ION DYNAMICS OF THE EULER-POISSON SYSTEM

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Abstract. We consider the Euler-Poisson system with Boltzmann relation that is a fundamental fluid model describing the dynamics of ions in an electrostatic plasma. It is often employed to study phenomena of plasma such as plasma sheaths and double layers. We show that the Euler-Poisson system admits a two-parameter family of solitary waves in the super-ion-sonic regime, and prove their the convective linear stability. We also propose a criterion for the singularity formation of the pressureless case, under which we prove that the smooth solutions develop a C^1 blow-up in a finite time and obtain their temporal blow-up rates. Our blow-up condition does not require the largeness of gradient of velocity.
Keywords: Euler-Poisson system; Boltzmann relation; Solitary waves; Linear stability, Singularity formation

1. Introduction

We consider the one-dimensional Euler-Poisson system with the Boltzmann relation, which is a fundamental fluid model describing the dynamics of ions in an electrostatic plasma. In a non-dimensionalized form, the system is given by

$$\begin{cases} \rho_t + (\rho u)_x = 0, & \text{(1.1a)} \\ u_t + u u_x + K(\log \rho)_x = -\phi_x, & \text{(1.1b)} \\ -\phi_{xx} = \rho - e^{\phi}, & \text{(1.1c)} \end{cases}$$

where $\rho > 0$, u and ϕ are the unknown functions of $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ representing the ion density, the fluid velocity for ions, and the electric potential, respectively, and $K = T_i/T_e \geq 0$ is a constant of the ratio of the ion temperature to the electron temperature. In the one-fluid model (1.1), the electron density ρ_e is assumed to satisfy the Boltzmann relation $\rho_e = e^{\phi}$, which can be formally derived from the two-fluid model under the massless electron assumption.[1] A rigorous justification of this zero mass limit is discussed in [15].

The system (1.1) is referred to as the pressureless Euler-Poisson system when $K = 0$, and the isothermal Euler-Poisson system when $K > 0$, respectively. In plasma physics, it is often assumed that $K = 0$, which is an ideal situation for the case $T_i \ll T_e$. In a mathematical point of view, the absence of the pressure term makes the system weakly coupled, so it enables to analyze certain properties of the system significantly easier. These include for instance, the existence of solitary waves and the finite time singularity formation.

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[1] The mass of ions is much heavier than that of electrons for plasma environments. Additionally, it is assumed that (i) (isothermal) T_i and T_e are constant, (ii) (electrostatic) the time variation of the magnetic field is negligible, (iii) (plane wave) the dynamics of ions and electrons occur only in one direction. We refer to [6, 9, 30] for more detailed physicality of the model.

Unlike the compressible Euler system, the Euler-Poisson system has a dispersive character due to the presence of the electric potential, and this aspect makes the system exhibit rich dynamics and interesting phenomena.

The system (1.1) has been employed to study various phenomena of plasma such as plasma sheaths [14, 25], KdV limit [4, 17], KP-II and Zakharov-Kuznetsov limits [21, 32], and NLS limit [24]. In particular, effort has been made to mathematically justify the phenomena of plasma 'solitons' by showing existence of solitary waves [8, 33] and studying their interactions [19, 22, 34].

Various analytical and numerical studies indicate that in certain physical regime, the KdV equation is a good approximation of the Euler-Poisson system (1.1). Moreover, as solutions of the KdV equation are dominated by their solitary waves, this gives hope of a similar result for the Euler-Poisson system with more general initial data. This motivation naturally leads us to the study of stability of solitary waves for the Euler-Poisson system. In fact, linear stability of traveling solitary waves for the system (1.1) has been studied in [5, 18].

A question of global existence or finite time blow-up of smooth solutions naturally arises in the study of large-time dynamics of the Euler-Poisson system (1.1). If the global existence of smooth solutions is guaranteed at least *near the solitary waves*, then it will be a first step to study the nonlinear stability. For the pressureless case, the solutions to (1.1) are shown to develop singularities in a finite time, [3, 23, 29]. For the isothermal case $K > 0$, the related questions concerning global existence of smooth solutions and finite-time singularity formation are widely open. To the best of our knowledge, no global well-posedness of smooth solutions is known except that of [16] for the 3D isothermal Euler-Poisson system, where the small and irrotational smooth solutions are shown to persist globally.

