NOTE ON THE SOLVABILITY OF THE FULLY FOURTH ORDER NONLINEAR BOUNDARY VALUE PROBLEMS

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1. Introduction

Many researchers have considered the following differential equation;

$$\left\{\begin{array}{l} u''''(t) = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{array}\right. \tag{1.1}$$

where f is a continuous mapping from $[0,1] \times \mathbb{R}$ into $\mathbb{R}$, we denote by $\mathbb{R}$ the set of all real numbers; see [9, 10, 14, 15, 21, 22, 23, 25, 26]. Equation (1.1) can be used to model the deformations of an Cantilever equation. In mechanics, the problem is called Cantilever beam equation, and in the equation, the physical meaning of the derivatives of the deformation function $u(t)$ is a follows: u'''' is the load density stiffness, u''' is the shear force stiffness, u'' is the bending moment stiffness, and u' is the slope ;see [9, 10]. In some practice, only its positive solution is significant.

Meanwhile, fractional differential equations have been studied by many researchers; see [3, 5, 6, 7, 8, 11, 12, 17, 18, 19, 24]. However, to the best of our knowledge, there are no results for the boundary value problem represented by (1.4) for $3 < \alpha \leq 4$, which we consider in the present paper. We use the method of order reduction and contaraction principle [27, 28] to prove the existence and uniqueness of solutions. In [20], we consider the boundary value problem for fractional order differential equation

$$\left\{\begin{array}{l} D^{\alpha}_{0+} u(t) = f(t, u(t)), t \in [0,1] \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{array}\right. \tag{1.2}$$

where D^{α}_{0+} is the Riemann-Liouville derivative of order α, $3 < \alpha \leq 4$, and f is a continuous function of $[0,1] \times \mathbb{R}$ into $\mathbb{R}$.

Several authors consider the following fourth order boundary value problems for the fully nonlinear with boundary conditions see; [27, 28, 29, 30, 31, 32, 33].

$$\left\{\begin{array}{l} u''''(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{array}\right. \tag{1.3}$$

Recently in [27], using an order reduction and contaraction method, authors give a existance and uniqueness of solution. In [28] authors using a contraction method and iteration method they give a existance and uniqueness of solutions of boundary value problems and also examples of non-linear function as a function f in (1.3).

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In this paper we propose the following differential equation of fractional order α, $3 < \alpha \leq 4$ with the two point boundary condition involving the form (1.2).

$$
(1.4) \qquad \begin{cases} D_{0+}^{\alpha}u(t) = f(t, u(t), D_{0+}^{\alpha-3}u(t), D_{0+}D_{0+}^{\alpha-3}u(t), D_{0+}D_{0+}D_{0+}^{\alpha-3}u(t)), \\ \quad 0 < t < 1, \\ u(0) = D_{0+}^{\alpha-3}u(0) = D_{0+}^{\alpha-2}u(1) = D_{0+}^{\alpha-1}u(1) = 0. \end{cases}
$$

In particular, using a order reduction and contaraction method, we give a unique solution compairing method in [27, 28], we also give a suitable Lipschitz constant through a study.

Let $\alpha > 0$. The Riemann-Liouville fractional integral of order α of u, denoted $I_{0+}^{\alpha}u$, is defined by

$$
I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}u(s)ds,
$$

provided the right-hand side exists. The Riemann-Liouville fractional derivative of order α of a function u of $(0, \infty)$ into $\mathbb{R}$ is given by

$$
D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1}u(s)ds,
$$

where $n = [\alpha] + 1$ ($[\alpha]$ denotes the integer part of α) and $\Gamma(\alpha)$ denotes the gamma function; see [11, 18]. Note that for $\alpha > \beta > 0$, we have

$$
D_{0+}^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}.
$$

A function $u \in C[0,1]$ is called a solution of problem (1.4) if $D^{\alpha}u \in C[0,1]$, $D^{\alpha-3}u \in L^1[0,1]$, $D^{\alpha-2}u \in L^1[0,1]$, $D^{\alpha-1}u \in L^1[0,1]$, u satisfies the boundary conditions and equality in (1.4) a.e. on [0, 1].

