Bilinear estimates for functions with the Dirichlet boundary condition and an application to SQG

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Abstract. We discuss the validity of the bilinear estimate of functions satisfying the Dirichlet boundary condition on the two dimensional half space. For two functions f, g, we compare two nonlinearity of the standard product fg and the gradient of f and the perpendicular component of the gradient of g, and we show that the first case needs a restriction for the regularity index, while the second case does not. We also introduce an application to the surface quasi-geostrophic equation with the critical dissipation. This paper is a survey of these results.

1. Introduction

Let us consider problems on the half space,

$$\mathbb{R}^2_+ := \{x \in (x_1, x_2) \in \mathbb{R}^2 \,|\, x_2 > 0\},$$

and we consider the Dirichlet Laplacian $-\Delta_D$,

$$\begin{cases} D(-\Delta_D) = \{f \in H^1_0(\Omega) \,|\, \Delta f \in L^2(\mathbb{R}^2_+)\}, \\ -\Delta_D f = -\Delta f = -\displaystyle\sum_{j=1}^{2} \partial^2_{x_j} f, \quad f \in D(-\Delta_D). \end{cases}$$

We also write Λ_D the square root of $-\Delta_D$,

$$\Lambda_D := \sqrt{-\Delta_D}.$$

The aim of this paper is to discuss a simple problem of partial differential equations on domains with the boundary. To this end, we start by the bilinear estimates in Besov spaces for product of two functions and for the nonlinear term appearing in the surface quasi-geostrophic equation.

When the domain is the whole space $\mathbb{R}^2$, then it is well-known that

$$\|fg\|_{\dot{B}^s_{p,q}} \leq C\Big(\|f\|_{\dot{B}^s_{p_1,q}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_3}} \|g\|_{\dot{B}^s_{p_4,q}}\Big),$$

where

$$s > 0, \quad 1 \leq p, p_j, q \leq \infty \ (j = 1, 2, 3, 4), \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

As for the nonlinearity for the surface quasi-geostrophic equattion, this kind of estimates for $(\nabla^\perp(-\Delta)^{-1/2} f \cdot \nabla) g$ is known, since the Riesz transform is bounded in the homogeneous Besov spaces, where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. We discuss the validity of such inequalities on the half space with regularity number s measured by the Dirichlet Laplacian, and we will find possible range of s.

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We introduce Besov spaces associated with the Dirichlet Laplacian along [10, 16]. It is known that $-\Delta_D$ is a self-adjoint operator and we can apply the spectral theorem. We then have a partition of identity $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ such that

$$f = \int_{-\infty}^{\infty} dE(\lambda)f \text{ in } L^2(\mathbb{R}^2_+), \quad f \in L^2(\mathbb{R}^2_+),$$

$$-\Delta_D f = \int_{-\infty}^{\infty} \lambda dE(\lambda)f \text{ in } L^2(\mathbb{R}^2_+), \quad f \in D(-\Delta_D).$$

Moreover, for every measurable function $\varphi : \mathbb{R} \to \mathbb{C}$, we can define $\varphi(-\Delta_D)$ by

$$\begin{cases} D(\varphi(-\Delta_D)) = \Big\{ f \in L^2(\mathbb{R}^2_+) \,\Big|\, \int_0^{\infty} |\varphi(\lambda)|^2 d\|E(\lambda)f\|^2_{L^2} < \infty \Big\}, \\ \varphi(-\Delta_D)f = \int_0^{\infty} \varphi(\lambda)dE(\lambda)f, \quad f \in D(\varphi(-\Delta_D)). \end{cases}$$

We next introduce a partition of unity $\{\phi_j\}_{j\in\mathbb{Z}} \subset C_0^{\infty}(\mathbb{R})$ such that

$$\operatorname{supp}\phi_0 \subset [2^{-1}, 2], \quad \phi_j(\lambda) = \phi_0(2^{-j}\lambda) \text{ for } \lambda \in \mathbb{R}, \quad \sum_{j\in\mathbb{Z}} \phi_j(\lambda) = 1 \text{ for } \lambda > 0.$$

