On tetrahedron type equations associated with B_3, C_3, F_4 and H_3

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Abstract

Tetrahedron equation is a three dimensional analogue of the Yang-Baxter equation. It allows a formulation in terms of the Coxeter group A_3. This short note includes miscellaneous remarks on the generalizations along B_3, C_3, F_4 and the non-crystallographic Coxeter group H_3. It is a supplement to the author's talk in the online workshop *Combinatorial Representation Theory and Connections with Related Fields* at RIMS, Kyoto University in November 2021.

1. Introduction

Yang-Baxter equation [7] plays a central role in solvable lattice models in two dimension [1] and integrable quantum field theories in $1+1$ dimensional space time [15]. Tetrahedron equation [16] is an analogue of the Yang-Baxter equation in three dimensional space. It is naturally endowed with the Coxeter group A_3 [8].

One can generalize the equations and solutions from the viewpoint of finite Coxeter groups [11, 12, 9]. Given a rank n Coxeter group X_n, a common feature is the correspondence [10]:

$$\begin{gathered}\text{basic operators} \leftrightarrow \text{Coxeter relations in } X_n,\\ \text{tetrahedron type equation} \leftrightarrow \text{inclusion } X_n \hookrightarrow X_{n+1} \text{ as a parabolic subgroup.}\end{gathered} \tag{1}$$

In contrast, the correspondence in two dimension [3] holds between the Yang-Baxter equation and the cubic Coxeter relation $s_i s_j s_i = s_j s_i s_j$ of the generators, the reflection equation and the quartic one $s_i s_j s_i s_j = s_j s_i s_j s_i$ and the G_2 reflection equation [9] and the sextic one $s_i s_j s_i s_j s_i s_j = s_j s_i s_j s_i s_j s_i$. They may be viewed as

$$\begin{gathered}\text{basic operators} \leftrightarrow \text{generators in } X_n,\\ \text{Yang-Baxter type equation} \leftrightarrow \text{Coxeter relation in } X_n.\end{gathered} \tag{2}$$

After reviewing the type A case in Section 2 and BC cases in Section 3, we treat F_4 in Section 4. Theorem 4.1 is new. It is presented with some details which were not included in [11, Sec.4]. A further generalization to the non-crystallographic Coxeter group H_3 is given in Section 5. Sections 2, 3 and 5 are examples of $(1)_{n=2}$. On the other hand, Section 4 correspond to $(1)_{n=3}$, and the main interest there is how the F_4 equation is decomposed into those from $B_3, C_3 \subset F_4$. Many details are omitted in this brief note. A full treatment can be found in the book [10].

2. $A_2 \hookrightarrow A_3$

We shall exclusively consider a version of the tetrahedron equation having the form:

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124}. \tag{3}$$

Here R is a linear operator $R \in \mathrm{End}(F_q^{\otimes 3})$ for some vector space F_q. The equality (3) is to hold in $\mathrm{End}(F_q^{\otimes 6}) = \mathrm{End}(\overset{1}{F}_q \otimes \overset{2}{F}_q \otimes \overset{3}{F}_q \otimes \overset{4}{F}_q \otimes \overset{5}{F}_q \otimes \overset{6}{F}_q)$, where R_{ijk} acts on the components $\overset{i}{F}_q \otimes \overset{j}{F}_q \otimes \overset{k}{F}_q$ as R and elsewhere as the identity.[1] We assume $R = R^{-1}$ except in Section 5. Recall that the Yang-Baxter equation corresponds to reversing a triangle which is a planar object. The tetrahedron equation (3) is a three dimensional analogue of it in the sense that the

[1]The set theoretical versions is obtained by replacing F_q by a set and $\otimes$ by the product of sets.

it is similarly associated with the inversion of a tetrahedron in 3D space. This can be seen by drawing a diagram for (3).

One can formally associate the operator R with the relation $s_1s_2s_1 = s_2s_1s_2$ of the generators of the Coxeter group A_2.[2] Then the tetrahedron equation (3) is a natural consequence of the embedding $A_2 \hookrightarrow A_3$ as a parabolic subgroup. To explain it, consider the reduced expression (rex) graph of the longest element of A_3:

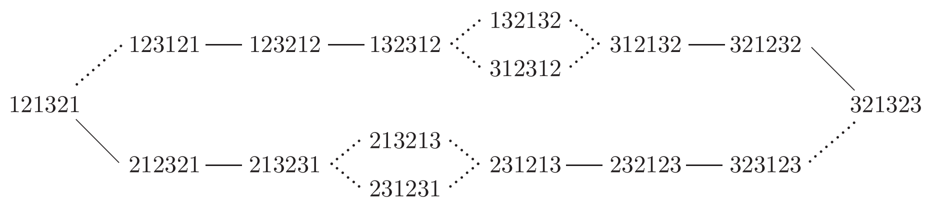

Figure 1. The rex graph for the longest element of A_3.

