A survey on the recent results for the compressible fluid model of Korteweg type

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1 Introduction

This article is to give a survey of our recent joint papers [8, 23, 29, 30]. We are concerned with compressible fluids endowed with internal capillarity, which aims to study the dynamics of a liquid-vapor mixture. The model originates from the 19th century work by Van der Waals [31] and Korteweg [25]. The rigorous derivation of the corresponding equations that we shall name the compressible Navier-Stokes-Korteweg system is due to Dunn and Serrin [15], where a capillary term related to surface tension is added to the classical compressible fluid equations. The Korteweg system is in fact based on an extended version of nonequilibrium thermodynamics, which assume that the energy of the fluid not only depends on standard variables (density, velocity and temperature) but also on the gradient of the density.

Let us consider the fluid of density $\rho \geq 0$ and velocity field $u \in \mathbb{R}^d$. The barotropic case is given by

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla \Pi = \mathcal{A}u + \mathrm{div}\mathcal{K}, \end{cases} \tag{1.1}$$

for $\mathbb{R} \times \mathbb{R}^d$, where the Korteweg tensor reads

$$\mathrm{div}\mathcal{K} = \nabla(\rho\kappa(\rho)\Delta\rho + \frac{1}{2}(\kappa(\rho) + \rho\kappa'(\rho))|\nabla\rho|^2) - \mathrm{div}(\kappa(\rho)\nabla\rho \otimes \nabla\rho).$$

The capillarity coefficient $\kappa > 0$ depends on ρ in general. The pressure $\Pi = P(\rho)$ is a suitable smooth function and the diffusion operator $\mathcal{A}u$ is denoted by $\mathcal{A}u \triangleq \mathrm{div}(2\mu D(u)) + \nabla(\lambda \mathrm{div} u)$, where $D(u) = \frac{1}{2}(\nabla u +^t \nabla u)$ is the symmetric gradient. The Lamé coefficients λ and μ (the *bulk* and *shear viscosities*) are density-dependent functions, which are supposed to be smooth enough and to satisfy

$$\lambda > 0, \quad \nu \triangleq \lambda + 2\mu > 0.$$

System (1.1) is supplemented with initial data

$$(\rho, u)|_{t=0} = (\rho_0, u_0) \tag{1.2}$$

and we investigate solutions going to the constant equilibrium $(\rho_\infty, 0)$ with $\rho_\infty > 0$, at infinity.

Clearly, System (1.1) reduces to the classical compressible Navier-Stokes equations if the capillarity coefficient $\kappa \equiv 0$. As we known, so far there is a huge literature on the existence and long time behavior of solutions to the compressible Navier-Stokes equations in different settings. Here, we focus on the non-capillary case of (1.1). The existence of smooth solutions to the Cauchy problem (1.1)-(1.2) is known in Sobolev space from those works by Hattori and Li [19, 20]. In comparison with the local existence, global smooth solutions are obtained for initial data close enough to the stable equilibrium $(\rho_\infty, 0)$ with convex pressure profiles. Inspired by the fact that (1.1) is invariant by the transformation

$$\rho(t,x) \rightsquigarrow \rho(l^2 t, lx), \quad u(t,x) \rightsquigarrow lu(l^2 t, lx), \quad l > 0$$

up to a change of the pressure term Π into $l^2\Pi$, Danchin and Desjardins [13] investigated the global well-posedness of strong solutions to (1.1)-(1.2) in critical Besov spaces provided that initial data close enough to $(\rho_\infty, 0)$ with $P'(\rho_\infty) > 0$. Bresch, Desjardins and Lin [5] established the global existence of weak solutions in a periodic or strip domain. However, the uniqueness problem of weak solutions still remains a great open problem. Kotschote [26] considered the initial-boundary value problem in bounded domain and proved the local existence and uniqueness of strong solutions in maximal L^p-regularity class. Tan and Wang [32] deduced various optimal time-decay rates of solutions and their spatial derivatives based on the detailed spectral analysis. Chen and Zhao [10] studied the global existence and nonlinear stability of stationary solutions to compressible Navier-Stokes-Korteweg system with the external force of general form. Charve [6] investigated the Korteweg compressible models (including (1.1) and the non-local system) for large initial data, and established the unique local in time solution in the situation that is not necessarily stable ($P'(\rho_\infty)$ is non-positive in fact). Bian, Yao and Zhu [4] performed the vanishing capillarity limit of smooth solutions to the initial value problem. Li and Yong [27] justified the zero Mach number limit in the regime of smooth solutions. Germain and Lefloch [16] developed the finite energy methods and validated the zero viscosity-capillarity limit associated with the Navier-Stokes-Korteweg system in one dimension. Specifically, they established the existence of finite energy solutions as well as their convergence toward entropy solutions to the Euler system. Chikami and Kobayashi [11] established the global existence and decay of strong solutions in the critical Besov spaces, where the assumption on the pressure law is not necessary monotone increasing. Huang, Hong and Shi [22] also considered the similar case and proved the local-in-time existence of smooth solutions to (1.1)-(1.2). The global-in-time existence of smooth solutions was also established in periodic domain. Murata and Shibata [28] addressed a different statement on the global existence and decay estimates of strong solutions, where the maximal L^p-L^q regularity to the linearized equation in $\mathbb{R}^d$ is mainly employed. Recently, Antonelli and Spirito [1] constructed the global existence of finite energy weak solutions for large initial data, where vacuum regions are allowed in the definition of weak solutions.