2. Lemmas

Let h be a continuous mapping $[0,1]$ into $\mathbb{R}$, we consider the following fractional differential boundary value problems defined by

$$
(2.1) \qquad \begin{cases} D_{0+}^{\alpha}u(t) = h(t), \quad 0 < t < 1, \\ u(0) = u(1) = 0, D_{0+}D_{0+}^{\alpha-3}u(0) = D_{0+}D_{0+}^{\alpha-3}u(1) = 0, \end{cases}
$$

where $3 < \alpha \leq 4$. In this section, we show the unique solution to the boundary value problem represented by (2.1). A mapping u of $[0,1]$ into $\mathbb{R}$ is a solution of that boundary value problems if u is continuous on $[0,1]$ and u satisfies (2.1).

The following lemma can be found in [6]; see also [11]. We denoted by $C(0,1)$ the set of all continuous mappings of $(0,1)$ into $\mathbb{R}$ and by $L(0,1)$ the set of all Lebesgue integrable mappings of $[0,1]$ into $\mathbb{R}$.

Lemma 1. *Let $\alpha > 0$. If $u(t) \in C(0,1) \cap L(0,1)$ satisfying $D_{0+}^{\alpha}u(t) \in C(0,1) \cap L(0,1)$, then there exist constants $C_1, C_2, \ldots, C_n \in \mathbb{R}$ such that*

$$
I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = u(t) + C_1t^{\alpha-1} + C_2t^{\alpha-2} + \cdots + C_nt^{\alpha-n},
$$

where $n = [\alpha] + 1$ and $I_{0+}^{\alpha}u$ is the Riemann-Liouville fractional integral of order α of a function u is defined by

$$
I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}u(s)ds.
$$

Using Lemma 1, we obtain the following.

Lemma 2. *Let $g \in C^n[0,1]$ be given. Then the unique solution to problem $D^\alpha y(t) = g(t)$ together with the boundary conditions in (1.4) is*

$$u(t) = \int_0^t G(t,s)g(s)ds,$$

where

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha)}\left((t-s)^{\alpha-1} - t^{\alpha-1} + (\alpha-1)st^{\alpha-2}\right). & \text{if } 0 \le s \le t \le 1, \\ \frac{1}{\Gamma(\alpha)}\left(-t^{\alpha-1} + (\alpha-1)st^{\alpha-2}\right) & \text{if } 0 \le t \le s < 1. \end{cases} \tag{2.2}$$

Proof. In order to have $G(t,s)$, by Lemma 1, we have

$$\begin{aligned} u(t) &= c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + c_4 t^{\alpha-4} + I_+^\alpha g(t) \\ &= c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + c_4 t^{\alpha-4} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(t) \end{aligned}$$

By the boundary condition in (1.4), since $u(0) = 0$, we have $c_4 = 0$.

$$D_{0+}^{\alpha-3}\left(\int_0^t (t-s)^{\alpha-1} g(s)ds\right) = \frac{\Gamma(\alpha)}{2}\int_0^t (t-s)^2 g(s)ds.$$

Since $D_{0+}^{\alpha-3}u(0) = 0$, we have $c_3 = 0$. Since

$$D_{0+}^{\alpha-2}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)ds\right) = \int_0^t (t-s)g(s)ds.$$

and $D_{0+}^{\alpha-2}u(1) = 0$, we have

$$0 = c_1\Gamma(\alpha) + c_2\Gamma(\alpha-1) + \int_0^1 (1-s)g(s)ds.$$

Also we have

$$D_{0+}^{\alpha-1}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)ds\right) = \int_0^t g(s)ds.$$

Then we have

$$c_1 = -\frac{1}{\Gamma(\alpha)}\int_0^1 g(s)ds.$$

Since

$$\begin{aligned} 0 &= c_1\Gamma(\alpha) + c_2\Gamma(\alpha-1) + \int_0^1 (1-s)g(s)ds. \\ &= -\int_0^1 g(s)ds + c_2\Gamma(\alpha-1) + \int_0^1 (1-s)g(s)ds, \end{aligned}$$

we have

$$\begin{aligned} c_2 &= -\frac{1}{\Gamma(\alpha-1)}\left(\int_0^1 (1-s)g(s)ds - \int_0^1 g(s)ds\right) \\ &= \frac{(\alpha-1)}{\Gamma(\alpha)}\left(\int_0^1 sg(s)ds\right). \end{aligned}$$

