

ON THE NILPOTENT ORBITS ARISING FROM ADMISSIBLE AFFINE VERTEX ALGEBRAS

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ABSTRACT. We give a simple description of the closure of the nilpotent orbits appearing as associated varieties of admissible affine vertex algebras in terms of primitive ideals.

1. INTRODUCTION

The notion of associated variety of a primitive ideal in a universal enveloping algebra is a widely used and very useful one in representation theory, permitting the application of geometric methods, see e.g., [V91]. Recently, a similar notion has been introduced for vertex algebras ([Ar12]) which has turned out to be useful not only in the representation theory of vertex algebras (e.g., [Ar15b, AM18, AK18, vEH, AvE]) but also in connection with four dimensional $\mathcal{N} = 2$ super-conformal field theories via the 4D/2D correspondence discovered in [BLL⁺15], see e.g., [SXY17, So17, BR18, Ar, BMR19, PP20, D21, XY].

Let \mathfrak{g} be a simple complex Lie algebra and G the corresponding adjoint algebraic group. For any choice of level k the associated variety $X_{\mathbb{L}_k(\mathfrak{g})}$ of the simple affine vertex algebra $\mathbb{L}_k(\mathfrak{g})$ is a conic, G -invariant subvariety of \mathfrak{g}^* . While the associated variety of a primitive ideal in $U(\mathfrak{g})$ is always contained in the nilpotent cone $\mathcal{N} \subset \mathfrak{g}^*$, this is not necessarily the case for the variety $X_{\mathbb{L}_k(\mathfrak{g})}$. Nevertheless, it was shown in [Ar15a] that $X_{\mathbb{L}_k(\mathfrak{g})}$ is the closure of a nilpotent orbit \mathbb{O}_q , which depends only on the denominator $q \in \mathbb{Z}_{\geq 1}$ of the rational number k whenever $\mathbb{L}_k(\mathfrak{g})$ is an admissible representation [KW89] of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$. More explicitly, we have

$$\overline{\mathbb{O}}_q = \begin{cases} \mathcal{N}_q & \text{if } (q, \check{r}) = 1, \\ {}^L\mathcal{N}_{q/\check{r}} & \text{if } (q, \check{r}) \neq 1, \end{cases}$$

where by definition

$$(1) \quad \begin{aligned} \mathcal{N}_q &= \{x \in \mathfrak{g} : (\text{ad } x)^{2q} = 0\}, \\ \text{and } {}^L\mathcal{N}_q &= \{x \in \mathfrak{g} : \pi_{\theta_s}(x)^{2q} = 0\}, \end{aligned}$$

and π_{θ_s} is the irreducible finite-dimensional representation of \mathfrak{g} with highest weight θ_s , θ_s is the highest short root of \mathfrak{g} and \check{r} is the lacing number of \mathfrak{g} , that is, the maximal number of edges in the Dynkin diagram of \mathfrak{g} . The irreducibility of the varieties of (1) was checked in [Ar15a] so that the nilpotent orbit \mathbb{O}_q is indeed well-defined. Prior to [Ar15a], the irreducibility of variety $\overline{\mathbb{O}}_q$ had been proved in [oGVAG04] in most cases and the orbits \mathbb{O}_q were studied in [EKV09] as exceptional nilpotent orbits for those of principal Levi type and $(q, \check{r}) = 1$.

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In connection with the 4D/2D correspondence mentioned above, the variety $\overline{\mathbb{O}}_q$ also appears as the Higgs branch of some Argyres-Douglas theories [XYY, SXY17, WX].

The purpose of this note is to give a simple description of the orbit \mathbb{O}_q in terms of primitive ideals. It is well known that the associated variety of the annihilator of an integral highest weight representation of \mathfrak{g} is the closure of a *special* [L79] nilpotent orbit ([BV82, BV83]). Since the nilpotent orbit \mathbb{O}_q is not special in general, we need to consider non-integral highest weight representations.

Let ρ be the half sum of positive roots of \mathfrak{g} and $\check{\rho}$ the half sum of positive coroots of \mathfrak{g} . We denote by $J_\lambda \subset U(\mathfrak{g})$ the annihilator of the simple highest weight representation $L(\lambda)$ of \mathfrak{g} with highest weight λ , and by $\text{Var}(I)$ the associated variety of a primitive ideal $I \subset U(\mathfrak{g})$. Our main result is the following.

