# On $L^1$ estimates of solutions of compressible viscoelastic system

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## 1 Introduction

This article is a summary of [8] on large time behavior of solutions of the following system:

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \tag{1.1}$$
$$\rho(\partial_t v + v \cdot \nabla v) - \nu \Delta v - (\nu + \nu')\nabla \mathrm{div} + \nabla P(\rho) = \beta^2 \mathrm{div}(\rho F^\top F), \tag{1.2}$$
$$\partial_t F + v \cdot \nabla F = (\nabla v) F \tag{1.3}$$

in $\mathbb{R}^3$. Here $\rho = \rho(x,t)$, $v = {}^\top(v^1(x,t), v^2(x,t), v^3(x,t))$, and $F = (F^{jk}(x,t))_{1 \le j,k \le 3}$ denote the unknown density, the velocity field, and the deformation tensor, respectively, at position $x \in \mathbb{R}^3$ and time $t \ge 0$; $P = P(\rho)$ is the given pressure; $\nu$ and $\nu'$ are the viscosity coefficients satisfying

$$\nu > 0, \ 2\nu + 3\nu' \ge 0;$$

$\beta > 0$ is the propergation speed of elastic wave. We assume that $P'(1) > 0$, and we set $\gamma = \sqrt{P'(1)}$.

The system (1.1)–(1.3) is considered under the initial condition

$$(\rho, v, F)|_{t=0} = (\rho_0, v_0, F_0). \tag{1.4}$$

We also impose the following conditions for $\rho_0$ and $F_0$:

$$\rho_0 \mathrm{det} F_0 = 1, \tag{1.5}$$
$$\sum_{m=1}^{3} (F_0^{ml} \partial_{x_m} F_0^{jk} - F_0^{mk} \partial_{x_m} F_0^{jl}) = 0, \ j,k,l = 1,2,3, \tag{1.6}$$
$$\mathrm{div}(\rho_0 {}^\top F_0) = 0. \tag{1.7}$$

It follows from [5, Appendix A] and [13, Proposition.1] that the quantities (1.5)–(1.7) are invariant for $t \geq 0$:

$$\rho \det F = 1, \tag{1.8}$$

$$\sum_{m=1}^{3} (F^{ml} \partial_{x_m} F^{jk} - F^{mk} \partial_{x_m} F^{jl}) = 0, \ j, k, l = 1, 2, 3. \tag{1.9}$$

$$\mathrm{div}(\rho^{\top} F) = 0. \tag{1.10}$$

The purpose of this article is to the large time behavior of solutions of the problem (1.1)–(1.7) around a motionless state $(1, 0, I)$, where $I$ is the $3 \times 3$ identity matrix.

The system (1.1)–(1.3) is treated as a basic model for compressible fluid with elastic effect in macroscopic scale. In fact, the elastic effect appear in the last term of the right-hand side of (1.2) (See [10, 15] for more detail of its physical background). We note that by setting $\beta = 0$ formally, the system becomes the usual compressible Navier-Stokes equations.

In the case $\beta = 0$, Hoff and Zumbrun [2] derived the following $L^p$ $(1 \leq p \leq \infty)$ decay estimates in $\mathbb{R}^n$, $n \geq 2$:

$$\|(\phi(t), m(t))\|_{L^p} \leq \begin{cases} C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})} L(t), & 1 \leq p < 2, \\ C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})}, & 2 \leq p \leq \infty, \end{cases}$$

where $m = \rho v$; $L(t) = \log(1+t)$ when $n = 2$, and $L(t) = 1$ when $n \geq 3$. Furthermore, the authors of [2] showed the following asymptotic property:

$$\left\| \left( (\phi(t), m(t)) - \left(0, \mathcal{F}^{-1}\left(e^{-\nu|\xi|^2 t} \hat{\mathcal{P}}(\xi) \hat{m}_0\right)\right)\right) \right\|_{L^p} \leq C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})}$$