The word 121321 for instance means $s_1s_2s_1s_1s_3s_2s_1$ with s_1, s_2, s_3 being the generators of A_3. Two reduced expressions are connected by a solid (resp. dotted) line if they are transformed by a single application of the cubic (resp. quadratic) Coxeter relation $121 = 212$ or $232 = 323$ (resp. $13 = 31$).[3] Every time they are applied, we attach an operator $\Phi = \Phi^{-1} \in \mathrm{End}(F_q^{\otimes 3})$ (resp. $P = P^{-1} \in \mathrm{End}(F_q^{\otimes 2})$) with indices signifying the positions of the changing letters. Here P is the exchange of components $P(u \otimes v) = v \otimes u$. Going from 121321 to the most distant 321323 via the lowest path in Figure 1 gives[4]

$$P_{34}\Phi_{123}\Phi_{345}P_{56}P_{23}\Phi_{345}\Phi_{123}. \tag{4}$$

Similarly the most upper path leads to

$$\Phi_{456}\Phi_{234}P_{12}P_{45}\Phi_{234}\Phi_{456}P_{34}. \tag{5}$$

Let us postulate that such a composition of operators along any nontrivial loop in the rex graph yields the identity. It amounts to setting (4) = (5). We further relate Φ to R by $\Phi_{ijk} = R_{ijk}P_{ik}$.[5] Substitute it into (4) = (5) and send all the P_{ij}' s to the right by using $P_{34}R_{123} = R_{124}P_{34}$ etc. The result reads $R_{124}R_{135}R_{236}R_{456}\sigma = R_{456}R_{236}R_{135}R_{124}\sigma'$ with $\sigma = P_{34}P_{13}P_{35}P_{56}P_{23}P_{35}P_{13}$ and $\sigma = P_{46}P_{24}P_{12}P_{45}P_{24}P_{46}P_{34}$. Since $\sigma = \sigma'$ is the reverse ordering of the six components, (3) follows. Different choices of the initial point and the branches of the paths in the rex graph lead to apparently different guises which are all equivalent to (3).

The formal connection of the tetrahedron equation (3) to A_3 explained so far is known to admit a concrete realization in the representation theory of quantized coordinate ring $A_q(A_3)$, which leads to a solution such that F_q is a q-oscillator Fock space [8].[6]

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[2]In this note, the symbol like A_2 will be used either to mean the classical simple Lie algebra or the Coxeter group arising as its Weyl group.

[3]The rex graph is connected [13].

[4]Indices of the operators referring to the *positions* should not be confused with the numbers in Figure 1 signifying the *labels* of the generators.

[5]$R = R^{-1}$ and $\Phi^{-1} = \Phi$ amount to assuming $P_{ik}R_{ijk}P_{ik} = R_{ijk}$ or equivalently $R_{ijk} = R_{kji}$.

[6]The solution in [2, eq.(30)] coincides with the one in [8, p194] (up to typo) as shown in [11, eq.(2.29)].

3. $B_2 \hookrightarrow B_3$ and $C_2 \hookrightarrow C_3$

Parallel results for the quantized coordinate rings $A_q(B_3)$ and $A_q(C_3)$ have been obtained in [11, 12].

Figure 2. Dynkin diagrams of C_3 (left) and B_3 (right). The operators associated with the Coxeter relations are also indicated under them. The both R and S correspond to the type A cubic one, and they are simply related as $S = R|_{q\to q^2}$ in the concrete realization by quantized coordinate rings. K corresponds to the quartic Coxeter relation $s_2s_3s_2s_3 = s_3s_2s_3s_2$ for C_3. For B_3, the role of the short and the long simple roots are exchanged, hence $K^\vee_{ijkl} = P_{il}P_{jk}K_{ijkl}P_{il}P_{jk} = K_{lkji}$.

We formally attach the operators $\Phi = \Phi_{123}$, $\Phi' = \Phi'_{123}$, $\Psi = \Psi_{1234}$ and $\Psi' = \Psi'_{1234}$ to the transformations of the subword in the reduced expressions as follows[7]:

$$C_3 : \Phi = RP_{13} : 121 \leftrightarrow 212, \qquad \Psi = KP_{14}P_{23} : 2323 \to 3232, \tag{6}$$

$$B_3 : \Phi' = SP_{13} : 121 \leftrightarrow 212, \qquad \Psi' = K^\vee P_{14}P_{23} : 2323 \to 3232. \tag{7}$$

Since the role of the short and the long simple roots are exchanged between B_2 and C_2, K and $K^\vee$ are related by $K^\vee_{ijkl} = P_{il}P_{jk}K_{ijkl}P_{il}P_{jk} = K_{lkji}$. We assume $K = K^{-1}$ in what follows. It formally implies $\Psi' = \Psi^{-1}$. We also attach P to $13 \leftrightarrow 31$ for the both of B_3 and C_3.

The rex graphs for the longest element of B_3 and C_3 are identical and consist of 42 reduced expressions. An example is 123121323 in terms of indices. Demanding again that the compositions of $P, \Phi, \Phi', \Psi, \Psi'$ along nontrivial loops in the rex graph becomes the identity, we get

$$\begin{aligned} C_3 :\ & R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456} = R_{456}K_{1236}K_{1478}R_{258}R_{249}K_{3579}R_{689} \\ & \in \mathrm{End}\big(\overset{1}{F}_{q^2} \otimes \overset{2}{F}_q \otimes \overset{3}{F}_{q^2} \otimes \overset{4}{F}_q \otimes \overset{5}{F}_q \otimes \overset{6}{F}_q \otimes \overset{7}{F}_{q^2} \otimes \overset{8}{F}_q \otimes \overset{9}{F}_q\big), \end{aligned} \tag{8}$$

$$\begin{aligned} B_3 :\ & S_{689}K_{9753}S_{249}S_{258}K_{8741}K_{6321}S_{456} = S_{456}K_{6321}K_{8741}S_{258}S_{249}K_{9753}S_{689} \\ & \in \mathrm{End}\big(\overset{1}{F}_q \otimes \overset{2}{F}_{q^2} \otimes \overset{3}{F}_q \otimes \overset{4}{F}_{q^2} \otimes \overset{5}{F}_{q^2} \otimes \overset{6}{F}_{q^2} \otimes \overset{7}{F}_q \otimes \overset{8}{F}_{q^2} \otimes \overset{9}{F}_{q^2}\big). \end{aligned} \tag{9}$$