Theorem 1.1. *Let q be a positive integer, and set*

$$\lambda_q = \begin{cases} \rho/q - \rho & \text{if } (q, \check{r}) = 1, \\ \check{\rho}/q - \rho & \text{if } (q, \check{r}) \neq 1. \end{cases}$$

Then

$$\text{Var}(J_{\lambda_q}) = \overline{\mathbb{O}}_q.$$

The proof of Theorem 1.1 goes as follows. We use the representation theory of admissible affine vertex algebras to show that $\text{Var}(J_{\lambda_q})$ is contained in $\overline{\mathbb{O}}_q$. Then we show that the dimension of $\text{Var}(J_{\lambda_q})$ is equal to that of $\overline{\mathbb{O}}_q$ using the formulas obtained in [EKV09]. Since both varieties are irreducible, the equality follows.

Let $k = -\hbar + p/q$ be an admissible level and $\mathbb{L}_k(\lambda)$ the irreducible representation of $\widehat{\mathfrak{g}}$ of highest weight $\lambda + k\Lambda_0$. By [Ar16], the irreducible $\widehat{\mathfrak{g}}$ -module $\mathbb{L}_k(\lambda)$ is a module over the vertex algebra $\mathbb{L}_k(\mathfrak{g})$ if and only if $\lambda + k\Lambda_0$ is an admissible weight whose integral root system is isomorphic to that of $k\Lambda_0$. This vertex algebra is quasi-lisse [Ar15a] (see [AK18] for the definition), and it is known that the characters, that is to say graded dimensions, of ordinary modules of lisse, and more generally quasi-lisse, vertex algebras have modular invariance properties [Z96, Mi04, AK18]. This structure, in turn, permits the application of techniques from the theory of modular forms to the representation theory of such vertex algebras. In our recent work [AvEM] we have applied such techniques to discover and prove isomorphisms between affine vertex algebras and affine W -algebras known as collapsing levels. In this context the minimal value among the conformal dimensions of ordinary modules of a quasi-lisse vertex algebra is an important invariant which controls asymptotic behaviour of the characters, and the results of the present paper relate to this circle of ideas through the following result:

Proposition 1.2 ([AvEM, Proposition 3.12]). *Assume that k is admissible and that $\lambda + k\Lambda_0$ is an admissible weight whose integral root system is isomorphic to that of $k\Lambda_0$. Then*

$$\text{Var}(J_\lambda) \subset \overline{\mathbb{O}}_q, \quad |\lambda + \rho| \geq |\lambda_q + \rho|,$$

and the equality holds if and only if $\text{Var}(J_\lambda) = \overline{\mathbb{O}}_q$ and $J_\lambda = J_{\lambda_q}$. In particular, $\mathbb{L}_k(\lambda_q)$ is a simple representation of the vertex algebra $\mathbb{L}_k(\mathfrak{g})$ and possesses the minimal conformal dimension among the simple $\mathbb{L}_k(\mathfrak{g})$ -modules that belong to the category \mathcal{O}_k .

We remark that Theorem 1.1 and Proposition 1.2 together are somewhat similar in spirit to [BV85, Proposition 5.10].

One sees from the statement of Proposition 1.2 that the weight $\lambda_q + k\Lambda_0$ is not the unique one that gives the minimal conformal dimension unless $k \in \mathbb{Z}_{\geq 0}$. The statement is in some ways more pleasant, however, after Drinfeld-Sokolov reduction. Let $f \in \mathfrak{g}$ be a nilpotent element, and let $H_{DS,f}^0(?)$ be the Drinfeld-Sokolov reduction functor associated with f . The following theorem is proven in [AvEM], too.

Theorem 1.3 ([AvEM, Theorem 4.11]). *Let k be admissible, and assume that f admits an even good grading defined by a semisimple element x^0 .*

- (1) *Let $f \in \overline{\mathbb{O}}_k$, so that $H_{DS,f}^0(\mathbb{L}_k(\mathfrak{g})) \cong \mathscr{W}_k(\mathfrak{g}, f)$ (see [AvE]). Then among the simple positive energy $\mathscr{W}_k(\mathfrak{g}, f)$ -modules, the minimal conformal dimension is given by h_{λ_q} where in general*

$$h_{\lambda} = \frac{|\lambda + \rho|^2 - |\rho|^2}{2(k + \check{h})} - \frac{k + \check{h}}{2} |x^0|^2 + (x^0, \rho)$$

is the conformal dimension of the $\mathscr{W}_k(\mathfrak{g}, f)$ -module $H_{DS,f}^0(\mathbb{L}_k(\lambda))$.