Hence we have

$$\begin{aligned}
u(t) &= c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + D_{0+}^{-\alpha} g(t) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 -g(s) ds t^{\alpha-1} + \frac{(\alpha-1)}{\Gamma(\alpha)} \left(\int_0^1 s g(s) ds \right) t^{\alpha-2} \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds \\
&= \frac{1}{\Gamma(\alpha)} \left(\int_0^t \left((t-s)^{\alpha-1} - t^{\alpha-1} + (\alpha-1) s t^{\alpha-2} \right) g(s) ds \right. \\
&+ \left. \int_t^1 \left(-t^{\alpha-1} + (\alpha-1) s t^{\alpha-2} \right) g(s) ds \right).
\end{aligned}$$

Then the green function $G(t,s)$ is defined by

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha)} \left((t-s)^{\alpha-1} - t^{\alpha-1} + (\alpha-1) s t^{\alpha-2} \right). \text{ if } 0 \leq s \leq t, \\ \frac{1}{\Gamma(\alpha)} \left(-t^{\alpha-1} + (\alpha-1) s t^{\alpha-2} \right) \text{ if } t \leq s < 1. \end{cases}$$

□

Next in order to estimate the order reducton, we consider the followings.

$$\int_0^t G(t,s) h(s) ds = [-G(t,s) v(s)]_0^t + \int_0^t G_1(t,s) v(s) ds,$$

and

$$\int_t^1 G(t,s) h(s) ds = [-G(t,s) v(s)]_t^1 + \int_t^1 G_1(t,s) v(s) ds,$$

where $v(t) = \int_t^1 h(s) ds$ and $G_1(t,s) = \frac{\partial G}{\partial s}(t,s)$;

$$G_1(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha-1)} \left(-(t-s)^{\alpha-2} + t^{\alpha-2} \right) \text{ if } 0 \leq s \leq t \leq 1 \\ \frac{1}{\Gamma(\alpha-1)} t^{\alpha-2} \text{ if } 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

Moreover

$$\int_0^1 G(t,s) h(s) ds = \int_0^1 G_1(t,s) v(s) ds.$$

In fact since $v(t) = \int_1^t h(s)$, $v(1) = 0$,

$$\begin{aligned}
&[-G(t,s) v(s)]_0^1 \\
&= -\frac{1}{\Gamma(\alpha)} \left(\left[(t-s)^{\alpha-1} v(s) \right]_0^t + \left[-t^{\alpha-1} + (\alpha-1) s t^{\alpha-2} v(s) \right]_0^1 \right) \\
&= \frac{1}{\Gamma(\alpha)} \left(t^{\alpha-1} \right) v(0) \\
&+ \frac{1}{\Gamma(\alpha)} \left(\left(-(t^{\alpha-1} + (\alpha-1) t^{\alpha-2} \right) v(1) - t^{\alpha-1} v(0) \right) = 0.
\end{aligned}$$

We also have

$$D_{0+}^{\alpha-3} \left(\int_0^1 G_1(t,s) v(s) ds \right) = \int_0^1 G_2(t,s) v(s) ds,$$

where

$$G_2(t,s)=\begin{cases} s\,(0\le s\le t\le 1),\\ t\,(0\le t\le s\le 1).\end{cases} \tag{2.4}$$

$$D_{0+}D_{0+}^{\alpha-3}\left(\int_0^1 G(t,s)h(s)ds\right)=D_{0+}\left(\int_0^1 G_2(t,s)v(s)ds\right)=\int_0^1 G_3(t,s)v(s)ds.$$

where

$$G_3(t,s)=\begin{cases} 0\,(0\le s\le t\le 1),\\ 1\,(0\le t\le s\le 1).\end{cases} \tag{2.5}$$

3. Estimates of integral equations

In this section in order to have the solution of boundary value problem (1.2), we tranfered it to interal equation and estimate its value. We consider the method of order reduction and Banach contarction to the integral equations.

Now we use the method of order reduction to transform (1.4) to a nonlinear integral equation; see [27]. To do this, let

$$T_1v(t)=I_{0+}^{\alpha-3}T_2v(t)=\int_0^1 G_1(t,s)v(s)ds, \tag{3.1}$$

$$T_2v(t)=\int_0^1 G_2(t,s)v(s)ds,\ T_3v(t)=\int_0^1 G_3(t,s)v(s)ds, \tag{3.2}$$

where $G_1(t,s)$, $G_2(t,s)$ and $G_3(t,s)$ are given by (2.3), (2.4) and (2.5). From the above formulas, it follows that