It is known in [21] (see also [15]) that the functions of the square root of the Dirichlet Laplacian is uniformly bounded.

$$\sup_{j\in\mathbb{Z}} \|\phi_j(\Lambda_D)\|_{L^p\to L^p} < \infty, \quad 1 \le p \le \infty.$$

We can then define the test function spaces of non-homogeneous type and homogeneous type.

$$\begin{aligned} \mathcal{X} :=& \{ f \in L^1 \cap L^2 \,|\, p_M(f) < \infty \text{ for all } M = 1, 2, \ldots \}, \\ & p_M(f) := \|f\|_{L^1} + \|\Lambda_D^M f\|_{L^1}, \\ \mathcal{Z} :=& \{ f \in L^1 \cap L^2 \,|\, q_M(f) < \infty \text{ for all } M = 1, 2, \ldots \}, \\ & q_M(f) := p_M(f) + \sup_{j\le 0} 2^{M|j|} \|\Lambda_D^{-M} \phi_j(\Lambda_D) f\|_{L^1}. \end{aligned}$$

It can be checked that $\mathcal{X}, \mathcal{Z}$ are Frechét spaces, and we denote by $\mathcal{X}', \mathcal{Z}'$ their topological duals. We then define Besov spaces as follows.

Definition. For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, we define

$$\dot{B}^s_{p,q} := \Big\{ f \in \mathcal{Z}' \,\Big|\, \|f\|_{\dot{B}^s_{p,q}} := \Big\{ \sum_{j\in\mathbb{Z}} \Big(2^{sj} \|\phi_j(\Lambda_D)f\|_{L^p} \Big)^q \Big\}^{\frac{1}{q}} < \infty \Big\}.$$

The following is our result for the bilinear estimates.

Theorem 1.1. ([13, 14]) *Let* $1 \le p, p_j, q \le \infty \, (j = 1, 2, 3, 4)$ *satisfy the condition of the Hölder inequality.*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

(1) *Let* $0 < s < 2 + 1/p$. *Then*

$$\|fg\|_{\dot{B}^s_{p,q}} \le C\Big(\|f\|_{\dot{B}^s_{p_1,q}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_3}} \|g\|_{\dot{B}^s_{p_3,q}} \Big)$$

for all $f \in \dot{B}^s_{p_1,q} \cap L^{p_3}$ *and* $g \in L^{p_2} \cap \dot{B}^s_{p_4,q}$. *if* $s = 2 + 1/p$ *and* $1 \le q < \infty$, *or* $s > 2 + 1/p$, *then it does not hold.*

(2) *Let* $s > 0$. *Then*

$$\|(\nabla^\perp \Lambda_D^{-1} f) \cdot \nabla g\|_{\dot{B}^s_{p,q}} \le C\Big(\|f\|_{\dot{B}^s_{p_1,q}} \|g\|_{\dot{B}^1_{p_2,1}} + \|f\|_{\dot{B}^0_{p_3,1}} \|g\|_{\dot{B}^{s+1}_{p_4,q}}\Big)$$

for all $f \in \dot{B}^s_{p_1,q} \cap \dot{B}^0_{p_3,1}$ *and* $g \in \dot{B}^1_{p_2,1} \cap \dot{B}^{s+1}_{p_4,q}$.

We give some comments about the optimality of $s = 2 + 1/p$ in Theorem 1.1 (1). For the sake of the simplicity, let us discuss the case when $p = 2$. For every smooth f, g such that $(-\Delta_D)^m f, (-\Delta_D)^m g \in L^2$ for all $m = 0, 1, 2, \ldots$, we easily see that the product fg is also in the domain of the Dirichlet Laplacian, since the value of fg on the boundary is zero and

