Asymptotic properties of steady 2D Navier-Stokes flows and of 3D axisymmetric Navier-Stokes flows with no swirl

Hideo Kozono, Waseda University/Tohoku University

Yutaka Terasawa, Nagoya University

Yuta Wakasugi, Hiroshima University

1 Introduction

In this survey article, we consider the stationary 2D Navier-Stokes flows with finite generalized Dirichlet integrals and investigate their asymptotic properties. Moreover, we also consider the stationary 3D axisymmetric Navier-Stokes flows with no swirl with finite generalized Dirichlet integrals and with some poinwise growth condition for the velocities. As those applications, we also get Liouville-type theorems, which assure the triviality of Navier-Stokes flows under a certain condition for the vorticities. This article is a summary of main results in [10] and [11].

In the rest of this introduction, we explain asymptotic properties for 2D Navier-Stokes flows with finite generalized Dirichlet integrals and its application to some Liouville-type theorem. In the second section, we explain the proof of these results. In the third section, we explain asymptotic properties for 3D axisymmetric Navier-Stokes flows with no swirl and its application to some Liouville-type theorem without any proof. Since the proof for 3D axisymmetric flows with no swirl is similar to that for 2D flows except some technical details, we omit it.

We first consider the 2D stationary Navier-Stokes equations

$$\begin{cases} -\Delta v + (v \cdot \nabla) v + \nabla p = 0 \\ \operatorname{div} v = 0 \end{cases} \tag{1.1}$$

in $\mathbb{R}^2$ or an exterior domain $B_{r_0}^c \equiv \{x = (x_1, x_2) \in \mathbb{R}^2; r = |x| > r_0\}$ with some constant $r_0 > 0$. Here, $v = v(x) = (v_1(x), v_2(x))$ and $p = p(x)$ are the velocity vector and the scalar pressure, respectively. In the pioneer paper by Leray [12], he constructed a solution v of (1.1) with

$$\int_{r>r_0} |\nabla v(x)|^2 dx < \infty \tag{1.2}$$

satisfying the nonhomogeneous boundary condition with zero-flux on $r = r_0$. It had long been an open question whether, under the condition (1.2), v behaves like

$$v(x) \to v_\infty \quad \text{as } |x| \to \infty \tag{1.3}$$

with some constant vector $v_\infty \in \mathbb{R}^2$. For this problem, Gilbarg–Weinberger [6] proved that if the solution v in the class (1.2) satisfies

$$v \in L^\infty(B_{r_0}^c) \tag{1.4}$$

then there is constant vector $v_\infty \in \mathbb{R}^2$ such that

$$\lim_{r\to\infty} \int_0^{2\pi} |v(r,\theta) - v_\infty| d\theta = 0 \tag{1.5}$$

with

$$\omega(r,\theta) = o(r^{-3/4}), \tag{1.6}$$

$$\nabla v(r,\theta) = o(r^{-3/4}(\log r)) \tag{1.7}$$

uniformly in $\theta \in (0, 2\pi)$ as $r \to \infty$, where ω is the vorticity $\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1$. Later on Amick [1] and Korobkov–Pileckas–Russo [8] proved that every solution v of (1.1) with the finite Dirichlet integral (1.2) is bounded as in (1.4), and so necessarily satisfies (1.5) and (1.6). Recently, Korobkov–Pileckas–Russo [9] succeeded to obtain a remarkable result which states that every solution v of (1.1) with (1.2) converges uniformly at infinity, i.e.,

$$\lim_{r\to\infty} \sup_{\theta\in(0,2\pi)} |v(r,\theta) - v_\infty| = 0, \tag{1.8}$$

where $v_\infty \in \mathbb{R}^2$ is the constant vector as in (1.5). On the other hand, for the small prescribed constant vector $v_\infty \in \mathbb{R}^2$, the existence of solutions v of (1.1) with (1.2) having the parabolic wake region has been fully investigated by Finn-Smith [5].

In this article, we consider a more generalized class

$$\int_{r>r_0} |\nabla v(r,\theta)|^q \, dx < \infty \quad \text{with some } 2 < q < \infty. \tag{1.9}$$

We call this a generalized Dirichlet integral, and call a solution in this class a solution with a finite generalized Dirichlet integral. Since our domain is unbounded and since we are interested in the asymptotic behavior of solutions v of (1.1), the larger q, the weaker the assumption on the decay of ∇v at the spatial infinity.