for $2 \leq p \leq \infty$. Here $\hat{\mathcal{P}}(\xi) = I - \frac{\xi^{\top}\xi}{|\xi|^2}$, $\xi \in \mathbb{R}^n$. The above estimates and asymptotic properties indicate the hyperbolic aspect of the system due to the diffusion wave property caused by interaction between viscous diffusion and sound wave. In fact, the obtained rate $(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})} L(t)$ of its $L^p$ norm are slower than the heat kernel when $1 \leq p < 2$. Moreover, if $2 < p \leq \infty$, the convergence rate is improved by cutting off $\left(0, \mathcal{F}^{-1}\left(e^{-\nu|\xi|^2 t} \hat{\mathcal{P}}(\xi) \hat{m}_0\right)\right)$, which provides the pure diffusion phenomena.

We next mention the related works in the case $\beta > 0$. The existence of the strong solution around the motionless state was proved by Hu and Wang [4, 5]. It was proved by Hu and Wu [6] and Li, Wei and Yao [9] that the solution satisfies the following $L^p$ decay estimates for the case $2 \le p \le \infty$ and the $L^2$ decay estimates of its higher order derivatives:

$$\|u(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)},$$

$$\|\nabla^k u(t)\|_{L^2} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}},\ k = 0, 1, \dots, N-1,$$

provided that $u_0 = (\rho_0 - 1, v_0, F_0 - I)$ belongs to $H^N$, $N \ge 3$, and is small in $L^1 \cap H^3$. Here $u(t) = (\phi(t), w(t), G(t)) = (\rho(t) - 1, v(t), F(t) - I)$ is the perturbation. The above $L^p$ estimates imply the diffusive aspect of the system (1.1)–(1.3). However, in expectation of the diffusion wave phenomena of the system (1.1)–(1.3) due to tripartite interaction of sound wave, viscous diffusion and elastic wave, the obtained $L^p$ decay estimate had room for improvement. Motivated by the work [2], the author of [7] improved and generalized the results of [6, 9] by showing the following estimates for $1 < p \le \infty$:

$$\|u(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}\left(1-\frac{2}{p}\right)},\ 1 < p \le \infty,$$

provided that $u_0 = (\rho_0 - 1, v_0, F_0 - I)$ is small in $L^1 \cap H^3$. This gives the hyperbolic aspect of the system which does not appear in the results in [6, 9] and the work [2] of the case $\beta = 0$ due to elastic wave. We also refer to [3, 11, 16] in recent progresses.

We point out that the case $p = 1$ was not discussed. In [7], the material coordinate transform and the non-local operator were used to get over the obstruction of applying the semigroup theory to the nonlinear problem. In fact, we made use of the following function to reformulate the nonlinear problem for $u$:

$$\Psi(x,t) := \nabla\psi(x,t),\ \psi(x,t) := \tilde{\psi}(x,t) - (-\Delta)^{-1}\mathrm{div}^\top(\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi})).$$

Here $\tilde{\psi}(x,t) = x - X(x,t)$ is the displacement vector; $X = X(x,t)$ is the inverse of the material coordinate $x = x(X,t)$ satisfying the flow map:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = v(x(X,t),t), \\ x(X,0) = X; \end{cases}$$

$(-\Delta)^{-1}$ denotes $(-\Delta)^{-1} = \mathcal{F}^{-1}|\xi|^{-2}\mathcal{F}$; $h(\nabla\tilde{\psi})$ is a function determined from $F - I = \nabla\tilde{\psi} + h(\nabla\tilde{\psi}) =$. We then obtained the linear condition $\phi + \mathrm{tr}\Psi = 0$ which eliminates non-decaying and $L^1$ unbounded terms in the expression of the semigroup $e^{-tL}\tilde{U}_0$, where $\tilde{U}_0 = (\phi_0, w_0, \Psi_0)$, $\Psi_0 = \Psi(0)$. Therefore we investigated the problem for $\tilde{U}(t) = (\phi(t), w(t), \Psi(t))$ and derived the desired estimate from the following integral equation:

$$\tilde{U}(t) = e^{-tL}\tilde{U}_0 + \int_0^t e^{-(t-s)L} N(\tilde{U}(s))\mathrm{d}s,$$

where $L$ is a linearized operator around the motionless state; $N(\tilde{U}(s)) = (N_1(\tilde{U}(s)), N_2(\tilde{U}(s)), N_3(\tilde{U}(s)))$ is a nonlinear term satisfying the linear condition $N_1(\tilde{U}(s)) + \mathrm{tr}N_3(\tilde{U}(s)) = 0$. However, we cannot conclude that the $L^1$ norm of $u(t)$ is controlled by $U(t)$ due to the $L^1$ unboundness of the Liesz operater appearing in the definition of $\Psi$.