$$(-\Delta)(fg) = (-\Delta f)g - 2\nabla f \cdot \nabla g + f(-\Delta g),$$

and each term in the right hand side is justified in L^1_{loc} at least. If we consider derivatives of higher order, we need to consider whether or not $(-\Delta)(fg)$ again belongs to the domain of the Dirichlet Laplacian. On the boundary value of $(-\Delta)(fg)$, it is easy to see that $(-\Delta f)g$ and $f(-\Delta g)$ have the boundary value zero, however, $\nabla f \cdot \nabla g$ does not necessarily satisfy such condition on the boundary. Therefore, we would not be able to justfy $(-\Delta_D)^2(fg)$ in general. On the other hand, it is still possible to apply the fractional Laplacian of small order close to zero. When $0 < \alpha < 1/2 = 1/p$, the multiplication by the sign function with respect to x_2 is bounded operator in the Sovolev spaces on the entire space (Lemma 2.2), which allows us to approximate the function $(-\Delta)(fg)$ by some functions with the zero boundary value. We can then deduce that $s = 2 + \alpha < 2 + 1/p$ should be the threshold to assure the bilinear estimate. In contrast, no restriction appears for the regularity number in Theorem 1.1 (2), since the derivative ∂_{x_2}, othogonal to the boundary, changes the boundary condition. In fact, we explain the Dirichlet condition by the odd extention with respect to x_2 and the Neumann condition by the even extention with respect to x_2 in this paper, and the derivative by x_2 changes the two conditions each other, which allows us to obtain that for instance $f\partial_{x_2} g$ satisfies the Dirichlet boundary condition (see Lemma 2.3 for more detail). We also refer to [11] for the relation between boundary value and the derivative of the orthogonal direction to the boundary.

We next apply Theorem 1.1 (2) to the surface quasi-geostrophic equation on the two dimensional half space.

$$\begin{aligned} \partial_t \theta + (\nabla^\perp \Lambda_D \theta) \cdot \nabla \theta + \Lambda_D \theta = 0, \qquad & t > 0, x \in \mathbb{R}^2_+, \\ \theta|_{\partial \mathbb{R}^2_+} = 0, \qquad & \theta(0, x) = \theta_0(x). \end{aligned}$$

The equations are known as an important model in geophysical fluid dynamics, which is derived from general quasi-geostrophic equations in the special case of constant potential vorticity and buoyancy frequency (see [18, 19]).

If the domain is the entire space $\mathbb{R}^2$, there are plenty of literature which studies the existence of global solutions with the fractional Laplacian $(-\Delta)^{\alpha/2}$, $0 < \alpha \le 2$. The global regularity for any smooth data is known in the subcritical case, $\alpha > 1$, and let us focus on the critical case, $\alpha = 1$. The global regularity with small data was proved by Constantin, Cordoba and Wu [2] (see also Constantin and Wu [8]). The poroblem for large data case was solved by Caffarelli and Vasseur [1], Kiselev, Nazarov and Volberg [17]. As another approach, Constantin and Vicol [7] established the nonlinear maximum proncigple to prove the global regularity. On the other hand, in the super-critical case, the regularity only for small data is known (see [9]), and blow-up for smooth solutions is an open problem. On the bounded domains with the smooth boundary, local exitence of strong solutions and small data global solutions are known, and was shown by by Constantin and Nguyen [6]. Related to weak solutions, we refer to the papers by Constantin and Ignatova [3] and Constantin and Nguyen [5]. The global solvability for large data is an important problem, but it has not been settled, and let us refer several recent papers by Constantin and Ignatova [4], Stokols and Vasseur [20].

In this paper, the purpose of our application is to give a simple example with the boundary, and by the help of the odd extention, we can handle the boundary value of functions with the Dirichlet boundary condition appropriately to obtain the existence of global solutions with arbitrary smooth data.

The following is our result for the surface quasi geostrophic equation.

Theorem 1.2. ([14]) *Let $\theta_0 \in \dot{B}^0_{\infty,1}$. Then the integral equation*

$$\theta(t) = e^{-t\Lambda_D}\theta_0 - \int_0^t e^{-(t-\tau)\Lambda_D}\Big((u\cdot\nabla)\theta\Big)\,d\tau, \quad u = \nabla^\perp \Lambda_D^{-1}\theta$$

posseses a unique global solution θ such that

$$\theta \in C([0,\infty), \dot{B}^0_{\infty,1}) \cap L^1(0,\infty; \dot{B}^1_{\infty,1}).$$

Furthermore, $\theta = \theta(t,x)$ is continuous for $t \geq 0$ and x in the closure of $\mathbb{R}^2_+$ and θ is identically zero on the boundary.

