

Mann type approximation theorem by using
 balanced mappings on geodesic spaces
 測地距離空間における balanced mapping を用いた
 Mann 型収束定理

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Abstract

In this paper, we show a fixed point approximation theorem with the form of balanced mapping using Mann's iterative method for nonexpansive mappings on admissible complete CAT(1) spaces and complete CAT(-1) spaces.

1 Introduction

The CAT(κ) space is one of the metric spaces with convex structures, and several researchers study the convex analysis on those spaces. Fixed point approximation is one of the topics of the convex analysis, and has been studied in CAT(κ) spaces as well. Mann's iterative scheme is the method to generate a sequence converging to a fixed point of mappings produced by Mann [10] in 1953. The most basic form of Mann's iterative scheme is expressed by

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) T x_n \quad (n \in \mathbb{N}),$$

where $\{x_n\}_{n \in \mathbb{N}}$ is an iterative sequence, $\{\alpha_n\}_{n \in \mathbb{N}} \subset [0, 1]$ is a real sequence, T is a mapping that has fixed points, and \oplus is a symbol of the convex combination. Mann [10] showed a convergence theorem with Mann's iterative scheme for a nonexpansive mapping in a Hilbert space. Thereafter, many researchers has been studied that scheme in several spaces. For instance, that scheme is studied on Banach spaces, complete CAT(0) spaces, complete CAT(1) spaces, and so on. Henceforth, we call

the theorem using Mann’s iterative scheme a Mann type approximation theorem.

In 2008, Dhompongsa and Panyanak [1] proved a Mann type approximation theorem in a complete CAT(0) space. Five years later, Kimura, Saejung, and Yotkaew [8] proved the same type theorem in a complete CAT(1) space. Recently, Kimura [5] showed the following Mann type approximation theorem.

Theorem 1.1 ([5]). *Let X be a complete CAT(0) space and $\{T_k\}_{k=1}^N$ nonexpansive mappings from X into itself such that $\bigcap_{k=1}^N F(T_k) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n^k\}_{k=1}^N \subset [a, b] \subset]0, 1[$ be real sequences such that $\sum_{k=1}^N \beta_n^k = 1$ for all $n \in \mathbb{N}$. Let $x_1 \in X$ arbitrarily and define a sequence $\{x_n\} \subset X$ by*

$$x_{n+1} = \operatorname{argmin}_{y \in X} \left(\alpha_n d(x_n, y)^2 + (1 - \alpha_n) \sum_{k=1}^N \beta_n^k d(T_k x_n, y)^2 \right)$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ Δ -converges to an element of $\bigcap_{k=1}^N F(T_k)$.

In the assumption of Theorem 1.1, put $\gamma_n^1 = \alpha_n$ and $\gamma_n^{k+1} = (1 - \alpha_n)\beta_n^k$ for each $n \in \mathbb{N}$ and $k \in \{1, 2, \dots, N\}$, and let $S_1: X \rightarrow X$ be the identity mapping and put $S_{k+1} = T_k$ for each $k \in \{1, 2, \dots, N\}$. Then we can describe the iteration of Theorem 1.1 as

(i)
$$x_{n+1} = \operatorname{argmin}_{y \in X} \sum_{k=1}^{N+1} \gamma_n^k d(S_k x_n, y)^2,$$

and $\sum_{k=1}^{N+1} \gamma_n^k = 1$ holds. The right-hand side of (i) has the form of the balanced mapping proposed by Hasegawa and Kimura [3].

In the above theorem, the equation on the right-hand side can be described as a balanced mapping. In this paper, we show a fixed point approximation theorem with the form of balanced mapping using Mann’s iterative method on admissible complete CAT(1) spaces and complete CAT(−1) spaces.

2 Preliminaries

Let X be a metric space with a metric d . For a mapping $T: X \rightarrow X$, the set of all fixed points of T is denoted by $F(T)$. A mapping $\gamma: [0, 1] \rightarrow X$ is called a geodesic joining $x \in X$ and $y \in X$ if $\gamma(0) = x$, $\gamma(1) = y$, and $d(\gamma(s), \gamma(t)) = |s - t| d(x, y)$ hold for any $s, t \in [0, 1]$. For $D \in]0, \infty]$, X is called a uniquely D -geodesic space if there exists a unique geodesic $\gamma: [0, 1] \rightarrow X$ for any $x, y \in X$ with $d(x, y) < D$. In particular, ∞ -geodesic space is merely said a geodesic space. For a uniquely D -geodesic space, the image of the geodesic joining x and y is denoted by $[x, y]$.