2. Solitary waves for the Euler-Poisson system

The Euler-Poisson system (1.1) with the far-field condition, $(\rho, u, \phi)(s,t) \to (1,0,0)$ as $s \to \pm\infty$, admits a two-parameter family of traveling solitary wave solutions

$$(\rho - 1, u, \phi)(s,t) = (n_c, u_c, \phi_c)(s - ct + \gamma)$$

for all $c \in (\sqrt{K+1}, \sqrt{K+1} + \varepsilon_K^*)$ and $\gamma \in \mathbb{R}$, where $\varepsilon_K^* > 0$ is some critical value depends on $K \geq 0$. See [8] for the isothermal case and [22, 33] for the pressureless case for more details. The solitary wave for the system (1.1) is *super-ion-sonic wave*. In fact, $\sqrt{1+K}$ is called the *ion sound speed* in the context of plasma physics.

The authors of this paper showed in [4] that (n_c, u_c, ϕ_c) converges to the rescaled solitary wave solution of the associated KdV equation as the amplitude parameter $\varepsilon > 0$ tends to zero. More specifically, in the Gardner-Morikawa scaling (also called as the KdV scaling)

$$\xi := \varepsilon^{1/2} x = \varepsilon^{1/2}(s - ct), \quad c = \sqrt{1+K} + \varepsilon, \tag{2.1}$$

it is shown that

$$\phi_c(\varepsilon^{-1/2}\xi) - \varepsilon \Psi_{KdV}(\xi) = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0,$$

where

$$\Psi_{KdV}(\xi) := \frac{3}{\sqrt{1+K}} \operatorname{sech}^2\left(\sqrt{\sqrt{1+K}/2}\, \xi \right).$$

Here $\Psi_{KdV}(\xi)$ satisfies

$$-\partial_\xi \Psi_{KdV} + \sqrt{1+K}\, \Psi_{KdV} \partial_\xi \Psi_{KdV} + \frac{1}{2\sqrt{1+K}} \partial_\xi^3 \Psi_{KdV} = 0.$$

It justifies that the solitary waves of (1.1) converges to those of the KdV equation as the amplitude parameter ε tends to zero. This leads us to the study of linear stability of the solitary waves of (1.1).

2.1. **Convective linear stability of small amplitude solitary waves.** We present the results in [5] regarding the convective linear stability of the solitary waves for the isothermal Euler-Poisson system (1.1). See [18] for the pressureless case. Since a standard energy method is not applicable for this case, a more detailed spectral analysis is required to study the asymptotic stability of traveling solitary waves. See Section 1 of [5] for the discussion regarding this issue.

We introduce the moving frame $x = s - ct$, in which the linearized system of (1.1) around the solitary wave $(n_c, u_c, \phi_c)(x)$ is given by

$$(\partial_t - \mathcal{L})(\dot{n}, \dot{u})^T = (0,0)^T, \tag{2.2}$$

where

$$\mathcal{L}\begin{pmatrix}\dot{n}\\ \dot{u}\end{pmatrix} := -\partial_x\left[\begin{pmatrix}-c+u_c & 1+\rho_c\\ \frac{K}{1+\rho_c} & -c+u_c\end{pmatrix}\begin{pmatrix}\dot{n}\\ \dot{u}\end{pmatrix} + \begin{pmatrix}0\\ (-\partial_x^2 + e^{\phi_c})^{-1}(\dot{n})\end{pmatrix}\right].$$

Due to the translation invariance and the fact that the speed c is a parameter, $\lambda = 0$ is an L^2-eigenvalue of the operator $\mathcal{L}$ with algebraic multiplicity at least two. Indeed, we have that that

$$\mathcal{L}\partial_x(n_c, u_c)^T = (0,0)^T \quad \text{and} \quad \mathcal{L}\partial_c(n_c, u_c)^T = -\partial_x(n_c, u_c)^T.$$

Thus

$$\partial_x(n_c, u_c)^T(x) \quad \text{and} \quad \partial_c(n_c, u_c)^T(x) - t\partial_x(n_c, u_c)^T(x)$$

are non-decaying (in time) solutions to (2.2).