- (2) *Suppose further that $f \in \mathbb{O}_k$, so that $\mathscr{W}_k(\mathfrak{g}, f)$ is lisse. Then the minimal conformal dimension h_{λ_q} is realised by a unique simple $\mathscr{W}_k(\mathfrak{g}, f)$ -module.*

It seems to be widely believed that a rational, lisse, simple, self-dual conformal vertex algebra V admits a unique simple module of minimal conformal dimension. Theorem 1.3 confirms this assertion for exceptional W -algebras ([KW03, Ar15a]) (in a wider sense, see [AvE]) that are lisse [Ar15a] and rational ([Ar13, Ar15a, CL18, AvE, F22, CL])¹.

In another direction, let us mention that one can show that

$$(2) \quad \overline{\mathbb{O}}_q \cong \text{Spec } H^\bullet(\mathfrak{u}_\zeta(\mathfrak{g}), \mathbb{C})$$

based on the works [GK93, BNPP14], where $\mathfrak{u}_\zeta(\mathfrak{g})$ is the small quantum group associated with \mathfrak{g} at the q -th root ζ of unity provided that q is odd and not a bad prime for \mathfrak{g} . Indeed, for $q > h$, with h the Coxeter number of \mathfrak{g} , $\text{Spec } H^\bullet(\mathfrak{u}_\zeta(\mathfrak{g}), \mathbb{C})$ is known to be the nilpotent cone \mathcal{N} of \mathfrak{g} by [GK93] and $\mathcal{N} = \overline{\mathbb{O}}_q$ by [Ar15a]. For $q \leq h$, the authors of [BNPP14] proved that $\text{Spec } H^\bullet(\mathfrak{u}_\zeta(\mathfrak{g}), \mathbb{C})$ is the closure $G \cdot \mathfrak{u}_{I_q}$ of some Richardson nilpotent orbit described in term of a set of simple roots $I_q \subset \Pi$ which only depends on q . Here Π is a set of simple roots of \mathfrak{g} , and \mathfrak{u}_I denotes the nilpotent radical of the standard parabolic subalgebra of \mathfrak{g} corresponding to a subset $I \subset \Pi$. It is straightforward to check that $G \cdot \mathfrak{u}_{I_q}$ coincides with $\overline{\mathbb{O}}_q$ from the tables in [Ar15a]. The equality (2) suggests a strong connection between the representations of $\mathbb{L}_k(\mathfrak{g})$ and of $\mathfrak{u}_\zeta(\mathfrak{g})$, which will be studied in a forthcoming paper [ACK].

Finally, we remark that the statement of Theorem 1.1 for $(q, \check{r}) = 1$, that is, the equality $\text{Var}(J_{\rho/q-\rho}) = \mathcal{N}_q$, seems to be true without the assumption that $(q, \check{r}) = 1$.

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¹The rationality of exceptional W -algebras has been recently proved in full generality by McRae [McR].

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2. ADMISSIBLE AFFINE VERTEX ALGEBRAS

In this section we prove the inclusions \subset in Theorem 1.1 using the representation theory of admissible affine vertex algebras.

Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, Δ the set of roots of \mathfrak{g} , Δ_+ a set of positive roots. Let $Q, P, \check{Q}, \check{P}$ be the root lattice, the weight lattice, the coroot lattice and the coweight lattice, respectively. Let $\alpha^\vee = 2\alpha/(\alpha|\alpha)$ denote the coroot corresponding to the root α . The Weyl vector and covector are given, respectively, by $\rho = \frac{1}{2} \sum_{\alpha \in \Delta} \alpha$ and $\check{\rho} = \frac{1}{2} \sum_{\alpha \in \Delta} \alpha^\vee$. Let h and \check{h} be the Coxeter number and the dual Coxeter number of \mathfrak{g} , respectively. We have

$$(3) \quad h = (\check{\rho}|\theta) + 1, \quad \check{h} = (\rho|\check{\theta}) + 1,$$

where θ is the highest root of \mathfrak{g} . We denote by $\check{\mathfrak{g}}$ the Langlands dual Lie algebra of \mathfrak{g} , characterized by its root system $\Delta(\check{\mathfrak{g}}) = \{\alpha^\vee/\check{r}: \alpha \in \Delta(\mathfrak{g})\}$.

Let $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ be the affine Kac-Moody algebra, with the commutation relations:

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}(x|y)K, \quad [K, \widehat{\mathfrak{g}}] = 0,$$

for all $x, y \in \mathfrak{g}$ and all $m, n \in \mathbb{Z}$, where $(\cdot|\cdot) = \frac{1}{2h} \times \text{Killing form}$ is the normalized invariant inner product of \mathfrak{g} and xt^n stands for $x \otimes t^n$, for $x \in \mathfrak{g}$, $n \in \mathbb{Z}$. Let $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+$ be the standard triangular decomposition, that is, $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K$ the Cartan subalgebra of $\widehat{\mathfrak{g}}$, $\widehat{\mathfrak{n}}_+ = \mathfrak{n}_+ + t\mathfrak{g}[t]$, $\widehat{\mathfrak{n}}_- = \mathfrak{n}_- + t^{-1}\mathfrak{g}[t^{-1}]$. Let $\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0$ be the dual of $\widehat{\mathfrak{h}}$, where $(\Lambda_0|K) = 1$, $(\Lambda_0|\mathfrak{h}) = 0$. The affine Weyl vector is $\widehat{\rho} = \rho + \check{h}\Lambda_0$.