Let (X, d) be a uniquely D -geodesic space. For two points $x, y \in X$, we define a convex combination $tx \oplus (1 - t)y$ by $\gamma(1 - t)$ for $t \in [0, 1]$, where γ is a geodesic joining x and y . A subset C of X is said to be convex if $[x, y] \subset C$ holds for any $x, y \in C$. For

points $x, y, z \in X$, we define a geodesic triangle $\Delta(x, y, z)$ with vertices $x, y, z \in X$ by the union $[x, y] \cup [y, z] \cup [z, x]$.

Let $\kappa \in \{-1, 0, 1\}$ and (X, d) a uniquely D_κ -geodesic space, where $D_{-1} = D_0 = \infty$, and $D_1 = \pi$. Let us denote a 2-dimensional model space with the constant curvature κ by (M_κ, d') , where d' is a metric on M_κ . To be accurate, M_{-1} is the 2-dimensional hyperbolic space, M_0 is the 2-dimensional Euclidean space, and M_1 is the 2-dimensional unit sphere. M_κ is one of the uniquely D_κ -geodesic spaces. Then for each geodesic triangle $\Delta(x, y, z)$ on X with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, there exists a geodesic triangle with vertices $\overline{\Delta}(\overline{x}, \overline{y}, \overline{z}) \subset M_\kappa$ such that $d(x, y) = d'(\overline{x}, \overline{y})$, $d(y, z) = d'(\overline{y}, \overline{z})$, and $d(z, x) = d'(\overline{z}, \overline{x})$. That triangle on M_κ is called a comparison triangle of $\Delta(x, y, z)$. For a triangle $\Delta(x, y, z) \subset X$ with its comparison triangle $\overline{\Delta}(\overline{x}, \overline{y}, \overline{z}) \subset M_\kappa$, and two vertices $s_1, s_2 \in \{x, y, z\}$ and these corresponding vertices \overline{s}_1 and \overline{s}_2 , there exists a point $\overline{p} \in [\overline{s}_1, \overline{s}_2]$ for each $p \in [s_1, s_2]$ such that $d(s_1, p) = d'(\overline{s}_1, \overline{p})$. Such a point $\overline{p} \in \overline{\Delta}(\overline{x}, \overline{y}, \overline{z})$ is called a comparison point of p . X is called a CAT(κ) space if, for any $\Delta(x, y, z) \subset X$ with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ and $\overline{\Delta}(\overline{x}, \overline{y}, \overline{z}) \subset M_\kappa$, the inequality $d(p, q) \leq d'(\overline{p}, \overline{q})$ always holds for $p, q \in \Delta(x, y, z)$ and their comparison points $\overline{p}, \overline{q} \in \overline{\Delta}(\overline{x}, \overline{y}, \overline{z})$. A CAT(κ) space is said to be admissible if the distance between any two points is less than $D_\kappa/2$. Every CAT(-1) space is admissible obviously.

Let $\kappa \in \{-1, 1\}$ and X a CAT(κ) space. Then the following inequalities hold for any $x, y, z \in X$ and $t \in]0, 1[$:

- if $\kappa = -1$, then

$$\begin{aligned} \cosh d(tx \oplus (1-t)y, z) \sinh d(x, y) \\ \leq \cosh d(x, z) \sinh td(x, y) + \cosh d(y, z) \sinh((1-t)d(x, y)), \end{aligned}$$

- if $\kappa = 1$, then

$$\begin{aligned} \cos d(tx \oplus (1-t)y, z) \sin d(x, y) \\ \geq \cos d(x, z) \sin td(x, y) + \cos d(y, z) \sin((1-t)d(x, y)). \end{aligned}$$

Considering the convexity or concavity of \sinh and \sin , we easily obtain the following:

- if $\kappa = -1$, then $\cosh d(tx \oplus (1-t)y, z) \leq t \cosh d(x, z) + (1-t) \cosh d(y, z)$,
- if $\kappa = 1$, then $\cos d(tx \oplus (1-t)y, z) \geq t \cos d(x, z) + (1-t) \cos d(y, z)$

for any $x, y, z \in X$ and $t \in]0, 1[$.

Let X be a set and f a function from X into \mathbb{R} . We write the set of all minimizers (resp. maximizers) of f as $\operatorname{argmin}_{x \in X} f(x)$ (resp. $\operatorname{argmax}_{x \in X} f(x)$). For a set X and a mapping $T: X \rightarrow X$, let us denote the set of all fixed points of T by $F(T)$.