Our first result is on the following pointwise decay estimates for the vorticity $\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1$ and ∇v.

Theorem 1.1. *Let* v *be a smooth solution of* (1.1) *in* $B_{r_0}^c = \{x \in \mathbb{R}^2; r > r_0\}$ *satisfying* (1.9) *with some* $q \in (2, \infty)$*. Then, we have*

$$|\omega(r,\theta)| = o(r^{-(1/q+1/q^2)}) \quad (r \to \infty), \tag{1.10}$$

$$|\nabla v(r,\theta)| = o(r^{-(\frac{1}{q}+\frac{1}{q^2})} \log r) \quad (r \to \infty) \tag{1.11}$$

uniformly in $\theta \in [0, 2\pi]$.

Remark 1.1. *From the assumption* (1.9) *with a density argument, we can see that the velocity* v *itself satisfies*

$$|v(r,\theta)| = o(r^{1-2/q}) \quad (r \to \infty) \tag{1.12}$$

(see Lemma 2.1).

Remark 1.2. *The assertion of the above theorem is also true when* $q = 2$*. This was obtained by Gilbarg–Weinberger [6, Theorem 6] and Korobkov–Plieckas–Russo [8], while they proved estimates for* ω *and* ∇v*:*

$$|\omega(r,\theta)| = o(r^{-3/4}), \quad |\nabla v(r,\theta)| = o(r^{-3/4} \log r) \quad (r \to \infty).$$

Theorem 1.1 and the maximum principle immediately give the following Liouville type theorem.

Corollary 1.2 (Liouville type theorem)**.** *Let* v *be a smooth solution of* (1.1) *in* $\mathbb{R}^2$ *satisfying*

$$\nabla v \in L^q(\mathbb{R}^2) \quad \big(\text{or } \omega \in L^q(\mathbb{R}^2)\big)$$

with some $q \in (2, \infty)$*. Then,* v *must be a constant vector.*

2 Proof of Theorem 1.1

2.1 Proof of the estimate of the vorticity and Corollary 1.2

We first prove (1.10). Hereafter, we denote by C positive constants depending only on the quantities appearing in parentheses.

The proof of Theorem 1.1 is similar to that of [6, Theorem 5]. Our idea is to control the quantity

$$\int |\nabla \omega|^2 |\omega|^{q-2} \, dx.$$

A similar quantity is also used in [2].

We first show the asymptotic behavior for a vector field satisfying (1.9).

Lemma 2.1. *If a vector field* $v = v(x) = (v_1, v_2) \in L^q_{\mathrm{loc}}(B^c_{r_0})$ *satisfies* (1.9) *with some* $2 < q < \infty$, *then*

$$\lim_{r\to\infty} r^{-(1-2/q)} \sup_{\theta\in(0,2\pi)} |v(r,\theta)| = 0. \tag{2.1}$$

Proof. We first take a cut-off function $\chi \in C^\infty(\mathbb{R}^2)$ so that $\chi(x) = 0$ $(|x| \le r_0)$ and $\chi(x) = 1$ $(|x| \ge r_0 + 1)$. Then, we have $\nabla(\chi v) \in L^q(\mathbb{R}^2)$. Therefore, by [7, Lemma 2.1], for any $\varepsilon > 0$, there exists $v_\varepsilon \in C_0^\infty(\mathbb{R}^2)$ satisfying $\|\nabla(\chi v) - \nabla v_\varepsilon\|_{L^q(\mathbb{R}^2)} < \varepsilon$. Let $w_\varepsilon = \chi v - v_\varepsilon$. Then, by the Sobolev embedding, we estimate