Since the solitary waves exponentially decay to zero as $|x| \to +\infty$, the essential spectrum of $\mathcal{L}$ in L^2-space coincides with that of the associated linear operator at the end state, that is confined in the imaginary axis in the complex plane, and the zero eigenvalue is embedded in the essential spectrum. Moreover, the point spectrum of $\mathcal{L}$ in L^2-space is empty.

For a Hilbert space $\mathcal{H}$, we denote $\mathcal{H} \times \mathcal{H}$ by $(\mathcal{H})^2$. We present the result for the spectral stability.

Proposition 2.1 (Spectrum of $\mathcal{L}$ in L^2-space, [5])**.** *Consider the operator $\mathcal{L} : (L^2)^2 \to (L^2)^2$ with dense domain $(H^1)^2$. Then, for all sufficiently small $\varepsilon > 0$, it holds that*

$$\sigma(\mathcal{L}) = \sigma_{\text{ess}}(\mathcal{L}) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = 0\}.$$

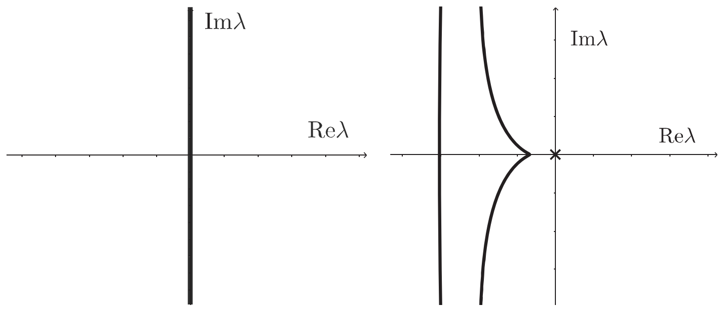
However, in terms of a standard semigroup approach, the spectral stability itself is not sufficient to conclude the asymptotic linear stability. We resolve this issue by employing the exponentially weighted spaces defined by

$$\|f(x)\|_{L^2_\beta(\mathbb{R})} := \|e^{\beta x} f(x)\|_{L^2(\mathbb{R})} \quad \text{and} \quad \|f(x)\|_{H^s_\beta(\mathbb{R})} := \|e^{\beta x} f(x)\|_{H^s(\mathbb{R})},$$

where $H^s(\mathbb{R})$ is the usual L^2-Sobolev norm, and $\beta \geq 0, s > 0$. Since $0 < \phi_c \in L^\infty(\mathbb{R})$, the linear operator $-\partial_x^2 + e^{\phi_c}$ is invertible on $L^2_\beta(\mathbb{R})$ for $\beta \in [0,1]$.

We first present some preliminary results for the linear asymptotic stability.

Proposition 2.2. *Consider the operator $\mathcal{L} : (L^2_\beta)^2 \to (L^2_\beta)^2$ with dense domain $(H^1_\beta)^2$. For any fixed $c_0 \in (0, \sqrt{2\sqrt{1+K}/3})$, let $\beta = c_0\varepsilon^{1/2}$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the following holds:*



(a) Spectrum in L^2 (b) Spectrum in L^2_β

FIGURE 1. The bold curves indicate the essential spectrums of $\mathcal{L}$. In (b), the zero eigenvalue of $\mathcal{L}$ is isolated in L^2_β for sufficiently small $\beta > 0$.

(i) *$\mathcal{L}$ generates a C_0-semigroup, $e^{\mathcal{L}t}$.*
(ii) *$\lambda = 0$ is an isolated eigenvalue of $\mathcal{L}$ with algebraic multiplicity two.*
(iii) *$(\lambda - \mathcal{L})^{-1}$ is uniformly bounded on* $\operatorname{Re}\lambda \geq 0$, *outside any small neighborhood of the origin.*

Our main result follows from Proposition 2.2 and the Gearhart-Prüss stability theorem [1, 31]. Let $\mathcal{P}_0$ be the spectral projection associated with the isolated eigenvalue $\lambda = 0$.

Theorem 2.3 (Asymptotic linear stability in weighted L^2-spaces, [5])**.** *Under the same assumptions as in Proposition 2.2, the following statement holds: for any given $(n_0, u_0)^T \in (L^2_\beta)^2$ satisfying $\mathcal{P}_0(n_0, u_0)^T = 0$, it holds*

$$\|e^{\mathcal{L}t}(n_0, u_0)^T\|_{(L^2_\beta)^2} \leq C_1 e^{-C_2 t}\|(n_0, u_0)^T\|_{(L^2_\beta)^2}, \quad \forall t \geq 0, \tag{2.3}$$

for some constants $C_1, C_2 > 0$ depending on ε.