The starting point of our research for the Cauchy problem of (1.1)-(1.2) is the global existence result achieved by Danchin and Desjardins [13]. To the best of our knowledge, there is few results on the global wellposedness theory to (1.1)-(1.2) in the general L^p critical framework (see Remark 1.1 below). For the convenience of readers, we would like to present the main statement in [13] first. Denote the functional space by

$$E=\Big\{(a,u)\Big|a\in\tilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{B}^{d/2-1}_{2,1}\cap\dot{B}^{d/2}_{2,1})\cap L^1(\mathbb{R}_+;\dot{B}^{d/2+1}_{2,1}\cap\dot{B}^{d/2+2}_{2,1});\\ u\in\tilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{B}^{d/2-1}_{2,1})\cap L^1(\mathbb{R}_+;\dot{B}^{d/2+1}_{2,1})\Big\}.$$

The reader is referred to [3] for the definition of Besov spaces.

Theorem 1.1. *([13]) Let $\rho_\infty>0$ be such that $P'(\rho_\infty)>0$. Suppose that the initial density fluctuation $\rho_0-\rho_\infty$ belongs to $\dot{B}^{\frac{d}{2}}_{2,1}\cap\dot{B}^{\frac{d}{2}-1}_{2,1}$ and that the initial velocity u_0 is in $\dot{B}^{\frac{d}{2}-1}_{2,1}$. There exists a constant $\eta>0$ depending only on $\kappa,\mu,\nu,\rho_\infty,P'(\rho_\infty)$ and d, such that, if*

$$\|\rho_0-\rho_\infty\|_{\dot{B}^{\frac{d}{2}}_{2,1}\cap\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|u_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\le\eta,$$

then the Cauchy problem (1.1)-(1.2) *has a unique global solution* (ρ, u) *such that* $(\rho - \rho_\infty, u) \in E$.

As a matter of fact, only the case of constant capillarity and viscosity coefficients has been considered in [13]. The case of smooth coefficients may be treated along the same lines (see also the work by Haspot in [17] concerning the polytropic case). Referring to [12] in the non-capillary case, we see that the internal capillarity can smooth out the density fluctuation in viscous compressible flows such that the solution behaves as the heat smoothing effect in all frequencies, which indicates that there is no loss of regularity for the high frequencies. Inspired by the smoothing property, one can prove the solution constructed in Theorem 1.1 is *Gevrey analytic.* Precisely,

Theorem 1.2 ([8]). *Let* p *fulfill*

$$2 \le p \le \min(4, d^*) \quad \textit{and, additionally,} \quad p \neq 4 \quad \textit{if} \quad d = 2, \tag{1.3}$$

where $d^* \triangleq 2d/(d-2)$. *There exists an integer* $k_0 \in \mathbb{N}$ *and a real number* $\eta > 0$ *depending only on the functions* κ, λ, μ *and* P, *and on* p *and* d, *such that if one defines the threshold between low and high frequencies as in Section 2, if* $a_0 \triangleq \rho_0 - \rho_\infty \in \dot{B}^{\frac{d}{p}}_{p,1}$ *and* $u_0 \in \dot{B}^{\frac{d}{p}-1}_{p,1}$ *with, besides,* (a_0^ℓ, u_0^ℓ) *in* $\dot{B}^{\frac{d}{2}-1}_{2,1}$ *satisfy*

$$\mathcal{X}_{p,0} \triangleq \|(a_0, u_0)\|^\ell_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|a_0\|^h_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|u_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \le \eta, \tag{1.4}$$

then (1.1)-(1.2) *has a unique global-in-time solution* (a, u) *in the space* X_p *defined by*

$$X_p \triangleq \{(a,u) | (a,u)^\ell \in \tilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1}_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}+1}_{2,1}),$$
$$a^h \in \tilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{p}+2}_{p,1}), \;\; u^h \in \tilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{p}-1}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{p}+1}_{p,1})\}$$

Furthermore, there exists a constant $c_0 > 0$ *so that* (a, u) *belongs to the space*

$$Y_p \triangleq \{(a,u) \in X_p | e^{\sqrt{c_0 t}\Lambda_1}(a,u) \in X_p\},$$

where Λ_1 *stands for the Fourier multiplier with symbol* $|\xi|_1 = \sum_{i=1}^d |\xi_i|$.

Remark 1.1. *In the physical dimensions* $d = 2, 3$, *Condition* (1.4) *allows us to consider the case* $p > d$, *and the velocity regularity exponent* $d/p - 1$ *thus becomes negative. Therefore, Theorem 1.2 applies to large highly oscillating initial velocities (see e.g., [7, 9] for explanations), which is the main motivation of* L^p *extension. In addition, Theorem 1.2 tells us if initial data* (1.2) *are sufficiently small in critical Besov spaces, then the solution of System* (1.1) *is globally in the Gevrey class, where the radius of uniform analyticity increases like* $\sqrt{t}$ *as* $t \to \infty$.

The proof of Theorem 1.2 can be finished by means of the standard fixed point theorem. To do this, *a priori* estimates are necessary, which mainly depend on the L^p energy argument and nonlinear estimates involving Gevrey regularity.