In this article, we solve the above difficulty by changing the reformulation, and show that the following $L^1$ estimate hold for $t \geq 0$ uniformly:

$$\|u(t)\| \leq C(1+t)^{\frac{1}{2}},$$

provided that $u_0$ is small in $L^1 \cap H^4$ and belongs to $W^{2,1}$. Moreover, the obtained $(1+t)^{\frac{1}{2}}$ is sharp under some low frequency assumption considered in [6].

We give an outline of the proof of the main result. We first notice that the constraint (1.9) is read as $\rho = \mathrm{det}F^{-1}$. Then we see from the definition of the determinant that $\phi$ is handled by $-\mathrm{div}\tilde{\psi}$

$$\phi = -\mathrm{div}\tilde{\psi} + O(|\nabla\tilde{\psi}|^2), \ \|\nabla\tilde{\psi}\|_{C(0,\infty;L^\infty)} \ll 1.$$

For simplicity, we omit the tilde $\tilde{\cdot}$ of $\tilde{\psi}$ here. We confirm that the $L^1$ norm of $u = (\phi, w, G)$ is estimated by $U = (-\mathrm{div}\psi, w, \nabla\psi)$. Therefore we arrive at the problem for $U = (\tilde{\phi}, w, \tilde{G}) = (-\mathrm{div}\psi, w, \nabla\psi)$:

$$\begin{cases} \partial_t U + LU = N(U), \\ \tilde{\phi} + \mathrm{tr}\tilde{G} = 0, \ \tilde{G} = \nabla\psi, \\ U|_{t=0} = U_0, \end{cases} \tag{1.11}$$

where $N(U) = (N_1(U), N_2(U), N_3(U))$ is the nonlinearity such that $N_1(U) + \mathrm{tr}N_3(U) = 0$. We point out that since $U$ and $N(U)$ hold the same linear constraint as in [7], the semigroup $e^{-tL}U_0$ and the Duammel term $\int_0^t e^{-(t-s)L}N(U(s))\mathrm{d}s$ do not have terms which are time-independent and unbounded in $L^1$. Consequently, the desired $L^1$ estimate follows the following integral equation of $U$:

$$U(t) = e^{-tL}U_0 + \int_0^t e^{-(t-s)L} N(U(s))\mathrm{d}s.$$

This article is organized as follows. In Section 2 we state the main result of this article. In Section 3 we give the outline of the proof of the main result.

## 2 Main Result

In this section we summerize the results in [8].

To state the main result, we first introduce the problem for the perturbation $u(t) = (\phi(t), w(t), G(t)) = (\rho(t) - 1, v(t), F(t) - I)$:

$$\begin{cases} \partial_t \phi + \mathrm{div} w = g_1(u), \\ \partial_t w - \nu \Delta w - \tilde{\nu} \nabla \mathrm{div} w + \gamma^2 \nabla \phi - \beta^2 \mathrm{div} G = g_2(u), \\ \partial_t G - \nabla w = g_3(u), \\ \nabla \phi + \mathrm{div}^\top G = g_4(\phi, G), \\ u|_{t=0} = u_0 = (\phi_0, w_0, G_0). \end{cases} \tag{2.1}$$