The semigroup estimate (2.3) holds for any solution to the linearized Euler-Poisson system (2.2) with no component of the non-decaying modes.

2.2. **Ingredients of the stability analysis.** The proof of Proposition 2.2 is broken down into the following steps. For more details, we refer to [5].

Step 1: Characterization of the essential spectrum. For appropriately chosen $\beta > 0$, the essential spectrum of $\mathcal{L}$ is strictly shifted into the open left-half plane (see Figure 1). This is due to the fact that the end state of solitary wave solutions lies in a *super-ion-sonic* regime, i.e., $c > \sqrt{1+K}$.

Step 2: Characterization of the point spectrum. One of the main tasks in our analysis is to characterize the eigenvalues of $\mathcal{L}$. We aim to show that $\lambda = 0$ is the only L^2_β-eigenvalue of $\mathcal{L}$ on some closed set containing the closed right-half plane, and its algebraic multiplicity is two. The corresponding eigenvector and the generalized eigenvector are given by $\partial_x(n_c, u_c)^T$ and $\partial_c(n_c, u_c)^T$, respectively.

For this part, we employ the Evans function techniques, [2, 10–13, 20, 26, 35]. The Evans function is an analytic function of the spectral parameter λ, and on the *natural domain*, the location and order of zeroes of the Evans function coincide with those of eigenvalues of $\mathcal{L}$. For the associated eigenvalue problem, the natural domain of the Evans function is $\{\lambda : \operatorname{Re}\lambda > 0\}$ in L^2 space, and the natural domain can be extended to contain $\{\lambda : \operatorname{Re}\lambda \geq 0\}$ in the exponentially weighted space L^2_β.

In general, an *explicit* form of the Evans function is not available except for a few cases. To overcome this issue, we make use of a specific scale, related to (2.1),

$$\xi = \varepsilon^{1/2} x, \quad \lambda = \varepsilon^{3/2}\Lambda, \tag{2.4}$$

and observe that as ε tends to zero, the rescaled eigenvalue problem for the Euler-Poisson system can be formally reduced to the eigenvalue problem for the associated KdV equation

$$\Lambda \dot{n}_* - \partial_\xi \dot{n}_* + \sqrt{1+K}\,\partial_\xi(\Psi_{KdV}\dot{n}_*) + \frac{1}{2\sqrt{1+K}}\partial_\xi^3 \dot{n}_* = 0, \quad (\dot{n}_*(\xi) := \dot{n}(x)), \tag{2.5}$$

for which an explicit form of the Evans function $D_{\mathrm{KdV}}(\Lambda)$ is established in [27]; it vanishes only at $\Lambda = 0$ with multiplicity of two.

We apply the approach developed in [28] to show that in the scaling (2.4), the Evans function $D(\lambda, \varepsilon)$ for the Euler-Poisson system converges to that for the associated KdV equation as ε tends to zero, and that the convergence is uniform on a domain containing the closed right-half plane. Together with some additional arguments, we deduce that $\lambda = 0$ is a zero of $D(\lambda, \varepsilon)$ with multiplicity two, and there is no other zero. The relations between the Evans functions and the associated eigenvalue problems are summarized in the following diagram:

$\frac{d\mathbf{y}}{dx} = A(x,\lambda,\varepsilon)\mathbf{y}$	$\Leftrightarrow$	$\frac{d\mathbf{p}}{d\xi} = A_*(\xi,\Lambda,\varepsilon)\mathbf{p}$,	$\to$	$\frac{d\mathbf{p}}{d\xi} = A_*(\xi,\Lambda,0)\mathbf{p}$	$\Leftrightarrow$	Eq. (2.5)
$\vert$	$\xi=\varepsilon^{1/2}x$, $\lambda=\varepsilon^{3/2}\Lambda$	$\vert$	as $\varepsilon \to 0$	$\vert$		$\vert$
$D(\lambda,\varepsilon)$	$=$	$D_*(\Lambda,\varepsilon)$	$\to$	$D_*(\Lambda,0)$	$=$	$D_{\mathrm{KdV}}(\Lambda)$