Let X be a metric space with a metric d . For $\{x_n\} \subset X$, define an asymptotic center of $\{x_n\}$ by $\operatorname{argmin}_{x \in X} \limsup_{n \rightarrow \infty} d(x, x_n)$, and denote it by $AC(\{x_n\})$. The sequence $\{x_n\}$ is said to Δ -converge to x_0 if $AC(\{x_{n_i}\}) = \{x_0\}$ holds for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, and then refer to x_0 as the Δ -limit of $\{x_n\}$. The Δ -convergence of $\{x_n\}$ to x_0 is expressed by the notation $x_n \xrightarrow{\Delta} x_0$.

Let X be a complete CAT(1) space. Then a sequence $\{x_n\} \subset X$ is said to be spherically bounded if $\inf_{y \in X} \limsup_{n \rightarrow \infty} d(x_n, y) < \pi/2$ holds. Then the asymptotic center of any spherically bounded sequence $\{x_n\}$ is exactly one point, and $\{x_n\}$ has Δ -convergent subsequences. See [2] for details. Incidentally, suppose X is a complete CAT(0) space, then the asymptotic center of any bounded sequence $\{x_n\}$ is exactly one point, and $\{x_n\}$ has Δ -convergent subsequences [9]. These natures contribute to prove theorems for Mann’s iterative schemes on CAT(κ) spaces.

3 Main results

In this section, we prove approximation theorems using a Mann’s iterative scheme with balanced mappings on a CAT(1) space and a CAT(−1) space.

Lemma 3.1 ([4]). *Let X be an admissible complete CAT(1) space and C a nonempty closed convex subset of X . For $u_1, u_2, \dots, u_N \in X$ and $\alpha^1, \alpha^2, \dots, \alpha^N \in [0, 1]$ with $\sum_{k=1}^N \alpha^k = 1$, define a function $g: X \rightarrow]0, 1]$ by $g(x) = \sum_{k=1}^N \alpha^k \cos d(u_k, x)$ for each $x \in X$. Then g has a unique maximizer on C .*

Let X be an admissible complete CAT(1) space and $\{\alpha^k\}_{k=1}^N$ a real sequence on $[0, 1]$ satisfying $\sum_{k=1}^N \alpha^k = 1$, and let $\{T_k\}_{k=1}^N$ be mappings from X into itself. Then a mapping $U: X \rightarrow X$ defined by

$$Ux = \operatorname{argmax}_{y \in X} \sum_{k=1}^N \alpha^k \cos d(T_k x, y) \quad (x \in X)$$

is well-defined as a single-valued mapping from Lemma 3.1. This mapping U is called a balanced mapping for $\{\alpha^k\}$ and $\{T_k\}$ on an admissible complete CAT(1) space.

Lemma 3.2 ([7]). *Let X be an admissible complete CAT(1) space and let $\{T_k\}_{k=1}^N$ be mappings from X into itself. Let $\{\alpha^k\}_{k=1}^N \subset]0, 1[$ such that $\sum_{k=1}^N \alpha^k = 1$ and $U: X \rightarrow X$ a balanced mapping for $\{\alpha^k\}$ and $\{T_k\}$. Then the following inequality holds for any $x, z \in X$:*

$$\sum_{k=1}^N \alpha^k \cos d(T_k x, Ux) \cos d(Ux, z) \geq \sum_{k=1}^N \alpha^k \cos d(T_k x, z).$$

Lemma 3.3 ([4]). *Let X be an admissible complete CAT(1) space and let $\{T_k\}_{k=1}^N$ be quasicontractive mappings from X into itself such that $\bigcap_{k=1}^N F(T_k) \neq \emptyset$. Let $\{\alpha^k\}_{k=1}^N \subset]0, 1[$ such that $\sum_{k=1}^N \alpha^k = 1$ and $U: X \rightarrow X$ a balanced mapping for $\{\alpha^k\}$ and $\{T_k\}$. Then $F(U) = \bigcap_{k=1}^N F(T_k)$ holds, and U is also quasicontractive.*

Using above lemmas, we prove the following convergence theorem on an admissible complete CAT(1) space.

Theorem 3.4. Let X be an admissible complete CAT(1) space and let $\{T_k\}_{k=1}^N$ be nonexpansive mappings from X into itself such that $\bigcap_{k=1}^N F(T_k) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n^k\}_{k=1}^N \subset [a, b] \subset]0, 1[$ be real sequences such that $\sum_{k=1}^N \beta_n^k = 1$ for all $n \in \mathbb{N}$. Let $x_1 \in X$ arbitrarily and define a sequence $\{x_n\} \subset X$ by

$$x_{n+1} = \operatorname{argmax}_{y \in X} \left(\alpha_n \cos d(x_n, y) + (1 - \alpha_n) \sum_{k=1}^N \beta_n^k \cos d(T_k x_n, y) \right)$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ Δ -converges to an element of $\bigcap_{k=1}^N F(T_k)$.