Lemma 1.1. *There exists some constant* C *such that for all* $t \geq 0$,

$$\mathcal{X}_p(t) \leq C(\mathcal{X}_{p,0} + \mathcal{X}_p^2(t) + \mathcal{X}_p^3(t)), \tag{1.5}$$

where

$$\begin{aligned}\mathcal{X}_p(t) \triangleq \|(a,u)\|^{\ell}_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|(a,u)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} \\ + \|a\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1}) \cap L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})} + \|u\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1}) \cap L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}.\end{aligned} \tag{1.6}$$

By denoting $A \triangleq e^{\sqrt{c_0 t}\Lambda_1} a$ and $U \triangleq e^{\sqrt{c_0 t}\Lambda_1} u$, furthermore, we have

Lemma 1.2. *If* $\|A\|_{\tilde{L}^\infty(\dot{B}^{\frac{d}{p}}_{p,1})}$ *is small enough, then the following a priori estimate holds true*

$$\mathcal{Y}_p(t) \leq C\big(\mathcal{X}_{p,0} + \mathcal{Y}_p^2(t)\big) \quad \textit{for all} \;\; t \geq 0, \tag{1.7}$$

where

$$\begin{aligned}\mathcal{Y}_p(t) \triangleq \|(A,U)\|^{\ell}_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|(A,U)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} \\ + \|A\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1}) \cap L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})} + \|U\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1}) \cap L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}.\end{aligned}$$

As the direct consequence of Gevrey smoothing, the regularizing decay for higher-order derivatives of solutions can be available.

Theorem 1.3. *Let (ϱ, u) be the solution constructed in Theorem 1.2. Then, for any $s \geq 0$, there exists a constant C_s such that for all $t > 0$, it holds that*

$$\|\rho(t)-\rho_\infty\|^{\ell}_{\dot B^{\frac{d}{2}-1+s}_{2,1}} \leq C_s \mathcal{X}_{p,0} t^{-\frac{s}{2}}, \qquad \|u(t)\|^{\ell}_{\dot B^{\frac{d}{2}-1+s}_{2,1}} \leq C_s \mathcal{X}_{p,0} t^{-\frac{s}{2}},$$
$$\|\rho(t)-\rho_\infty\|^{h}_{\dot B^{\frac{d}{p}+s}_{p,1}} \leq C_s \mathcal{X}_{p,0} t^{-\frac{s}{2}} e^{-c\sqrt{t}}, \qquad \|u(t)\|^{h}_{\dot B^{\frac{d}{p}-1+s}_{p,1}} \leq C_s \mathcal{X}_{p,0} t^{-\frac{s}{2}} e^{-c\sqrt{t}}.$$

Theorem 1.3 exhibits algebraic time-decay estimates in critical Besov spaces (and even exponential decay for the high frequencies) for arbitrary derivatives of the solution. However, those decay rates of solutions in low frequencies are not *optimal* in contrast to that of the heat kernel. It is found that there are no existing papers on the optimal time-decay estimates of solutions addressed by Theorem 1.2. Generally speaking, the elaborate spectral analysis may be always effective. By exploring the parabolic diffusion of (1.1), here, we developed more elementary energy argument (independent of spectral analysis), which leads to desired time-decay estimates of L^q-L^r type.

Denote the pseudo differential operator Λ^s by $\Lambda^s f \triangleq \mathcal{F}^{-1}\left(|\xi|^s \mathcal{F} f\right)$ for $s \in \mathbb{R}$.

Theorem 1.4. *([23]) Let (ρ, u) be the global solution of* (1.1)-(1.2) *in Theorem 1.2. Let the real number σ_1 satisfy*

$$1-\frac{d}{2} < \sigma_1 \leq \sigma_0 \quad \textit{with} \quad \sigma_0 \triangleq \frac{2d}{p}-\frac{d}{2}. \tag{1.8}$$

There exists a positive constant $c = c(p, d, \lambda, \mu, P, \kappa)$ such that if in addition the initial data (a_0, u_0) satisfy

$$\mathcal{D}_{p,0} \triangleq \|(a_0, u_0)\|^{\ell}_{\dot B^{-\sigma_1}_{2,\infty}} \leq c, \tag{1.9}$$

then the solution (ρ, u) fulfills

$$\|\Lambda^l (a, u)\|_{L^r} \lesssim \vartheta(t)^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{r})-\frac{l+\sigma_1}{2}} \tag{1.10}$$

for $t \geq 0, p \leq r \leq \infty$ and $l \in \mathbb{R}$, where $\vartheta(t) := t$ if $-\tilde\sigma_1 < l + d(\frac{1}{p}-\frac{1}{r}) \leq \frac{d}{p}+1$ and $\vartheta(t) := \langle t\rangle (\triangleq \sqrt{1+t^2})$ if $-\tilde\sigma_1 < l + d(\frac{1}{p}-\frac{1}{r}) \leq \frac{d}{p}-1$ satisfying $\tilde\sigma_1 \triangleq \sigma_1 + d(1/2-1/p)$.