They are called 3D reflection equations and represent a "factorization of the three string scattering amplitude" in the presence of a reflecting plane. See also [6]. Solutions of (8) and (9) associated with the quantized coordinate rings $A_q(C_3)$ and $A_q(B_3)$ have been obtained in [11, 12]. At $q = 0$ they yield the set theoretical versions where F_q and F_{q^2} are replaced by $\mathbb{Z}_{\geq 0}$.[8] The both R and S reduce to the map on $(\mathbb{Z}_{\geq 0})^3$ as

$$(a, b, c) \mapsto (b + (a - c)_+, \min(a, c), b + (c - a)_+), \tag{10}$$

where $(x)_+ = \max(x, 0)$. Similarly K becomes a map on $(\mathbb{Z}_{\geq 0})^4$ defined by

$$\begin{aligned} & K : (a, b, c, d) \mapsto (a', b', c', d'), \\ & a' = x + a + b - d, \quad b' = c - x + d - \min(a, c + x), \\ & c' = \min(a, c + x), \quad d' = b + (c + x - a)_+, \quad x = (c - a + (d - b)_+)_+. \end{aligned} \tag{11}$$

[7]For K one needs to specify the direction of the map as opposed to the cubic Coxeter relation because the two sides are not invariant under the reverse ordering.

[8]It also emerges by a tropical variable change from another set theoretical version where the R, S, K become birational maps.

Let us present an example of the set theoretical version of (8) and (9), which are equalities of maps on $(\mathbb{Z}_{\geq 0})^9$.

Type C

$$\begin{tikzcd}[column sep=small]
 & (211202341) \arrow[dl, "R_{456}"'] \arrow[dr, "R_{689}"] & \\
(211020341) \arrow[d, "K_{1236}"'] & & (211205314) \arrow[d, "K_{3579}"] \\
(301021341) \arrow[d, "K_{1478}"'] & & (213205118) \arrow[d, "R_{249}"] \\
(301021341) \arrow[d, "R_{258}"'] & & (223105119) \arrow[d, "R_{258}"] \\
(321001361) \arrow[d, "R_{249}"'] & & (213115109) \arrow[d, "K_{1478}"] \\
(311101360) \arrow[d, "K_{3579}"'] & & (313015119) \arrow[d, "K_{1236}"] \\
(313101164) \arrow[dr, "R_{689}"'] & & (313015119) \arrow[dl, "R_{456}"] \\
 & (313106119) &
\end{tikzcd}$$

Type B

$$\begin{tikzcd}[column sep=small]
 & (211202341) \arrow[dl, "S_{456}"'] \arrow[dr, "S_{689}"] & \\
(211020341) \arrow[d, "K_{6321}"'] & & (211205314) \arrow[d, "K_{9753}"] \\
(401021341) \arrow[d, "K_{8741}"'] & & (213205116) \arrow[d, "S_{249}"] \\
(301021431) \arrow[d, "S_{258}"'] & & (223105117) \arrow[d, "S_{258}"] \\
(321001451) \arrow[d, "S_{249}"'] & & (213115107) \arrow[d, "K_{8741}"] \\
(311101450) \arrow[d, "K_{9753}"'] & & (413015117) \arrow[d, "K_{6321}"] \\
(314101153) \arrow[dr, "S_{689}"'] & & (314014117) \arrow[dl, "S_{456}"] \\
 & (314105117) &
\end{tikzcd}$$

FIGURE 3. Examples of the set theoretical solution of the 3D reflection equations for type B and C on $(\mathbb{Z}_{\geq 0})^9$. R and S are given by (10). K_{ijkl} is given by (11) if $i < j < k < l$ and by $P_{il}P_{jk}(K_{lkji}$ by (11)$)P_{il}P_{jk}$ if $i > j > k > l$.

4. $B_3 \hookrightarrow F_4 \hookleftarrow C_3$

Let us consider F_4 which contains B_3 and C_3 as parabolic subgroups. Compare Figure 2 and Figure 4.

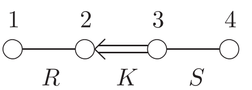

FIGURE 4. The Dynkin diagram of F_4. The operators R, S and K are associated according to Figure 2.

As before we attach P to the quadratic Coxeter relation $ij = ji$ (in terms of indices of the generators) with $|i - j| \geq 2$. Furthermore in view of (6) and (7), we set

$$\Phi = RP_{13} : 121 \leftrightarrow 212, \quad \Upsilon = SP_{13} : 343 \leftrightarrow 434, \quad \Psi = KP_{14}P_{23} : 2323 \to 3232. \tag{12}$$

We have used the symbol Υ refreshing Φ' in (7) since the relevant letters are now 3 and 4 instead of 1 and 2. We keep assuming $\Phi = \Phi^{-1}, \Upsilon = \Upsilon^{-1}$ and $R = R^{-1}, S = S^{-1}, K = K^{-1}$. They imply $R_{ijk} = R_{kji}, S_{ijk} = S_{kji}$ and $\Psi^{-1}_{ijkl} = K^{\vee}_{ijkl}P_{il}P_{jk} = K_{lkji}P_{il}P_{jk}$ as before.