$$\begin{aligned}
\limsup_{r\to\infty} r^{-(1-2/q)} \sup_{\theta\in(0,2\pi)} |v(r,\theta)| &\le \limsup_{r\to\infty} r^{-(1-2/q)} \sup_{\theta\in(0,2\pi)} |w_\varepsilon(r,\theta) - w_\varepsilon(0)| \\
&\quad + \limsup_{r\to\infty} r^{-(1-2/q)} |w_\varepsilon(0)| \\
&\quad + \limsup_{r\to\infty} r^{-(1-2/q)} \sup_{\theta\in(0,2\pi)} |v_\varepsilon(r,\theta)| \\
&\le C\|\nabla w_\varepsilon\|_{L^q(\mathbb{R}^2)} \\
&\le C\varepsilon,
\end{aligned}$$

where the constant $C > 0$ is independent of ε. Since ε is arbitrary, we conclude

$$\limsup_{r\to\infty} r^{-(1-2/q)} \sup_{\theta\in(0,2\pi)} |v(r,\theta)| = 0.$$

□

Lemma 2.2. *Let* $r_0 > 0$ *and assume that* $v = (v_1, v_2)$ *is a smooth solution of* (1.1) *in* $r > r_0$, *and there exists* $q \in (2, \infty)$ *such that*

$$\int_{r>r_0} |\nabla v|^q \, dx < \infty.$$

Then, the vorticity $\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1$ *satisfies*

$$\int_{r>r_1} r^{2/q} |\nabla \omega|^2 |\omega|^{q-2} \, dx < \infty$$

for any $r_1 > r_0$.

Proof. Let $R > r_1 > r_0$ and take $\xi_1, \xi_2 \in C^\infty((0,\infty))$ so that

$$\xi_1(r) = \begin{cases} 0 & (r \le \frac{r_0+r_1}{2}), \\ 1 & (r \ge r_1), \end{cases} \qquad \xi_2(r) = \begin{cases} 1 & (r \le 1), \\ 0 & (r \ge 2). \end{cases}$$

We define

$$\eta_R(r) = r^{2/q} \xi_1(r) \xi_2\left(\frac{r}{R}\right).$$

We easily see that

$$|\nabla \eta_R| \le C r^{-1+2/q}, \quad |\Delta \eta_R| \le C$$

hold with some constant $C > 0$.

Let $h = h(\omega)$ be a C^1 and piecewise C^2 function, which is determined later. Then, using the condition $\operatorname{div} v = 0$, we have the following identity [6, p. 385]

$$\operatorname{div}\left[\eta_R \nabla h - h \nabla \eta_R - \eta_R h v\right] = \eta_R h'' |\nabla \omega|^2 - h\left[\Delta \eta_R + v \cdot \nabla \eta_R\right] + \eta_R h' \left[\Delta \omega - v \cdot \nabla \omega\right].$$

Since ω satisfies the vorticity equation $\Delta\omega - v \cdot \nabla \omega = 0$ and $\eta_R = 0$ near $r = r_0$ and $r = \infty$, integrating the above identity on $\{r > r_0\}$, we have

$$\int_{r>r_0} \eta_R h''(\omega) |\nabla \omega|^2 \, dx = \int_{r>r_0} h(\omega)(\Delta \eta_R + v \cdot \nabla \eta_R) \, dx. \tag{2.2}$$

Taking $h(\omega) = |\omega|^q$, we obtain

$$\int_{r_1<r<R} r^{2/q} |\nabla \omega|^2 |\omega|^{q-2} \, dx \le C \int_{r>r_0} |\omega|^q |\Delta \eta_R + v \cdot \nabla \eta_R| \, dx.$$

From Lemma 2.1, we see $r^{-1+2/q}|v(r,\theta)| \le C$ and hence,

$$\int_{r_1<r<R} r^{2/q} |\nabla \omega|^2 |\omega|^{q-2} \, dx \le C \int_{r>r_0} |\omega|^q \, dx.$$

The right-hand side is finite and so is the left-hand side. Letting $R \to \infty$, we complete the proof. □

From the above lemma, we obtain the following decay estimate of ω, which is the assertion of Theorem 1.1.

