

Representation theory of generalized quantum algebras using Weyl groupoids

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Hamiltonian circuit

Let $\Gamma = (V, E)$ be a graph, where V (resp. E) is the set of vertices (resp. edges).

Let $\varphi : \{1, 2, \dots, |V|\} \rightarrow V$ be a bijection.

We call φ a *Hamiltonian circuit* of Γ if

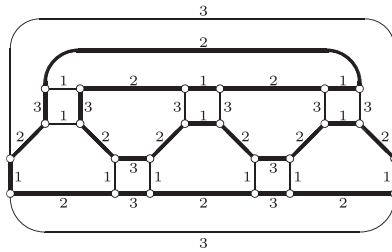
$$\{\varphi(i), \varphi(i + 1)\} \in E \quad (1 \leq i \leq |E| - 1)$$

and $\{\varphi(|E|), \varphi(1)\} \in E$.

In a very beautiful and easy way,

J.H. Conway, N.J.A. Sloane and Allan R. Wilks, (Gray codes for reflection groups, *Graphs and Combinatorics* 5 (1989) 315–325) showed an existence of a Hamiltonian circuit of the Caley graph of every finite Coxeter group $W = \langle s_i | i \in I \rangle$.

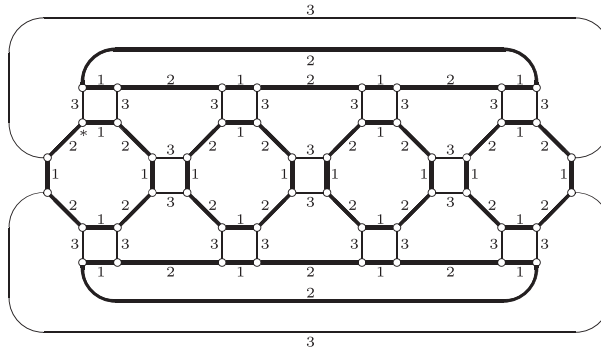
Example 1: Caley graph of Weyl group $W(A_3)$ of type A_3



$$W(A_3) = \langle s_1, s_2, s_3 \rangle$$

$$s_i^2 = e, s_1s_3 = s_3s_1, s_1s_2s_1 = s_2s_1s_2, s_2s_3s_2 = s_3s_2s_3.$$

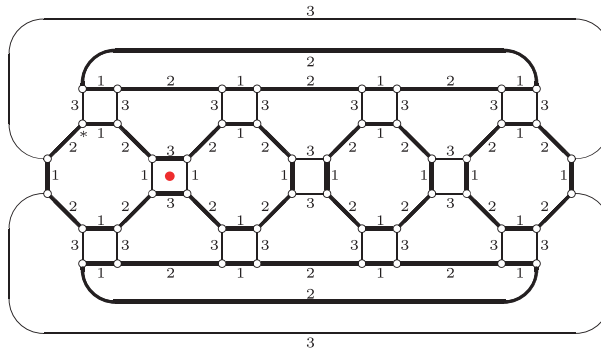
Example 2-process 0: Caley graph of Weyl group $W(B_3)$ of type B_3



$$W(B_3) = \langle s_1, s_2, s_3 \rangle$$

$$s_i^2 = e, s_1 s_3 = s_3 s_1, (s_1 s_2)^2 = (s_2 s_1)^2, s_2 s_3 s_2 = s_3 s_2 s_3.$$

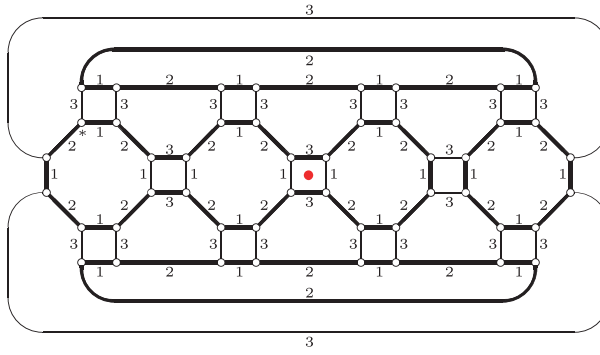
Example 2-process 1: Caley graph of Weyl group $W(B_3)$ of type B_3



$$W(B_3) = \langle s_1, s_2, s_3 \rangle$$

$$s_i^2 = e, s_1 s_3 = s_3 s_1, (s_1 s_2)^2 = (s_2 s_1)^2, s_2 s_3 s_2 = s_3 s_2 s_3.$$

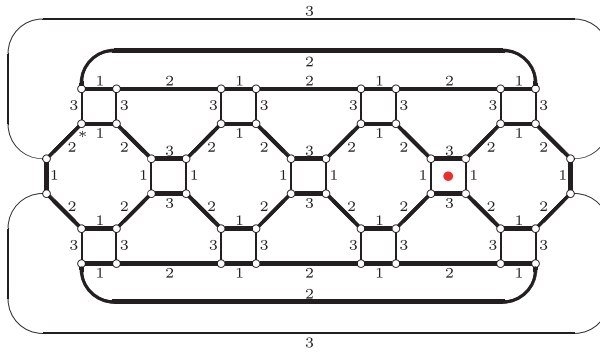
Example 2-process 2: Caley graph of Weyl group $W(B_3)$ of type B_3



$$W(B_3) = \langle s_1, s_2, s_3 \rangle$$

$$s_i^2 = e, s_1 s_3 = s_3 s_1, (s_1 s_2)^2 = (s_2 s_1)^2, s_2 s_3 s_2 = s_3 s_2 s_3.$$

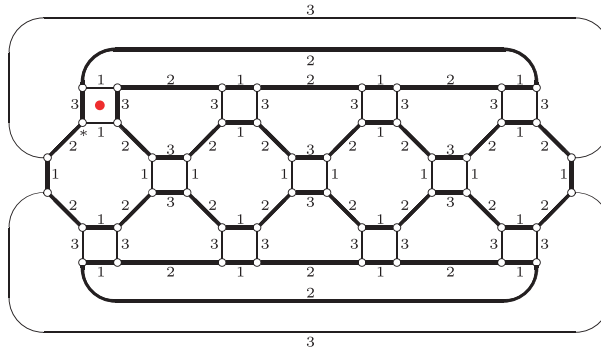
Example 2-process 3: Caley graph of Weyl group $W(B_3)$ of type B_3



$$W(B_3) = \langle s_1, s_2, s_3 \rangle$$

$$s_i^2 = e, s_1 s_3 = s_3 s_1, (s_1 s_2)^2 = (s_2 s_1)^2, s_2 s_3 s_2 = s_3 s_2 s_3.$$

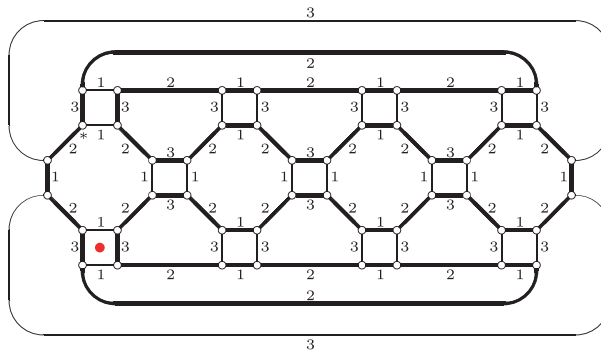
Example 2-process 4: Caley graph of Weyl group $W(B_3)$ of type B_3



$$W(B_3) = \langle s_1, s_2, s_3 \rangle$$

$$s_i^2 = e, s_1 s_3 = s_3 s_1, (s_1 s_2)^2 = (s_2 s_1)^2, s_2 s_3 s_2 = s_3 s_2 s_3.$$

Example 2-process 5-completed: Caley graph of Weyl group $W(B_3)$ of type B_3

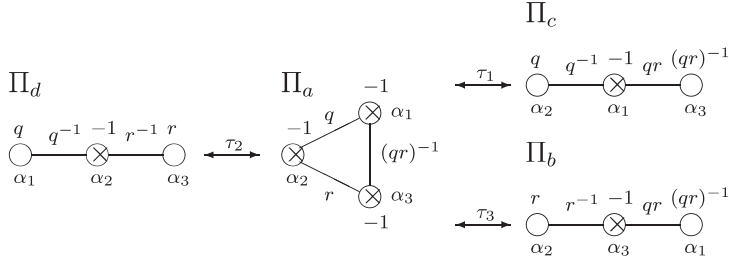


$$W(B_3) = \langle s_1, s_2, s_3 \rangle$$

$s_i^2 = e, s_1 s_3 = s_3 s_1, (s_1 s_2)^2 = (s_2 s_1)^2, s_2 s_3 s_2 = s_3 s_2 s_3$. In a similar but non-easy way, the speaker showed an existence of a Hamiltonian circuit of the Caley graph of the Weyl groupoids associated to the generalized quantum groups by [Hiroyuki Yamane, Hamilton circuits of Cayley graphs of Weyl groupoids of generalized quantum groups, arXiv.2103.16126].

