper, we try to remove the condition that $\inf_{n\in\mathbb{N}} \alpha_n > 0$ in Theorem 1. In order to this, we modify the coefficient condition.

2 Preliminaries

Let λ be a positive real number. A metric space X is said to be λ-geodesic if for each $x, y \in X$ with $d(x, y) < \lambda$, there exists a mapping $c : [0, l] \to X$ such that $c(0) = x, c(l) = y$, and

$$d(c(t_1), c(t_2)) = |t_1 - t_2|$$

for all $t_1, t_2 \in [0, l]$, where $l = d(x, y)$. The mapping c is called a geodesic from x to y. If a geodesic c from x to y is unique, the geodesic segment $[x, y]$ is defined by

$$[x, y] = \{c(t) : 0 \leq t \leq l\}.$$

Let X be a uniquely geodesic space, and $x, y, z \in X$. Let $x, y, z \in X$ with $d(x, y) + d(y, z) + d(z, x) < 2\pi$. The set $\Delta = \Delta(x, y, z)$ is defined by $\Delta = [x, y] \cup [y, z] \cup [z, x]$. We take $\bar{x}, \bar{y}, \bar{z} \in \mathbb{S}^2$ such that $d(x, y) = d_{\mathbb{S}^2}(\bar{x}, \bar{y}), d(y, z) = d_{\mathbb{S}^2}(\bar{y}, \bar{z}), d(z, x) = d_{\mathbb{S}^2}(\bar{z}, \bar{x})$. The set $\bar{\Delta} = \bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$ is defined by $\bar{\Delta} = [\bar{x}, \bar{y}] \cup [\bar{y}, \bar{z}] \cup [\bar{z}, \bar{x}]$. X is called a CAT(1) space, if for all Δ and $p, q \in \Delta$ with $\bar{p}, \bar{q} \in \bar{\Delta}$,

$$d(p, q) \leq d_{\mathbb{S}^2}(\bar{p}, \bar{q}).$$

We say that a CAT(1) space X is admissible if

$$d(w, w') < \frac{\pi}{2}$$

for all $w, w' \in X$.

The definition of the resolvent of f is as follows [4]:

$$J_f x = \operatorname*{argmin}_{y\in X}\{f(y) + \tan d(y, x) \sin d(y, x)\},$$

where $f : X \to]-\infty, \infty]$ is a proper lower semicontinuous function. We denote by $\operatorname{argmin}_X f$ the set of all $u \in X$ such that $f(u) = \inf f(X)$. For a bounded sequence $\{x_n\} \subset X$, the asymptotic center $\mathcal{A}(\{x_n\})$ of $\{x_n\}$ is defined by

$$\mathcal{A}(\{x_n\}) = \left\{ u \in X \;\middle|\; \limsup_{n\to\infty} d(u, x_n) = \inf_{y\in X} \limsup_{n\to\infty} d(y, x_n) \right\}.$$

A sequence $\{x_n\}$ is said to be Δ-convergent to a point $p \in X$ if

$$\mathcal{A}(\{x_{n_i}\}) = \{p\}$$

for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

We know the following lemmas.

Lemma 1 ([4]). *Let X be an admissible complete* CAT(1) *space, f a proper convex lower semicontinuous functions of X into $]-\infty,\infty]$. Let $J_{\eta f}$ be the resolvent of ηf for all $\eta > 0$ and $C_{\eta,z}$ the real number given by*

$$C_{\eta,z} = \cos d(J_{\eta f} z, z)$$

for all $\eta > 0$ and $z \in X$. Then

$$\begin{aligned}(\lambda C_{\lambda,x}^2(1+C_{\mu,y}^2)C_{\mu,y} + \mu C_{\mu,y}^2(1+C_{\lambda,x}^2)C_{\lambda,x}) \cos d(J_{\lambda f}x, J_{\mu f}y) \\ \geq \lambda C_{\lambda,x}^2(1+C_{\mu,y}^2) \cos d(J_{\lambda f}x, y) + \mu C_{\mu,y}^2(1+C_{\lambda,x}^2) \cos d(J_{\mu f}y, x)\end{aligned}$$

for all $x, y \in X$ and $\lambda, \mu > 0$. Further,

$$\frac{\pi}{2}\left(\frac{1}{C_{\lambda,x}^2}+1\right)(C_{\lambda,x} \cos d(u, J_{\lambda f}x) - \cos d(u,x)) \geq \lambda(f(J_{\lambda f}x) - f(u))$$

and

$$\cos d(J_{\lambda f}x, x) \cos d(u, J_{\lambda f}x) \geq \cos d(u,x) \tag{1}$$

for all $x \in X, u \in \operatorname{argmin}_X f$ and $\lambda > 0$.

Let X be a metric space such that $d(v,v') < \pi/2$ for all $v, v' \in X$, T a mapping of X into itself, and C_z the real number given by

$$C_z = \cos d(Tz, z)$$

for all $z \in X$. The mapping T is said to be vicinal [5] if

$$\begin{aligned}(C_x^2(1+C_y^2) + C_y^2(1+C_x^2)) \cos d(Tx, Ty) \\ \geq C_x^2(1+C_y^2) \cos d(Tx, y) + C_y^2(1+C_x^2) \cos d(Ty, x)\end{aligned}$$

for all $x, y \in X$.