Let $\widetilde{\mathfrak{h}} = \widehat{\mathfrak{h}} \oplus \mathbb{C}D$ be the extended Cartan subalgebra of $\widehat{\mathfrak{g}}$, $\widetilde{\mathfrak{h}}^* = \widehat{\mathfrak{h}}^* \oplus \mathbb{C}\delta$ the dual of $\widetilde{\mathfrak{h}}$, with $(\delta|D) = 1$, $(\delta|\mathfrak{h}) = 0$. Let $\widehat{\Delta}^{\text{re}} \subset \widetilde{\mathfrak{h}}^*$ be the real root system, $\widehat{\Delta}_+^{\text{re}}$ the standard set of positive roots in $\widehat{\Delta}^{\text{re}}$. We have

$$\begin{aligned} \widehat{\Delta}^{\text{re}} &= \{\alpha + n\delta \mid n \in \mathbb{Z}, \alpha \in \Delta\} = \widehat{\Delta}_+^{\text{re}} \sqcup \left(-\widehat{\Delta}_+^{\text{re}}\right), \\ \widehat{\Delta}_+^{\text{re}} &= \Delta_+ \sqcup \{\alpha + n\delta \mid \alpha \in \Delta, n > 0\}. \end{aligned}$$

Let $\widehat{W} = W \ltimes \check{Q}$ be the affine Weyl group of $\widehat{\mathfrak{g}}$ and $\widetilde{W} = W \ltimes \check{P}$ the extended affine Weyl group. Here the action of the element $\beta \in \check{Q}$ or \check{P} is via the translation t_β defined as

$$t_\beta(\lambda) = \lambda + (\lambda|\delta)\beta - \left[(\lambda|\beta) + \frac{1}{2}|\beta|^2(\lambda|\delta)\right]\delta.$$

For $\lambda \in \widehat{\mathfrak{h}}^*$, let

$$\widehat{\Delta}(\lambda) = \{\alpha \in \widehat{\Delta}^{\text{re}} \mid \langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \in \mathbb{Z}\},$$

the set of roots integral with respect to λ . A weight $\lambda \in \widehat{\mathfrak{h}}^*$ is said to be *admissible* if

- (1) λ is regular dominant, that is, $\langle \lambda + \widehat{\rho}, \alpha^\vee \rangle > 0$ for all $\alpha \in \widehat{\Delta}_+(\lambda) = \widehat{\Delta}(\lambda) \cap \widehat{\Delta}_+^{\text{re}}$,
- (2) $\mathbb{Q}\widehat{\Delta}^{\text{re}} = \mathbb{Q}\widehat{\Delta}(\lambda)$.

We shall denote by Adm^k the set of admissible weights of level k . The irreducible highest weight representation $\mathbb{L}(\lambda)$ is called admissible if λ is admissible.

Given any $k \in \mathbb{C}$, let

$$\mathbb{V}^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}_k,$$

where \mathbb{C}_k is the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts by 0 and K acts as a multiplication by the scalar k . There is a unique vertex algebra structure on $\mathbb{V}^k(\mathfrak{g})$ such that $|0\rangle$ is the image of $1 \otimes 1$ in $\mathbb{V}^k(\mathfrak{g})$ and

$$x(z) := (x_{(-1)}|0\rangle)(z) = \sum_{n \in \mathbb{Z}} (xt^n)z^{-n-1}$$

for all $x \in \mathfrak{g}$, where we regard \mathfrak{g} as a subspace of $\mathbb{V}^k(\mathfrak{g})$ through the embedding $x \in \mathfrak{g} \mapsto x_{(-1)}|0\rangle \in \mathbb{V}^k(\mathfrak{g})$. The vertex algebra $\mathbb{V}^k(\mathfrak{g})$ is called the *universal affine vertex algebra* associated with \mathfrak{g} at level k .

Any graded quotient of $\mathbb{V}^k(\mathfrak{g})$ inherits a vertex algebra structure from $\mathbb{V}^k(\mathfrak{g})$. In particular, the unique simple graded quotient $\mathbb{L}_k(\mathfrak{g})$ of $\mathbb{V}^k(\mathfrak{g})$ is a vertex algebra, isomorphic to $\mathbb{L}(k\Lambda_0)$ as a $\widehat{\mathfrak{g}}$ -module.