An example of reduced expressions of the longest element of the Coxeter group F_4 is $\mathbf{w}_0 =$ 434234232123423123412321 in terms of indices. It has length 24. The rex graph for it consists of 2144892 vertices. Let $\tilde{\mathbf{w}}_0$ be the reverse ordering of $\mathbf{w}_0$, which is most distant from it in the graph. One way to go from $\mathbf{w}_0$ to $\tilde{\mathbf{w}}_0$ is shown below, where the underlines indicate the changing part and the relevant operators are given on the right for each step.

$$
\begin{array}{lll}
\mathbf{w}_0: & \underline{434}23\underline{42}3212342312\underline{341}2321 & P_{6,7}P_{18,19}P_{19,20}\Upsilon_{1,2,3} \\
& 34\underline{3232}432123423\underline{121}342321 & \Psi^{-1}_{3,4,5,6}\Phi_{16,17,18} \\
& 3\underline{42}32\underline{343212}3423212342321 & P_{2,3}P_{10,11}P_{9,10}P_{8,9}\Upsilon_{6,7,8} \\
& 3243\underline{24}3212\underline{434}232123\underline{42}321 & P_{5,6}P_{20,21}\Upsilon_{11,12,13} \\
& 32\underline{434}2321234\underline{32321}2324321 & \Upsilon_{3,4,5}P_{16,17}\Psi^{-1}_{13,14,15,16} \\
& 3234\underline{32321}23\underline{42}321\underline{3232}4321 & P_{8,9}\Psi^{-1}_{5,6,7,8}P_{12,13}\Psi^{-1}_{17,18,19,20} \\
& 323\underline{42}321\underline{3232}432123\underline{234321} & P_{4,5}\Psi^{-1}_{9,10,11,12}P_{19,20}P_{23,24}P_{22,23}\Upsilon_{20,21,22} \\
& \underline{3232}4321232\underline{343212}3423214 & \Psi^{-1}_{1,2,3,4}P_{16,17}P_{15,16}P_{14,15}\Upsilon_{12,13,14} \\
& 232\underline{343212}3\underline{24}3212\underline{434}23214 & \Upsilon_{17,18,19}P_{11,12}P_{8,9}P_{7,8}P_{6,7}\Upsilon_{4,5,6} \\
& 23243212\underline{434}232123\underline{432321}4 & \Upsilon_{9,10,11}P_{18,19}P_{22,23}\Psi^{-1}_{19,20,21,22} \\
& 23243\underline{2123432321}232432134 & P_{9,10}P_{8,9}\Phi_{6,7,8}P_{14,15}\Psi^{-1}_{11,12,13,14} \\
& 2324\underline{31}23412321\underline{3232}432134 & P_{5,6}\Psi^{-1}_{15,16,17,18} \\
& 232\underline{41}3234123\underline{2123}2\underline{34321}34 & P_{4,5}P_{15,16}\Phi_{13,14,15}P_{21,22}P_{20,21}\Upsilon_{18,19,20} \\
& 23214323412\underline{31}23124321434 & P_{12,13} \\
& 232143234\underline{121}323124321434 & \Phi_{10,11,12} \\
& 23214323\underline{4212323}124321434 & P_{10,11}P_{9,10}\Psi_{12,13,14,15} \\
& 23214\underline{32321}4323\underline{21243}21434 & P_{9,10}\Psi^{-1}_{6,7,8,9}P_{18,19}P_{17,18}\Phi_{15,16,17} \\
& 2321\underline{42}321\underline{3432}3\underline{124}3\underline{121}434 & P_{5,6}P_{12,13}\Upsilon_{10,11,12}P_{15,16}P_{16,17}\Phi_{19,20,21} \\
& 23\underline{212}432\underline{143}2\underline{4341}23212434 & \Phi_{3,4,5}P_{10,11}P_{9,10}P_{15,16}\Upsilon_{13,14,15} \\
& 2\underline{31}2143\underline{24}312\underline{34132321}2434 & P_{2,3}P_{8,9}P_{13,14}P_{14,15}P_{19,20}\Psi^{-1}_{16,17,18,19} \\
& 21321\underline{434}23\underline{121}3\underline{42}32132434 & \Upsilon_{6,7,8}\Phi_{11,12,13}P_{15,16} \\
& 2132\underline{13432321}232432132434 & P_{6,7}P_{5,6}P_{11,12}\Psi^{-1}_{8,9,10,11} \\
& 21323412321\underline{3232}432132434 & \Psi^{-1}_{12,13,14,15} \\
& 213234123\underline{2123}2\underline{34321}32434 & P_{12,13}\Phi_{10,11,12}P_{18,19}P_{17,18}\Upsilon_{15,16,17} \\
& 21323412\underline{31}23124321432434 & P_{9,10} \\
& 213234\underline{121}323124321432434 & \Phi_{7,8,9} \\
& 21323\underline{4212323}124321432434 & P_{7,8}P_{6,7}\Psi_{9,10,11,12} \\
& 21\underline{32321}4323\underline{21243}21432434 & P_{6,7}\Psi^{-1}_{3,4,5,6}P_{15,16}P_{14,15}\Phi_{12,13,14} \\
& \underline{2123}21\underline{3432}3\underline{124}3\underline{121}432434 & P_{3,4}\Phi_{1,2,3}P_{9,10}\Upsilon_{7,8,9}P_{12,13}P_{13,14}\Phi_{16,17,18} \\
& 123\underline{1214}32\underline{4341}23212432434 & P_{6,7}\Phi_{4,5,6}P_{12,13}\Upsilon_{10,11,12} \\
& 123214\underline{23234}1\underline{32321}2432434 & P_{10,11}\Psi_{7,8,9,10}P_{16,17}\Psi^{-1}_{13,14,15,16} \\
& 12321432\underline{342123}2132432434 & P_{9,10}P_{10,11}P_{13,14}\Phi_{11,12,13} \\
& 1232143213\underline{42}3\underline{121}32432434 & P_{11,12}\Phi_{14,15,16} \\
\tilde{\mathbf{w}}_0: & 123214321324321232432434. &
\end{array}
\tag{13}
$$