Let us give few remarks to prove Theorem 1.2. The local solvability follows from an analogous argument to [22] throught the odd extention and the bilinear estimate in Theorem 1.1 (2). We there need maximal regularity estimate proved in [10]. To extend the local solution, we can apply the nonlinear maximum principle by [7] to guarantee the uniform boundedness of the Hölder space with the order α sufficiently smaller than $1/\|\theta_0\|_{L^\infty}$, which allows us to solve the equation in a certain length of the time interval any number of times. We refer to the paper [14] for the proof of Theorem 1.2. We give a comment that the half space case is settled naturally by the argument above and moreover the analyticity in spacetime is obtained in [12, 14].

In the next section, we give proof outline of Theorem 1.1. We refer to the paper [13] for the detail of the bilinear estimate of the standard product, fg, the paper [14] for $(\nabla^\perp\Lambda_D^{-1}f)\cdot\nabla g$ with the application to the critical surface quasi-geostrophic equation.

Notation. We denote by L^p the Lebesgue spaces, $\dot{H}^s_p$ the Sovolev spaces associated with the Dirichlet Laplacian, and $\dot{B}^s_{p,q}$ the Besov spaces associated with the Dirichlet Laplacian. When the domain is the entire space $\mathbb{R}^2$, we clarify the domain of the function spaces to write explicitly, $L^p(\mathbb{R}^2)$, $\dot{H}^s_p(\mathbb{R}^2)$, $\dot{B}^s_{p,q}(\mathbb{R}^2)$. We also write $-\Delta_D$ the Dirichlet Laplacian, Λ_D its square root on the half space, $-\Delta_{\mathbb{R}^2}$ the Laplacian and Λ_D its square root on the entire space as an operators on $\mathcal{S}'(\mathbb{R}^2)$.

2. Proof ourline of Theorem 1.1

We investigate the behavior of functions with the Dirichlet boundary condition throught the odd extention with respect to x_2.

$$f_{odd}(x_1,x_2) := \begin{cases} f(x_1,x_2) & \text{for } x_2 > 0, \\ -f(x_1,x_2) & \text{for } x_2 < 0. \end{cases}$$

We write the Laplacian on $\mathbb{R}^2$, $-\Delta_{\mathbb{R}^2}$, and its square root,

$$\Lambda_{\mathbb{R}^2} := \sqrt{-\Delta_{\mathbb{R}^2}}.$$

Let us focus on the case when $2 \le s < 2 + 1/p$. We will argue as follows. To consider the norm of the product fg with the regularity number s, we write $\Lambda_D^s(fg)$ as

$$\Big(\text{odd extention of } \Lambda_D^s(fg)\Big) = \Lambda_{\mathbb{R}^2}^s(fg)_{odd} = \Lambda_{\mathbb{R}^2}^{s-2}(-\Delta_{\mathbb{R}^2})\Big(\operatorname{sign} x_2 \cdot f_{odd} g_{odd}\Big).$$

If f, g satisfy the Dirichlet boundary condition, then we can suppose that

$$f_{odd}g_{odd}, \nabla f_{odd}g_{odd} = 0 \text{ on } \partial\mathbb{R}^2_+,$$

which implies that

$$\Lambda_{\mathbb{R}^2}^{s-2}(-\Delta_{\mathbb{R}^2})\Big(\operatorname{sign} x_2 \cdot f_{odd}g_{odd}\Big) = \Lambda_{\mathbb{R}^2}^{s-2}\Big(\operatorname{sign} x_2(-\Delta_{\mathbb{R}^2})(f_{odd}g_{odd})\Big).$$

Here it will be proved in Lemma 2.2 below that the multiplication by $\operatorname{sign} x_2$ is a bounded operator in $\dot{H}_p^{s-2}(\mathbb{R}^2)$, where the norm is defined by

$$\|f\|_{\dot{H}_p^{s-2}(\mathbb{R}^d)} := \|\Lambda_{\mathbb{R}^2}^{s-2} f\|_{L^p(\mathbb{R}^2)}.$$

We can then apply the standart bilinear estimate to obtain the first inequality. In what follows, we introduce two lemmas on the relation of the Laplacian between the entire space and the half space with the Dirichlet condition, and finally we explain our idea of the proof of Theorem 1.1.