Proof. Define a mapping $U_n: X \rightarrow X$ by $U_n x = \operatorname{argmax}_{y \in X} (\alpha_n \cos d(x_n, y) + (1 - \alpha_n) \sum_{k=1}^N \beta_n^k \cos d(T_k x_n, y))$ for each $x \in X$ and $n \in \mathbb{N}$. The well-definedness of the mapping U_n is guaranteed by Lemma 3.1. Then $x_{n+1} = U_n x_n$ holds for any $n \in \mathbb{N}$. From Lemma 3.3, we have $F(U_n) = \bigcap_{k=1}^N F(T_k)$ for all $n \in \mathbb{N}$, and we also get the quasiconvexity of U_n .

Let $p \in \bigcap_{k=1}^N F(T_k)$. Then we have $d(x_{n+1}, p) = d(U_n x_n, p) \leq d(x_n, p)$ and hence $\{d(x_n, p)\}$ is decreasing. It means that the real sequence $\{d(x_n, p)\}$ has the limit value for any $p \in \bigcap_{k=1}^N F(T_k)$.

We show $\lim_{n \rightarrow \infty} d(T_k x_n, x_n) = 0$ for any $k = 1, 2, \dots, N$. Let $t \in]0, 1[$. Using Lemma 3.2, we have

$$\begin{aligned} & \left(\alpha_n \cos d(x_n, U_n x_n) + (1 - \alpha_n) \sum_{k=1}^N \beta_n^k \cos d(T_k x_n, U_n x_n) \right) \cos d(U_n x_n, p) \\ & \geq \alpha_n \cos d(x_n, p) + (1 - \alpha_n) \sum_{k=1}^N \beta_n^k \cos d(T_k x_n, p) \\ & \geq \cos d(x_n, p) \end{aligned}$$

and thus

$$(ii) \quad \alpha_n \cos d(x_n, U_n x_n) + (1 - \alpha_n) \sum_{k=1}^N \beta_n^k \cos d(T_k x_n, U_n x_n) \geq \frac{\cos d(x_n, p)}{\cos d(U_n x_n, p)}.$$

From (ii), we obtain

$$\alpha_n \cos d(x_n, U_n x_n) + 1 - \alpha_n \geq \frac{\cos d(x_n, p)}{\cos d(U_n x_n, p)}.$$

Since $\alpha_n \geq a$, we get

$$-a(1 - \cos d(x_n, U_n x_n)) + 1 \geq \frac{\cos d(x_n, p)}{\cos d(U_n x_n, p)}$$

and thus

$$1 - \cos d(x_n, U_n x_n) \leq \frac{1}{a} \left(1 - \frac{\cos d(x_n, p)}{\cos d(x_{n+1}, p)} \right) \rightarrow 0 \quad (n \rightarrow \infty).$$

Therefore we have $\lim_{n \rightarrow \infty} d(x_n, U_n x_n) = 0$.

Moreover, from (ii), we get

$$\alpha_n + (1 - \alpha_n) \sum_{k=1}^N \beta^k \cos d(T_k x_n, U_n x_n) \geq \frac{\cos d(x_n, p)}{\cos d(U_n x_n, p)}$$

and it implies that

$$(1 - \alpha_n) \sum_{k=1}^N \beta^k (1 - \cos d(T_k x_n, U_n x_n)) \leq 1 - \frac{\cos d(x_n, p)}{\cos d(U_n x_n, p)}.$$

Fix $k \in \{1, 2, \dots, N\}$. Then we obtain

$$(1 - \alpha_n) \beta^k (1 - \cos d(T_k x_n, U_n x_n)) \leq 1 - \frac{\cos d(x_n, p)}{\cos d(U_n x_n, p)}$$

and hence

$$1 - \cos d(T_k x_n, U_n x_n) \leq \frac{1}{(1 - b)a} \left(1 - \frac{\cos d(x_n, p)}{\cos d(U_n x_n, p)} \right) \rightarrow 0 \quad (n \rightarrow \infty).$$

Thus we have $\lim_{k \rightarrow \infty} d(T_k x_n, U_n x_n) = 0$ for any $k = 1, 2, \dots, N$ and hence we obtain $\lim_{n \rightarrow \infty} d(T_k x_n, x_n) = 0$ for any $k = 1, 2, \dots, N$.