Let us write (13) schematically as

$$
\mathbf{w}_0 \xrightarrow{O_1} \mathbf{w}_1 \xrightarrow{O_2} \cdots \xrightarrow{O_{N-1}} \mathbf{w}_{N-1} \xrightarrow{O_N} \mathbf{w}_N = \tilde{\mathbf{w}}_0, \tag{14}
$$

where $O_m \in \{P_{k,k+1}, \Phi_{k,k+1,k+2}, \Psi^{\pm 1}_{k,k+1,k+2,k+3}, \Upsilon_{k,k+1,k+2,k+3}\}$ and $N = 126$. For instance $O_1 = \Upsilon_{1,2,3}$ and $O_{126} = P_{11,12}$. Considering the inverse procedure reversing the length 24 arrays at every stage, one finds another route going from $\mathbf{w}_0$ to $\tilde{\mathbf{w}}_0$ as

$$\mathbf{w}_0 = \tilde{\mathbf{w}}_N \xrightarrow{\tilde{O}_N^{-1}} \tilde{\mathbf{w}}_{N-1} \xrightarrow{\tilde{O}_{N-1}^{-1}} \cdots \xrightarrow{\tilde{O}_2^{-1}} \tilde{\mathbf{w}}_1 \xrightarrow{\tilde{O}_1^{-1}} \tilde{\mathbf{w}}_0, \tag{15}$$

where $\tilde{\mathbf{w}}_r$ denotes the reverse word of $\mathbf{w}_r$. The operators $\tilde{O}_m$ is chosen according to O_m as

$$\begin{aligned} O_m &: P_{k,k+1}, \quad \Phi_{k,k+1,k+2}, \quad \Upsilon_{k,k+1,k+2}, \quad \Psi^{\pm 1}_{k,k+1,k+2,k+3}, \\ \tilde{O}_m &: P_{j+1,j+2}, \quad \Phi_{j,j+1,j+2}, \quad \Upsilon_{j,j+1,j+2}, \quad \Psi^{\mp 1}_{j-1,j,j+1,j+2} \end{aligned} \tag{16}$$

with $j + k = 23$. The reason for exceptionally inverting Ψ is that the reverse ordering of 2323 into 3232 interchanges the role of two sides in (12). As in the preceding cases of A_3, B_3, C_3, we define the F_4 analogue of the tetrahedron equation to be the condition that the composition of the operators along the nontrivial loops in the rex graph for the longest element is the identity:

$$O_N \cdots O_2 O_1 = \tilde{O}_1^{-1} \tilde{O}_2^{-1} \ldots \tilde{O}_N^{-1}. \tag{17}$$

From (13) one obtains, after cancelling the product of $P_{i,j}$'s, the following equation:

$$\begin{aligned} &R_{14,15,16} R_{9,11,16} K_{7,8,10,16} K_{17,15,13,9} R_{4,5,16} S_{7,12,17} R_{1,2,16} S_{6,10,17} R_{9,14,18} \\ &\times K_{17,5,3,1} R_{11,15,18} K_{6,8,12,18} R_{1,4,18} R_{1,8,15} S_{7,13,19} K_{19,11,6,1} K_{19,15,12,4} S_{3,10,19} \\ &\times R_{4,8,11} K_{20,14,7,1} R_{2,5,18} S_{6,13,20} S_{3,12,20} R_{1,9,21} K_{20,15,10,2} R_{4,14,21} K_{3,8,13,21} \\ &\times R_{2,11,21} R_{2,8,14} S_{6,7,22} K_{22,4,3,2} R_{5,15,21} K_{22,14,13,11} S_{10,12,22} K_{23,9,6,2} S_{3,7,23} \\ &\times S_{19,20,22} K_{22,18,17,16} S_{10,13,23} K_{23,14,12,5} S_{3,6,24} K_{23,21,19,16} K_{24,9,7,4} S_{17,20,23} \\ &\times K_{24,11,10,5} S_{12,13,24} S_{17,19,24} K_{24,21,20,18} R_{5,8,9} S_{22,23,24} \\ = &\ \text{product in reverse order}, \end{aligned} \tag{18}$$

where the reverse ordering does not change the indices of $K_{i,j,k,l}$ internally into $K_{l,k,j,i}$. There are 50 operators in total on each side; 16 R's, 16 S's and 18 K's. They all have distinct set of indices.

Given a reduced expression of the longest element $\mathbf{w}_0 = i_1 \ldots i_{24}$, one can get another one by $\mathbf{w}_0' = (5 - i_1) \ldots (5 - i_{24})$. Suppose the F_4 analogue of the tetrahedron equation for $\mathbf{w}_0$ is $Z_1 \cdots Z_{50} = Z_{50} \cdots Z_1$ where Z_r is one of R_{ijk}, S_{ijk} and K_{ijkl} for some $i,j,k,l \in \{1, \ldots, 24\}$. Then the equation for $\mathbf{w}_0'$ takes the form $Z_1' \cdots Z_{50}' = Z_{50}' \cdots Z_1'$, where $R'_{ijk} = S_{ijk}$, $S'_{ijk} = R_{ijk}$ and $K'_{ijkl} = K_{lkji}$. The F_4 analogue of the tetrahedron equation which appeared first in [11, eq.(48)] is related to (18) by this transformation.