Lemma 2 ([5]). *Let X be a metric space such that $d(v,v') < \pi/2$ for all $v, v' \in X$, T a vicinal mapping of X into itself, p an element of X, and $\{x_n\}$ a sequence in X such that $\mathcal{A}(\{x_n\}) = \{p\}$ and $d(Tx_n, x_n) \to 0$. Then p is a fixed point of T.*

Lemma 3 ([3]). *Let X be a complete* CAT(1) *space such that $d(v,v') < \pi/2$ for all $v, v' \in X$. Let f be a proper lower semicontinuous convex function of X into $]-\infty,\infty]$, J_f the resolvent of f. Then $F(J_f) = \operatorname{argmin}_X f$.*

3 Main result

The following theorem is the main result of this paper.

Theorem 2. *Let X be an admissible complete* CAT(1) *space. Let $f : X \to]-\infty, \infty]$ be a proper convex lower semicontinuous function such that* $\operatorname{argmin} f \neq \emptyset$. *Let $\{\gamma_n\}$ be a real sequence in $[0,1]$ converging to 0. For an initial point $x_1 \in X$ such that $f(x_1) < \infty$, generate a sequence $\{x_n\}$ as follows:*

$$\begin{aligned} y_n &= J_f x_n, \\ \alpha_n &\in \left[\frac{1}{2}d(x_n, y_n) - \gamma_n, 1\right] \cap [0,1], \\ x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n y_n. \end{aligned}$$

Then, $x_n \stackrel{\Delta}{\rightharpoonup} x_0 \in \operatorname{argmin} f$.

Proof. Take $\{d(x_{n_i}, y_{n_i})\} \subset \{d(x_n, y_n)\}$. There exists $\{\alpha_{n_{i_j}}\} \subset \{\alpha_{n_i}\}$ such that $\alpha_{n_{i_j}} \to \alpha_0 \in [0,1]$. If $\alpha_0 = 0$, then

$$\frac{1}{2}d(x_{n_{i_j}}, y_{n_{i_j}}) \leq \alpha_{n_{i_j}} + \gamma_{n_{i_j}} \to 0.$$

We get $d(x_{n_{i_j}}, y_{n_{i_j}}) \to 0$.

If $\alpha_0 \in]0,1]$, for $u \in \operatorname{argmin} f$, using (1) in Lemma 1, we have

$$\begin{aligned} \cos d(u, x_{n+1}) &\geq (1-\alpha_n)\cos d(u, x_n) + \alpha_n \cos d(u, y_n) \\ &\geq (1-\alpha_n)\cos d(u, x_n) + \alpha_n \frac{\cos d(u, x_n)}{\cos d(y_n, x_n)} \\ &= \cos d(u, x_n) + \alpha_n \cos d(u, x_n)\left(\frac{1}{\cos d(y_n, x_n)} - 1\right). \end{aligned}$$

It implies

$$\frac{\cos d(u, x_{n+1})}{\cos d(u, x_n)} - 1 \geq \alpha_n \left(\frac{1}{\cos d(y_n, x_n)} - 1\right).$$

We know that $d(u, x_n) \to [0, \pi/2[$. In fact, since

$$\cos d(u, x_{n+1}) \geq (1-\alpha_n)\cos d(u, x_n) + \alpha_n \cos d(u, y_n) \geq \cos d(u, x_n),$$

we have

$$d(u, x_{n+1}) \leq d(u, x_n).$$

Thus $d(u, x_n) \to [0, \pi/2[$. Hence we get,

$$0 \leq \alpha_{n_{i_j}} \left(\frac{1}{\cos d(y_{n_{i_j}}, x_{n_{i_j}})} - 1\right) \leq \frac{\cos d(u, x_{n_{i_j}+1})}{\cos d(u, x_{n_{i_j}})} - 1 \to 0.$$

We get $d(x_{n_{i_j}}, y_{n_{i_j}}) \to 0$. It means that $d(x_n, y_n) \to 0$.

Let $\{x_{n_i}\} \subset \{x_n\}$ with $w = \mathcal{A}(\{x_{n_i}\})$. There exists $\{x_{n_{i_j}}\} \subset \{x_{n_i}\}$ such that $x_{n_{i_j}} \overset{\Delta}{\rightharpoonup} z$. In fact, since

$$d(u, x_{n+1}) \leq d(u, x_n) \leq d(u, x_1) < \frac{\pi}{2},$$

we have $\limsup_{n\to\infty} d(u, x_n) < \pi/2$. Using Lemma 2 and Lemma 3, we have $z \in F(J_f) = \operatorname{argmin} f$. We put $v = \mathcal{A}(\{x_n\})$. Then,

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, z) &= \lim_{i\to\infty} d(x_{n_i}, z) \\
&= \lim_{j\to\infty} d(x_{n_{i_j}}, z) \\
&\leq \limsup_{j\to\infty} d(x_{n_{i_j}}, w) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, w) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, v) \\
&\leq \limsup_{n\to\infty} d(x_n, v).
\end{aligned}$$

We get $v = w = z$. Hence, $x_n \overset{\Delta}{\rightharpoonup} v = z \in \operatorname{argmin} f$. □

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