Here $\tilde{\nu} = \nu + \nu'$; $g_j(u), j = 1, 2, 3,\ g_4(\phi, G)$ denote the nonlinear terms;

$$\begin{aligned} g_1(u) &= -\mathrm{div}(\phi w), \\ g_2(u) &= -w \cdot \nabla w + \frac{\phi}{1+\phi}(-\nu \Delta w - \tilde{\nu} \nabla \mathrm{div} w + \gamma^2 \nabla \phi) - \frac{1}{1+\phi} \nabla Q(\phi) \\ &\quad - \frac{\beta^2 \phi}{1+\phi} \mathrm{div} G + \frac{\beta^2}{1+\phi} \mathrm{div}(\phi G + G^\top G + \phi G^\top G), \\ g_3(u) &= -w \cdot \nabla G + (\nabla w) G, \\ g_4(\phi, G) &= -\mathrm{div}(\phi^\top G), \end{aligned}$$

where

$$Q(\phi) = \phi^2 \int_0^1 P''(1 + s\phi) \mathrm{d}s.$$

The main result of this article is stated as follows.

**Theorem 2.1.** *(i) Assume that $\phi_0$ and $G_0$ satisfy $\nabla \phi_0 + \mathrm{div}^\top G_0 = g_4(\phi_0, G_0)$ and $(I + G_0)^{-1} = \nabla X_0$ for some vector field $X_0$. There is a positive number $\epsilon$ such that if $u_0 = (\phi_0, w_0, G_0)$ satisfies $\|u_0\|_{H^4} + \|u_0\|_{L^1} \le \epsilon$ and $u_0 \in W^{2,1}$, then there exists a unique solution $u(t) \in C([0, \infty); H^4)$ of the problem (2.1) satisfying*

$$\|u(t)\|_{L^1} \le C(1 + t)^{\frac{1}{2}} (\|u_0\|_{W^{2,1}} + \|u_0\|_{H^4})$$

*uniformly for $t \ge 0$. Here $C$ is a positive constant.*

*(ii)In addition, if there exists a positive number $r > 0$ such that the following low frequency condition holds for $0 \le |\xi| \le r$:*

$$|\hat{\phi}_0(\xi)| \ge c_0, |\hat{m}_0(\xi)| + |\hat{\mathcal{G}}_0(\xi) - {}^\top \hat{\mathcal{G}}_0(\xi)| \le c_1 |\xi|^{\eta_0}, \tag{2.2}$$

*where* $(m_0, \mathcal{G}_0) := (\rho_0 v_0, \rho_0 F_0 - I)$*;* $c_0$*,* $c_1$ *and* $\eta_0$ *are positive numbers independent of* $t$*,* $u(t)$ *satisfies the following lower* $L^2$ *estimate uniformly for* $t \geq R$*:*

$$\|u(t)\|_{L^1} \geq c(1+t)^{\frac{1}{2}}. \tag{2.3}$$

*Here* $R$ *is a large positive number, and* $c$ *is a positive number independent of* $t$*.*

**Remark 2.2.** In a similar manner to the following proof and the previous result [7], we can generalize the above $L^1$ estimates as

$$\|u(t)\|_{L^p} \leq C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}\left(1-\frac{2}{p}\right)}(\|u_0\|_{L^1} + \|u_0\|_{H^3}),\ t \geq 0$$
$$\|u(t)\|_{L^p} \geq c(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}\left(1-\frac{2}{p}\right)},\ t \geq M.$$

for $1 < p \leq \infty$. We also note that the condition $\nabla\phi_0 - \mathrm{div}^\top(I+G_0)^{-1} = 0$ imposed in [7] is not necessary.

# 3 Outline of the proof of Theorem 2.1

In this section, we briefly explain the outline of the proof of Theorem 2.1.

We only show the upper $L^1$ estimate (i). The lower $L^1$ estimate (ii) immediately follows from the interpolation inequality and the lower $L^2$ estimate

$$\|u(t)\|_{L^2} \geq c(1+t)^{-\frac{3}{4}},\ t \geq M,$$

provided that the low frequency condition (2.2) holds (See [6] for the proof). We obtain the following integral equation from (2.1) by using the Duammel's principle:

$$u(t) = e^{-tL}u_0 + \int_0^t e^{-(t-s)L} g(u(s))\mathrm{d}s.$$

Here $L$ is a linearized operator defined as

$$L = \begin{pmatrix} 0 & \mathrm{div} & 0, \\ \gamma^2\nabla & -\nu\Delta - \tilde{\nu}\nabla\mathrm{div} & -\beta^2\mathrm{div} \\ 0 & -\nabla & 0 \end{pmatrix}.$$