It is well-known that the low-frequency assumption usually plays a key role in the large-time asymptotics of solutions. To capture the optimal decay rates of L^q-L^r type, the low-frequency assumption in (1.4) is reasonably strengthened by (1.9). In fact, (1.9) can be regarded as a natural generalization of the L^1 assumption due to the Sobolev embedding $L^1 \hookrightarrow \dot{B}^{-d/2}_{2,\infty}$ (if taking $\sigma_1 = \sigma_0 = d/2$ and $p = 2$). The proof of Theorem 1.4 lies in the following time-weighted inequality

$$\mathcal{D}_p(t) \lesssim (\mathcal{D}_{p,0} + \|(\nabla a_0, u_0)\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}}) \quad \text{for all} \quad t \geq 0, \tag{1.11}$$

where

$$\begin{aligned}\mathcal{D}_p(t) \triangleq \sup_{\sigma\in[\varepsilon-\sigma_1,\frac{d}{2}+1]} \|\langle\tau\rangle^{\frac{\sigma_1+\sigma}{2}}(a,u)\|^{\ell}_{L^\infty_t(\dot{B}^{\sigma}_{2,1})} \\ + \|\tau^{\beta}(\nabla a, u)\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{p}+1}_{p,1})} + \|\langle\tau\rangle^{\beta}(\nabla a, u)\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\end{aligned} \tag{1.12}$$

for $\beta = \sigma_1 + \frac{d}{2} + \frac{1}{2} - \varepsilon$ ($\varepsilon > 0$ sufficiently small). Proving (1.11) consists of two steps. The first step (bounding the low-frequency part of $\mathcal{D}_p$ in (1.12)) is devoted to refined time-weighted estimates. In the second step, we establish gain of regularity and decay altogether for the high frequencies of solutions. The step strongly relies on the elementary L^p energy approach, since the capillarity tensor behaves like the heat diffusion of density fluctuation. The strategy is in the spirit of Hoff's viscous effective flux (see [21]), which was developed by Haspot [18] in the critical framework.

Furthermore, the smallness requirement of low frequencies in terms of $\mathcal{D}_{p,0}$ can be removed by using energy methods of Lyapunov type.

Theorem 1.5. *([23]) Let the real number σ_1 satisfy*

$$1 - \frac{d}{2} < \sigma_1 \leq \sigma_0 \quad \textit{with} \quad \sigma_0 \triangleq \frac{2d}{p} - \frac{d}{2}.$$

If $\mathcal{D}_{p,0} \triangleq \|(a_0, u_0)\|^{\ell}_{\dot{B}^{-\sigma_1}_{2,\infty}}$ is bounded, then the solution (a, u) constructed in Theorem 1.2 fulfills

$$\|\Lambda^l(a,u)\|_{L^r} \lesssim (1+t)^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{r})-\frac{l+\sigma_1}{2}} \tag{1.13}$$

for $t \geq 0$, $p \leq r \leq \infty$ and $l \in \mathbb{R}$ satisfying $-\tilde{\sigma}_1 < l + d(\frac{1}{p} - \frac{1}{r}) \leq \frac{d}{p} - 1$.

In comparison with the time-weighted energy method in Theorem 1.4, the proof of Theorem 1.5 is totally different and resorts to a Lyapunov-type inequality in time for critical energy norms:

$$\frac{d}{dt}\Big(\|(a,u)^\ell\|_{\dot B^{\frac d2-1}_{2,1}}+\|(\nabla a,u)\|^h_{\dot B^{\frac dp-1}_{p,1}}\Big)
+c_0\Big(\|(a,u)^\ell\|_{\dot B^{\frac d2-1}_{2,1}}+\|(\nabla a,u)\|^h_{\dot B^{\frac dp-1}_{p,1}}\Big)^{1+\frac{2}{d/2-1+\sigma_1}}\le 0. \quad (1.14)$$

for some constant $c_0>0$. Solving (1.14) yields the desired optimal decay estimates directly.

In order to show (1.14), the main task is to establish the nonlinear evolution of Besov norm $\dot B^{-\sigma_1}_{2,\infty}$ (restricted in the low-frequency of solutions) for both non oscillation case ($2\le p\le d$) and oscillation case ($p>d$). That is, it suffices to bound

$$\|(a,u)(t,\cdot)\|^\ell_{\dot B^{-\sigma_1}_{2,\infty}}\le C_0 \quad (1.15)$$

for all $t\ge 0$, where $C_0>0$ depends on the norm $\|(a_0,u_0)\|^\ell_{\dot B^{-\sigma_1}_{2,\infty}}$ and $\mathcal{X}_{p,0}$. The crucial inequality is included in the following lemma.

Lemma 1.3. *Let* $1-\frac d2<\sigma_1\le\sigma_0$ *and* p *satisfy* (1.3)*. It holds that*

$$\Big(\|(a,u)(t)\|^\ell_{\dot B^{-\sigma_1}_{2,\infty}}\Big)^2\lesssim\Big(\|(a_0,u_0)\|^\ell_{\dot B^{-\sigma_1}_{2,\infty}}\Big)^2
+\int_0^t\Big(\mathcal{N}^1_p(\tau)+\mathcal{N}^2_p(\tau)\Big)\Big(\|(a,u)(\tau)\|^\ell_{\dot B^{-\sigma_1}_{2,\infty}}\Big)^2d\tau+\int_0^t\mathcal{N}^3_p(\tau)\|(a,u)(\tau)\|^\ell_{\dot B^{-\sigma_1}_{2,\infty}}d\tau, \quad (1.16)$$

where

$$\mathcal{N}^1_p(t)\triangleq\|(a,u)\|^\ell_{\dot B^{\frac d2+1}_{2,1}}+\|a\|^h_{\dot B^{\frac dp+2}_{p,1}}+\|u\|^h_{\dot B^{\frac dp+1}_{p,1}},\qquad \mathcal{N}^2_p(t)\triangleq\|a\|^2_{\dot B^{\frac dp}_{p,1}},$$

$$\mathcal{N}^3_p(t)\triangleq\Big(\|(a,u)\|^\ell_{\dot B^{\frac d2-1}_{2,1}}+\|a\|^h_{\dot B^{\frac dp}_{p,1}}+\|u\|^h_{\dot B^{\frac dp-1}_{p,1}}\Big)\Big(\|a\|^h_{\dot B^{\frac dp+2}_{p,1}}+\|u\|^h_{\dot B^{\frac dp+1}_{p,1}}\Big).$$