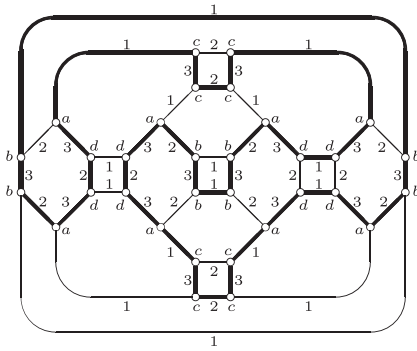
In rank 3 and 4 cases, he exactly obtained Hamiltonian circuits.

Weyl groupoid of the Lie superalgebra $D(2, 1; \alpha)$



$$R_+^{\Pi_d} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3\}. \quad R_+^{\Pi_a} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}.$$

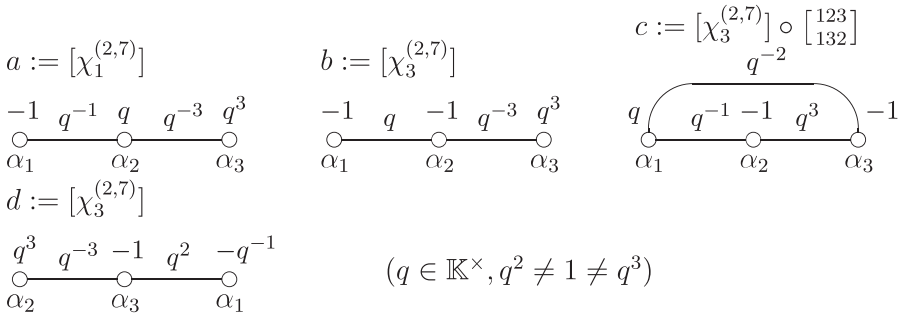
Example 3: Caley graph of Weyl groupoid $W(D(2, 1; \alpha))$ of the Lie superalgebra $D(2, 1; \alpha)$

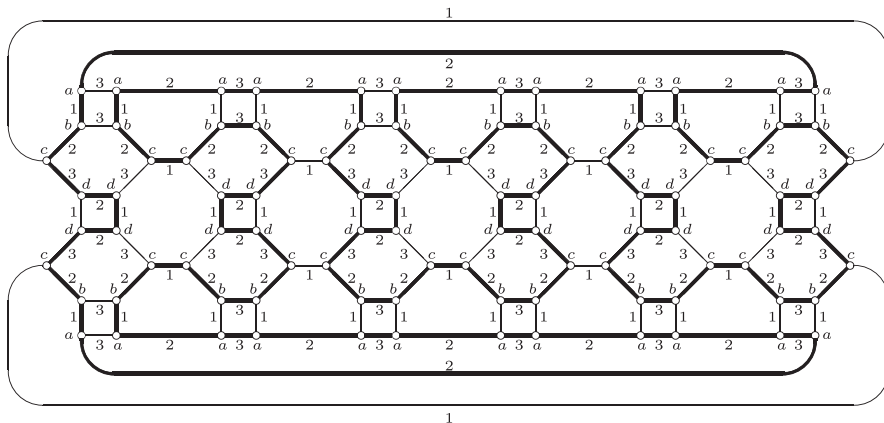


$W = W(D(2, 1; \alpha)) = \langle 0, e^x, s_1^x, s_2^x, s_3^x (x \in \{a, b, c, d\}) \rangle$
 is the semigroup with
 $0w = w0 = 0 (w \in W)$,
 $(e^x)^2 = e^x, e^x e^y = 0 (x \neq y)$,
 $s_1^c s_1^a = e^a, s_2^b s_2^a = e^a, s_3^d s_3^a = e^a$,
 $s_1^b s_1^b = e^b, s_2^a s_2^b = e^b, s_3^c s_3^b = e^b$,
 $s_1^a s_1^c = e^c, s_2^c s_2^c = e^c, s_3^c s_3^c = e^c$,
 $s_1^d s_1^d = e^d, s_2^d s_2^d = e^d, s_3^a s_3^d = e^d$,

$$\begin{aligned} s_1^c s_2^c s_1^a &= s_2^b s_1^b s_2^a, & s_2^b s_3^b s_2^a &= s_3^d s_2^d s_3^a, & s_1^c s_3^c s_1^a &= s_3^d s_1^d s_3^a, \\ s_1^a s_2^b s_1^b &= s_2^c s_1^c s_2^b, & s_2^d s_3^a s_2^b &= s_3^c s_2^c s_3^b, & s_3^c s_1^b &= s_1^b s_3^c, \\ s_1^b s_2^a s_1^c &= s_2^c s_1^c s_2^a, & s_3^c s_2^c &= s_2^c s_3^c, & s_1^d s_3^a s_1^c &= s_3^c s_1^c s_3^a, \\ s_2^d s_1^d &= s_1^d s_2^d, & s_2^a s_3^d s_2^d &= s_3^b s_2^b s_3^d, & s_1^a s_3^d s_1^d &= s_3^b s_1^b s_3^d \end{aligned}$$

Generalized quantum group $U_q(G(3))$ of Lie superalgebra $G(3)$





Generalized root system

Let $n \in \mathbb{N}$. Let \mathfrak{B} be the set of all the \mathbb{Z} -bases of \mathbb{Z}^n .

Let R be a nonempty subset of \mathbb{Z}^n .

Let $\tilde{\mathfrak{B}}$ be the subset of \mathfrak{B} formed by all $\Pi \in \mathfrak{B}$ with $R = R_+^\Pi \cup (-R_+^\Pi)$, where $R_+^\Pi := R \cap \text{Span}_{\mathbb{Z}_{\geq 0}} \Pi$.

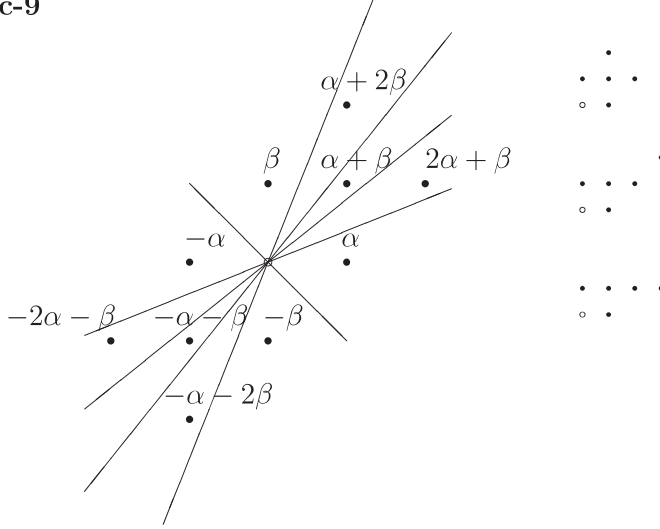
We call R a *finite generalized root system* if