Lemma 2.3. *Under the assumption of Lemma 2.2, we have*

$$\lim_{r\to\infty} r^{\frac{1}{q}+\frac{1}{q^2}} \sup_{\theta \in (0,2\pi)} |\omega(r,\theta)| = 0.$$

Proof. We first note that when $2^n > r_0$, the inequality

$$\begin{aligned}
&\int_{2^n}^{2^{n+1}} \frac{dr}{r} \int_0^{2\pi} |\omega|^{q-2} \left(r^2 |\omega|^2 + r^{1+1/q} |\omega| |\omega_\theta|\right) \, d\theta \\
&\le C \int_{2^n<r<2^{n+1}} |\omega|^{q-2} \left(|\omega|^2 + r^{1/q} |\omega| |\nabla \omega|\right) \, dx \\
&\le C \int_{r>2^n} |\omega|^{q-2} \left(|\omega|^2 + r^{2/q} |\nabla \omega|^2\right) \, dx
\end{aligned}$$

holds. From the mean value theorem for the integration, there exists a sequence $r_n \in (2^n, 2^{n+1})$ such that

$$\int_{2^n}^{2^{n+1}} \frac{dr}{r} \int_0^{2\pi} |\omega|^{q-2} \left(r^2|\omega|^2 + r^{1+1/q}|\omega||\omega_\theta| \right) d\theta$$
$$= \log 2 \int_0^{2\pi} |\omega(r_n,\theta)|^{q-2} \left(r_n^2|\omega(r_n,\theta)|^2 + r_n^{1+1/q}|\omega(r_n,\theta)||\omega_\theta(r_n,\theta)| \right) d\theta.$$

Therefore, we have

$$\begin{aligned}&\int_0^{2\pi} |\omega(r_n,\theta)|^{q-2} \left(r_n^2|\omega(r_n,\theta)|^2 + r_n^{1+1/q}|\omega(r_n,\theta)||\omega_\theta(r_n,\theta)| \right) d\theta \\ &\leq C \int_{r>2^n} |\omega|^{q-2} \left(|\omega|^2 + r^{2/q}|\nabla\omega|^2 \right) dx.\end{aligned} \tag{2.3}$$

On the other hand, integrating the identity

$$|\omega(r_n,\theta)|^q - |\omega(r_n,\varphi)|^q = \int_\varphi^\theta \frac{\partial}{\partial\theta'}|\omega(r_n,\theta')|^q \, d\theta'$$

with respect to $\varphi \in [0, 2\pi]$, we have

$$\begin{aligned}2\pi|\omega(r_n,\theta)|^q - \int_0^{2\pi} |\omega(r_n,\varphi)|^q \, d\varphi &\leq 2\pi \int_0^{2\pi} \left| \frac{\partial}{\partial\theta'}|\omega(r_n,\theta')|^q \right| d\theta' \\ &\leq C \int_0^{2\pi} |\omega(r_n,\theta')|^{q-1}|\omega_\theta(r_n,\theta')| \, d\theta'.\end{aligned}$$

Multiplying it by r_n and using (2.3) with $r_n^{1+\frac{1}{q}} \leq r_n^2$ which is implied by $q > 2$, we see that

$$\begin{aligned}r_n^{1+1/q}|\omega(r_n,\theta)|^q &\leq Cr_n^{1+1/q} \int_0^{2\pi} |\omega(r_n,\varphi)|^q \, d\varphi \\ &\quad + Cr_n^{1+1/q} \int_0^{2\pi} |\omega(r_n,\theta')|^{q-1}|\omega_\theta(r_n,\theta')| \, d\theta' \\ &\leq C \int_{r>2^n} |\omega|^{q-2} \left(|\omega|^2 + r^{2/q}|\nabla\omega|^2 \right) dx.\end{aligned}$$

By Lemma 2.3, the right-hand side tends to 0 as $n \to \infty$. Hence, we have

$$\lim_{n\to\infty} r_n^{1+1/q} \sup_{\theta\in(0,2\pi)} |\omega(r_n,\theta)|^q = 0. \tag{2.4}$$

By noting $r_{n+1} < 4r_n$ and the maximum principle, we have for $r \in (r_n, r_{n+1})$ that

$$\begin{aligned}r^{1+1/q} \sup_{\theta\in(0,2\pi)} |\omega(r,\theta)|^q &\leq r_{n+1}^{1+1/q} \max\{ \sup_{\theta\in(0,2\pi)} |\omega(r_n,\theta)|^q, \sup_{\theta\in(0,2\pi)} |\omega(r_{n+1},\theta)|^q \} \\ &\leq \max\{ 4r_n^{1+1/q} \sup_{\theta\in(0,2\pi)} |\omega(r_n,\theta)|^q, r_{n+1}^{1+1/q} \sup_{\theta\in(0,2\pi)} |\omega(r_{n+1},\theta)|^q \}.\end{aligned}$$