Indeed, based on (1.16), it follows from Young's inequality that

$$\Big(\|(a,u)(t)\|^{\ell}_{\dot{B}^{-\sigma_1}_{2,\infty}}\Big)^2 \lesssim \Big(\|(a_0,u_0)\|^{\ell}_{\dot{B}^{-\sigma_1}_{2,\infty}}\Big)^2 + \int_0^t \mathcal{N}^3_p(\tau)d\tau + \int_0^t \Big(\mathcal{N}^1_p(\tau)+\mathcal{N}^2_p(\tau)+\mathcal{N}^3_p(\tau)\Big)\Big(\|(a,u)(\tau)\|^{\ell}_{\dot{B}^{-\sigma_1}_{2,\infty}}\Big)^2 d\tau. \quad (1.17)$$

Furthermore, according to the definition of $\mathcal{X}_p$ in (1.6) and Theorem 1.2, we arrive at

$$\int_0^t \Big(\mathcal{N}^1_p(\tau)+\mathcal{N}^3_p(\tau)\Big)d\tau \le \mathcal{X}_p + \mathcal{X}_p^2 \le C\mathcal{X}_{p,0},$$

since $\mathcal{X}_{p,0} \ll 1$. On the other hand, we use the interpolation inequality and get

$$\|a^\ell\|^2_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})} \lesssim \|a^\ell\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|a^\ell\|_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})} \lesssim \|a\|^\ell_{L^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\|a\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}$$

and

$$\|a^h\|^2_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})} \lesssim \|a\|^h_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|a\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})},$$

which lead to

$$\int_0^t \mathcal{N}^2_p(\tau)d\tau \lesssim \mathcal{X}_p^2 \le C\mathcal{X}_{p,0}.$$

Therefore, (1.15) is followed by Gronwall's inequality.

2 Zero sound speed $P'(\rho_\infty)=0$

Most of previous efforts are dedicated to the stable case $P'(\rho_\infty)>0$ with $\rho_\infty>0$, except for [11, 22, 26]. It is well-known that the Navier-Stokes-Korteweg system (NSK) was deduced by using Van der Waals potential ([15, 25, 31]), where the pressure law is not necessary monotone increasing. Therefore, it is interesting to investigate more physical case $P'(\rho_\infty)=0$ (zero sound speed) and $P'(\rho_\infty)<0$. In those cases, the pressure term couldn't provide any dissipation.

For simplicity, let us consider the case of zero sound speed with the far-field $\rho_\infty>0$, which indicates vacuum is ruled out on this stage. We recall briefly the Fourier study of the corresponding linearized system as in [13]. A simple calculation leads to the following linear perturbation system

$$\begin{cases} \partial_t a + \mathrm{div} m = 0, \\ \partial_t m - \mu_\infty \mathrm{div} D(m) - (\mu_\infty + \lambda_\infty)\nabla \mathrm{div} m - \kappa_\infty \nabla \Delta a = 0, \end{cases} \tag{2.1}$$

where m is the scaled momentum. Denote $\mathcal{P} = \mathrm{Id} - \frac{\nabla}{\Delta}\mathrm{div}$ (Leray Projector). Hence, $m = \mathcal{P}m + \mathcal{Q}m$ where $\mathcal{P}m$ is the divergence-free part and $\mathcal{Q}m$ is the compressible part. Consequently,

$$\begin{cases} \partial_t a + \mathrm{div}\mathcal{Q}m = 0, \\ \partial_t \mathcal{Q}m - \nu_\infty \Delta \mathcal{Q}m - \kappa_\infty \nabla \Delta a = 0, \\ \partial_t \mathcal{P}m - \mu_\infty \Delta \mathcal{P}m = 0, \end{cases} \tag{2.2}$$

where $\nu_\infty = \lambda_\infty + 2\mu_\infty > 0$. Clearly, $\mathcal{P}m$ just satisfies an ordinary heat equation. Regarding for $\mathcal{Q}m$, it is convenient to introduce

$$\mathcal{V} \triangleq \Lambda^{-1}\mathrm{div} m.$$

Consequently, the new variable $(a, \mathcal{V})$ satisfies the coupling 2×2 system:

$$\begin{cases} \partial_t a + \Lambda \mathcal{V} = 0, \\ \partial_t \mathcal{V} - \bar{\nu}\Delta \mathcal{V} - \kappa \Lambda^3 a = 0. \end{cases} \tag{2.3}$$