- (R0) $|R| < \infty$ and $\mathfrak{B} \neq \emptyset$.
- (R1) $\forall \Pi \in \tilde{\mathfrak{B}}, \Pi \subset R$
- (R2) $\forall \Pi \in \tilde{\mathfrak{B}}, \forall \alpha \in \Pi, \mathbb{Z}\alpha \cap R = \{\pm\alpha\}$
- (R3) $\forall \Pi \in \tilde{\mathfrak{B}}, \forall \alpha \in \Pi, \exists \Pi^{(\alpha)} \in \tilde{\mathfrak{B}}, R_+^{\Pi^{(\alpha)}} \cap (-R_+^\Pi) = \{-\alpha\}$

Namely $\Pi^{(\alpha)} = \{-\alpha\} \cup \{\beta + N_{\alpha,\beta}^\Pi \alpha \mid \beta \in \Pi \setminus \{\alpha\}\}$,

where $N_{\alpha,\beta}^\Pi = \max\{k \in \mathbb{Z}_{\geq 0} \mid \beta + k\alpha \in R_+^\Pi\}$. Let $N_{\alpha,\alpha}^\Pi := -2$. Then $\Pi^{(\alpha)} = \{\beta + N_{\alpha,\beta}^\Pi \alpha \mid \beta \in \Pi\}$.

Rank-2-Hec-9



Let $\Pi^{(\beta_1, \dots, \beta_k)}$ denote $\Pi^{(\beta_1)}$ (if $k = 1$) and $(\Pi^{(\beta_1, \dots, \beta_{k-1})})^{(\beta_k)}$ (if $k \geq 2$).

There exist bijections $\pi_\Pi : \{1, 2, \dots, n\} \rightarrow \Pi$ ($\Pi \in \mathfrak{B}$) such that for $1 \leq i \leq n$, $\pi_{\Pi(\pi(\Pi)(i))}(j) = \pi_\Pi(j) + N_{\pi(\Pi)(i), \pi(\Pi)(j)}^\Pi \pi_\Pi(i)$.

In particular, if $\Pi^{(\beta_1, \dots, \beta_k)} = \Pi$ happens, then $\pi_{\Pi^{(\beta_1, \dots, \beta_k)}} = \pi_\Pi$ as a map, that is, $\pi_{\Pi^{(\beta_1, \dots, \beta_k)}}(i) = \pi_\Pi(i)$ for all $1 \leq i \leq n$. Let $\mathfrak{B}^\sharp := \{\pi_\Pi \mid \Pi \in \mathfrak{B}\}$. For $1 \leq i \leq n$, define the map $\tau_i : \mathfrak{B}^\sharp \rightarrow \mathfrak{B}^\sharp$ by $\tau_i(\pi_\Pi) := \pi_{\Pi(\pi(\Pi)(i))}$. Then $\tau_i^2 = \text{id}$. Moreover

$$\underbrace{\tau_i \tau_j \tau_i \cdots}_{m_{ij}}(\pi_\Pi) = \underbrace{\tau_j \tau_i \tau_j \cdots}_{m_{ij}}(\pi_\Pi),$$

where $m_{ij} = m_{ij}^\Pi := |(\mathbb{Z}_{\geq 0} \pi_\Pi(i) \oplus \mathbb{Z}_{\geq 0} \pi_\Pi(j)) \cap R| < \infty$, for $1 \leq i \neq j \leq n$.

The Cayley graph of the Weyl groupoid $W(R)$ (defined by the next slide) is the graph $\Gamma(R) = (V(R), E(R))$ with $V(R) := \mathfrak{B}^\sharp$ and $E(R) := \{\{\pi, \tau_i(\pi)\} \mid \pi \in \mathfrak{B}^\sharp, 1 \leq i \leq n\}$. The (universal) Weyl groupoid $W(R)$ is the groupoid ‘formed’ by the base change matrices

$$S_i^{\pi, \tau_i(\pi)} \in \text{GL}_n(\mathbb{Z}) \quad (1 \leq i \leq n, \pi \in \mathfrak{B}^\sharp)$$

defined by

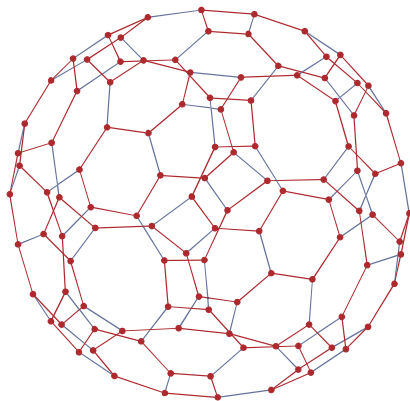
$$[\tau_i(\pi)(1), \dots, \tau_i(\pi)(n)] = [\pi(1), \dots, \pi(n)] S_i^{\pi, \tau_i(\pi)}.$$

We have $S_i^{\pi, \tau_i(\pi)} = (S_i^{\pi, \tau_i(\pi)})^{-1} = S_i^{\tau_i(\pi), \pi}$, $S_i^{\pi, \tau_i(\pi)} S_i^{\tau_i(\pi), \pi} = E$ and

$$\underbrace{S_i^{\pi, \tau_i(\pi)} S_j^{\tau_i(\pi), \tau_j \tau_i(\pi)} \dots}_{m_{ij}} = \underbrace{S_j^{\pi, \tau_j(\pi)} S_i^{\tau_j(\pi), \tau_i \tau_j(\pi)} \dots}_{m_{ij}} \quad (i \neq j).$$

These are exactly the defining relations of $W(R)$ (I. Heckenberger, H. Yamane, A generalization of Coxeter groups, root systems, and Matsumoto's theorem, *Math. Z.* 259 (2008), 255-276, see also H. Yamane, Generalized root systems and the affine Lie superalgebra $G^{(1)}(3)$, *São Paulo J. Math. Sci.* 10 (2016), no. 1, 9-19.). Cuntz-Heckenberger classified Weyl groupoids. (M. Cuntz and I. Heckenberger, Finite Weyl groupoids, *J. Reine Angew. Math.* 702 (2015) 77-108.) (M. Cuntz and I. Heckenberger, Finite Weyl groupoids of rank three, *Trans. Amer. Math. Soc.* 364 (2012), 1369-1393) In Rank=3, 55-cases exists; among them, 3-cases are associated with simple complex Lie algebras, 3-cases are associated with simple complex Lie superalgebras of type $A - G$, and 18-cases are associated with irreducible generalized quantum groups.

Quite recently, I found Hamiltonian circuits for some of their classification of Rank=3 using Wolfram Mathematica 12.3.



This is made by Wolfram Mathematica 12.3. The 15-th generalized root system
of the list of the Cuntz-Heckenberger's paper.

Generalized Quantum Group $U(\chi)$

Let $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. Let I be a nonempty set. Let $\mathbb{Z}\Pi$ be a free \mathbb{Z} -module with a base $\Pi = \{\alpha_i | i \in I\}$. Then the rank of $\mathbb{Z}\Pi$ is $|I|$. Let $\chi : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \rightarrow \mathbb{K}^\times$ be a map with

$$\chi(\lambda + \mu, \nu) = \chi(\lambda, \nu)\chi(\mu, \nu), \chi(\lambda, \mu + \nu) = \chi(\lambda, \mu)\chi(\lambda, \nu) \quad (\lambda, \mu, \nu \in \mathbb{Z}\Pi).$$

If $\chi(\alpha_i, \alpha_j) = q^{a_{ij}}$, it is usual quantum groups.

If $\chi(\alpha_i, \alpha_j) = (-1)^{p(i)p(j)}q^{a_{ij}}$, it is usual quantum supergroups.

To such χ , we can associate an associative \mathbb{K} -algebra (with 1) defined by the following axioms (U1)-(U6).

(U1) U has generators K_λ, L_λ (=almost $K_{-\lambda}$ ($\lambda \in \mathbb{Z}\Pi$), E_i, F_i ($i \in I$).