$\Downarrow$

$0 = D(0,\varepsilon)$ $= \partial_\lambda D(0,\varepsilon)$	$\Rightarrow$	$0 \neq \partial_\lambda^2 D(0,\varepsilon)$, $0 \neq D(\lambda,\varepsilon)$ for $\lambda \neq 0$	$\Leftarrow$	$0 = D_{\mathrm{KdV}}(0) = \partial_\Lambda D_{\mathrm{KdV}}(0) \neq \partial_\Lambda^2 D_{\mathrm{KdV}}(0)$, $0 \neq D_{\mathrm{KdV}}(\Lambda)$ for $\Lambda \neq 0$

Step 3: Uniform resolvent bounds. Another key ingredient in the analysis of the linear asymptotic stability is establishing Proposition 2.2.(iii). To accomplish this, we consider the Green function for the first-order ODE system associated with the eigenvalue problem for the Euler-Poisson system and apply a perturbation argument involving the so-called *roughness of exponential dichotomies*, [7].

2.3. **(Linear) Instability criterion for large amplitude solitary waves.** From some observations using the Evans function, we obtain the following instability criterion.

Proposition 2.4 (L^2-instability criterion, [5])**.** *Let*

$$Q(c) := \int_{-\infty}^{\infty} (\rho_c u_c)(x)\, dx.$$

If $\partial_c Q(c) < 0$ for some c, the operator $\mathcal{L}$ on $(L^2)^2$ has a positive eigenvalue.

Seeking unstable solitary waves of large amplitude in accordance with this criterion, we numerically evaluated the integral $Q(c)$. In fact, it turned out that the instability criterion is inconclusive. More precisely, our numerical data show that $Q(c)$ is strictly increasing, so one cannot conclude that there is a positive eigenvalue; this is in contrast to the numerical result found in [18] for the pressureless case. See Section 8 of [5] for more details. The questions regarding instability will be investigated in the future study.

2.4. **Large amplitude solitary wave profiles of the Euler-Poisson system.** One of the remarkable differences between the pressureless and the isothermal Euler-Poisson systems lies in their density profiles of large amplitude solitary wave solutions. More precisely, from the proof of existence of the solitary waves in [4], one can see that while $n_c^* := \sup_x n_c(x)$ remains bounded above for the case $K > 0$, n_c^* approaches to infinity as $\varepsilon \nearrow \varepsilon_0^*$. Several numerical experiments for the large amplitude solitary waves are presented in Section 8 of [5].

The feature of L^∞-*blow-up* of density profile for the case $K = 0$ may be closely related to the fact that the pressureless Euler-Poisson system can develop the *delta shock* in finite time. Specifically, if the initial data $(n_0, u_0)(x)$ satisfies

$$\partial_x u_0(x) \le -\sqrt{2\rho_0(x)} \tag{2.6}$$

at some point $x \in \mathbb{R}$, then the maximal existence time of the smooth solution is finite, see [23]. For the singularity formation at finite time T_*, one can further show, by a simple comparison technique for ODE, that the gradient of velocity blows up in a non-integrable way in time, i.e.,

$$-\partial_x u \gtrsim (T_* - t)^{-1} \text{ as } t \nearrow T_*. \tag{2.7}$$

From (2.7), together with the continuity equation (1.1a), we see that L^∞ norm of density becomes unbounded as $t \nearrow T_*$. This non-physical singular behavior emerges since the pressure term is artificially ignored. As we discussed earlier, this is not the case in the presence of the pressure, in general. We suspect that it would be due to this singular behavior if the *large amplitude* solitary waves for the pressureless case are unstable.

Another interesting observation is that our numerical experiments demonstrate that

$$\inf_{x\in\mathbb{R}} (\partial_x u_c/\sqrt{\rho_c}) \searrow -\sqrt{2} \quad \text{as } \varepsilon \nearrow \varepsilon_0^* \approx 0.5852.$$

See Figure 2 for the numerical plot of $\partial_x u_c/\sqrt{\rho_c}$ with $\varepsilon = 0.585 < \varepsilon_0^*$. From this numerical experiment together with the above mentioned study of [23], one may expect that there may be a certain *critical threshold phenomena* in the pressureless Euler-Poisson system. However, the situation is not so simple as we will see in the next section.