Taking the Fourier transform with respect to $x \in \mathbb{R}^d$ implies that

$$\frac{d}{dt}\begin{pmatrix} \hat{a} \\ \hat{\mathcal{V}} \end{pmatrix} = A(\xi)\begin{pmatrix} \hat{a} \\ \hat{\mathcal{V}} \end{pmatrix} \quad \text{with} \quad A(\xi) = \begin{pmatrix} 0 & -|\xi| \\ \kappa|\xi|^3 & -\bar{\nu}|\xi|^2 \end{pmatrix},$$

where $\xi \in \mathbb{R}^d$ is the Fourier variable. It is not difficult to check that

(i): If $\nu_\infty^2 \geq 4\kappa_\infty$, then $A(\xi)$ has two real eigenvalues:

$$\lambda_\pm = \frac{-\nu_\infty \pm \sqrt{\nu_\infty^2 - 4\kappa_\infty}}{2}|\xi|^2;$$

(ii): If $\nu_\infty^2 < 4\kappa_\infty$, then $A(\xi)$ has two complex conjugated eigenvalues:

$$\lambda_\pm = \frac{-\nu_\infty \pm \mathrm{i}\sqrt{4\kappa_\infty - \nu_\infty^2}}{2}|\xi|^2,$$

where $\mathrm{i} = \sqrt{-1}$ is the unit imaginary number.

Remark 2.1. *Let us underline that the case* (i) *or* (ii) *is of "regularity-gain type" according to the dissipation notion for general hyperbolic-parabolic system with dispersion formulated in [24] recently, which implies that the solution admits parabolic regularization in all frequency space. In particular, the Korteweg system is purely dissipative in the case* (i)*, and is a dissipative-dispersive hybrid in the case* (ii)*.*

Also, we would like to survey the recent results on the case (i). Firstly, we give the definition of hybrid Besov spaces as follows.

Definition 2.1. *Let* $s,t\in\mathbb{R}$*,* $p,q,r_1,r_2\in[1,\infty]$*. We denote* $\dot{B}^{s,t}_{(p,r_1),(q,r_2)}$ *by the space of functions* $f\in\mathcal{S}'_0$ *(the subspace of those tempered distributions module polynomials) equipped with norm:*

$$\|f\|_{\dot{B}^{s,t}_{(p,r_1),(q,r_2)}}=\Big\{\sum_{j\geq j_0}2^{sjr_1}\|\dot{\Delta}_jf\|^{r_1}_{L^p}\Big\}^{\frac{1}{r_1}}+\Big\{\sum_{j<j_0}2^{tjr_2}\|\dot{\Delta}_jf\|^{r_2}_{L^q}\Big\}^{\frac{1}{r_2}},$$

for some integer j_0*. For convenience, we write* $\|f\|_{\dot{B}^{s,t}_{(p,r_1),(q,r_2)}}\triangleq\|f\|^h_{\dot{B}^s_{p,r_1}}+\|f\|^\ell_{\dot{B}^t_{q,r_2}}$*.*

Moreover, one can define the hybrid Chemin-Lerner spaces $\tilde{L}^{\rho_1,\rho_2}_T(\dot{B}^{s,t}_{(p,r_1),(q,r_2)})$ with norm:

$$\|f\|_{\tilde{L}^{\rho_1,\rho_2}_T(\dot{B}^{s,t}_{(p,r_1),(q,r_2)})}=\Big\{2^{sj}\|\dot{\Delta}_jf\|_{L^{\rho_1}_TL^p}\Big\}_{l^{r_1}_{j\geq j_0}}+\Big\{2^{tj}\|\dot{\Delta}_jf\|_{L^{\rho_2}_TL^q}\Big\}_{l^{r_2}_{j<j_0}}$$

for $T>0$.

Theorem 2.1. *([29]) Let* $\rho_\infty>0$ *such that* $P'(\rho_\infty)=0$*. Let* $\nu^2_\infty\geq4\kappa_\infty$ *and* $1\leq q\leq p\leq\min\{d,2q\}$ *with*

$$\frac{1}{q}\leq\frac{1}{p}+\frac{1}{d}. \tag{2.4}$$

There exists a positive $\eta>0$ *depending on functions* κ,λ,μ *and* P *and on* p,q *and* d *such that if* $(a_0,m_0)\in\dot{B}^{\frac{d}{p}}_{p,1}\times\dot{B}^{\frac{d}{p}-1}_{p,1}$*, besides,* $(a^\ell_0,u^\ell_0)\in\dot{B}^{\frac{d}{q}-2}_{q,\infty}\times\dot{B}^{\frac{d}{q}-3}_{q,\infty}$ *satisfying*

$$\|(\nabla a_0,m_0)\|_{\dot{B}^{\frac{d}{p}-1,\frac{d}{q}-3}_{(p,1),(q,\infty)}}\leq\eta,$$

then (1.1)-(1.2) *admits a unique global-in-time solution* (a,m) *in the space* $E^{p,q}$ *satisfying*

$$\|(a,m)\|_{E_T^{p,q}} \lesssim \|(\nabla a_0, m_0)\|_{\dot{B}^{\frac{d}{p}-1,\frac{d}{q}-3}_{(p,1)(q,\infty)}} \tag{2.5}$$

for any $T>0$*, where*

$$\|(a,m)\|_{E_T^{p,q}} \triangleq \|(\nabla a, m)\|_{\tilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1,\frac{d}{q}-3}_{(p,1),(q,\infty)})} + \|(\nabla a, m)\|_{\tilde{L}^1_T(\dot{B}^{\frac{d}{p}+1,\frac{d}{q}-1}_{(p,1),(q,\infty)})}.$$