(U2) The following equations hold. $K_0 = L_0 = 1, K_\lambda K_\mu = K_{\lambda+\mu}, L_\lambda L_\mu = L_{\lambda+\mu}, K_\lambda L_\mu = L_\mu K_\lambda, K_\lambda E_i = \chi(\lambda, \alpha_i) E_i K_\lambda,$
 $K_\lambda F_i = \chi(\lambda, -\alpha_i) F_i K_\lambda, L_\lambda E_i = \chi(-\alpha_i, \lambda) E_i L_\lambda, L_\lambda F_i = \chi(\alpha_i, \lambda) F_i L_\lambda, E_i F_j - F_j E_i = \delta_{ij}(-K_{\alpha_i} + L_{\alpha_i}).$

(U3) We have an injection $\varsigma_1 : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \rightarrow U, (\lambda, \mu) \mapsto K_\lambda L_\mu,$ and $\varsigma_1(\mathbb{Z}\Pi \times \mathbb{Z}\Pi)$ is an linearly independent set.

(U4) Let $U^0 := \text{Span}_{\mathbb{K}} \varsigma_1(\mathbb{Z}\Pi \times \mathbb{Z}\Pi)$. Let U^+ (resp. U^-) be the \mathbb{K} -subalgebra (with 1) of U generated by E_i (resp. F_i). Then we have the \mathbb{K} -linear isomorphism $\varsigma_2 : U^- \otimes U^0 \otimes U^+ \rightarrow U$ defined by $\varsigma_2(Y \otimes Z \otimes X) := YZX$.

(U5) We have the \mathbb{K} -linear subspaces U_λ ($\lambda \in \mathbb{Z}\Pi$) of U . $U^0 \subset U_0, E_i \in U_{\alpha_i}, F_i \in U_{-\alpha_i}$ ($i \in I$), $U = \bigoplus_{\lambda \in \mathbb{Z}\Pi} U_\lambda, U_\mu U_\nu \subset U_{\mu+\nu}$ ($\mu, \nu \in \mathbb{Z}\Pi$).

(U6) Let $U_\lambda^\pm := U_\lambda \cap U^\pm$ ($\lambda \in \mathbb{Z}\Pi$). Let $\mathbb{Z}\Pi^+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ ($\subset \mathbb{Z}\Pi$). Then for any $\lambda \in \mathbb{Z}\Pi^+ \setminus \{0\}$, we have $\{X \in U_\lambda^+ | \forall i \in I, XF_i = F_i X\} = \{0\}$ and $\{Y \in U_{-\lambda}^- | \forall i \in I, YE_i = E_i Y\} = \{0\}$. (We have $\dim U_\lambda^+ = \dim U_{-\lambda}^-$ ($\lambda \in \mathbb{Z}\Pi^+$)).

Let $q_\lambda := \chi(\lambda, \lambda)$ ($\lambda \in \mathbb{Z}\Pi$). Define the map $\text{ord} : \mathbb{Z}\Pi \rightarrow \mathbb{N} \cup \{\infty\}$ by $\text{ord}(\lambda) := |\{\sum_{r=0}^k q_\lambda^r | k \in \mathbb{Z}_{\geq 0}\}|$. Let $U^{+'} := (\sum_{\lambda \in \mathbb{Z}\Pi^+ \setminus \{0\}} U_\lambda^+) \setminus \{0\}$. Define the map $\text{deg} : U^{+'} \rightarrow \mathbb{Z}\Pi^+ \setminus \{0\}$ by $\text{deg}(X) := \lambda$ ($X \in U_\lambda^+ \cap U^{+'}, \lambda \in \mathbb{Z}\Pi^+ \setminus \{0\}$).

Kharchenko-PBW Theorem (1999) There exist a nonempty subset S of $U^{+'}$ and a total order \preceq satisfying:

$\{X_1^{n_1} X_2^{n_2} \cdots X_k^{n_k} | X_t \in S, 0 \leq n_t < \text{ord}(\text{deg}(X_t)) (1 \leq t \leq k), X_1 < X_2 < \cdots < X_k\}$ form a \mathbb{K} -basis of U^+ . Note that $\alpha_i \in R_\chi^+$ ($i \in I$).

Moreover the set $R_\chi^+ := \text{deg}(S) (\subset \mathbb{Z}\Pi^+ \setminus \{0\})$ is independent from S .

From now on, we assume $|R_\chi^+| < \infty$. Then $\text{deg}_{i,S}$ is injective.

Weyl-Kac-type typical irreducible character formulas of $U(\chi)$

Assume that $q_\beta := \chi(\beta, \beta) \neq 1$ for all $\beta \in R_\chi$, where this condition is not essential since $\text{Char}(\mathbb{C}) = 0$. Let $\text{ord}(\beta) := \min\{k \in \mathbb{N} | 1 + q_\beta + \dots + q_\beta^k = 0\}$; if $\text{ord}(\beta)$ can not be defined, let $\text{ord}(\beta) := \infty$.

Let $R_\chi^{+, \text{fin}} := \{\beta \in R_\chi^+ | \text{ord}(\beta) < \infty\}$. Let $R_\chi^{+, \text{inf}} := \{\beta \in R_\chi^+ | \text{ord}(\beta) = \infty\}$. Then $R_\chi^+ = R_\chi^{+, \text{fin}} \cup R_\chi^{+, \text{inf}}$.

For $\beta \in R_\chi^{+, \text{inf}}$ and $\lambda \in \mathbb{Z}\Pi$, we have $k_\lambda^\beta \in \mathbb{Z}$ with $\chi(\beta, \lambda)\chi(\lambda, \beta) = q_\beta^{k_\lambda^\beta}$.

For $\beta \in R_\chi^{+, \text{inf}}$, define the \mathbb{Z} -module $s_\beta : \mathbb{Z}\Pi \rightarrow \mathbb{Z}\Pi$ by $s_\beta(\lambda) := \lambda - k_\lambda^\beta \beta$ ($\lambda \in \mathbb{Z}\Pi$).

Let $W := \langle s_\beta | \beta \in R_\chi^{+, \text{inf}} \rangle$. In fact, W is a Weyl group. We have the group homomorphism $\text{sgn} : W \rightarrow \{\pm 1\} \subset \mathbb{Z}$ with $\text{sgn}(s_\beta) = -1$.

Define the \mathbb{Z} -module homomorphism $\hat{\rho} : \mathbb{Z}\Pi \rightarrow \mathbb{K}^\times$ by $\hat{\rho}(\alpha_i) := q_{\alpha_i}$ ($i \in I$).

For $\beta \in R_\chi^{+, \text{inf}}$, we have $r_\beta \in \mathbb{Z}$ with $\hat{\rho}(\beta) = q_\beta^{r_\beta}$.

Let $\Lambda : U^0 \rightarrow \mathbb{C}$ be a \mathbb{C} -algebra homomorphism.

Let $\mathcal{L}(\Lambda)$ be a left irreducible U -module for which we have $v_\Lambda \in \mathcal{L}(\Lambda) \setminus \{0\}$ with $E_i v_\Lambda = 0$ ($i \in I$), $Z v_\Lambda = \Lambda(Z) v_\Lambda$ ($Z \in U^0$) and letting $\mathcal{L}(\Lambda)_\lambda := U_\lambda v_\Lambda$ ($\lambda \in \mathbb{Z}\Pi$), we also have $\mathcal{L}(\Lambda) = \bigoplus_{\lambda \in \mathbb{Z}\Pi} \mathcal{L}(\Lambda)_\lambda$.

From now on, assume $\dim \mathcal{L}(\Lambda) < \infty$.

For $\beta \in R_\chi^{+, \text{inf}}$, we have $n_\beta^\Lambda \in \mathbb{Z}_{\geq 0}$ with $\Lambda(K_\beta L_{-\beta}) = q_\beta^{n_\beta^\Lambda}$.