Lemma 2.1. *Let $s \ge 0$ and $1 \le p \le \infty$. Then $\Lambda_D^s f \in L^p(\mathbb{R}^2_+)$ if and only if $(-\Delta_{\mathbb{R}^2})^{s/2} f \in L^p(\mathbb{R}^2)$. We also have that*

$$2^{\frac{1}{p}}\|\Lambda_D^s f\|_{L^p(\mathbb{R}^2_+)} = \|(-\Delta_{\mathbb{R}^2})^{s/2} f\|_{L^p(\mathbb{R}^2)}$$

provided that $\Lambda_D^s f \in L^p(\mathbb{R}^2_+)$.

Proof outline. We write the kernle of the semigroup generated by the fractional Laplacian on $\mathbb{R}^2$,

$$P_s(t) = P_s(t,x) = \mathcal{F}^{-1}[e^{-t|\xi|^s}](x), \quad t > 0, x \in \mathbb{R}^2.$$

We write

$$\begin{aligned} e^{-t\Lambda_D^s} f(x) &= \int_{\mathbb{R}^2_+} \Big(P_s(t, x-y) - P_s(t, x_1 - y_1, x_2 + y_2)\Big) f(y)dy \\ &= \int_{\mathbb{R}^2} P_s(t, x-y) f_{odd}(y)\, dy \\ &= P_s(t) * f_{odd}(x), \quad x \in \mathbb{R}^2_+, \end{aligned}$$

and

$$\Lambda_D^s f = \lim_{t\to 0} \frac{e^{-t\Lambda_D^s} f - f}{t} = \lim_{t \to 0} \frac{P_s(t) * f_{odd}|_{\mathbb{R}^2_+} - f}{t} = (-\Delta_{\mathbb{R}^2})^{s/2} f_{odd}|_{\mathbb{R}^2_+},$$

if the limit exists in $L^p(\mathbb{R}^2_+)$. This allows us to have the equivalency of $\Lambda_D^s f \in L^p(\mathbb{R}^2_+)$ and $(-\Delta_{\mathbb{R}^2})^{s/2} f \in L^p(\mathbb{R}^2)$. The norm relation follows from

$$\|\Lambda_D^s f\|^p_{L^p(\mathbb{R}^2_+)} = \frac{1}{2}\|(\Lambda_D^s f)_{odd}\|^p_{L^p(\mathbb{R}^2)} = \frac{1}{2}\|\Lambda^s_{\mathbb{R}^2} f_{odd}\|^p_{L^p(\mathbb{R}^2)}.$$

□

Lemma 2.2. *Let $0 < s < 1/p$ and $1 < p < \infty$. Then on the entire space*

$$\|(\operatorname{sign} x_2) f\|_{\dot H^s_p(\mathbb{R}^2)} \le C\|f\|_{\dot H^s_p(\mathbb{R}^2)}$$

for all $f \in \dot H^s_p(\mathbb{R}^2)$.

Proof outline. We introduce φ_ε an approximation of the sign function with respect to x_2 defined by for an odd function $\varphi \in C_0^\infty(\mathbb{R}^)$ with $\varphi(x_2) = 1\,(x_2 \ge 1)$,

$$\varphi_\varepsilon(x_2) := \varphi(\varepsilon^{-1}x_2), \quad x_2 \in \mathbb{R}, \varepsilon > 0.$$

We start by proving that

$$\|\varphi_\varepsilon f\|_{\dot H^s_p(\mathbb{R}^2)} \le C\|f\|_{\dot H^s_p(\mathbb{R}^2)} \text{ for all } \varepsilon > 0.$$