$$D_{0+}D_{0+}D_{0+}^{\alpha-3}T_1v(t)=D_{0+}D_{0+}T_2v(t)=D_{0+}T_3v(t)=-v(t)$$

Note that since

$$T_1v(t)=\int_0^1 G_1(t,s)v(s)ds=\int_0^1 G(t,s)f(s)ds,$$

we have

$$T_1v(0)=T_1v(1)=0.$$

Moreover by definition,

$$T_2v(1)=\int_0^1 G_2(1,s)v(s)ds=0,\ T_3v(1)=\int_0^1 G_3(1,s)v(s)ds=0.$$

Boundary value problem (1.4) can be converted into a terminal value problem

$$D_{0+}v(t)=-f(t,T_1v(t),T_2v(t),T_3v(t),-v(t)),v(1)=0.$$

From the above formulas, it follows that

$$D_{0+}v(t)=f(t,T_1v(t),T_2v(t),T_3v(t),-v(t))$$

where

$$D_{0+}T_3v(t)=-v(t),\ D_{0+}T_2v(t)=T_3v(t),\ D_{0+}^{\alpha-3}T_1v(t)=T_2v(t).$$

Then we have the following lemmas.

Lemma 3. *Let $3 < \alpha \leq 4$. The boundary problem (1.4) is equivalent to the following integral equations forms;*

$$\begin{cases} v(t) = \int_t^1 f(s, T_1v(s), T_2v(s), T_3v(s), -v(s))ds, \\ T_1v(t) = \int_0^1 G_1(t,s)v(s)ds, \\ T_2v(t) = \int_0^1 G_2(t,s)v(s)ds, \\ T_3v(t) = \int_0^1 G_3(t,s)v(s)ds, \end{cases}$$

where $G_1(t,s)$, $G_2(t,s)$ and $G_3(t,s)$ are given by (2.3), (2.4) and (2.5).

Next we define an operator A from $C[0,1]$ into $C[0,1]$ by

$$Av(t) = \int_t^1 f(s, T_1v(s), T_2v(s), T_3v(s), v(s))ds, \tag{3.3}$$

where $v \in C[0,1]$. Also let

$$(T_1v)(t) = \int_0^1 G_1(t,s)v(s)ds,\ (T_2v)(t) = \int_0^1 G_2(t,s)v(s)ds, \tag{3.4}$$

$$(T_3v)(t) = \int_0^1 v(s)ds,\ (T_4v)(t) = v(t). \tag{3.5}$$

Then the solution of boundary value problem (1.4) is equivalent to the fixed point of A on $C[0,1]$. Take $u_0(t) = 1 - t$. By (3.4), we get

$$\begin{aligned} &\int_t^1 (T_1u_0)(t)dt \\ &= \frac{1}{\Gamma(\alpha)}\left(\frac{1}{\alpha(\alpha+1)}\frac{(1-t^{\alpha+1})}{1-t} - \frac{1}{2\alpha}\frac{(1-t^{\alpha})}{1-t} + \frac{1}{2\alpha}\frac{(1-t^{\alpha-2})}{1-t}\right)u_0(t) \end{aligned} \tag{3.6}$$

In fact take $u_0(t) = 1 - t$. By (3.4), we get

$$\begin{aligned} (T_1u_0)(t) &= \int_0^t \frac{1}{\Gamma(\alpha-1)}\left(-(t-s)^{\alpha-2}(1-s) + t^{\alpha-2}(1-s)\right)ds \\ &+ \int_t^1 \frac{1}{\Gamma(\alpha-1)}\left(t^{\alpha-2} - t^{\alpha-2}s\right)ds \\ &= \int_0^t \frac{1}{\Gamma(\alpha-1)}\left(-(t-s)^{\alpha-1} + (t-s)^{\alpha-2}t - (t-s)^{\alpha-2}\right. \\ &+ \int_0^1 \frac{1}{\Gamma(\alpha-1)}\left(t^{\alpha-2} - t^{\alpha-2}s\right)ds \\ &= \frac{-1}{\alpha\Gamma(\alpha-1)}t^{\alpha} + \frac{1}{(\alpha-1)\Gamma(\alpha-1)}t^{\alpha} - \frac{1}{(\alpha-1)\Gamma(\alpha-1)}t^{\alpha-1} \\ &+ \frac{1}{\Gamma(\alpha-1)}\left(t^{\alpha-2} - \frac{1}{2}t^{\alpha-2}\right) \\ &= \frac{1}{\Gamma(\alpha+1)}t^{\alpha} - \frac{1}{\Gamma(\alpha)}t^{\alpha-1} + \frac{1}{2\Gamma(\alpha-1)}t^{\alpha-2} \end{aligned}$$