Then W has another action \cdot on $\mathbb{Z}\Pi$ defined by $s_\beta \cdot \lambda := s_\beta(\lambda) - (r_\beta + n_\beta^\Lambda)\beta$ ($\beta \in R_\chi^{+, \text{inf}}$, $\lambda \in \mathbb{Z}\Pi$).

Theorem (H. Yamane, J. Algebra Appl. 20(1) (2021) 2140014) Assume that $\hat{\rho}(\beta)\Lambda(K_\beta L_{-\beta}) \notin \{q_\beta^t | 1 \leq t \in \text{ord}(\beta) - 1\}$ for all $\beta \in R_\chi^{+, \text{fin}}$. Then we have

$$\dim \mathcal{L}(\Lambda)_\lambda = \sum_{w \in W} \text{sgn}(w) \dim U_{\lambda - w \cdot 0}^-$$

for $\lambda \in \mathbb{Z}\Pi$.

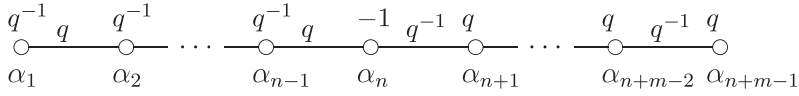
Classification of finite dimensional $\mathcal{L}(\Lambda)$

Let $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. Let $\mathbb{C}_{<\infty}^\times := \{\zeta \in \mathbb{C}^\times | 2 \leq \exists k < \infty, \zeta^k = 1\}$ and $\mathbb{C}_\infty^\times := \{q \in \mathbb{C}^\times | \forall k < \infty, q^k \neq 1\}$. Let $R_\chi^{+, \text{fin}} := \{\alpha \in R_\chi^+ | \chi(\alpha, \alpha) \in \mathbb{C}_{<\infty}^\times\}$ and $R_\chi^{+, \text{inf}} := \{\alpha \in R_\chi^+ | \chi(\alpha, \alpha) \in \mathbb{C}_\infty^\times\}$. Let $q \in \mathbb{C}_{<\infty}^\times$.

(1) Let U be the one associated with the Lie superalgebra $A(m-1, n-1)$. Then $\Pi = \{\alpha_i := \epsilon_i - \epsilon_{i+1} | 1 \leq i \leq n+m-1\}$, $R_\chi^{+, \text{fin}} = \{\epsilon_i - \epsilon_j | 1 \leq i \leq n, n+1 \leq j \leq n+m\}$ and $R_\chi^{+, \text{inf}} = \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq n\}$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions (a1)-(a2) is satisfied.

(a1) $1 \leq \forall i \leq n-1, \exists k_i \in \mathbb{Z}_{\geq 0}, \Lambda(K_{\alpha_i} L_{-\alpha_i}) = q^{-k_i}$

(a2) $n+1 \leq \forall i \leq n+m-1, \exists k_i \in \mathbb{Z}_{\geq 0}, \Lambda(K_{\alpha_i} L_{-\alpha_i}) = q^{k_i}$



Generalized Dynkin diagram of $A(m-1, n-1)$

(2) Let U be the one associated with the Lie superalgebra $B(m, n)$. Then $\Pi = \{\alpha_i := \epsilon_i - \epsilon_{i+1} | 1 \leq i \leq n+m-1\} \cup \{\alpha_{n+m} := \epsilon_{n+m}\}$, $R_\chi^{+, \text{fin}} = \{\epsilon_i \pm \epsilon_j | 1 \leq i \leq n, n+1 \leq j \leq n+m\}$ and $R_\chi^{+, \text{inf}} = \{\epsilon_i | 1 \leq i \leq n+m\} \cup \{\epsilon_i \pm \epsilon_j | 1 \leq i < j \leq n\} \cup \{\epsilon_i \pm \epsilon_j | n+1 \leq i < j \leq n+m\}$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions (b1)-(b5) is satisfied. Let $\beta := \alpha_n + \alpha_{n+1} + \dots + \alpha_{m+n}$.

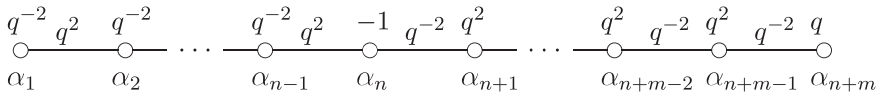
(b1) $1 \leq \forall i \leq n-1, \exists k_i \in \mathbb{Z}_{\geq 0}, \Lambda(K_{\alpha_i} L_{-\alpha_i}) = q^{-2k_i}$

(b2) $n+1 \leq \forall i \leq n+m-1, \exists k_i \in \mathbb{Z}_{\geq 0}, \Lambda(K_{\alpha_i} L_{-\alpha_i}) = q^{2k_i}$

(b3) $\exists k \in \mathbb{Z}_{\geq 0}, \Lambda(K_{\alpha_{m+n}} L_{-\alpha_{m+n}}) = q^k$

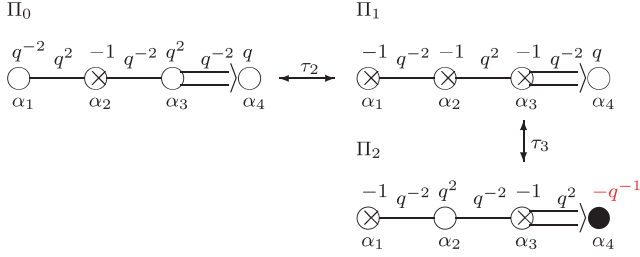
(b4) $\Lambda(K_\beta L_\beta^{-1}) \in \{q^{-2k} | 0 \leq k \leq m-1\} \cup \{(-q)^{-t-2m} | t \in \mathbb{Z}_{\geq 0}\}$

(b5) If $\Lambda(K_\beta L_\beta^{-1}) = q^{-2k}$ for some $0 \leq k \leq m-1$, then $\Lambda(K_{\alpha_i} L_{\alpha_i}^{-1}) = 1$ for all $n+k+1 \leq i \leq n+m$.



Generalized Dynkin diagram of $B(m, n)$

If t of (b4) is an odd integer, we cannot take $q \rightarrow 1$. In that case Λ is typical.

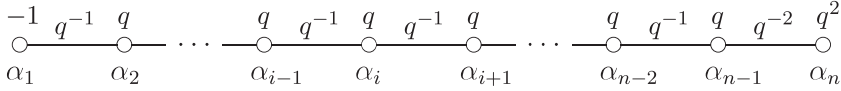


$$w_0 = s_3^{B_0} s_4^{B_0} s_3^{B_0} s_4^{B_0} s_1^{B_0} s_2^{B_0} s_3^{B_1} s_4^{B_2} s_3^{B_2} s_4^{B_2} s_3^{B_1} s_2^{B_0} s_1^{B_0} s_2^{B_1} s_3^{B_2} s_4^{B_2} s_3^{B_1} s_2^{B_0}$$

(3) Let U be the one associated with the Lie superalgebra $C(n)$. Then $\Pi = \{\alpha_i := \epsilon_i - \epsilon_{i+1} | 1 \leq i \leq n-1\} \cup \{\alpha_n := 2\epsilon_n\}$, $R_\chi^{+, \text{fin}} = \{\epsilon_1 \pm \epsilon_j | 2 \leq j \leq n\}$ and $R_\chi^{+, \text{inf}} = \{\epsilon_i \pm \epsilon_j | 2 \leq i < j \leq n\} \cup \{2\epsilon_i | 2 \leq i \leq n\}$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions (a1)-(a2) is satisfied.

(c1) $2 \leq \forall i \leq n-1, \exists k_i \in \mathbb{Z}_{\geq 0}, \Lambda(K_{\alpha_i} L_{-\alpha_i}) = q^{k_i}$.

(c2) $\exists k_n \in \mathbb{Z}_{\geq 0}, \Lambda(K_{\alpha_n} L_{-\alpha_n}) = q^{2k_n}$.