By the decomposition of the frequency of φ_ε and f,

$$\varphi_\varepsilon f = \Big(\sum_{k \le l+3} + \sum_{k > l+3}\Big)\Big(\phi_k(\Lambda_{\mathbb{R}^2})\varphi_\varepsilon\Big)\Big(\phi_l(\Lambda_{\mathbb{R}^2}) f\Big) =: (\varphi_\varepsilon f)_I + (\varphi_\varepsilon f)_{II}.$$

The first term is handled by the standard bilinear estimate,

$$\|(\varphi_\varepsilon f)_I\|_{\dot H^s_p(\mathbb{R}^2)} \le C\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^2)}\|f\|_{\dot H^s_p(\mathbb{R}^2)} \le C\|f\|_{\dot H^s_p(\mathbb{R}^2)},$$

since the frequency of f is higher than that of ε. On the other hand, we apply the bilinear estimate for x_2 variable for the second term. Let $1/p = 1/p_1 + 1/p_2$, $s = 1/p$, from which we have $s = 1/p - 1/p_2$. It follows from the bilinear estimate and the Sobolev embedding that

$$\begin{aligned}
\|(\varphi_\varepsilon f)_{II}\|_{\dot H^s_p(\mathbb{R}^2)} \le& C\|\varphi_\varepsilon\|_{\dot H^s_{p_1}(\mathbb{R})}\Big\|\|f\|_{L^{p_2}(\mathbb{R}_{x_2})}\Big\|_{L^p(\mathbb{R}_{x_1})}\\
\le& C\|\varphi_\varepsilon\|_{\dot H^{\frac{1}{p_1}}_{p_1}(\mathbb{R})}\Big\|\|f\|_{\dot H^s_p(\mathbb{R}_{x_2})}\Big\|_{L^p(\mathbb{R}_{x_1})}\\
\le& C\|\varphi_1\|_{\dot H^{\frac{1}{p_1}}_{p_1}(\mathbb{R})}\|\mathcal{F}^{-1}|\xi_2|^s\mathcal{F}f\|_{\dot H^s_p(\mathbb{R}^2)}.
\end{aligned}$$

By applying the Fourier multiplier theorem to a multiplier $|\xi_2|^s/|\xi|^s$, we have

$$\|\mathcal{F}^{-1}|\xi_2|^s\mathcal{F}f\|_{\dot H^s_p(\mathbb{R}^2)} \le C\|\mathcal{F}^{-1}|\xi|^s\mathcal{F}f\|_{\dot H^s_p(\mathbb{R}^2)} \le C\|f\|_{\dot H^s_p(\mathbb{R}^2)},$$

which proves the inequality.

By considering the limit as $\varepsilon \to 0$ with taking a subsequence if necessary, we conclude that

$$\|(\operatorname{sign} x_2) f\|_{\dot H^s_p(\mathbb{R}^2)} \le \liminf_{\varepsilon\to 0}\|\varphi_\varepsilon f\|_{\dot H^s_p(\mathbb{R}^2)} \le C\|f\|_{\dot H^s_p(\mathbb{R}^2)}.$$

□

Proof of Theorem 1.1 (1) when $2 \le s < 2 + 1/p$, $1 < p, p_j < \infty$. Lemmas 2.1 ,2.2 imply that

$$\begin{aligned} 2^{\frac{1}{p}} \|\Lambda_D^s (fg)\|_{L^p(\mathbb{R}^2_+)} =& \Big\| \Lambda_{\mathbb{R}^2}^{s-2} \Big(\operatorname{sign} x_2 (-\Delta_D)(f_{odd} g_{odd}) \Big) \Big\|_{L^p(\mathbb{R}^2)} \\ \le& C \|\Lambda_{\mathbb{R}^2}^s (f_{odd} g_{odd})\|_{L^p(\mathbb{R}^2)}. \end{aligned}$$