3. Singularity formation for the pressureless case

In this section, we establish a criterion for singularity formation of (1.1) with $K = 0$, under which we show the smooth solutions develop a C^1 blow-up in a finite time along with the temporal blow-up rates. In general, it is known that smooth solutions to nonlinear hyperbolic equations fail to exist globally in time when the gradient of initial velocity is negatively large. Specifically, if the initial data satisfies (2.6), then the smooth solution breaks down in a finite time, [23]. Roughly speaking, this means that if the given initial data is near the shock waves, then the corresponding solutions develops into the shock waves. In contrast, our blow-up condition does not require the largeness of gradient of the initial velocity. In particular, our results demonstrate that C^1 norm of velocity blows up even if the initial velocity has trivial gradient. From a physical point of view, this phenomenon

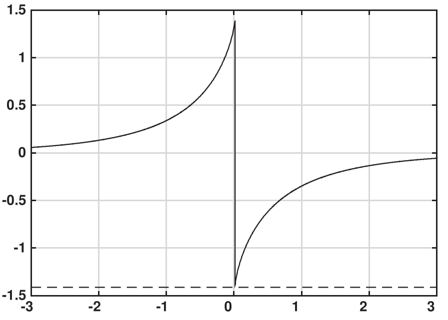


FIGURE 2. The graph (solid) of $\partial_x u_c(x)/\sqrt{\rho_c(x)}$ for $\varepsilon = 0.585 < \varepsilon_0^*$. The horizontal (dashed) line represents $-\sqrt{2}$.

is caused by the effect of the electrostatic repulsive force. For instance, when the initial density is locally lower than the background density, i.e., ion density is locally rarefied, the electrostatic potential is determined in a way that the fluid momentum with negative gradient is generated at later times, resulting in the finite-time singularity formation. See a numerical simulation in Figure 3.

We consider the pressureless Euler-Poisson system (1.1) around a constant state, i.e., $(\rho, u, \phi) \to (1, 0, 0)$ as $|x| \to \infty$. It is known that the system (1.1) with $K = 0$ admits a unique smooth solution locally in time for the smooth initial data, for instance, $(\rho_0 - 1, u_0) \in H^2(\mathbb{R}) \times H^3(\mathbb{R})$. Furthermore, as long as the smooth solution exists, the energy

$$H(t) := \int_{\mathbb{R}} \frac{1}{2}\rho u^2 + \frac{1}{2}|\partial_x \phi|^2 + (\phi - 1)e^{\phi} + 1\, dx$$

is conserved, that is, $H(t) = H(0)$ for all $t \in [0, T]$. We refer to [21] for more details.

We state our main theorem of this section.

Theorem 3.1 ([3]). *Let $f_- : (-\infty, 0] \to [0, \infty)$ be a strictly increasing function defined in* (3.2). *For the initial data satisfying*

$$\text{(3.1)} \qquad exp\left(f_-^{-1}(H(0))\right) > 2\rho_0(\alpha) \text{ for some } \alpha \in \mathbb{R},$$

the maximal existence time T_* for the smooth solution to the Euler-Poisson system (1.1) *is finite. In particular,*

$$\lim_{t \nearrow T_*} \sup_{x \in \mathbb{R}} \rho(x, t) = +\infty \qquad \text{and} \qquad \inf_{x \in \mathbb{R}} u_x(x, t) \approx \frac{1}{t - T_*}$$

for all $t < T_*$ sufficiently close to T_*.

Theorem 3.1 demonstrates that singularities in solutions to (1.1) can occur in a finite time if the initial density at some point is small compared to the initial energy. In fact, the negativity of the initial velocity gradient is not required. Moreover, there is a fairly wide class of the initial data satisfying the condition (3.1). From the elliptic estimates for the Poisson equation (1.1c), we have

$$0 \le H(0) \le \frac{\sup_{x \in \mathbb{R}} \rho_0}{2} \int_{\mathbb{R}} |u_0|^2\, dx + \frac{1}{K_0} \int_{\mathbb{R}} |\rho_0 - 1|^2\, dx =: C(\rho_0, u_0),$$

where $K_0 := (1 - \inf \rho_0)/(-\log \inf \rho_0)$. On the other hand, since $\lim_{\zeta \searrow 0} f_-^{-1}(\zeta) = 0$, for any given constant $0 < c < 1/2$, there is $\delta_c > 0$ such that $\zeta < \delta_c$ implies $\exp(f_-^{-1}(\zeta)) > 2c$. Thus, (3.1) holds for all initial data satisfying $\inf \rho_0 = c \in (0, 1/2)$ and $C(\rho_0, u_0) < \delta_c \ll 1$. In particular, one can take $u_0 \equiv 0$.