In Figure 4 one observes the mixture of the B_3 and C_3 structures in Figure 2. This is made precise in

Theorem 4.1. [10, Th.7.2] *The F_4 analogue of the tetrahedron equation (18) is reduced to a composition of the 3D reflection equations for B_3 (9) and C_3 (8) twelve times for each.*

Proof. Let X_0 denote the expression in the LHS of (18) which consists of 16 R's, 16 S's and 18 K's. It can be transformed to the RHS along the following steps:

$$X_0 \to Y_0 \to X_1 \to Y_1 \to \cdots \to X_{24} \to Y_{24} = \text{reverse ordering of } X_0.$$

Here rewriting $X_i \to Y_i$ only uses trivial commutativity of operators having totally distinct indices. On the other hand, the step $Y_i \to X_{i+1}$ indicates an application of a 3D reflection equation, which reverses seven consecutive factors somewhere in the length 50 array Y_i. Let us label the 50 operators in X_0 with $1, 2, \ldots, 50$ by saying that $X_0 = 1 \cdot 2 \cdot \cdots \cdot 50$. Thus for instance $1 = R_{14,15,16}$, $2 = R_{9,11,16}$, $3 = K_{7,8,10,16}$ and $50 = S_{22,23,24}$. To save the space, we specify a length 50 array in two rows. Thus X_0 is expressed as $\begin{pmatrix} 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25 \\ 26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50 \end{pmatrix}$.

The intermediate forms $Y_0, Y_1, \ldots, Y_{23}$ are listed below in such a notation.[9]

$$\begin{pmatrix} 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25 \\ 26,27,28,29,30,31,32,33,34,35,36,39,40,41,43,45,46,37,38,42,44,47,48,50,49 \end{pmatrix}$$

$$\begin{pmatrix} 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25 \\ 26,27,28,29,30,31,32,35,36,41,43,33,34,39,40,45,46,50,48,47,44,42,38,37,49 \end{pmatrix}$$

$$\begin{pmatrix} 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25 \\ 26,27,28,29,32,30,31,35,36,41,43,50,46,45,40,49,39,34,33,48,47,44,42,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,1,2,3,4,5,6,15,7,8,9,10,11,12,13,14,16,17,19,18,20,21,22,23,24 \\ 25,26,27,28,29,43,41,36,35,31,30,46,32,45,40,49,39,34,33,48,47,44,42,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,1,2,3,5,19,4,6,15,7,8,17,10,16,14,9,13,20,24,26,43,18,12,22,23 \\ 27,41,46,36,11,21,25,28,32,45,48,29,35,40,49,47,39,44,42,31,30,34,33,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,1,2,3,5,19,4,6,15,7,8,17,10,16,14,9,13,20,24,26,43,18,12,22,23 \\ 27,41,46,48,36,45,47,32,28,25,21,29,35,40,49,39,44,42,31,30,34,11,33,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,1,2,3,5,19,4,6,15,7,8,17,10,16,14,9,13,20,24,26,43,48,46,18,41 \\ 27,23,22,36,45,47,32,28,25,12,21,29,35,40,49,39,44,42,31,30,34,11,33,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,48,1,2,3,5,19,4,6,15,17,43,46,7,8,10,16,14,26,24,20,13,18,41,45 \\ 27,23,22,36,47,32,28,25,9,12,21,29,35,40,49,39,44,42,31,30,34,11,33,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,48,1,2,3,5,19,4,6,15,17,43,26,46,7,8,10,16,18,41,45,47,14,24,27 \\ 32,49,20,23,36,40,28,22,25,35,39,29,13,21,44,42,31,12,30,34,9,11,33,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,48,1,2,3,5,19,4,6,15,17,43,46,47,26,45,41,18,7,16,10,14,24,27,32 \\ 49,20,23,36,40,28,8,22,25,35,39,44,29,13,21,42,31,12,30,34,9,11,33,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,48,47,46,1,2,3,5,19,43,45,41,17,15,6,26,18,7,16,4,10,14,24,27,32 \\ 49,20,23,36,40,28,8,22,25,35,39,44,29,13,21,42,31,12,30,34,9,11,33,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,48,47,46,1,2,3,5,19,43,45,49,41,17,15,6,26,18,7,16,32,27,24,14,10 \\ 20,23,36,40,28,4,8,22,25,35,39,44,29,13,21,42,31,12,30,34,9,11,33,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,48,47,46,1,2,3,5,19,43,45,49,41,17,15,26,18,7,16,32,27,24,14,6,10 \\ 20,23,36,40,44,39,28,35,42,25,22,29,13,21,31,8,12,30,34,4,9,11,33,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,48,47,44,46,1,2,3,5,19,43,45,49,41,17,15,26,18,7,16,32,27,24,14,40 \\ 36,23,20,10,6,39,28,35,42,25,22,29,13,21,31,8,12,30,34,4,9,11,33,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,49,48,47,46,45,44,43,41,19,1,5,17,3,15,26,32,18,40,27,36,39,2,7,16 \\ 24,28,35,42,14,23,20,25,22,29,10,13,21,31,6,8,12,30,34,4,9,11,33,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,49,48,47,46,45,44,43,41,19,1,5,17,26,32,40,3,15,18,27,36,39,42,35,28 \\ 24,16,7,14,23,20,25,22,29,10,13,21,31,6,8,12,30,34,2,4,9,11,33,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,49,48,47,46,45,44,43,41,19,1,5,17,26,32,40,42,39,36,27,18,15,35,28,24 \\ 16,3,7,14,23,20,25,22,29,10,13,21,31,6,8,12,30,34,2,4,9,11,33,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,49,48,47,46,45,44,42,40,39,43,41,36,19,32,26,35,27,28,24,17,18,23,15,16 \\ 5,1,3,7,14,20,25,29,22,10,13,21,31,6,8,12,30,34,2,4,9,11,33,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,49,48,47,46,45,44,42,40,39,43,41,36,19,32,26,35,27,28,24,17,18,23,15,16 \\ 5,29,25,20,14,7,3,22,10,13,21,31,6,8,12,30,34,1,2,4,9,11,33,38,37 \end{pmatrix}$$