It follows from the bilinear estimate on the entire space $\mathbb{R}^2$ that

$$\|\Lambda_D^s (fg)\|_{L^p(\mathbb{R}^2_+)} \le C \Big(\|\Lambda_{\mathbb{R}^2}^s f_{odd}\|_{L^{p_1}(\mathbb{R}^2)} \|g_{odd}\|_{L^{p_2}(\mathbb{R}^2)} + \|f_{odd}\|_{L^{p_3}(\mathbb{R}^2)} \|\Lambda_{\mathbb{R}^2}^s g_{odd}\|_{L^{p_4}(\mathbb{R}^2)} \Big),$$

and by Lemma 2.1

$$\|\Lambda_D^s (fg)\|_{L^p(\mathbb{R}^2_+)} \le C \Big(\|\Lambda_D^s f\|_{L^{p_1}(\mathbb{R}^2_+)} \|g\|_{L^{p_2}(\mathbb{R}^2)} + \|f\|_{L^{p_3}(\mathbb{R}^2)} \|\Lambda_D^s g\|_{L^{p_4}(\mathbb{R}^2_+)} \Big).$$

For the proof of the inequality in Besov spaces, we apply the Bony paraproduct formula and the above inequality in the Sobolev spaces to obtain the bilinear estimates in Besov spaces. □

Optimality of $s = 2 + 1/p$ in Theorem 1.1 (1). We can see that the optimality is independent of dimensions and let us focus on the case when the space dimension is one. The reason is due to the boundary value of the function, and the crucial point is the boundary value of the function with the x_2 direction orthogonal to the boundary.

Let us consider the half line $\mathbb{R}_+$ and we construct f, g such that

$$f, g, \Lambda_D^s f, \Lambda_D^s g \in L^p(\mathbb{R}_+) \text{ for all } s > 0, \text{ but } \Lambda_D^{s+1}(fg) \notin L^p(\mathbb{R}_+).$$

Let φ be such that

$$\varphi \in C_0^\infty([0,\infty)), \quad 0 \le \varphi \le 1, \quad \varphi(x) = \begin{cases} 1 & \text{for } 0 \le x \le \dfrac{1}{2}, \\ 0 & \text{for } x \ge 1, \end{cases}$$

and we define

$$f(x) = g(x) = x\varphi(x).$$

We notice that $f, g, \Lambda_D^s f, \Lambda_D^s g \in L^p(\mathbb{R}_+)$ for all $s > 0$,

$$\partial_x^2 (fg) = (\partial_x^2 f) g + 2 \partial_x f \cdot \partial_x g + f \partial_x^2 g,$$

and

$$(\partial_x^2 f) g, f \partial_x^2 g \in C_0^\infty((0,\infty)).$$

On the other hand,

$$\partial_x f \cdot \partial_x g = \varphi^2 + 2x\varphi\varphi' + x^2 (\varphi')^2, \quad 2x\varphi\varphi' + x^2(\varphi')^2 \in C_0^\infty((0,\infty)),$$

and we need to investigate φ^2, and will prove that $\Lambda^{1/p}(\varphi^2) \notin L^p(\mathbb{R}_+)$. We write

$$\Lambda_D^{\frac{1}{p}} \varphi^2(x) = \Lambda_{\mathbb{R}}^{\frac{1}{p}} \varphi_{odd}^2(x) = C \int_{\mathbb{R}} \frac{\varphi_{odd}^2(x) - \varphi_{odd}^2(y)}{|x-y|^{1+\frac{1}{p}}} dy.$$

By a direct calculation, there exist $c, \delta > 0$ such that

$$\begin{aligned} \Lambda_{\mathbb{R}}^{\frac{1}{p}} \varphi_{odd}^2(x) \ge& \frac{c}{|x|^{\frac{1}{p}}} \text{ for } 0 < x < \delta, \\ \Lambda_{\mathbb{R}}^{\frac{1}{p}} \varphi_{odd}^2(x) \le& -\frac{c}{|x|^{\frac{1}{p}}} \text{ for } -\delta < x < 0, \end{aligned}$$

which imply that $\Lambda_{\mathbb{R}}^{\frac{1}{p}}\varphi_{odd}^2 \in L^p(\mathbb{R})$ and $\Lambda_D^{\frac{1}{p}}\varphi^2 \in L^p(\mathbb{R}_+)$. We then conclude that

$$\Lambda_D^{\frac{1}{p}}(\partial_x f \partial_x g) \notin L^p(\mathbb{R}_+),$$

and therefore, $\Lambda_D^{2+\frac{1}{p}}(fg) \notin L^p(\mathbb{R}_+)$. □

To prove Theorem 1.1 (2), we need the following lemma.