and

$$\int_t^1 (T_1u_0)(s)ds$$
$$= \int_t^1 \left(\frac{1}{\Gamma(\alpha+1)} s^{\alpha} - \frac{1}{\Gamma(\alpha)} s^{\alpha-1} + \frac{1}{2\Gamma(\alpha-1)} s^{\alpha-2} \right) dt$$
$$= \frac{1}{\Gamma(\alpha+2)}(1-t^{\alpha+1}) - \frac{1}{\Gamma(\alpha+1)}(1-t^{\alpha}) + \frac{1}{2\Gamma(\alpha)}(1-t^{\alpha-1})$$
$$= \left(\frac{1}{\Gamma(\alpha+2)} \frac{1-t^{\alpha+1}}{1-t} - \frac{1}{\Gamma(\alpha+1)} \frac{1-t^{\alpha}}{1-t} + \frac{1}{2\Gamma(\alpha)} \frac{1-t^{\alpha-1}}{1-t} \right) u_0(t).$$

Since $1-t^3 < 1-t^{\alpha} < 1-t^4$, we have $\left(\frac{1-t^{\alpha+1}}{1-t}\right) < \frac{1-t^5}{1-t} = 1+t+t^2+t^3+t^4 \le 5$. Thus we have

$$\int_t^1 (T_1u_0)(t)dt \le C_1u_0(t)$$

where

$$C_1 = \left(\frac{5}{\Gamma(\alpha+2)} - \frac{3}{\Gamma(\alpha+1)} + \frac{3}{2\Gamma(\alpha)} \right). \tag{3.7}$$

We also have

$$(T_2u_0)(t) = \int_0^t s(1-s)ds + \int_t^1 t(1-s)ds = \frac{t}{2} - \frac{t^2}{2} + \frac{t^3}{6}$$

$$\int_t^1 (T_2u_0)(t)dt$$
$$= \left(\frac{1}{4}(1+t+t^2) - \frac{1}{6}(1+t+t^2+t^3) + \frac{1}{24}(1+t+t^2+t^3+t^4) \right)(1-t)$$
$$\le \frac{1}{6}u_0(t)$$

$$(T_3u_0)(t) = \int_0^1 \frac{1}{2}(1-s)^2 ds = \frac{1}{2}\left(-t^2+3t-1)\right)$$

$$\int_t^1 (T_3u_0)(s)ds = \int_t^1 \frac{1}{2}(1-s)^2 ds = \frac{1}{6}(1-t)^3 \le \frac{1}{6}u_0(t).$$

Thus we have

$$\int_t^1 (T_1u_0)(t)dt \le M_1u_0(t), \int_t^1 (T_2u_0)(t)dt \le M_2u_0(t), \int_t^1 (T_3u_0)(s)ds \le M_3u_0(t), \tag{3.8}$$

where

$$M_1 = \left(\frac{5}{\Gamma(\alpha+2)} - \frac{3}{\Gamma(\alpha+1)} + \frac{3}{2\Gamma(\alpha)} \right), \ M_2 = \frac{1}{6}, \ M_3 = \frac{1}{6}. \tag{3.9}$$

Next in order to use the method in [28], first we give the setting. For each number $M > 0$ we denote

$$D_M = \{(t,u,w,y,z) | 0 \le t \le 1, \|u\| \le M_1M, \|y\| \le M_2M, \|v\| \le M_3M, \|z\| \le M\}$$

and by $B[O, M]$ we also denote a closed ball centered at O with the radius M in the space of continuous functions $C[0,1]$ with the norm $\|\varphi\| = \max_{0\le t\le 1}|\varphi(t)|$. In this case, by [28, Theorem 2.2], we have the following lamma;

Lemma 4. *Assume that there exist numbers M, $C_i > 0$ where $(i = 1, 2, 3, 4)$, such that*

$$|f(t, u, w, y, z)| \le M, \tag{3.10}$$

$$|f(t, x_1, x_2, x_3, x_4,) - f(t, y_1, y_2, y_3, y_4)| \le \sum_{i=1}^{4} C_i|x_i - y_i| \tag{3.11}$$

for any $(t, u, w, y, z), (t, x_1, x_2, x_3, x_4), (t, y_1, y_2, y_3, y_4) \in D_M$. Then the operator A defined by (3.3), where v_φ, u_φ are the solutions of the problems (1.4), maps the closed ball $B[O, M]$ into itself. Moreover, if

$$q := M_1C_1 + M_2C_2 + M_3C_3 + C_4 < 1 \tag{3.12}$$

then the operator A is contractive operator in $B[O, M]$.