The first term of the right hand side is formally estimated as

$$\|e^{-tL}u_0\|_{L^1} \leq C(1+t)^{\frac{1}{2}}\|u_0\|_{L^1} + Ce^{-ct}\|u_0\|_{W^{2,1}}, t \geq 0,$$

provided that the conditions $\phi_0 + \mathrm{tr}G_0 = 0$ and $G_0 = \nabla\psi_0$ hold. Here $\psi_0$ is some vector field. On the other hand, since we cannot determine whether the nonlinear terms $g_1(u)$ and $g_3(u)$ satisfy these same conditions for $\phi_0$ and $G_0$ or not, the estimate of the Duammel term $\int_0^t e^{-(t-s)L} g(u(s))\mathrm{d}s$ becomes difficult. Therefore, inspired from the idea of [14, 15], we rewrite the problem into the suitable form by using the material coordinate transform to enable the application of the linear semigroup theory.

We define $X = X(x,t)$ as the inverse of $x = x(X,t)$ which is the solution of the following flow map:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t}(X,t) = v(x(X,t),t), \\ x(X,0) = X \in \mathbb{R}^3. \end{cases}$$

According to the continuum theory, the deformation tensor $F$ and its inverse $F^{-1}$ are given by $F = \frac{\partial x}{\partial X}$ and $F^{-1} = \frac{\partial X}{\partial x} = \nabla X$, respectively. We next set the displacement vector $\psi(x,t) := x - X(x,t)$. We see that $\psi(x,t)$ satisfies the following properties:

$$\partial_t\psi - v = -v\cdot\nabla\psi,$$

$$G = F - I = (I - \nabla\psi)^{-1} - I.$$

In addition, since the constraint (1.8) is rewritten as $\rho = \det F^{-1}$, we have

$$\phi = \rho - 1 = \det F^{-1} - 1 = -\mathrm{div}\psi + (\mathrm{tr}(\nabla\psi)^2 - (\mathrm{tr}(\nabla\psi))^2) + \det(\nabla\psi).$$

These impliy that the behavior of $\phi$ and $G$ are confirmed from the first order derivatives of $\psi$.

We next set $\psi_0 = \psi|_{t=0}$, $\tilde{\phi} = -\mathrm{div}\psi$, $\tilde{G} = \nabla\psi$, and $U(t) := {}^\top(\tilde{\phi}(t), w(t), \tilde{G}(t)) = {}^\top(-\mathrm{div}\psi(t), w(t), \nabla\psi(t))$. We then see from the following lemma that $\|u(t)\|_{L^1}$ is estimated by $\|U(t)\|_{L^1}$.

**Lemma 3.1.** *There exists $\epsilon_1 < 1$ such that if $\|G\|_{C([0,\infty);H^3)} \le \epsilon_1$, then the following estimates hold uniformly for $t \ge 0$:*

(*i*) $C^{-1}\|\nabla\psi(t)\|_{L^1} \le \|G(t)\|_{L^1} \le C\|\nabla\psi(t)\|_{L^1}$,

(*ii*) $$\|\nabla^{k+1}\psi(t)\|_{L^2} \le \begin{cases} C\|\nabla G(t)\|_{L^2}, & k=1, \\ C(\|\nabla G(t)\|_{H^1}^2 + \|\nabla^2 G(t)\|_{L^2}), & k=2, \\ C(\|\nabla G(t)\|_{H^1}\|\nabla^2 G(t)\|_{H^1} + \|\nabla^3 G(t)\|_{L^2}), & k=3, \end{cases}$$

(*iii*) $\|\nabla^{k+1}\psi_0\|_{L^1} \le C\|\nabla^k G_0\|_{L^1}$, $k = 1,2$,

(*vi*) $\|\phi(t)\|_{L^1} \le \|\mathrm{div}\psi(t)\|_{L^1} + C(\|\nabla\psi(t)\|_{L^\infty} + \|\nabla\psi(t)\|_{L^\infty}^2)\|\nabla\psi(t)\|_{L^1}$.