3.1. **Ingredients of the blow-up analysis.** The proof of Theorem 3.1 is broken down into the following steps. Detailed proofs of the results in this subsection can be found in [3].

Step 1: Uniform boundedness of ϕ. One of our key observations is that as long as the solution exists, $\phi(x,t)$ is uniformly bounded in x and t. Let us define the functions

$$f(z) := \begin{cases} f_+(z) := \displaystyle\int_0^z \sqrt{2\left((s-1)e^s + 1\right)}\, ds & \text{for } z \geq 0, \\ f_-(z) := \displaystyle\int_z^0 \sqrt{2\left((s-1)e^s + 1\right)}\, ds & \text{for } z \leq 0. \end{cases} \tag{3.2}$$

f_+ and f_- have the inverse functions $f_+^{-1} : [0, +\infty) \to [0, +\infty)$ and $f_-^{-1} : [0, +\infty) \to (-\infty, 0]$, respectively. Furthermore, f is of $C^1(\mathbb{R})$. Then, using the energy conservation, we obtain

Lemma 3.2. *As long as the smooth solution to* (1.1) *exists for* $t \in [0, T]$,

$$f_-^{-1}\left(H(0)\right) \leq \phi(x,t) \leq f_+^{-1}\left(H(0)\right) \quad \textit{for all } (x,t) \in \mathbb{R} \times [0, T].$$

Step 2: Second order ODE. For $u \in C^1$, the characteristic curves $x(\alpha, t)$ are defined as the solution to the ODE

$$x' = u(x(\alpha,t),t), \quad x(\alpha, 0) = \alpha \in \mathbb{R}, \quad t \geq 0,$$

where $' := d/dt$ and the initial position α is considered as a parameter. From (1.1), one can easily obtain that

$$\dot{\rho} = -u_x \rho, \quad \dot{u}_x = -u_x^2 + \rho - e^{\phi},$$

where $\dot{} := \partial_t + u\partial_x$. We define

$$w(\alpha, t) := \frac{\partial x}{\partial \alpha}(\alpha, t)$$

and show that w satisfies a certain second-order ordinary differential equation:

$$w'' + e^{\phi(x(\alpha,t),t)} w = \rho_0(\alpha), \quad w(\alpha, 0) = 1, \quad w'(\alpha, 0) = u_{0x}(\alpha). \tag{3.3}$$

Using Lemma 3.2 for (3.3), one has

$$w'' + e^{f_-^{-1}(H(0))} w \leq \rho_0(\alpha), \quad w(\alpha, 0) = 1, \quad w'(\alpha, 0) = u_{0x}(\alpha). \tag{3.4}$$

Step 3: Blow-up criterion. It is obvious that for each $\alpha \in \mathbb{R}$,

$$\begin{aligned} 0 < w(\alpha,t) < +\infty \quad &\Longleftrightarrow \quad 0 < \rho(x(\alpha,t),t) < +\infty, \\ \lim_{t \nearrow T_*} w(\alpha,t) = 0 \quad &\Longleftrightarrow \quad \lim_{t \nearrow T_*} \rho(x(\alpha,t),t) = +\infty. \end{aligned}$$

We show that $\sup_{x \in \mathbb{R}} |\rho(x,t)|$ and $\sup_{x \in \mathbb{R}} |u_x(x,t)|$ blow up at the same time, if one of them blows up at a finite time T_*. Moreover, $u_x \searrow -\infty$ as $t \nearrow T_*$ at a non-integrable order in time t. Using Lemma 3.2, we obtain

Lemma 3.3. *Suppose that the smooth solution to* (1.1) *exists for all* $0 \leq t < T_* < +\infty$. *Then the following statements hold.*