Let $x_0 \in AC(\{x_n\})$ and $\{x_{n_i}\}$ a subsequence of $\{x_n\}$, and let $u \in AC(\{x_{n_i}\})$. For $k = 1, 2, \dots, N$, we get

$$\begin{aligned} \limsup_{i \rightarrow \infty} d(x_{n_i}, u) &\leq \limsup_{i \rightarrow \infty} d(x_{n_i}, T_k u) \\ &\leq \limsup_{i \rightarrow \infty} (d(x_{n_i}, T_k x_{n_i}) + d(T_k x_{n_i}, T_k u)) \\ &\leq \limsup_{i \rightarrow \infty} d(x_{n_i}, u) \end{aligned}$$

and thus $\limsup_{i \rightarrow \infty} d(x_{n_i}, u) = \limsup_{i \rightarrow \infty} d(x_{n_i}, T_k u)$. By the uniqueness of the asymptotic center of $\{x_{n_i}\}$, we have $u \in \bigcap_{k=1}^N F(T_k)$. Then we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(x_n, x_0) &\leq \lim_{n \rightarrow \infty} d(x_n, u) = \lim_{i \rightarrow \infty} d(x_{n_i}, u) \\ &\leq \limsup_{i \rightarrow \infty} d(x_{n_i}, x_0) \leq \limsup_{n \rightarrow \infty} d(x_n, x_0). \end{aligned}$$

Therefore we get $\limsup_{n \rightarrow \infty} d(x_n, x_0) = \lim_{n \rightarrow \infty} d(x_n, u)$. By the uniqueness of the asymptotic center of $\{x_n\}$, we have $x_0 = u$. From the definition of the Δ -convergence, we get $x_n \xrightarrow{\Delta} x_0 \in \bigcap_{k=1}^N F(T_k)$, which is the desired result. \square

Next, we consider the balanced mapping on complete CAT(-1) spaces. Let X be a complete CAT(-1) space and $\{\alpha^k\}_{k=1}^N$ a real sequence on $[0, 1]$ satisfying $\sum_{k=1}^N \alpha^k = 1$, and let $\{T_k\}_{k=1}^N$ be mappings from X into itself. Then a mapping $U: X \rightarrow X$ defined by

$$Ux = \operatorname{argmin}_{y \in X} \sum_{k=1}^N \alpha^k \cosh d(T_k x, y) \quad (x \in X)$$

is well-defined as a single-valued mapping from the following Lemma 3.5. This mapping U is called a balanced mapping for $\{\alpha^k\}$ and $\{T_k\}$ on a complete CAT(-1) space.

Lemma 3.5. *Let X be a complete CAT(-1) space and C a nonempty closed convex subset of X . For $u_1, u_2, \dots, u_N \in X$ and $\alpha^1, \alpha^2, \dots, \alpha^N \in [0, 1]$ with $\sum_{k=1}^N \alpha^k = 1$, define a function $g: X \rightarrow [1, \infty[$ by $g(x) = \sum_{k=1}^N \alpha^k \cosh d(u_k, x)$ for all $x \in X$. Then g has a unique minimizer on C .*

The above theorem can be proved in the same way as Lemma 3.1.

Definition 3.6 ([6]). Let X be a uniquely geodesic space. For each $u, v \in X$ and $t \in [0, 1]$, we put $tu \oplus^{-1} (1-t)v = \operatorname{argmin}_{x \in X} (t \cosh d(u, x) + (1-t) \cosh d(v, x))$. If $u \neq v$, then $tu \oplus^{-1} (1-t)v$ coincides with

$$\left(\frac{1}{D} \tanh^{-1} \frac{t \sinh D}{1-t+t \cosh D} \right) u \oplus \left(\frac{1}{D} \tanh^{-1} \frac{(1-t) \sinh D}{t+(1-t) \cosh D} \right) v,$$

where $D = d(u, v)$. It obviously holds that $tu \oplus^{-1} (1-t)v = tu \oplus (1-t)v$ if $u = v$.

Lemma 3.7 ([6]). *Let X be a CAT(-1) space and let $x, y, z \in X$. Then*

$$\cosh d(\alpha x \oplus^{-1} (1-\alpha)y, z) \leq \frac{\alpha \cosh d(x, z) + (1-\alpha) \cosh d(y, z)}{\sqrt{\alpha^2 + 2\alpha(1-\alpha) \cosh d(x, y) + (1-\alpha)^2}}$$

holds for any $\alpha \in [0, 1]$.