For a graded quotient V of $\mathbb{V}^k(\mathfrak{g})$, the Zhu C_2 -algebra [Z96] can be defined as the quotient

$$R_V = V/t^{-2}\mathfrak{g}[t^{-1}]V.$$

There is a surjective linear map $S(\mathfrak{g}) \rightarrow R_V$ that sends the monomial $x_1 \dots x_r$, $x_i \in \mathfrak{g}$, to $(x_1 t^{-1}) \dots (x_r t^{-1})|0\rangle + t^{-2}\mathfrak{g}[t^{-1}]V$, and the kernel of this map is a Poisson ideal of $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$. Hence, R_V is a Poisson algebra. The associated variety of V is by definition the Poisson variety

$$X_V = \text{Specm } R_V,$$

which is a G -invariant and conic subvariety of \mathfrak{g}^* .

The simple vertex algebra $\mathbb{L}_k(\mathfrak{g})$ is called the admissible affine vertex algebra if $\mathbb{L}_k(\mathfrak{g})$ is admissible as representation over the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ associated with \mathfrak{g} , or equivalently, $k\Lambda_0$ is the admissible weight. If this is the case, the level k is called an admissible number for $\widehat{\mathfrak{g}}$. By [KW08, Proposition 1.2], k is an admissible number if and only if

$$(4) \quad k + \check{h} = \frac{p}{q}, \quad p, q \in \mathbb{Z}_{\geq 1}, \quad (p, q) = 1, \quad p \geq \begin{cases} \check{h} & \text{if } (q, \check{r}) = 1, \\ h & \text{if } (q, \check{r}) \neq 1. \end{cases}$$

If this is the case we have

$$\widehat{\Delta}(k\Lambda_0) = \begin{cases} \{\alpha + nq\delta \mid \alpha \in \Delta, n \in \mathbb{Z}\} & \text{if } (q, \check{r}) = 1, \\ \{\alpha + nq\delta \mid \alpha \in \Delta_{\text{long}}, n \in \mathbb{Z}\} \\ \sqcup \{\alpha + \frac{nq}{\check{r}}\delta \mid \alpha \in \Delta_{\text{short}}, n \in \mathbb{Z}\} & \text{if } (q, \check{r}) \neq 1, \end{cases}$$

where Δ_{long} and Δ_{short} are the sets of long roots and short roots, respectively.

We now recall some results on admissible affine vertex algebras, their associated varieties and modules. To state the results we recall the following subset of admissible weights

$$\text{Adm}_\circ^k = \{\lambda \in \text{Adm}^k \mid \widehat{\Delta}(\lambda) = w(\widehat{\Delta}(k\Lambda_0)) \text{ for some } w \in \widetilde{W}\}.$$

Theorem 2.1 ([Ar15a]). *Let k be an admissible number with denominator q as in (4). Then*

$$X_{\mathbb{L}_k(\mathfrak{g})} = \overline{\mathbb{O}}_q.$$

Theorem 2.2 ([Ar16]). *Let k be an admissible number, $\lambda \in \widehat{\mathfrak{h}}^*$. Then $\mathbb{L}(\lambda)$ is a $\mathbb{L}_k(\mathfrak{g})$ -module if and only if $\lambda \in \text{Adm}_\circ^k$.*

Proposition 2.3. *For a non-negative integer q define $\hat{\lambda}_q \in \widehat{\mathfrak{h}}^*$ by*

$$\hat{\lambda}_q + \hat{\rho} = \begin{cases} \frac{1}{q}(\rho + p\Lambda_0) & \text{if } (q, \check{r}) = 1, \\ \frac{1}{q}(\check{\rho} + p\Lambda_0) & \text{if } (q, \check{r}) \neq 1. \end{cases}$$

If k is an admissible number with denominator q , as in (4), then $\hat{\lambda}_q \in \text{Adm}_\circ^k$.