Proof. We only need to prove that A defined by (3.3) has a unique fixed point in $B[O, M]$. We introduce a linear operator T on $C[0, 1]$ as

$$(Tu)(t) = \int_t^1 (M_1(T_1u)(s) + M_2(T_2u)(s) + M_3(T_3u)(s) + M_4(T_4u)(s))\, ds. \tag{3.13}$$

As a first step, we show that for all $v \in C[0, 1]$ with $v(t) > 0$ $(t \in [0, 1])$, there exists $N = N(v)$ such that

$$(Tv)(t) \le Nu_0(t), t \in [0, 1]. \tag{3.14}$$

In fact, we take $N = M_1||T_1v|| + M_2||T_2v|| + M_3||T_3v|| + M_4||v||$. Then by (3.13), we obtain the result. Moreover, it follows from (3.8) that

$$(Tu_0)(t) \le qu_0(t), t \in [0, 1], \tag{3.15}$$

where

$$q = M_1C_1 + M_2C_2 + M_3C_3 + C_4 < 1.$$

For any given $v_0 \in B[O, M]$, let

$$v_n(t) = (Av_{n-1})(t), w_n(t) = |v_n(t) - v_{n-1}(t)|\,, n = 1, 2, \ldots.$$

Since A is a operator on $B(O, M)$, for $t \in [0, 1]$, we have

$$\begin{aligned}
&w_{n+1}(t) = |v_{n+1}(t) - v_n(t)| = |(Av_n)(t) - (Av_{n-1})(t)|\\
&\le \int_t^1 |f(s, (T_1v_n)(s), (T_2v_n)(s), (T_3v_n)(s), -v_n(s))\\
&-f(s, (T_1v_{n-1})(s), (T_2v_{n-1})(s), (T_3v_{n-1})(s), -v_{n-1}(s))|\, ds\\
&\le \int_t^1 (M_1T_1(|v_n - v_{n-1}|)(s) + M_2T_2(|v_n - v_{n-1}|)(s)\\
&+ M_3T_3(|v_n - v_{n-1}|)(s) + M_4\,|v_n(s) - v_{n-1}(s)|)ds = (T\,|v_n - v_{n-1}|)(t) = (Tw_n)(t).
\end{aligned}$$

By (3.14), (3.15), and the method of induction, there exists $N = N(w_1)$ such that

$$w_{n+1}(t) \le (Tw_n)(t) \le \cdots \le (T^nw_1)(t) \le N(T^{n-1}u_0)(t) \le NM^{n-1}u_0(t), t \in [0, 1].$$

Thus for all $m, n \in N$ and $t \in [0,1]$,

$$\begin{aligned}
&|v_{n+m+1}(t) - v_n(t)| = |v_{n+m+1}(t) - v_{n+m}(t) + \cdots + v_{n+1}(t) - v_n(t)| \\
&\leq w_{n+m+1}(t) + \cdots + w_{n+1}(t) \\
&\leq NM^{n+m-1}u_0(t) + \cdots + Nq^{n-1}u_0(t) \\
&= \frac{NM^{n-1}(1-q^{m+1})}{1-q}u_0(t) < \frac{Nq^{n-1}}{1-q}.
\end{aligned}$$

This shows that $\{v_n\}$ is a uniform Cauchy sequence in $B[O,M]$ and since $C[0,1]$ is complete and $B[O,M]$ is a clozed subspce of $C[0,1]$, $B[O,M]$ is also complete. Then there exists $v^* \in B[O,M]$ such that $\lim_{n\to\infty} v_n = v^*$. Moreover, v^* is a fixed point of A that follows from the continuity of A. Next we show that A has at most one fixed point. Suppose that there are two elements $x, y \in B[O,M]$ with $x = Ax$ and $y = Ay$. By (3.14), there exists N such that $(T(|x-y|))(t) \leq Nu_0(t)$, $t \in [0,1]$. Then for $n \in N$, we have

$$\begin{aligned}
&|x(t) - y(t)| = |(A^n x)(t) - (A^n y)(t)| \leq (T^n(|x-y|))(t) \\
&\leq N(T^{n-1}u_0)(t) \leq Nq^{n-1}u_0(t), t \in [0,1].
\end{aligned}$$

Consequently, we assert that $x = y$. This means that A has at most one fixed point. This completes the proof. □

Then we have the following theorem.