Lemma 2.3. ([14]) *Let $f \in \dot{B}^0_{\infty,1} \cap \dot{B}^1_{\infty,1}$. Then $f, \nabla f$ are regarded as continuous functions and we have the following relation between the odd extention and the even extention.*

$$(\partial_{x_1} f)_{odd} = \partial_{x_1} f_{odd}, \quad (\partial_{x_2} f)_{even} = \partial_{x_2} f_{odd},$$

where

$$f_{even}(x_1, x_2) := \begin{cases} f(x_1, x_2) & \text{for } x_2 > 0, \\ f(x_1, -x_2) & \text{for } x_2 < 0. \end{cases}$$

Proof of Theorem 1.1 (2). We have

$$\Big(\nabla^\perp \Lambda_D^{-1} f \cdot \nabla\Big) g = -(\partial_{x_2}\Lambda_D^{-1} f)\partial_{x_1} g + (\partial_{x_2}\Lambda_D^{-1} f)\partial_{x_2} g.$$

By Lemma 2.3, we write the first term,

$$\begin{aligned} \Big((\partial_{x_2}\Lambda_D^{-1} f)\partial_{x_1} g\Big)_{odd} =& (\partial_{x_2}\Lambda_D^{-1} f)_{even}(\partial_{x_1} g)_{odd} \\ =& (\partial_{x_2}(\Lambda_D^{-1} f)_{odd})(\partial_{x_1} g_{odd}) \\ =& (\partial_{x_2}\Lambda_{\mathbb{R}^2}^{-1} f_{odd})(\partial_{x_1} g_{odd}) \end{aligned}$$

and the second term

$$\begin{aligned} \Big((\partial_{x_1}\Lambda_D^{-1} f)\partial_{x_2} g\Big)_{odd} =& (\partial_{x_1}\Lambda_D^{-1} f)_{odd}(\partial_{x_2} g)_{even} \\ =& (\partial_{x_1}(\Lambda_D^{-1} f)_{odd})(\partial_{x_2} g_{odd}) \\ =& (\partial_{x_1}\Lambda_{\mathbb{R}^2}^{-1} f_{odd})(\partial_{x_2} g_{odd}). \end{aligned}$$

We here notice that there does not appear the sign function with respect to x_2, the bilinear estimate in the entire space is possible to be applied, and we then deduce from the bilinear estimate that

$$\begin{aligned} \Big\|(\nabla^\perp \Lambda_D^{-1} f)\cdot \nabla g\Big\|_{\dot{B}^s_{p,q}} \le& C\Big\|(\nabla^\perp \Lambda_{\mathbb{R}^2}^{-1} f_{odd})\cdot \nabla g_{odd}\Big\|_{\dot{B}^s_{p,q}(\mathbb{R}^2)} \\ \le& C\Big(\|f_{odd}\|_{\dot{B}^s_{p,q}(\mathbb{R}^2)}\|g_{odd}\|_{\dot{B}^1_{p_2,1}(\mathbb{R}^2)} + \|f_{odd}\|_{\dot{B}^0_{p_3,1}(\mathbb{R}^2)}\|g_{odd}\|_{\dot{B}^{s+1}_{p_4,q}} \\ \le& C\Big(\|f\|_{\dot{B}^s_{p,q}}\|g\|_{\dot{B}^1_{p_2,1}} + \|f\|_{\dot{B}^0_{p_3,1}}\|g\|_{\dot{B}^{s+1}_{p_4,q}}\Big), \end{aligned}$$

where we have applied the relation similarly to Lemma 2.1, between the Besov spaces associated with the Dirichlet Laplacian and the Besov spaces on the entire space through the odd extention. □

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