$$\begin{pmatrix} 50,49,48,47,46,45,44,42,40,39,43,41,36,19,32,26,35,27,28,24,17,18,23,29,25 \\ 15,16,20,14,5,7,22,10,13,21,31,3,6,8,12,30,34,38,33,37,11,9,4,2,1 \end{pmatrix}$$

$$\begin{pmatrix} 50,49,48,47,46,45,44,42,40,39,43,41,36,19,32,26,35,27,28,24,17,18,23,29,25 \\ 15,16,20,22,14,5,7,10,13,21,31,38,34,30,33,37,12,11,8,6,3,9,4,2,1 \end{pmatrix}$$

$$\begin{pmatrix} 50,49,48,47,46,45,44,42,38,40,39,43,41,36,19,32,26,35,27,28,17,18,23,29,25 \\ 31,34,24,15,16,20,22,30,33,37,21,14,13,12,11,9,10,8,7,5,6,3,4,2,1 \end{pmatrix}$$

$$\begin{pmatrix} 50,49,48,47,46,45,44,42,38,40,39,43,41,36,19,32,26,35,27,28,29,17,18,23,25 \\ 31,34,37,33,30,22,24,20,16,15,21,14,13,12,11,9,10,8,7,5,6,3,4,2,1 \end{pmatrix}$$

$$\begin{pmatrix} 50,49,48,47,46,45,43,41,44,42,40,39,38,37,34,36,35,32,19,26,27,28,29,31,33 \\ 25,23,18,30,22,24,20,17,16,15,21,14,13,12,11,9,10,8,7,5,6,3,4,2,1 \end{pmatrix}$$

[9] Thus the first one Y_0 already differs from X_0 slightly.

The blue (resp. red)[10] neighboring seven numbers specify the place and operators to which the B_3 (resp. C_3) reflection equation is applied. For example X_1 is obtained from Y_0 by replacing $37{\cdot}38{\cdot}42{\cdot}44{\cdot}47{\cdot}48{\cdot}50 = S_{19,20,22}K_{22,18,17,16}K_{23,21,19,16}S_{17,20,23}S_{17,19,24}K_{24,21,20,18}S_{22,23,24}$ with the reverse ordered form $S_{22,23,24}K_{24,21,20,18}S_{17,19,24}S_{17,20,23}K_{23,21,19,16}K_{22,18,17,16}S_{19,20,22} = 50 \cdot 48 \cdot 47 \cdot 44 \cdot 42 \cdot 38 \cdot 37$ by (9). The numbers of red and blue sequences are both twelve. □

Theorem 4.1 confirms that the triad (R, S, K) satisfying the 3D reflection equations also yield a solution to the F_4 analogue of the tetrahedron equation (18). We remark that the tetrahedron equations $RRRR = RRRR$ and $SSSS = SSSS$ have not been used. R and S act as catalysts for the main reactions which are 3D reflection equations (8) and (9) involving K. According to [14, Th.(2.17)], for any element of a Coxeter group, loops in its rex graph are generated by the loops in the rex graph of the longest element in finite parabolic subgroups of rank 3. Theorem 4.1 is consistent with it and provides a finer information distinguishing B_3 and C_3 structures within F_4.

An analogue of Theorem 4.1 for $A_3 \hookrightarrow A_4$ can be found in [10, eq.(3.101)].

5. $H_2 \hookrightarrow H_3$

This section is a supplement concerning the non-crystallographic Coxeter groups H_2, H_3, H_4 and presents the H_3 analogue of the tetrahedron equation. Although there is no associated quantized coordinate ring, they are treated formally in a similar manner to the preceding cases of A_3, B_3, C_3 and F_4. The Coxeter diagrams of H_2, H_3, H_4 are given in Figure 5.

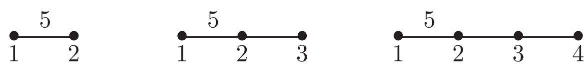

FIGURE 5. The Coxeter diagrams for non-crystallographic Coxeter groups H_2, H_3 and H_4. H_2 is customarily denoted also by $I_2(5)$, which is the $m = 5$ case of the dihedral groups $I_2(m)\,(m \geq 3)$.

They indicate that H_n is generated by $s_1, s_2, \ldots, s_n$ obeying the relations $s_i^2 = 1\,(1 \leq i \leq n)$, $(s_1s_2)^5 = 1$, $(s_is_{i+1})^3 = 1, (1 < i < n)$ and $(s_is_j)^2 = 1\,(|i-j| > 1)$. The groups $H_2 \subset H_3 \subset H_4$ are of order 10,120, 14400 with longest elements of length 5,15, 60, respectively. H_3 is known as the symmetry of the icosahedron or equivalently the dual dodecahedron [5]. The relations of the generators s_1, s_2, s_3 are given as $s_1^2 = s_2^2 = s_3^2 = 1$ and

$$s_1s_3 = s_3s_1, \quad s_1s_2s_1s_2s_1 = s_2s_1s_2s_1s_2. \tag{19}$$

Following the preceding examples, we attach the operators to (19) as

$$P = P^{-1} : 13 \leftrightarrow 31, \tag{20}$$
$$\Phi : 232 \to 323, \quad \Phi_{ijk} = R_{ijk}P_{ik}, \tag{21}$$
$$\Omega : 21212 \to 12121, \quad \Omega_{ijklm} = Y_{ijklm}P_{im}P_{jl}, \tag{22}$$

where, as before, the indices $i, j, k, \ldots$ specify the components that they act non-trivially.[11] The operators Ω and Y are the characteristic ones which emerges from H_2.