Proof. By (3), we have $\langle \hat{\lambda}_q + \hat{\rho}, \alpha^\vee \rangle > 0$ for $\alpha \in \widehat{\Delta}_+^{\text{re}}$. Hence $\hat{\lambda}_q$ is regular dominant. It remains to show that $\widehat{\Delta}(\hat{\lambda}_q) = y(\widehat{\Delta}(k\Lambda_0))$ for some $y \in \widehat{W}$. Firstly we suppose that $(q, \check{r}) = 1$, so that $(\check{r}p, q) = 1$. Take $c, d \in \mathbb{Z}$ such that $c\check{r}p + dq = -1$, and set

$$\mu = c\check{r}\rho \in \check{P}.$$

Then we have $t_{-\mu}(\alpha + n\delta) = \alpha + [n + c\check{r}(\rho|\alpha)]\delta$ and it follows that

$$\langle \hat{\lambda}_q + \hat{\rho}, t_{-\mu}(\alpha + n\delta)^\vee \rangle = \frac{1}{q} \left(\frac{2np}{(\alpha|\alpha)} + (1 + c\check{r}p)(\rho|\alpha^\vee) \right) \equiv \frac{2}{(\alpha|\alpha)} \frac{np}{q} \pmod{\mathbb{Z}}.$$

Thus $t_{-\mu}(\alpha + n\delta) \in \widehat{\Delta}(\hat{\lambda}_q)$ if and only if $n \equiv 0 \pmod{q}$, and so we conclude that

$$(5) \quad \widehat{\Delta}(\hat{\lambda}_q) = t_{-\mu}(\widehat{\Delta}(k\Lambda_0)).$$

Next we suppose that $(q, \check{r}) > 1$ so that $\check{r}|q$. Take $c, d \in \mathbb{Z}$ such that $cp - dq = -1$, and set

$$\mu = c\check{\rho} \in \check{P}.$$

Then we have $t_{-\mu}(\alpha + n\delta) = \alpha + [n + c(\check{\rho}|\alpha)]\delta$ and it follows that

$$\langle \hat{\lambda}_q + \hat{\rho}, t_{-\mu}(\alpha + n\delta)^\vee \rangle = \frac{1}{q} \left(\frac{2np}{(\alpha|\alpha)} + (1 + cp)(\check{\rho}|\alpha^\vee) \right) \equiv \frac{2}{(\alpha|\alpha)} \frac{np}{q} \pmod{\mathbb{Z}}.$$

Thus, $t_{-\mu}(\alpha + n\delta) \in \widehat{\Delta}(\hat{\lambda}_q)$ if and only if $n \equiv \begin{cases} 0 \pmod{q} & \text{if } \alpha \in \Delta_{\text{long}} \\ 0 \pmod{q/\check{r}} & \text{if } \alpha \in \Delta_{\text{short}} \end{cases}$. We

have thus shown that $\widehat{\Delta}(\hat{\lambda}_q) = t_{-\mu}(\widehat{\Delta}(k\Lambda_0))$. \square

Let λ_q be the restriction of a weight $\hat{\lambda}_q \in \widehat{\mathfrak{h}}^*$ to \mathfrak{h} . Then

$$(6) \quad \lambda_q = \begin{cases} \rho/q - \rho & \text{if } (q, \check{r}) = 1, \\ \check{\rho}/q - \rho & \text{if } (q, \check{r}) \neq 1 \end{cases}$$

as in introduction.

Proposition 2.4. *We have $\text{Var}(J_{\lambda_q}) \subset \overline{\mathbb{O}}_q$.*

Proof. Let k be an admissible number with denominator q , and let $\text{Zhu}(\mathbb{L}_k(\mathfrak{g}))$ be the Zhu algebra of $\mathbb{L}_k(\mathfrak{g})$. Then $\text{Zhu}(\mathbb{L}_k(\mathfrak{g})) \cong U(\mathfrak{g})/I_k$ for some two-sided ideal I_k of $U(\mathfrak{g})$. By [Ar15b, Theorem 9.5], we have

$$\text{Var}(I_k) = X_{\mathbb{L}_k(\mathfrak{g})} = \overline{\mathbb{O}}_q.$$

By Proposition 2.3 we have $\lambda_q \in \text{Adm}_\circ^k$, and so by Theorem 2.2 in turn, $\mathbb{L}(\hat{\lambda}_q)$ is an $\mathbb{L}_k(\mathfrak{g})$ -module. Hence, $L(\lambda_q)$ is a $\text{Zhu}(\mathbb{L}_k(\mathfrak{g}))$ -module. This immediately implies $I_k \subset J_{\lambda_q}$, and therefore $\text{Var}(J_{\lambda_q}) \subset \text{Var}(I_k) = \overline{\mathbb{O}}_q$.