Moreover, if those functions λ, μ, κ *and* P *are assumed to be real analytic near zero, then for* $d \geq 3$ *and* $1 < q \leq p \leq \min\{d, 2q\}$ *with* $1/q \leq 1/p + 1/d$*, the solution* (a,m) *fulfills* $e^{\sqrt{c_0 t}\Lambda_1}(a,m) \in E^{p,q}$*, where* $c_0 = c_0(d, \mu_\infty, \lambda_\infty, \kappa_\infty, \rho_\infty)$ *is some positive constant.*

Clearly, Theorem 2.1 indicates that the Korteweg system (1.1) is purely dissipative in the case of $\nu_\infty^2 \geq 4\kappa_\infty$ and acoustic waves are not available. Consequently, the usual L^2-type bounds on the low frequencies of solutions are improved to the L^p framework in contrast to the priori study of compressible Navier-Stokes equations ([7, 9, 12, 18]) or compressible Navier-Stokes-Korteweg equations ([8, 11, 13]). Similar to Theorem 1.2, the system with zero sound speed enjoys the Gevrey analyticity too, where the radius of uniform analyticity increases like $\sqrt{t}$ as $t \to \infty$. As a next step, one wonder what the global strong solutions constructed in Theorem 2.1 look like for large times. For that end, we develop an idea (see [30]) in Besov framework as follows:

$$\|\Lambda^l u\|_{\dot{B}^0_{2,1}} \lesssim t^{-\frac{l}{2}-\frac{\sigma}{2}} \|e^{\sqrt{t}\Lambda_1} u\|_{\dot{B}^{-\sigma}_{2,\infty}} \quad \text{for} \quad l > -\sigma. \tag{2.6}$$

The key estimate lies in uniform bounds on the growth of the radius of analyticity in negative Besov norms

$$\|e^{\sqrt{t}\Lambda_1} v\|_{\dot{B}^{-\sigma}_{2,\infty}} \leq C \quad \text{for} \quad t > 0.$$

Consequently, choosing a suitable regularity (for instance, $\sigma = d/2$) enables us to get the same time-decay estimates as heat kernel.

Theorem 2.2. *([30]) Let* (a,m) *be the global solution addressed by Theorem 2.1. Suppose that the real number* σ_1 *fulfills* $2 - \frac{d}{q} \leq \sigma_1 < d - \frac{d}{q}$*, if* $1 < p \leq 2$ *and* $2 - \frac{d}{q} \leq \sigma_1 \leq \frac{2d}{p} - \frac{d}{q}$*, if* $p > 2$*. If in addition initial norm*

$\|(\nabla a_0, m_0)\|^{\ell}_{\dot{B}^{-\sigma_1-1}_{q,\infty}}$ *is bounded, then the solution* (a,u) *satisfies the following decay estimates*

$$\|\Lambda^l a\|_{L^r} \leq C\langle t-t_0\rangle^{-\frac{\tilde{\sigma}_1}{2}-\frac{l}{2}}, \quad l > -\tilde{\sigma}_1; \tag{2.7}$$

$$\|\Lambda^l m\|_{L^r} \leq C\langle t-t_0\rangle^{-\frac{\tilde{\sigma}_1}{2}-\frac{1}{2}-\frac{l}{2}}, \quad l > -\tilde{\sigma}_1 - 1, \tag{2.8}$$

for all $t \geq t_0$ *and* $r \geq p$, *where* $t_0 > 0$ *is some certain transient (sufficiently small) time,* $\tilde{\sigma}_1 \triangleq \sigma_1 - \frac{d}{r} + \frac{d}{q}$.

In the case of zero sound speed, we see that the density decays at a slower time-rate than the velocity owing to the absence of a lower-order dissipation arising from the pressure. It is worth noting that those decay rates for $1 < r < 2$ are totally new, which provide a hint for long-time behaviors of compressible fluids. The mathematical analysis for another case (ii) is under working, the elaborate dissipative-dispersive coupling structure need to be treated. The vacuum mechanism (for instance, $\rho_\infty \geq 0$) is ruled out in our present analysis. In the presence of vacuum, the mathematical theory for viscous fluids is still far away from well known in critical spaces, which may be of interest. In addition, the study for the unstable case $P'(\rho_\infty) < 0$ is also left to the future consideration.

3 Appendix

In the last section, we would like to present useful notations and nonlinear tools for this survey. The reader is also referred to [3] for the definitions of the Littlewood-Paley decomposition and Besov spaces.