Lemma 3.8. *Let X be a complete CAT(-1) space and $\{T_k\}_{k=1}^N$ mappings from X into itself. Let $\{\alpha^k\}_{k=1}^N \subset [0, 1]$ such that $\sum_{k=1}^N \alpha^k = 1$ and let U be a balanced mapping for $\{\alpha^k\}$ and $\{T_k\}$. Then the following inequality holds for any $x, z \in X$:*

$$\sum_{k=1}^N \alpha^k \cosh d(T_k x, Ux) \cosh d(Ux, z) \leq \sum_{k=1}^N \alpha^k \cosh d(T_k x, z).$$

Proof. Let $t \in]0, 1[$ and put $D = d(Ux, z)$. By Lemma 3.7 we have

$$\begin{aligned} \sum_{k=1}^N \alpha^k \cosh d(T_k x, Ux) &\leq \sum_{k=1}^N \alpha^k \cosh d(T_k x, tUx \overset{-1}{\oplus} (1-t)z) \\ &\leq \sum_{k=1}^N \frac{\alpha^k t \cosh d(T_k x, Ux) + \alpha^k (1-t) \cosh d(T_k x, z)}{\sqrt{t^2 + 2t(1-t) \cosh D + (1-t)^2}}. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \left(1 - \frac{t}{\sqrt{t^2 + 2t(1-t) \cosh D + (1-t)^2}}\right) \sum_{k=1}^N \alpha^k \cosh d(T_k x, Ux) \\ \leq \frac{1-t}{\sqrt{t^2 + 2t(1-t) \cosh D + (1-t)^2}} \sum_{k=1}^N \alpha^k \cosh d(T_k x, z) \end{aligned}$$

and it implies that

$$\begin{aligned} \frac{2t(1-t) \cosh D + (1-t)^2}{t + \sqrt{t^2 + 2t(1-t) \cosh D + (1-t)^2}} \sum_{k=1}^N \alpha^k \cosh d(T_k x, Ux) \\ \leq (1-t) \sum_{k=1}^N \alpha^k \cosh d(T_k x, z). \end{aligned}$$

Dividing both sides by $1-t$ and letting $t \rightarrow 1$, we obtain the desired result. □

The proof of above theorem used the different type of convex combination $\overset{-1}{\oplus}$, but actually it can be proven by using ordinal convex combination. The reason we used that convex combination is that it is simpler to calculate than the ordinal one.

Lemma 3.9. *Let X be a complete CAT(-1) space and $\{T_k\}_{k=1}^N$ quasicontractive mappings from X into itself such that $\bigcap_{k=1}^N F(T_k) \neq \emptyset$. Let $\{\alpha^k\}_{k=1}^N \subset]0, 1[$ such that $\sum_{k=1}^N \alpha^k = 1$ and $U: X \rightarrow X$ a balanced mapping for $\{\alpha^k\}$ and $\{T_k\}$. Then $F(U) = \bigcap_{k=1}^N F(T_k)$ holds, and U is also quasicontractive.*

Proof. Let $z \in \bigcap_{k=1}^N F(T_k)$. Then we have

$$Uz = \operatorname{argmin}_{y \in X} \sum_{k=1}^N \alpha^k \cosh d(T_k z, y) = \operatorname{argmin}_{y \in X} \sum_{k=1}^N \alpha^k \cosh d(z, y) = z$$

and thus $F(U) \supset \bigcap_{k=1}^N F(T_k)$ holds.

We show $F(U) \subset \bigcap_{k=1}^N F(T_k)$. Let $z \in F(U)$ and $w \in \bigcap_{k=1}^N F(T_k)$. By Lemma 3.8, we have

$$\begin{aligned} \sum_{k=1}^N \alpha^k \cosh d(T_k z, z) \cosh d(z, w) &= \sum_{k=1}^N \alpha^k \cosh d(T_k z, Uz) \cosh d(Uz, w) \\ &\leq \sum_{k=1}^N \alpha^k \cosh d(T_k z, w) \leq \cosh d(z, w). \end{aligned}$$

Hence we get $\sum_{k=1}^N \alpha^k \cosh d(T_k z, z) \leq 1$ and it implies $z \in \bigcap_{k=1}^N F(T_k)$.

Finally we show the quasiconvexity of U . Let $x \in X$ and $z \in F(U)$. Then we have $z \in \bigcap_{k=1}^N F(T_k)$. From Lemma 3.8, we get

$$\begin{aligned} \cosh d(Ux, z) &\leq \sum_{k=1}^N \alpha^k \cosh d(T_k x, Ux) \cosh d(Ux, z) \\ &\leq \sum_{k=1}^N \alpha^k \cosh d(T_k x, z) \leq \cosh d(x, z) \end{aligned}$$

and it implies $d(Ux, z) \leq d(x, z)$. Therefore we get the conclusion. \square

Using above lemmas, we get the following result.