Combining this with (2.4) yields

$$\lim_{r\to\infty} r^{1+1/q} \sup_{\theta\in(0,2\pi)} |\omega(r,\theta)|^q = 0,$$

which completes the proof. Since (1.11) is obtained from (1.10) using the Cauchy-Pompeiu formula analogously with [6], we omit it. We refer to [10] for details of its proof. □

Next, we give the proof of Corollary 1.2.

Proof of Corollary 1.2. We first note that, by the Calderón–Zygmund inequality, $\omega \in L^q(\mathbb{R}^2)$ implies $\nabla v \in L^q(\mathbb{R}^2)$, and hence, we may assume $\nabla v \in L^q(\mathbb{R}^2)$. Then, by Theorem 1.1, we have $\omega \to 0$ as $|x| \to \infty$. Since ω satisfies the maximum principle, we have $\omega \equiv 0$. □

3 Results on 3D axisymmetric flows with no swirl

We next state results on 3D Navier-Stokes axisymmetric flows with no swirl. We consider the following 3D stationary Navier-Stokes equations.

$$\begin{cases} (v\cdot\nabla)v + \nabla p = \Delta v, \\ \nabla\cdot v = 0, \end{cases} \qquad x \in D, \tag{3.1}$$

where D is outside of a cylinder in $\mathbb{R}^3$ specified later, and $v = v(x) = (v_1(x), v_2(x), v_3(x))$ and $p = p(x)$ denote the velocity vector field and the scalar pressure at the point $x = (x_1, x_2, z) \in D$, respectively.

To state our main results, we prepare the following notations. We consider the cylindrical domain $D = \{(r,\theta,z) \in \mathbb{R}_+ \times [0,2\pi) \times \mathbb{R}; r > r_0\}$ with some constant $r_0 > 0$, and let $D_0 = \{(r,z) \in \mathbb{R}_+ \times \mathbb{R}; r > r_0\}$. For the axisymmetric velocity with no swirl, it holds that v^r and v^z are independent of θ and $v^\theta \equiv 0$. From this, we rewrite the equation (3.1) as

$$\begin{cases} (v^r\partial_r + v^z\partial_z)v^r + \partial_r p = \left(\partial_r^2 + \dfrac{1}{r}\partial_r + \partial_z^2 - \dfrac{1}{r^2}\right) v^r, \\ (v^r\partial_r + v^z\partial_z)v^z + \partial_z p = \left(\partial_r^2 + \dfrac{1}{r}\partial_r + \partial_z^2 - \dfrac{1}{r^2}\right) v^z, \\ \partial_r v^r + \dfrac{v^r}{r} + \partial_z v^z = 0, \end{cases} \qquad (r,z) \in D_0. \tag{3.2}$$

Let $\omega^\theta = \partial_z v^r - \partial_r v^z$. Then, ω^θ satisfies the vorticity equation

$$(v^r\partial_r + v^z\partial_z)\omega^\theta - \frac{v^r}{r}\omega^\theta = \left(\partial_r^2 + \frac{1}{r} + \partial_z^2 - \frac{1}{r^2}\right)\omega^\theta. \tag{3.3}$$

Moreover, $\Omega = \omega^\theta/r$ satisfies

$$-\left(\partial_r^2 + \partial_z^2 + \frac{3}{r}\partial_r\right)\Omega + (v^r\partial_r + v^z\partial_z)\,\Omega = 0. \tag{3.4}$$

The equation (3.4) has similar properties as the vorticity equation of the 2-dimensional Naver-Stokes equations. In particular, the solution to (3.4) satisfies the maximum principle.

We suppose that

$$\int_D |\nabla v(x)|^q \, dx < +\infty \tag{3.5}$$

for some $q \in (2, \infty)$. We also suppose that there exist $k \in \mathbb{R}$ and $C > 0$ such that

$$|v(r,z)| \le C(1+r)^k \tag{3.6}$$

holds for all $(r, z) \in D_0$.