Let $\dot{\Delta}_k$ and $\dot{S}_k$ be the Fourier cut-off operators (see [3]). Fixed $k_0 \in \mathbb{Z}$ (the value of which follows from the proof of the high-frequency estimates in fact), we denote $z^\ell \triangleq \dot{S}_{k_0} z$ and $z^h \triangleq z - z^\ell$. Restricting Besov norms to the low or high frequencies parts of distributions will be fundamental in our methods. For instance, we put[1]

$$\|f\|^{\ell}_{\dot{B}^{\sigma}_{p,1}} \triangleq \sum_{k \leq k_0} 2^{k\sigma}\|\dot{\Delta}_k f\|_{L^p} \text{ and } \|f\|^{h}_{\dot{B}^{\sigma}_{p,1}} \triangleq \sum_{k \geq k_0 - 1} 2^{k\sigma}\|\dot{\Delta}_k f\|_{L^p},$$

[1]Note that for technical reasons, we need a small overlap between low and high frequencies.

$$\|f\|^{\ell}_{\tilde{L}^\infty_T(\dot{B}^\sigma_{p,1})} \triangleq \sum_{k\leq k_0} 2^{k\sigma}\|\dot{\Delta}_k f\|_{L^\infty_T(L^p)} \text{ and } \|f\|^{h}_{\tilde{L}^\infty_T(\dot{B}^\sigma_{p,1})} \triangleq \sum_{k\geq k_0-1} 2^{k\sigma}\|\dot{\Delta}_k f\|_{L^\infty_T(L^p)}.$$

In what follows, we give nonlinear estimates in the proofs of Theorems 1.2, 1.4 and 1.5. Firstly, by using Bony's decomposition and Fourier multiplier theorems, one may deduce the following Gevrey product estimates in Besov spaces. The interesting reader is referred to [8] for more details.

Proposition 3.1. *Let* $1 < p < \infty$, $s_1, s_2 \leq d/p$ *with* $s_1 + s_2 > d\max(0, -1 + 2/p)$. *There exists a constant* C *such that the following estimate holds true:*

$$\|e^{\sqrt{ct}\Lambda_1}(fg)\|_{\dot{B}^{s_1+s_2-\frac{d}{p}}_{p,1}} \leq C\|F\|_{\dot{B}^{s_1}_{p,1}}\|G\|_{\dot{B}^{s_2}_{p,1}}\cdot \tag{3.1}$$

Remark 3.1. *Proposition 3.1 ensures that the space* $\big\{f \in \dot{B}^{\frac{d}{p}}_{p,1},\ e^{\sqrt{ct}\Lambda_1}f \in \dot{B}^{\frac{d}{p}}_{p,1}\big\}$ *is an algebra whenever* $1 < p < \infty$.

The product estimates (3.1) also holds in the framework of Chemin-Lerner's spaces, whereas the time exponent just fulfills Hölder inequality.

Proposition 3.2. *Let* $1 < p < \infty$ *and* $1 \leq q, q_1, q_2 \leq \infty$ *such that* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}\cdot$ *If* $s_1, s_2 \leq d/p$ *and* $s_1 + s_2 > d\max(0, -1 + 2/p)$, *then there exists a constant* $C > 0$ *such that for all* $T \geq 0$,

$$\|e^{\sqrt{ct}\Lambda_1}(fg)\|_{\tilde{L}^q_T(\dot{B}^{s_1+s_2-\frac{d}{p}}_{p,1})} \leq C\|F\|_{\tilde{L}^{q_1}_T(\dot{B}^{s_1}_{p,1})}\|G\|_{\tilde{L}^{q_2}_T(\dot{B}^{s_2}_{p,1})}\cdot \tag{3.2}$$

Secondly, System (1.1) also involves compositions of functions (through κ, λ and μ) and they can be bounded according to the following composition estimates by real analytic functions.

Proposition 3.3. *Let* Φ *be a real analytic function in a neighborhood of* 0, *such that* $\Phi(0) = 0$. *Let* $1 < p < \infty$ *and* $-\min\big(\frac{d}{p}, \frac{d}{p'}\big) < s \leq \frac{d}{p}$ *with* $\frac{1}{p'} = 1 - \frac{1}{p}\cdot$ *There exist two constants* R_0 *and* D *depending only on* p, d *and* Φ *such that if for some* $T > 0$,

$$\|e^{\sqrt{ct}\Lambda_1}z\|_{\tilde{L}^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})} \leq R_0, \tag{3.3}$$

then we have for all $q \in [1, \infty]$,

$$\|e^{\sqrt{ct}\Lambda_1}\Phi(z)\|_{\tilde{L}^q_T(\dot{B}^s_{p,1})} \leq D\|e^{\sqrt{ct}\Lambda_1}z\|_{\tilde{L}^q_T(\dot{B}^s_{p,1})}\cdot \tag{3.4}$$

Finally, we end the section with the endpoint maximal regularity property of the heat equation, which is adapted to the case of *complex* diffusion coefficient. The proof is similar to the case of real coefficient as in [3].

Proposition 3.4. *Let* $T > 0$, $s \in \mathbb{R}$ *and* $1 \le \rho_2, p, r \le \infty$. *Let* u *satisfy*

$$\begin{cases} \partial_t u - \nu \Delta u = f, \\ u|_{t=0} = u_0(x), \end{cases} \tag{3.5}$$

where $\nu \in \mathbb{C}$ *is a complex number with* $\operatorname{Re}\nu > 0$. *Then, there exists a constant* C *depending only on* d *and such that for all* $\rho_1 \in [\rho_2, \infty]$, *one has*

$$(\operatorname{Re}\nu)^{\frac{1}{\rho_1}} \|u\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,r}^{s+\frac{2}{\rho_1}})} \le C\Big(\|u_0\|_{\dot{B}_{p,r}^s} + (\operatorname{Re}\nu)^{\frac{1}{\rho_2}-1}\|f\|_{\tilde{L}_T^{\rho_2}(\dot{B}_{p,r}^{s-2+\frac{2}{\rho_2}})}\Big). \tag{3.6}$$

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