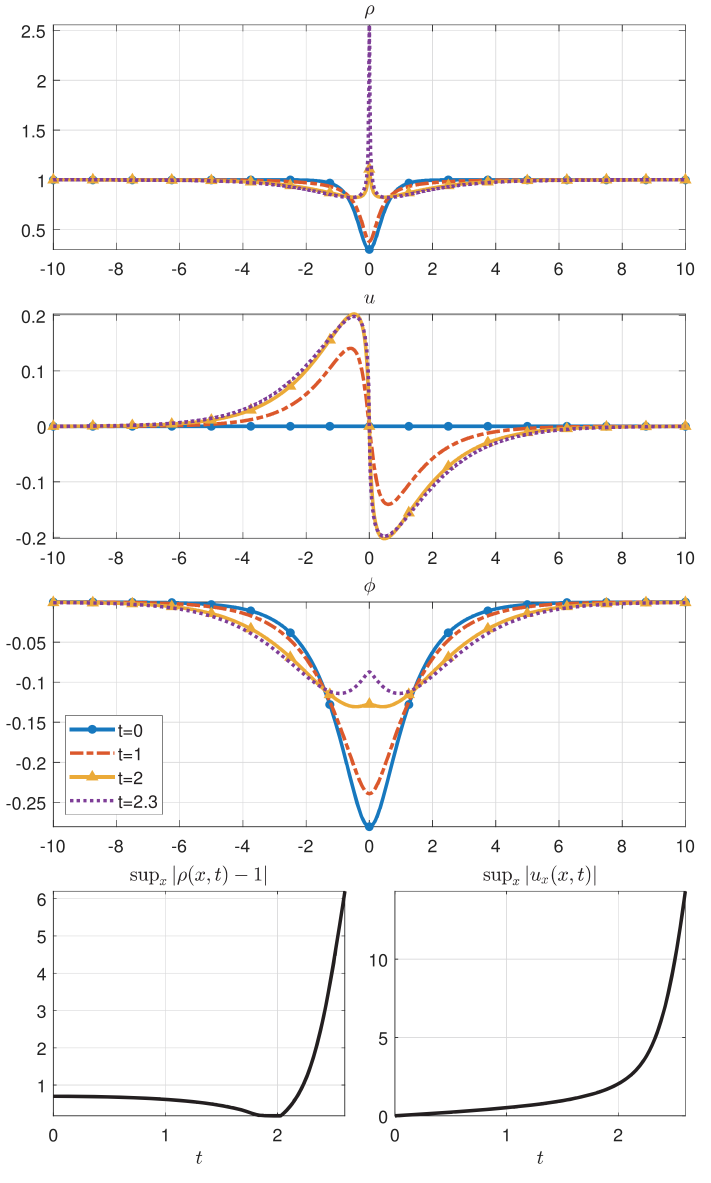


FIGURE 3. Numerical solution to the pressureless (1.1). $\rho(0,t)$ and $u_x(0,t)$ are getting larger as time t goes by.

(1) For each $\alpha \in \mathbb{R}$, the following holds true:

$$\lim_{t \nearrow T_*} w(\alpha, t) = 0 \tag{3.5}$$

if and only if

$$\liminf_{t \nearrow T_*} u_x\left(x(\alpha, t), t\right) = -\infty. \tag{3.6}$$

(2) *If one of* (3.5)–(3.6) *holds for some* $\tilde{\alpha} \in \mathbb{R}$, *then there are uniform constants* $c_0, c_1 > 0$ *such that*

$$\frac{c_0}{t - T_*} < u_x\left(x(\tilde{\alpha}, t), t\right) < \frac{c_1}{t - T_*} \tag{3.7}$$

for all $t < T_*$ *sufficiently close to* T_*.

Step 4: Zeros of the second-order ordinary differential inequality. The observation in Step 3 leads that one may apply some comparison arguments to study the existence of zeros of w for (3.4). We prove the following lemma.

Lemma 3.4. *Let* a *and* b *be positive constants. Suppose* $w(t)$ *satisfies*

$$w'' + aw \leq b \tag{3.8}$$

for all $t \geq T_0$ *and* $w(T_0) \geq 1$. *If* $a/2 > b$ *and*

$$\frac{a|w(T_0)|^2}{2} - w(T_0)b + \frac{|w'(T_0)|^2}{2} > 0, \tag{3.9}$$

then $w(t)$ *has a zero on the interval* $(T_0, +\infty)$.

Now by applying Lemma 3.4 for (3.4), and by Lemma 3.3, we prove the assertion of Theorem 3.1.

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