Generalized Dynkin diagram of $C(n)$

(4) Let U be the one associated with the Lie superalgebra $D(m, n)$. Then $\Pi = \{\alpha_i := \epsilon_i - \epsilon_{i+1} | 1 \leq i \leq n+m-1\} \cup \{\alpha_{n+m} := \epsilon_{n+m-1} + \epsilon_{n+m}\}$, $R_\chi^{+, \text{fin}} = \{\epsilon_i \pm \epsilon_j | 1 \leq i \leq n, n+1 \leq j \leq n+m\}$ and $R_\chi^{+, \text{inf}} = \{\epsilon_i \pm \epsilon_j | 1 \leq i < j \leq n\} \cup \{\epsilon_i \pm \epsilon_j | n+1 \leq i < j \leq n+m\}$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions (d1)-(d5) is satisfied. Let $\beta := 2\alpha_n + 2\alpha_{n+1} + \dots + 2\alpha_{n+m-2} + \alpha_{n+m-1} + \alpha_{m+n}$ and $\gamma := \alpha_n + \alpha_{n+1} + \dots + \alpha_{n+m-1}$.

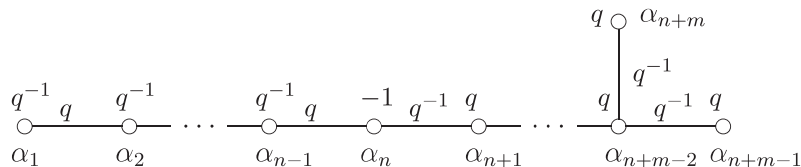
(d1) $1 \leq \forall i \leq n-1, \exists k_i \in \mathbb{Z}_{\geq 0}, \Lambda(K_{\alpha_i} L_{-\alpha_i}) = q^{-k_i}$

(d2) $n+1 \leq \forall i \leq n+m, \exists k_i \in \mathbb{Z}_{\geq 0}, \Lambda(K_{\alpha_i} L_{-\alpha_i}) = q^{k_i}$

(d3) $\exists k \in \mathbb{Z}_{\geq 0}, \Lambda(K_\beta L_{-\beta}) = q^{-2k}$

(d4) If $\Lambda(K_\beta L_\beta^{-1}) = q^{-2k}$ for some $0 \leq k \leq m-2$, then $\Lambda(K_\gamma L_\gamma^{-1}) = q^{-k}$ and $\Lambda(K_{\alpha_i} L_{\alpha_i}^{-1}) = 1$ for all $n+k+1 \leq i \leq n+m$.

(d5) If $\Lambda(K_\beta L_\beta^{-1}) = q^{-2(m-1)}$, then $\Lambda(K_\gamma L_\gamma^{-1}) = q^{-(m-1)}$.



Generalized Dynkin diagram of $D(m, n)$

Assume $k \geq m$ for (d3). Then $\Lambda(K_{\alpha_n} L_{-\alpha_n}) = \pm q^{-r}$ for some $r \in \mathbb{N}$. If $\Lambda(K_{\alpha_n} L_{-\alpha_n}) = -q^{-r}$, we cannot take $q \rightarrow 1$, and Λ is typical. Let $k \in \mathbb{N}$ and $u \in \{0, 1\}$. Assume $\Lambda(K_{\alpha_i} L_{-\alpha_i}) = q^{-d_i}$ ($1 \leq i \leq N - m - 1$, $\exists d_i \in \mathbb{Z}_{\geq 0}$). Assume $\Lambda(K_{\alpha_{N-m+1}} L_{-\alpha_{N-m+1}}) = \dots = \Lambda(K_{\alpha_N} L_{-\alpha_N}) = q^k$. Assume $\Lambda(K_{\alpha_{N-m}} L_{-\alpha_{N-m}}) = (-1)^u q^{-m(k+1)}$.

Let $\lambda := \alpha_{N-m} + \alpha_{N-m+1} + \dots + \alpha_{N-2} + \alpha_{N-1} + \alpha_N$.

If $u = 0$, then $\mathcal{L}(\Lambda)$ is NOT typical and $\dim \mathcal{L}(\Lambda)_{-\lambda} < \dim \mathcal{M}(\Lambda)_{-\lambda}$. If $u = 1$, then $\mathcal{L}(\Lambda)$ is typical and $\dim \mathcal{L}(\Lambda)_{-\lambda} = \dim \mathcal{M}(\Lambda)_{-\lambda}$. (5) Let U be the one associated with the Lie superalgebra $D(2, 1; \alpha)$. Let $q \in \mathbb{C}_\infty^\times$. Let $q_\beta := \chi(\beta, \beta)$. Let $\Lambda_\beta := \Lambda(K_\beta L_\beta^{-1})$.

(i) $r \in \mathbb{C}_\infty^\times$ and $qr \in \mathbb{C}_\infty^\times$.

(ii) $r \in \mathbb{C}_\infty^\times$ and $qr \in \mathbb{C}_{<\infty}^\times$.

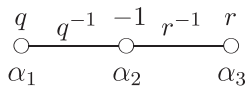
(iii) $r \in \mathbb{C}_{<\infty}^\times$ and $qr \in \mathbb{C}_\infty^\times$.

Let $\alpha_0 := \alpha_1 + 2\alpha_2 + \alpha_3$. Then $q_{\alpha_0} = q^{-1}r^{-1}$, $q_{\alpha_1} = q$, $q_{\alpha_2} = -1$, $q_{\alpha_3} = r$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions (da1)-(da3) is satisfied.

(da1) For $i \in \{0, 1, 3\}$, if $q_{\alpha_i} \in \mathbb{C}_\infty^\times$, then $\Lambda_{\alpha_i} = q_{\alpha_i}^{k_i}$ for some $k_i \in \mathbb{Z}_{\geq 0}$.

(da2) If $q_{\alpha_i} \in \mathbb{C}_\infty^\times$ and $k_0 = 0$, then $\Lambda_{\alpha_j} = 1$ for $j \in \{1, 2, 3\}$.

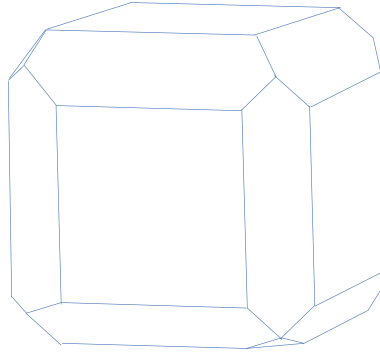
(da3) If $q_{\alpha_i} \in \mathbb{C}_\infty^\times$ and $k_0 = 1$, then $\Lambda_{\alpha_2} = 1$ or $\Lambda_{\alpha_1+\alpha_2} = q^{-1}$.



Generalized Dynkin diagram of $D(2, 1; \alpha)$

The following a shape of a finite-dimensional typical irreducible module of U

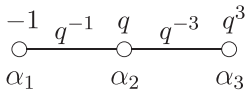
of $D(2, 1; \alpha)$.



(6) Let U be the one associated with the Lie superalgebra $G(3)$.

Let $\alpha_0 := \alpha_1 + 2\alpha_2 + \alpha_3$. Then $q_{\alpha_0} = -q^{-1}$, $q_{\alpha_1} = -1$, $q_{\alpha_2} = q$, $q_{\alpha_3} = q^3$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions (g1)-(g3) is satisfied.

- (g1) For $i \in \{0, 2, 3\}$, then $\Lambda_{\alpha_i} = q^{k_i}$ for some $k_i \in \mathbb{Z}_{\geq 0}$.
- (g2) $k_0 \in \{0, 4\}$ or $k_0 \geq 6$.
- (g3) If $k_0 = 0$, then $\Lambda_{\alpha_j} = 1$ for $j \in \{1, 2, 3\}$.
- (g4) If $k_0 = 4$, then $\Lambda_{\alpha_2} = 1$.



Generalized Dynkin diagram of $G(3)$

(7) Let U be the one associated with the Lie superalgebra $F(4)$.