Under the assumptions (3.5) and (3.6), we have the following asymptotic behavior of the vorticity Ω and ω^θ.

Theorem 3.1. *Let* (v, p) *be a smooth axisymmetric solution of* (3.1) *with no swirl satisfying* (3.5) *for some* $q \in [2, \infty)$ *and* (3.6). *Then, we have*

$$\lim_{r\to\infty} r^{1+\frac{3}{q}-\frac{1}{2q}\max\{0,1+k\}} \sup_{z\in\mathbb{R}} |\Omega(r,z)| = 0, \tag{3.7}$$

$$\lim_{r\to\infty} r^{\frac{3}{q}-\frac{1}{2q}\max\{0,1+k\}} \sup_{z\in\mathbb{R}} |\omega^\theta(r,z)| = 0 \tag{3.8}$$

as $r \to \infty$.

The following are related results to Theorem 3.1. Choe and Jin [4], Weng [15], and Carrillo, Pan, and Zhang [3] showed that an axisymmetric solution of (3.1) with $D = \mathbb{R}^3$, $v_\infty = 0$, where $v_\infty = \lim_{|x|\to\infty} v(x)$ and with finite Dirichlet integral satisfies

$$v(x) = O\left(\frac{(\log r)^{1/2}}{r^{1/2}}\right), \quad |\omega^\theta| = O\left(\frac{(\log r)^{3/4}}{r^{5/4}}\right), \quad |\omega^r| + |\omega^z| = O\left(\frac{(\log r)^{11/8}}{r^{9/8}}\right) \tag{3.9}$$

uniformly in z as $r \to \infty$, where $\omega^r, \omega^\theta, \omega^z$ are the components of the vorticity ω defined by

$$\omega^r = -\partial_z v^\theta, \quad \omega^\theta = \partial_z v^r - \partial_r v^z, \quad \omega^z = \frac{1}{r}\partial_r(rv^\theta).$$

Recently, Li and Pan [13] studied a similar asymptotic behavior under the assumption of 3.5 for some $q \in (2, \infty)$. They first showed

$$\begin{cases} \exists v_\infty^z \in \mathbb{R},\ |v(r,z) - (0,0,v_\infty^z)| = O(r^{1-3/q}), & q \in (2,3), \\ \text{for any } r_0 > 0 \text{ and } r > r_0,\ |v(r,z) - v(r_0,z)| = O\left(\log \frac{r}{r_0}\right), & q = 3, \\ \text{for any } r_0 \ge 0 \text{ and } r > r_0,\ |v(r,z) - v(r_0,z)| = O(r^{1-3/q}), & q \in (3,\infty) \end{cases} \tag{3.10}$$

as $r \to \infty$ uniformly in $z \in \mathbb{R}$. Then, for the behavior of ω, they obtained for any $\varepsilon > 0$,

$$|\omega^\theta(r,z)| = O(r^{-(1/q+3/q^2)+\varepsilon}), \quad |\omega^r(r,z)| + |\omega^z(r,z)| = O(r^{-(1/q+1/q^2+3/q^3)+\varepsilon}), \quad (3.11)$$

provided that $q \in [3,\infty)$ and $\sup_{z\in\mathbb{R}} |u(r_0,z)| \le C$ for some $r_0 > 0$ hold, or, $q \in (2,3)$ and $v^z_\infty = 0$ hold. Besides them, they also showed for any $\varepsilon > 0$,

$$|\omega^\theta(r,z)| = O(r^{-2/q+\varepsilon}), \quad |\omega^r(r,z)| + |\omega^z(r,z)| = O(r^{-(1/q+2/q^2)+\varepsilon}), \quad (3.12)$$

provided that $q \in (2,3)$ and $v^z_\infty \ne 0$ hold.

Finally we state the following Liouville-type theorem, which might be obtained by modifying the method of proof of Theorem 3.1.