[10] Even if not visible, they can be distinguished as explained below.

[11] Unlike the R for A_3, B_3, C_3, F_4, we do not assume $R^{-1} = R$ nor $R_{ijk} = R_{kji}$ in this section.

A reduced expression of the longest element of H_3 is 121213212132123 in terms of indices of s_i. Now a process analogous to (13) reads as

$$
\begin{array}{ll}
1212\underline{13}212132123 & P_{5,6} \\
12123\underline{12121}32123 & \Omega^{-1}_{6,7,8,9,10} \\
121\underline{232}121\underline{232}123 & \Phi_{4,5,6}\Phi_{10,11,12} \\
12\underline{13}2\underline{31}2\underline{13}2\underline{31}23 & P_{3,4}P_{6,7}P_{9,10}P_{12,13} \\
123121\underline{323}121\underline{323} & \Phi^{-1}_{7,8,9}\Phi^{-1}_{13,14,15} \\
12312123\underline{21212}32 & \Omega_{9,10,11,12,13} \\
1231212\underline{31}212\underline{13}2 & P_{8,9}P_{13,14} \\
123\underline{12121}3212312 & \Omega^{-1}_{4,5,6,7,8} \\
1\underline{232}121\underline{232}12312 & \Phi_{2,3,4}\Phi_{8,9,10} \\
132\underline{31}2\underline{13}2\underline{31}2312 & P_{4,5}P_{7,8}P_{10,11} \\
\underline{13}21\underline{323}121\underline{323}12 & P_{12}\Phi^{-1}_{567}\Phi^{-1}_{11,12,13} \\
312123\underline{21212}3212 & \Omega^{-1}_{7,8,9,10,11} \\
31212\underline{31}212\underline{13}212 & P_{6,7}P_{11,12} \\
3\underline{12121}321231212 & \Omega^{-1}_{2,3,4,5,6} \\
32121\underline{232}1231212 & \Phi_{6,7,8} \\
3212\underline{13}2\underline{31}231212 & P_{5,6}P_{8,9} \\
32123121\underline{323}1212 & \Phi^{-1}_{9,10,11} \\
3212312123\underline{21212} & \Omega_{11,12,13,14,15} \\
321231212312121. &
\end{array}
\tag{23}
$$

It reverses the initial reduced expression. There is another route achieving the reverse ordering in the same manner as explained in (15) for F_4. Equating the two ways, substituting (20) – (22) and using $P_{4,7}Y_{1,3,4,9} = Y_{1,3,7,9}P_{4,7}$ etc, we get the H_3 analogue of the tetrahedron equation [10, eq.(9.12)]:

$$
\begin{aligned}
& Y_{11,12,13,14,15}R^{-1}_{15,10,9}R_{5,7,15}Y^{-1}_{15,6,4,3,2}Y_{2,5,8,10,14}R^{-1}_{14,7,3}R^{-1}_{13,9,2}R_{1,6,14} \\
& \times R_{3,8,13}Y^{-1}_{13,10,7,4,1}Y_{1,3,5,9,12}R^{-1}_{12,8,4}R^{-1}_{11,2,1}R_{6,10,12}R_{4,5,11}Y^{-1}_{11,9,8,7,6} \\
& = Y_{6,7,8,9,11}R^{-1}_{11,5,4}R^{-1}_{12,10,6}R_{1,2,11}R_{4,8,12}Y^{-1}_{12,9,5,3,1}Y_{1,4,7,10,13}R^{-1}_{13,8,3} \\
& \times R^{-1}_{14,6,1}R_{2,9,13}R_{3,7,14}Y^{-1}_{14,10,8,5,2}Y_{2,3,4,6,15}R^{-1}_{15,7,5}R_{9,10,15}Y^{-1}_{15,14,13,12,11}.
\end{aligned}
\tag{24}
$$

The two sides have the form of the inverse of each other if the indices within each operator were reversed. There are 3 Y's, 3 Y^{-1}'s, 5 R's and 5 R^{-1}'s on each side.

If $Y^{-1}_{ijklm} = Y_{ijklm} = Y_{mlkji}$ and $R^{-1}_{ijk} = R_{ijk} = R_{kji}$ are valid, the above equation reduces to [10, eq.(9.13)]:

$$
\begin{aligned}
& Y_{11,12,13,14,15}R_{9,10,15}R_{5,7,15}Y_{2,3,4,6,15}Y_{2,5,8,10,14}R_{3,7,14}R_{2,9,13}R_{1,6,14} \\
& \times R_{3,8,13}Y_{1,4,7,10,13}Y_{1,3,5,9,12}R_{4,8,12}R_{1,2,11}R_{6,10,12}R_{4,5,11}Y_{6,7,8,9,11} \\
& = \text{product in reverse order.}
\end{aligned}
\tag{25}
$$

It remains a challenge to construct a solution of (24) or the reduced version (25). A planar graphical representation of (25) has appeared in [4, eq.(4.9)]. In view of $H_3, A_3 \subset H_4$, the H_4 equation should be decomposed into the tetrahedron equation and the H_3 equation similarly to Theorem 4.1.

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