Theorem 3.10. *Let X be a complete CAT(-1) space and $\{T_k\}_{k=1}^N$ nonexpansive mappings from X into itself such that $\bigcap_{k=1}^N F(T_k) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n^k\}_{k=1}^N \subset [a, b] \subset]0, 1[$ be real sequences such that $\sum_{k=1}^N \beta_n^k = 1$ for all $n \in \mathbb{N}$. Let $x_1 \in X$ arbitrarily and define a sequence $\{x_n\} \subset X$ by*

$$x_{n+1} = \operatorname{argmin}_{y \in X} \left(\alpha_n \cosh d(x_n, y) + (1 - \alpha_n) \sum_{k=1}^N \beta_n^k \cosh d(T_k x_n, y) \right)$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ Δ -converges to an element of $\bigcap_{k=1}^N F(T_k)$.

Proof. Define a mapping $U_n: X \rightarrow X$ by $U_n x = \operatorname{argmin}_{y \in X} (\alpha_n \cosh d(x_n, y) + (1 - \alpha_n) \sum_{k=1}^N \beta_n^k \cosh d(T_k x_n, y))$ for each $x \in X$ and $n \in \mathbb{N}$. The well-definedness of the mapping U_n is guaranteed by Lemma 3.5. Then $x_{n+1} = U_n x_n$ holds for any $n \in \mathbb{N}$. From Lemma 3.9, we have $F(U_n) = \bigcap_{k=1}^N F(T_k)$ for all $n \in \mathbb{N}$, and we also get the quasiconvexity of U_n .

Let $p \in \bigcap_{k=1}^N F(T_k)$. Then we have $d(x_{n+1}, p) = d(U_n x_n, p) \leq d(x_n, p)$ and hence $\{d(x_n, p)\}$ is decreasing. It means that the real sequence $\{d(x_n, p)\}$ has the limit value for any $p \in \bigcap_{k=1}^N F(T_k)$.

We show $\lim_{n \rightarrow \infty} d(T_k x_n, x_n) = 0$ for any $k = 1, 2, \dots, N$. Let $t \in]0, 1[$. Using Lemma 3.8, we have

$$\begin{aligned} & \left(\alpha_n \cosh d(x_n, U_n x_n) + (1 - \alpha_n) \sum_{k=1}^N \beta^k \cosh d(T_k x_n, U_n x_n) \right) \cosh d(U_n x_n, p) \\ & \leq \alpha_n \cosh d(x_n, p) + (1 - \alpha_n) \sum_{k=1}^N \beta^k \cosh d(T_k x_n, p) \\ & \leq \cosh d(x_n, p) \end{aligned}$$

and thus

$$(iii) \quad \alpha_n \cosh d(x_n, U_n x_n) + (1 - \alpha_n) \sum_{k=1}^N \beta^k \cosh d(T_k x_n, U_n x_n) \leq \frac{\cosh d(x_n, p)}{\cosh d(U_n x_n, p)}.$$

From (iii), we obtain

$$\alpha_n \cosh d(x_n, U_n x_n) + 1 - \alpha_n \leq \frac{\cosh d(x_n, p)}{\cosh d(U_n x_n, p)}.$$

Since $\alpha_n \geq a$, we get

$$a(\cosh d(x_n, U_n x_n) - 1) + 1 \leq \frac{\cosh d(x_n, p)}{\cosh d(U_n x_n, p)}$$

and thus

$$\cosh d(x_n, U_n x_n) - 1 \leq \frac{1}{a} \left(\frac{\cosh d(x_n, p)}{\cosh d(x_{n+1}, p)} - 1 \right) \rightarrow 0 \quad (n \rightarrow \infty).$$

Therefore we have $\lim_{n \rightarrow \infty} d(x_n, U_n x_n) = 0$.

Moreover, from (iii), we get

$$\alpha_n + (1 - \alpha_n) \sum_{k=1}^N \beta^k \cosh d(T_k x_n, U_n x_n) \leq \frac{\cosh d(x_n, p)}{\cosh d(U_n x_n, p)}$$

and it implies that

$$(1 - \alpha_n) \sum_{k=1}^N \beta^k (\cosh d(T_k x_n, U_n x_n) - 1) \leq \frac{\cosh d(x_n, p)}{\cosh d(U_n x_n, p)} - 1.$$

Fix $k \in \{1, 2, \dots, N\}$. Then we obtain

$$(1 - \alpha_n) \beta^k (\cosh d(T_k x_n, U_n x_n) - 1) \leq \frac{\cosh d(x_n, p)}{\cosh d(U_n x_n, p)} - 1$$

and hence

$$\cosh d(T_k x_n, U_n x_n) - 1 \leq \frac{1}{(1-b)a} \left(\frac{\cosh d(x_n, p)}{\cosh d(U_n x_n, p)} - 1 \right) \rightarrow 0 \quad (n \rightarrow \infty).$$

Thus we have $\lim_{k \rightarrow \infty} d(T_k x_n, U_n x_n) = 0$ for any $k = 1, 2, \dots, N$ and hence we obtain $\lim_{n \rightarrow \infty} d(T_k x_n, x_n) = 0$ for any $k = 1, 2, \dots, N$.