Theorem 3.2. *Let $D = \mathbb{R}^3$ and let (v,p) be a smooth axisymmetric solution of* (3.1) *with no swirl having the finite generalized Dirichlet integral as in (3.5) for some $q \in [2,\infty)$. Moreover, suppose that v satisfies*

$$v^r(r,z) \ge -C(1+r)^k, \quad (\operatorname{sign} z)v^z(r,z) \ge -C(1+r)^k \quad (3.13)$$

for any $(r,z) \in \mathbb{R}_+ \times \mathbb{R}$ with some $k \le q+1$ and C. Here, $\operatorname{sign} z = 1$ $(z > 0), 0$ $(z = 0), -1$ $(z < 0)$. *In addition, we assume that Ω is bounded. Then, we have $\omega^\theta \equiv 0$ and v is a constant vector.*

Pan and Li [14] recently showed some Liouville-type theorem for ancient axisymmetric solutions with no swirl. Their result implies some Liouville-type theorem for stationary axisymmetric solutions with no swirl. There is no implication between Theorem 3.2 and their result.

References

[1] Amick, J.C., *On Leray's problem of steady Navier-Stokes flow past a body in the plane.* Acta Math **161** (1988), 71–130.

[2] Bildhauer, M., Fuchs, M., Zhang, G., *Liouville-type theorems for steady flows of degenerate power law fluids in the plane.* J. Math. Fluid Mech. **15** (2013), 583–616.

[3] Carrillo, B., Pan, X., Zhang, Q.-S., *Decay and vanishing of some axially symmetric D-solutions of the Navier-Stokes equations*, J. Funct. Anal. (2020), https://doi.org/10.1016/j.jfa.2020.108504.

[4] Choe, H.-J., Jin, B.-J., *Asymptotic Properties of Axis-Symmetric D-Solutions of the Navier-Stokes Equations*, J. Math. Fluid Mech. **11** (2009), 208–232.

[5] Finn, R., Smith, D.R., *On the stationary solutions of the Navier-Stokes equations in two dimensions.* Arch. Rational Mech. Anal. **25** (1967), 26–39.

[6] Gilbarg, D., Weinberger, H. F., *Asymptotic properties of steady plane solutions of the Navier-Stokes equations with bounded Dirichlet integral.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5** (1978), 381–404.

[7] Kozono, H., Sohr, H., *On a new class of generalized solutions for the Stokes equation in exterior domains.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **19** (1992), 155–181.

[8] Korobkov, M.V., Pileckas, K., Russo, R., *On the steady Navier-Stokes equations in 2D exterior domains.* J. Differential Equations **269** (2020), 1796–1828.

[9] Korobkov, M.V., Pileckas, K., Russo, R., *On convergence of arbitrary D-solution of steady Navier-Stokes system in 2D exterior domains.* Arch. Ration. Mech. Anal. **233** (2019), 385–407.

[10] Kozono, H., Terasawa, Y., Wakasugi, Y., *Asymtotic properties of steady and nonsteady Navier-Stokes equations to the 2D Navier-Stokes equations with Finite Generalized Dirichlet Integral.* arXiv:1903.09969, to appear in Indiana University Mathematics Journal.

[11] Kozono, H., Terasawa, Y., Wakasugi, Y., *Asymptotic properties of steady solutions to the 3D axisymmetric Navier-Stokes Equations with no swirl.* J. Funct. Anal., 282, Paper No. 109289 (2022).

[12] Leray, J., *Étude de diverses équations intégrals non linéaires et de quelques problèmes que pose l'Hydrodynamique.* J. Math. Pures Appl. **12**, 1–82 (1933).

[13] Li, Z., Pan, X., *Asymptotic properties of generalized D-solutions to the stationary axially symmetric Navier-Stokes equations.* arXiv:2003.10087v1.

[14] Pan, X., Li, Z., *Liouville theorem of axially symmetric Navier-Stokes equations with growing velocity at infinity.* Nonlinear Analysis: Real World Applications, **56** (2020), 103159.

[15] Weng, S., *Decay properties of smooth axially symmetric D-solutions to the steady Navier-Stokes equations.* J. Math. Fluid Mech. **20** (2018), 7–25.

Graduate School of Mathematics
Nagoya University

Nagoya 458-0015
JAPAN
E-mail address: yutaka@math.nagoya-u.ac.jp

名古屋大学・多元数理科学研究科　寺澤　祐高