Let $\alpha_0 := 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$. Then $q_{\alpha_0} = q^{-3}$, $q_{\alpha_1} = -1$, $q_{\alpha_2} = q$, $q_{\alpha_3} = q_{\alpha_4} = q^2$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions (f1)-(f4) is satisfied.

- (f1) For $i \in \{0, 2, 3, 4\}$, then $\Lambda_{\alpha_i} = q^{k_i}$ for some $k_i \in \mathbb{Z}_{\geq 0}$.

- (f2) $k_0 \in \{0, 2, 3\}$ or $k_0 \geq 6$.
 (f3) If $k_0 = 0$, then $\Lambda_{\alpha_j} = 1$ for $j \in \{1, 2, 3, 4\}$.
 (f4) If $k_0 = 2$, then $\Lambda_{\alpha_2} = \Lambda_{\alpha_4} = 1$ and $\Lambda_{\alpha_1+\alpha_3} = q^{-3}$.
 (f4) If $k_0 = 3$, then $\Lambda_{\alpha_1+\alpha_2+2\alpha_4} = q^{-6}$ and $\Lambda_{\alpha_2} = q\Lambda_{\alpha_4}$.

$$\begin{array}{ccccccc} & -1 & & q & & q^{-2} & & q^2 & & q^{-2} & & q^2 \\ & \circ & & \circ & & \circ & & \circ & & \circ & & \circ \\ \alpha_1 & & & \alpha_2 & & \alpha_3 & & \alpha_4 & & & & \end{array}$$

Generalized Dynkin diagram of $F(4)$

- (8) Let U be the one associated with the Nichols algebra Hec-14.

Let $\alpha_0 := \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$. Then $q_{\alpha_0} = -q^{-1}$, $q_{\alpha_1} = q_{\alpha_2} = q$, $q_{\alpha_3} = -1$, $q_{\alpha_4} = -q^{-1}$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions

(h1)-(h4) is satisfied.

- (h1) For $i \in \{0, 1, 2, 4\}$, then $\Lambda_{\alpha_i} = q_{\alpha_i}^{k_i}$ for some $k_i \in \mathbb{Z}_{\geq 0}$.
 (h2) If $k_0 = 0$, then $\Lambda_{\alpha_j} = 1$ for $j \in \{1, 2, 3, 4\}$.
 (h3) If $k_0 = 1$, then $\Lambda_{\alpha_2} = \Lambda_{\alpha_3} = 1$.
 (h4) If $k_0 = 1$, then $\Lambda_{\alpha_3} = 1$ or $\Lambda_{\alpha_1+\alpha_3+\alpha_4} = 1$

$$\begin{array}{ccccccc} & q & & q^{-1} & & q & & q^{-1} & & -1 & & -q & & -q^{-1} \\ & \circ & & \circ & & \circ & & \circ & & \circ & & \circ & & \circ \\ \alpha_1 & & & \alpha_2 & & \alpha_3 & & \alpha_4 & & & & & & \end{array}$$

Generalized Dynkin diagram of Hec-14

- (9) Let U be the one associated with the Nichols algebra $\mathbb{Z}/3\mathbb{Z}$ -Quantum Group.

Let $\zeta \in \mathbb{C}^\times$ be such that $\zeta^2 + \zeta + 1 = 0$.

Let $\alpha_0 := 2\alpha_2 + \alpha_1$. Then $q_{\alpha_0} = \zeta q^{-1}$, $q_{\alpha_0} = q$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions (z1)-(z2) is satisfied.

- (z1) For $i \in \{0, 2\}$, then $\Lambda_{\alpha_i} = q_{\alpha_i}^{k_i}$ for some $k_i \in \mathbb{Z}_{\geq 0}$.
 (z2) $k_0 \neq 1$.
 (z2) If $k_0 = 0$, then $\Lambda_{\alpha_j} = 1$ for $j \in \{1, 2\}$.

$$\begin{array}{ccc} \zeta & & \\ \circ & \xrightarrow{q^{-1}} & \circ \\ \alpha_1 & & \alpha_2 \end{array} \quad \zeta^2 + \zeta + 1 = 0$$

Generalized Dynkin diagram of $\mathbb{Z}/3\mathbb{Z}$ -Quantum Group

Theorem (Angiono-Y. 2015) Explicit construction of the universal R-matrix $\mathcal{R} \in U \otimes U$. ($\mathcal{R}\Delta(X)\mathcal{R}^{-1} = \Delta'(X)$ ($X \in U$), $(\Delta \otimes I)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$, $(I \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$. $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$.) “ $\mathcal{R} = (\prod_{k=1}^r \exp_q(\dot{E}_k \otimes \dot{F}_k)) \exp(ht_0)$ ”

Shapovalov determinants (Heckenberger-Y. 2010)

Best result from Weyl groupoids In 2010, Heckenberger and I gave a factorization formula of the Shapovalov determinants of **the Lusztig’s small quantum groups at ANY root of unity**, and the quantum superalgebras, the generalized quantum groups.

$$S : U^+ \otimes U^- \xrightarrow{\text{‘identity’}} U = U^- \otimes U^0 \otimes U^+ \xrightarrow{\text{‘projection’}} U^0$$

Let $S_\lambda := S_{|U_\lambda^+ \otimes U_{-\lambda}^-}$.

$$\det \text{Im} S_\lambda = z \cdot \prod_{\alpha \in R^+} \prod_{t_\alpha=1}^{\infty} (-\hat{\rho}(\alpha) \chi(\alpha, \alpha)^{-t_\alpha} K_\alpha + L_\alpha)^{P_\lambda(\alpha, t_\alpha)},$$

where $\hat{\rho} = \hat{\rho}^{\chi, \Pi} : \mathbb{Z}\Pi \rightarrow \mathbb{C}^\times$ be the group homomorphism defined by $\hat{\rho}(\alpha_i) := \chi(\alpha_i, \alpha_i)$ ($i \in I$).

Example for $\lambda = 2\alpha_1 + \alpha_2$ Denote ‘projection’ by Ω . Let $K_i := K_{\alpha_i}$ and $L_i := L_{\alpha_i}$. $S(E_1 \otimes F_1) = \Omega(E_1 F_1) = \Omega(F_1 E_1 - K_1 + L_1) = -K_1 + L_1$. $S(E_1 \otimes F_2) = \Omega(E_1 F_2) = \Omega(F_2 E_1) = 0$. $\Omega(E_1 E_2 F_1 F_2) = \Omega((-K_1 + L_1) E_2 F_2) = (-K_1 + L_1)(-K_2 + L_2)$. $\Omega(E_1 E_2 F_2 F_1) = \Omega(E_1(-K_2 + L_2) F_1) = \Omega(E_1 F_1(-q_{21}^{-1} K_2 + q_{12} L_2)) = (-K_1 + L_1)(-q_{21}^{-1} K_2 + q_{12} L_2)$.

$$\begin{aligned} & \begin{vmatrix} \Omega(E_1 E_2 F_1 F_2) & \Omega(E_1 E_2 F_2 F_1) \\ \Omega(E_2 E_1 F_1 F_2) & \Omega(E_2 E_1 F_2 F_1) \end{vmatrix} \\ &= \begin{vmatrix} (-K_1 + L_1)(-K_2 + L_2) & (-K_1 + L_1)(-q_{21}^{-1} K_2 + q_{12} L_2) \\ (-K_2 + L_2)(-q_{12}^{-1} K_1 + q_{21} L_1) & (-K_2 + L_2)(-K_1 + L_1) \end{vmatrix} \\ &= (-K_1 + L_1)(-K_2 + L_2) \begin{vmatrix} -K_2 + L_2 & -q_{21}^{-1} K_2 + q_{12} L_2 \\ -q_{12}^{-1} K_1 + q_{21} L_1 & -K_1 + L_1 \end{vmatrix} \\ &= (-K_1 + L_1)(-K_2 + L_2) ((1 - q_{12}^{-1} q_{21}^{-1}) K_1 K_2 + (1 - q_{12} q_{21}) L_1 L_2) \\ &= (1 - q_{12} q_{21}) (-K_1 + L_1)(-K_2 + L_2) (-q_{12}^{-1} q_{21}^{-1} K_1 K_2 + L_1 L_2). \end{aligned}$$