Theorem 5. *Under the assumption of lemma 4, the boundary value problem (1.4) has a unique solution there hold the estimates*

$$\|u\| \leq M_1 M, \|D_{0+}^{\alpha-3}u\| \leq M_2 M, \|D_{0+}^{\alpha-2}u\| \leq M_3 M, \|D_{0+}^{\alpha-1}u\| \leq M. \tag{3.16}$$

Proof. The proof is similar to that of [28, Theorem 2.2]. □

Next for the positive solution case, we have the following, For each number $M > 0$ we denote

$$D_M^+ = \left\{ (t,u,y,v,z) \,\middle|\, \begin{array}{l} 0 \leq t \leq 1, 0 \leq u \leq M_1 M,\ 0 \leq y \leq M_2 M, \\ 0 \leq v \leq M_3 M, -M \leq z \leq 0 \end{array} \right\}$$

and

$$S_M = \{\varphi \in C[0,1] \mid 0 < \varphi(t) \leq M \text{ for any } t \in [0,1]\}.$$

Then as the special case of Theorem 5, we have the following theorem.

Theorem 6. *Suppose that in D_M^+ the function f is such that*

$$0 \leq f(x,u,y,v,z) \leq M, \tag{3.17}$$

and satisfies the Lipschitz condition (3.11) and condtion (3.12). Then, the problem (1.4) has a unique nonnegative solution.

4. Examples

For the examples in this section, as the constant value we apply the following. If $\alpha = 3.1$, then $M_1 = \frac{5}{\Gamma(3.1+2)} - \frac{3}{\Gamma(3.1+1)} + \frac{3}{2\Gamma(3.1)} = 0.421205$, $M_2 = \frac{1}{6} = 0.166667$, $M_3 = \frac{1}{6} = 0.166667$, and $M_4 = 1$.

Example 7. Next we consider the examples of fractional order given by a non-linear f.

$$\begin{cases} D^{3.1}_{0+}u(t) = -\frac{1}{3}D^{0.1}_{0+}u(t)D^{2.1}_{0+}u(t) + u(t)(D^{1.1}_{0+}u(t))^2 + \frac{1}{2}e^{-u(t)}, 0 < t < 1, \\ u(0) = D^{0.1}u(0) = D^{1.1}_{0+}u(1) = D^{2.1}_{0+}u(1) = 0,, \end{cases}$$

In this example

$$f(x,u,y,v,z) = -\frac{1}{3}yz + uv^2 + \frac{1}{2}e^{-u},$$

which maps $[0,1] \times R^3_+ \times R^- \to R_+$.

If we choose $M = 1$, then we have $\frac{1}{3}M_2MM_4M + M_1MM_3^2M^2 + \frac{1}{2}e^{-M_1M} \leq M$.

Therefore, if $\alpha = 3.1$, the Lipschitz coefficients in Lemma **??** are $C_1 = |f_u| = |v^2 - \frac{1}{2}e^{-u}| \leq |\left(\frac{1}{6}M\right)^2 - \frac{1}{2}e^{-M_1M}| \approx 0.30035$, $C_2 = |f_y| = \frac{1}{3}z = \frac{1}{3}M_4M = \frac{1}{3}$, $C_3 = |f_v| = 2uv \leq \frac{1}{3}M_1M^2 \approx 0.140402$, $C_4 = |f_z| = \frac{1}{3}y = \frac{1}{3}M_2M \approx .0555556$. Then we have

$$q := (C_1 * M_1 + C_2 * M_2 + C_3 * M_3 + C_4 * M_4) \approx 0.26102 < 1.$$

Then the conditions of Lemma 4 and Theorem 5 are satisfied.

Next we consider the examples of fractional order given by a non-linear f. They are versions in [28].