□

3. PROOF OF THEOREM 1.1

In view of Proposition 2.4, it suffices to show that

$$(7) \quad \dim \text{Var}(J_{\lambda_q}) = \dim \overline{\mathbb{O}}_q.$$

Since λ_q is a regular dominant weight, we have

$$\dim \text{Var}(J_{\lambda_q}) = \dim \mathcal{N} - |\Delta(\lambda_q)|$$

by [J78, Corollary 3.5], where

$$\Delta(\lambda_q) = \{\alpha \in \Delta \mid \langle \lambda_q + \rho, \alpha^\vee \rangle \in \mathbb{Z}\} = \widehat{\Delta}(\hat{\lambda}_q) \cap \Delta.$$

Therefore it is sufficient to show that

$$(8) \quad |\Delta(\lambda_q)| = \dim \mathcal{N} - \dim \overline{\mathbb{O}}_q,$$

or rather (substituting an equivalent expression for the right hand side), that

$$|\Delta(\lambda_q)| = \dim \mathfrak{g}^f - \text{rk}(\mathfrak{g}),$$

for $f \in \mathbb{O}_q$, where \mathfrak{g}^f is the centralizer of f in \mathfrak{g} . The rest of the section is devoted to a case-by-case proof of (8).

3.1. Proof of (8) for the simple classical Lie algebras. In this paragraph, we show that (8) holds for all simple Lie algebra \mathfrak{g} of classical type. Let $n \in \mathbb{Z}_{>0}$ and assume that \mathfrak{g} is either \mathfrak{sl}_n , \mathfrak{so}_n or \mathfrak{sp}_n .

First we set up notation. We denote by $\mathcal{P}(n)$ the set of partitions of n and, unless otherwise specified, we write an element λ of $\mathcal{P}(n)$ as a decreasing sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ of positive integers. Thus,

$$\lambda_1 \geq \dots \geq \lambda_r \geq 1 \quad \text{and} \quad \lambda_1 + \dots + \lambda_r = n.$$

We write λ^t for the dual partition of λ .

• By [CM, Theorem 5.1.1], nilpotent orbits of \mathfrak{sl}_n are parametrized by $\mathcal{P}(n)$. For $\lambda \in \mathcal{P}(n)$, we shall denote by \mathbb{O}_λ the corresponding nilpotent orbit of \mathfrak{sl}_n . For $f \in \mathbb{O}_\lambda$ the dimension of the centralizer of f in $\mathfrak{g} = \mathfrak{sl}_n$ is given by

$$\dim \mathfrak{g}^f = \sum_{i=1}^t \mu_i^2 - 1,$$

where $(\mu_1, \dots, \mu_t) = \lambda^t$.

• Set

$$\mathcal{P}_1(n) := \{\lambda \in \mathcal{P}(n) : \text{multiplicity of each even part is even}\}.$$

By [CM, Theorems 5.1.2 and 5.1.4], nilpotent orbits of \mathfrak{so}_n are parametrized by $\mathcal{P}_1(n)$, with the exception that each *very even* partition $\lambda \in \mathcal{P}_1(n)$ (i.e., λ has only even parts) corresponds to two nilpotent orbits. For $\lambda \in \mathcal{P}_1(n)$, not very even, we shall denote by \mathbb{O}_λ the corresponding nilpotent orbit of \mathfrak{so}_n . (Very even

nilpotent orbits do not appear in this work.) For $f \in \mathbb{O}_\lambda$ the dimension of the centralizer of f in $\mathfrak{g} = \mathfrak{so}_n$ is given by

$$\dim \mathfrak{g}^f = \frac{1}{2} \left(\sum_{i=1}^t \mu_i^2 - \#\{i : \lambda_i \text{ is odd}\} \right),$$

where $(\mu_1, \dots, \mu_t) = \lambda^t$.

- Set

$$\mathcal{P}_{-1}(n) := \{\lambda \in \mathcal{P}(n) : \text{multiplicity of each odd part is even}\}.$$

By [CM, Theorem 5.1.3], nilpotent orbits of \mathfrak{sp}_n are parametrized by $\mathcal{P}_{-1}(n)$. For $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{P}_{-1}(n)$, we shall denote by \mathbb{O}_λ the corresponding nilpotent orbit of \mathfrak{sp}_n . For $f \in \mathbb{O}_\lambda$ the dimension of the centralizer of f in $\mathfrak{g} = \mathfrak{sp}_n$ is given by

$$\dim \mathfrak{g}^f = \frac{1}{2} \left(\sum_{i=1}^t \mu_i^2 + \#\{i : \lambda_i \text{ is odd}\} \right),$$

where $(\mu_1, \dots, \mu_t) = \lambda^t$.