Skew center (Batra-Y. 2018)

Let $\varpi : \mathbb{Z}\Pi \rightarrow \mathbb{K}^\times$ be a \mathbb{Z} -module homomorphism. Define the \mathbb{K} -subspace $\mathfrak{Z}_\varpi = \mathfrak{Z}_\varpi(\chi, \Pi)$ of U_0 by

$$\mathfrak{Z}_\varpi = \mathfrak{Z}_\varpi(\chi, \Pi) := \{ Z \in U_0 \mid \forall \alpha \in \mathbb{Z}\Pi, \forall X \in U_\alpha, ZX = \varpi(\alpha)XZ \},$$

where $U = \bigoplus_{\lambda \in \mathbb{Z}\Pi} U_\lambda$ is the $\mathbb{Z}\Pi$ -grading with $K_\lambda, L_\mu \in U_0, E_i \in U_{\alpha_i}$ and $F_i \in U_{-\alpha_i}$. Let

$$\mathfrak{H}\mathfrak{C}_\varpi = \mathfrak{H}\mathfrak{C}_\varpi^{\chi, \Pi} : \mathfrak{Z}_\varpi(\chi, \Pi) \rightarrow U^0 = \bigoplus_{\lambda, \mu \in \mathbb{Z}\Pi} \mathbb{K}K_\lambda L_\mu.$$

be the Harish-Chandra map, which is injective.

$\text{Im}\mathfrak{H}\mathfrak{C}_\varpi$ is formed by the elements

$$\sum_{(\lambda, \mu) \in \mathbb{Z}\Pi \times \mathbb{Z}\Pi} a_{(\lambda, \mu)} K_\lambda L_\mu$$

with $a_{(\lambda, \mu)} \in \mathbb{K}$ satisfying the following equations (e1)-(e4).

Let $q_\alpha := \chi(\alpha, \alpha)$, $\varpi_{\lambda, \mu; \beta}^x := \varpi(\beta) \cdot \frac{\chi(\beta, \mu)}{\chi(\lambda, \beta)}$ for $\beta, \lambda, \mu \in \mathbb{Z}\Pi$. Let

$$\kappa(q_\beta) := \begin{cases} \min\{x \in \mathbb{N} \mid q_\beta^x = 1\} & \text{if } \exists y \in \mathbb{N}, q_\beta^y = 1, \\ 0 & \text{if } \forall z \in \mathbb{N}, q_\beta^z \neq 1. \end{cases}$$

(e1) For $(\lambda, \mu) \in \mathbb{Z}\Pi \times \mathbb{Z}\Pi$, $\beta \in R^+$ and $t \in \mathbb{Z} \setminus \{0\}$, if $q_\beta \neq 1$, $\kappa(q_\beta) = 0$ and $\varpi_{\lambda, \mu; \beta} = q_\beta^t$, then the equation $a_{(\lambda+t\beta, \mu-t\beta)} = \hat{\rho}(\beta)^t \cdot a_{(\lambda, \mu)}$ holds.

(e2) For $(\lambda, \mu) \in \mathbb{Z}\Pi \times \mathbb{Z}\Pi$, if there exists $\beta \in R^+$ satisfying the condition that $\kappa(q_\beta) = 0$ and $\varpi_{\lambda, \mu; \beta} \neq q_\beta^t$ for all $t \in \mathbb{Z}$, then the equation $a_{(\lambda, \mu)} = 0$ holds.

(e3) For $(\lambda, \mu) \in \mathbb{Z}\Pi \times \mathbb{Z}\Pi$, $\beta \in R^+$ and $1 \leq t \leq \kappa(q_\beta) - 1$, if $\kappa(q_\beta) \geq 2$ and $\varpi_{\lambda, \mu; \beta} = q_\beta^t$, then the equation

$$\begin{aligned} & \sum_{x=-\infty}^{+\infty} a_{(\lambda+(\kappa(q_\beta)x+t)\beta, \mu-(\kappa(q_\beta)x+t)\beta)} \hat{\rho}(\beta)^{-(\kappa(q_\beta)x+t)} \\ &= \sum_{y=-\infty}^{+\infty} a_{(\lambda+\kappa(q_\beta)y\beta, \mu-\kappa(q_\beta)y\beta)} \hat{\rho}(\beta)^{-\kappa(q_\beta)y} \end{aligned}$$

holds.

(e4) For $(\lambda, \mu) \in \mathbb{Z}\Pi \times \mathbb{Z}\Pi$ and $\beta \in R^+$, if $\kappa(q_\beta) \geq 2$ and $\varpi_{\lambda, \mu; \beta} \neq q_\beta^m$ for all $0 \leq m \leq \kappa(q_\beta) - 1$, then the $\kappa(q_\beta) - 1$ equations

$$\begin{aligned} & \sum_{x=-\infty}^{+\infty} a_{(\lambda+(\kappa(q_\beta)x+t)\beta, \mu-(\kappa(q_\beta)x+t)\beta)} \hat{\rho}(\beta)^{-(\kappa(q_\beta)x+t)} \\ &= \sum_{y=-\infty}^{+\infty} a_{(\lambda+\kappa(q_\beta)y\beta, \mu-\kappa(q_\beta)y\beta)} \hat{\rho}(\beta)^{-\kappa(q_\beta)y} \quad (1 \leq t \leq \kappa(q_\beta) - 1) \end{aligned}$$

hold.

For $(\lambda, \mu) \in \mathbb{Z}\Pi \times \mathbb{Z}\Pi$, define $\Lambda_{\lambda, \mu; \varpi}^\chi \in \text{Hom}_{\mathbb{K}\text{-alg}}(U^0, \mathbb{K})$ by

$$\Lambda_{\lambda, \mu; \varpi}^\chi(K_{\lambda'} L_{\mu'}) := \chi(\lambda, \mu') \chi(\alpha, \mu) \varpi(\lambda') \quad (\lambda', \mu' \in \mathbb{Z}\Pi).$$

Let $\text{Fin}_{\varpi}^\chi := \{(\lambda, \mu) \in \mathbb{Z}\Pi \times \mathbb{Z}\Pi \mid \dim \mathcal{L}(\Lambda_{\lambda, \mu; \varpi}^\chi) < \infty\}$. For $(\lambda, \mu) \in \text{Fin}_{\varpi}^\chi$, there exists a unique $Z_{\lambda, t\mu; \varpi}^\chi \in \mathfrak{Z}_{\varpi}(\chi, \Pi)$ such that

$$\mathfrak{H}\mathfrak{C}_{\varpi}(Z_{\lambda, t\mu; \varpi}^\chi) = \sum_{\nu \in \mathbb{Z}_{\geq 0}\Pi} \hat{\rho}^{\chi, \Pi}(\nu) \check{m}_\alpha K_{\lambda+\nu} L_{\mu-\nu},$$

where $\check{m}_\nu := \dim \mathcal{L}(\Lambda_{\lambda, \mu; \varpi}^\chi)_{-\nu}$.

Conjecture: (Batra-Y. 2019) The elements $Z_{\lambda, t\mu; \varpi}^\chi \in \mathfrak{Z}_{\varpi}(\chi, \Pi)$ with $(\lambda, \mu) \in \text{Fin}_{\varpi}^\chi$ form a \mathbb{K} -basis of $\mathfrak{Z}_{\varpi}(\chi, \Pi)$.

This fits into:

A. Sergeev, A. Veselov, Grothendieck rings of basic classical Lie superalgebras, *Ann. Math.* 173 (2011), 663-703.

The conjecture might be proved via Radford-Schneider theory for the correspondence between bicharacters and irreducible modules of the generalized quantum groups.

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