Let $x_0 \in AC(\{x_n\})$ and $\{x_{n_i}\}$ a subsequence of $\{x_n\}$, and let $u \in AC(\{x_{n_i}\})$. For $k = 1, 2, \dots, N$, we get

$$\begin{aligned} \limsup_{i \rightarrow \infty} d(x_{n_i}, u) &\leq \limsup_{i \rightarrow \infty} d(x_{n_i}, T_k u) \\ &\leq \limsup_{i \rightarrow \infty} (d(x_{n_i}, T_k x_{n_i}) + d(T_k x_{n_i}, T_k u)) \\ &\leq \limsup_{i \rightarrow \infty} d(x_{n_i}, u) \end{aligned}$$

and thus $\limsup_{i \rightarrow \infty} d(x_{n_i}, u) = \limsup_{i \rightarrow \infty} d(x_{n_i}, T_k u)$. By the uniqueness of the asymptotic center of $\{x_{n_i}\}$, we have $u \in \bigcap_{k=1}^N F(T_k)$. Then we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(x_n, x_0) &\leq \lim_{n \rightarrow \infty} d(x_n, u) = \lim_{i \rightarrow \infty} d(x_{n_i}, u) \\ &\leq \limsup_{i \rightarrow \infty} d(x_{n_i}, x_0) \leq \limsup_{n \rightarrow \infty} d(x_n, x_0). \end{aligned}$$

Therefore we get $\limsup_{n \rightarrow \infty} d(x_n, x_0) = \lim_{n \rightarrow \infty} d(x_n, u)$. By the uniqueness of the asymptotic center of $\{x_n\}$, we have $x_0 = u$. From the definition of the Δ -convergence, we get $x_n \xrightarrow{\Delta} x_0 \in \bigcap_{k=1}^N F(T_k)$, which is the desired result. \square

Theorem 3.4 is a convergence theorem on admissible complete CAT(1) spaces, and it targets at nonexpansive mappings. Since we have not found many examples of nonexpansive mappings in CAT(1) spaces, we consider Theorem 3.4 to be less useful. On the other hand, there exists many examples of nonexpansive mappings in CAT(-1) space and thus we consider Theorem 3.10 to be useful.

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References

- [1] S. Dhompongsa and B. Panyanak, *On Δ -convergence theorems in CAT(0) spaces*, Comput. Math. Appl. **56** (10) (2008), 2572–2579.
- [2] R. Espínola and A. Fernández-León, *CAT(k)-spaces, weak convergence and fixed points*, J. Math. Anal. Appl. **353** (2009), no. 1, 410–427.
- [3] T. Hasegawa and Y. Kimura, *Convergence to a fixed point of a balanced mapping by the Mann algorithm in a Hadamard space*, Linear Nonlinear Anal. **4** (2018), 405–412.

- [4] T. Kajimura, K. Kasahara, Y. Kimura, and K. Nakagawa, *Balanced mappings and an iterative scheme in complete geodesic spaces*, J. Nonlinear Anal. Optim. **12** (2021), 1–11.
- [5] Y. Kimura, *Explicit and implicit iterative schemes with balanced mappings in Hadamard spaces*, Thai Journal of Mathematics, **18** (2020), 425–434.
- [6] Y. Kimura and K. Sasaki, *A Halpern type iteration with multiple anchor points in complete geodesic spaces with negative curvature*, Fixed Point Theory **21** (2020), 631–646.
- [7] Y. Kimura and K. Sasaki, *A Halpern Type Iterative Scheme for a Finite Number of Mappings in Complete Geodesic Spaces with Curvature Bounded above*, Numer. Funct. Anal. Optim., to appear. DOI: 10.1080/01630563.2021.2009510.
- [8] Y. Kimura, S. Saejung and P. Yotkaew, *The Mann algorithm in a complete geodesic space with curvature bounded above*, Fixed Point Theory Appl. **2013** (2013), 13pages.
- [9] W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal. **68** (2008), no. 12, 3689–3696.
- [10] W. R. Mann, *Mean value methods in iteration*, Proc. Am. Math. Soc. **4** (1953), 506–510.