Example 8.

$$\begin{cases} D^{3.1}_{0+}u(t) = \frac{u(t)}{6}(u(t) + D^{0.1}_{0+}u(t) + D^{1.1}_{0+}u(t) - D^{2.1}_{0+}u(t)) + 1, 0 < t < 1, \\ u(0) = D^{0.1}u(0) = D^{1.1}_{0+}u(1) = D^{2.1}_{0+}u(1) = 0, \end{cases}$$

In this example

$$f(x,u,y,v,z) = \frac{u}{6}(u + y + v - z) + 1,$$

which maps $[0,1] \times R^3_+ \times R^- \to R_+$. We can choose $M = 2$, and therefore, if $\alpha = 3.1$, the Lipschitz coefficients in Lemma **??** are satisfied. By the definition of f, since

$$|f_u| = \frac{1}{3} * u < \frac{2}{3} * M_1 \approx 0.280803,$$
$$f_y = f_v = f_z = \frac{2}{6} * M_1 \approx 0.140402,$$

we have

$$\begin{aligned} q &:= M_1C_1 + M_2C_2 + M_3C_3 + M_4C_4 \\ &= M_1 * \frac{2}{3}M_1 + M_2 * \frac{1}{3}M_1 + \frac{1}{2} * \frac{1}{3}M_1 + \frac{1}{3}M_1 \\ &\approx 0.305478 < 1. \end{aligned}$$

Thus the conditions of Lemma 2 and Theorem 2 are satisfied.

Example 9.

$$\begin{cases} D^{3.1}_{0+}u(t) = \frac{u(x) + D^{1.1}_{0+}u(x)}{1 + u(x)^2 + D^{2.1}_{0+}u(x)^2 + D^{1.1}_{0+}u(x)^2 + D^{0.1}_{0+}u(x)^2} + e^{-u^2}, 0 < x < 1, \\ u(0) = D^{0.1}u(0) = D^{1.1}_{0+}u(1) = D^{2.1}_{0+}u(1) = 0, \end{cases}$$

In this example

$$f(x,u,y,v,z) = \frac{u+y}{1+u^2+y^2+v^2+z^2} + e^{-u^2},$$

which maps $[0,1] \times R_+^3 \times R^- \to R_+$. Analogously we can choose $M = 1$ and Following argument, the function $f(x,u,y,v,z)$ satisfies the condition of Lemma 4 and Theorem 5. Hence, the problem has a unique solution. In fact if we choose $\alpha = 3.1$, then the Lipschitz coefficients in Lemma 4 are given by the following;

$$\begin{aligned}
|f_u| &= \left| \frac{(1+u^2+y^2+v^2+z^2) - 2u(u+y)}{(1+u^2+y^2+v^2+z^2)^2} - 2ue^{-u^2} \right| \\
&= \left| \frac{1}{(1+u^2+y^2+v^2+z^2)} - \frac{2u(u+y)}{(1+u^2+y^2+v^2+z^2)^2} - 2ue^{-u^2} \right| \\
&= \left| \frac{1}{(1+(M_1^2+M_2^2+M_3^2+M_4^2)M^2)} - \frac{2M_1(M_1+M_2)M^2}{(1+(M_1^2+M_2^2+M_3^2+M_4^2)M^2)^2} \right. \\
&\quad \left. -2M_1Me^{-(M_1M)^2} \right|,
\end{aligned}$$

$$\begin{aligned}
|f_y| &= \left| \frac{1}{(1+u^2+y^2+v^2+z^2)} - \frac{2y(u+y)}{(1+u^2+y^2+v^2+z^2)^2} \right| \\
&= \left| \frac{1}{((1+(M_1^2+M_2^2+M_3^2+M_4^2)M^2)} - \frac{2M_2(M_1+M_2)M^2}{(1+(1+M_1^2+M_2^2+M_3^2+M_4^2)M^2)^2} \right|,
\end{aligned}$$

$$|f_v| = \left| -\frac{-2v(u+y)}{(1+u^2+y^2+v^2+z^2)^2} \right| = \left| \frac{2M_3(M_1+M_2)M^2}{(1+(1+M_1^2+M_2^2+M_3^2+M_4^2)M^2)^2} \right|,$$

$$|f_z| = \left| -\frac{-2z(u+y)}{(1+u^2+y^2+v^2+z^2)^2} \right| = \left| \frac{2M_4(M_1+M_2)M^2}{(1+(1+M_1^2+M_2^2+M_3^2+M_4^2)M^2)^2} \right|.$$

As we take $M = 2$, we have

$$|f_u| \approx 0.4430161,\ |f_y| \approx 0.360078,\ |f_v| \approx 0.0877564,\ |f_z| \approx 0.526538,$$

$$q := M_1 * |f_u| + M_2 * |f_y| + M_3 * |f_v| + M_4 * |f_z| \approx 0.787778 < 1.$$

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