The proof can be found inn [7, 8].

**Remark 3.2.** We note that the non-local operator $(-\Delta)^{-1}$ does not appear in this reformulation. Therefore we do not face the difficulty caused by the Liesz operator.

In view of Lemma 3.1, we focus on the problem for $U(t)$ instead of $u(t)$:

$$\begin{cases} \partial_t U + LU = N(U), \\ \tilde{\phi} + \mathrm{tr}\tilde{G} = 0, \ \tilde{G} = \nabla\psi, \\ U|_{t=0} = U_0 = (\tilde{\phi}_0, w_0, \tilde{G}_0), \ \tilde{\phi}_0 + \mathrm{tr}\tilde{G}_0 = 0, \ \tilde{G}_0 = \nabla\psi_0. \end{cases} \tag{3.1}$$

Here $N(U) = {}^{\top}(N_1(U), N_2(U), N_3(U))$ is the nonlinear term given by:

$$\begin{aligned} N_1(U) &= \mathrm{div}(w \cdot \nabla\psi), \\ N_2(U) &= g_2(u) - \frac{\gamma^2}{2}\nabla(\mathrm{tr}(\nabla\psi)^2 - (\mathrm{tr}(\nabla\psi))^2) - \gamma^2\nabla\mathrm{det}(\nabla\psi) \\ &\quad + \beta^2\mathrm{div}\{(I - \nabla\psi)^{-1} - I - \nabla\psi\}, \\ N_3(U) &= -\nabla(w \cdot \nabla\psi). \end{aligned}$$

We obtain the following integral equation from (3.1) by using the Duammel's principle:

$$U(t) = e^{-tL}U_0 + \int_0^t e^{-(t-s)L}N(U(s))\mathrm{d}s. \tag{3.2}$$

By direct calculation, $N_1(U)$ and $N_3(U)$ also hold the same linear condition as $\tilde{\phi}$ and $\tilde{G}$:

$$N_1(U) + \mathrm{tr}N_3(U) = 0. \tag{3.3}$$

Therefore we see from the condition (3.3) that the right-hand side of (3.2) is estimated as follows.

**Lemma 3.3.** *The following estimates hold uniformly for $t \geq 0$:*

$$\begin{aligned} &\|e^{-tL}U_0\|_{L^1} \leq C(1+t)^{\frac{1}{2}}\|U_0\|_{L^1} + Ce^{-ct}\|U_0\|_{W^{2,1}}, \\ &\left\|\int_0^t e^{-(t-s)L}N(U(s))\mathrm{d}s\right\|_{L^1} \\ &\leq C\int_0^t (1+t-s)^{\frac{1}{2}}\|N(U(s))\|_{L^1}\mathrm{d}s + C\int_0^t e^{-c(t-s)}\|N(U(s))\|_{W^{2,1}}\mathrm{d}s. \end{aligned}$$

The proof can be found in [7, 8]. It remains to estimate $U_0$ and $N(U(s))$ in $L^1$ and $W^{2,1}$. Thanks to Lemma 3.1 and the results in [6, 9], we have

$$\begin{aligned}
&\|U_0\|_{L^1} + \|U_0\|_{W^{2,1}} \le C(\|u_0\|_{W^{2,1}} + \|u_0\|_{H^4}),\\
&\|N(U(s))\|_{L^1} \quad \le C(1+s)^{-2}(\|u_0\|_{L^1} + \|u_0\|_{H^3}),\\
&\|N(U(s))\|_{W^{2,1}} \le C\|u_0\|_{H^4}.
\end{aligned}$$

These inequalities and Lemma 3.3 yield

$$\|u(t)\|_{L^1} \le C\|U(t)\|_{L^1} \le C(1+t)^{\frac{1}{2}}(\|u_0\|_{W^{2,1}} + \|u_0\|_{H^4}).$$

This completes the proof of Theorem 2.1 (i).

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