We recall that the *height* of an element $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ of the root lattice Q is by definition the integer $\text{ht}(\beta) = \sum_{\alpha \in \Delta} k_\alpha$. Equivalently $\text{ht}(\beta) = \langle \check{\rho}, \beta \rangle$. In order to show that (8) holds in the classical cases, we exploit a combinatorial formula for $\#\{\alpha \in \Delta : q | \text{ht}(\alpha) \in \mathbb{Z}\}$ proved in [EKV09, §3.2]. To state the formula we first write

$$n = qm_0 + s_0,$$

with $1 \leq s_0 \leq q$ if $\mathfrak{g} = \mathfrak{so}_n$ and n even, and $0 \leq s_0 \leq q - 1$ in all other cases. We consider the partition (q^{m_0}, s_0) of n , and we set

$$K_n(q) = m_0^2(q - s_0) + (m_0 + 1)^2 s_0.$$

In fact if $\mathfrak{g} = \mathfrak{sl}_n$ then $K_n(q) - 1$ is the dimension of the centralizer of a nilpotent element $f \in \mathfrak{g}$ associated with the partition (q^{m_0}, s_0) . By [EKV09, §3.2],

$$(9) \quad \#\{\alpha \in \Delta : q | \text{ht}(\alpha) \in \mathbb{Z}\} = d_{\mathfrak{g}}(q) - \text{rk}(\mathfrak{g}),$$

where $d_{\mathfrak{g}}(q)$ is expressed in terms of n and q as follows:

- (1) for $\mathfrak{g} = \mathfrak{sl}_n$,

$$d_{\mathfrak{g}}(q) = K_n(q) - 1.$$

- (2) for $\mathfrak{g} = \mathfrak{sp}_n$,

$$2d_{\mathfrak{g}}(q) = K_n(q) + \begin{cases} m_0, & \text{if } q \text{ is odd, } m_0 \text{ is even,} \\ m_0 + 1, & \text{if } q, m_0 \text{ are odd,} \\ 0, & \text{if } q \text{ is even.} \end{cases}$$

- (3) for $\mathfrak{g} = \mathfrak{so}_n$, n odd,

$$2d_{\mathfrak{g}}(q) = K_n(q) - \begin{cases} m_0, & \text{if } q, m_0 \text{ are odd,} \\ m_0 + 1, & \text{if } q \text{ is odd, } m_0 \text{ is even,} \\ 2m_0 + 1, & \text{if } q \text{ is even.} \end{cases}$$

(4) for $\mathfrak{g} = \mathfrak{so}_n$, n even,

$$2d_{\mathfrak{g}}(q) = K_n(q) - \begin{cases} m_0, & \text{if } q \text{ is odd, } m_0 \text{ is even,} \\ m_0 + 1, & \text{if } q, m_0 \text{ are odd,} \\ 2m_0, & \text{if } q, m_0 \text{ are even,} \\ 2(m_0 + 1), & \text{if } q \text{ is even, } m_0 \text{ is odd.} \end{cases}$$

3.1.1. *First case:* $(q, \check{r}) = 1$. In this case we have

$$|\Delta(\lambda_q)| = \#\{\alpha \in \Delta: \langle \rho, \alpha^\vee \rangle \in q\mathbb{Z}\}.$$

From (9), using $\text{rk}(\mathfrak{g}) = \text{rk}(\check{\mathfrak{g}})$, we have

$$\begin{aligned} \#\{\alpha \in \Delta(\check{\mathfrak{g}}): \langle \rho^\vee, \alpha \rangle \in q\mathbb{Z}\} &= \#\{\alpha \in \Delta(\check{\mathfrak{g}}): q \mid \text{ht}(\alpha)\} \\ &= d_{\check{\mathfrak{g}}}(q) - \text{rk}(\mathfrak{g}), \end{aligned}$$

(this holds whether or not $(q, \check{r}) = 1$). Using Lemma 3.1 below, we conclude that it is enough to verify the equality

$$(10) \quad \dim \mathfrak{g}^f = d_{\mathfrak{g}}(q)$$

for $f \in \mathbb{O}_q$.

Lemma 3.1. *For any integer q , we have $d_{\check{\mathfrak{g}}}(q) = d_{\mathfrak{g}}(q)$.*

Proof. Notice that \mathfrak{g} and $\check{\mathfrak{g}}$ share the same rank ℓ and the same exponents $m_1 \leq m_2 \leq \dots \leq m_\ell$. But it is known that, denoting the number of positive roots of fixed height i by p_i , the sequence $p_1 \geq p_2 \geq \dots \geq p_{h-1}$, where h is the Coxeter number of \mathfrak{g} , gives a partition of Δ_+ which is the dual partition to the partition $m_\ell \geq \dots \geq m_2 \geq m_1$ given by the exponents ([St59, Theorem 9.1]). Then the statement follows from the equality $d_{\check{\mathfrak{g}}}(q) = \#\{\alpha \in \Delta(\check{\mathfrak{g}}): q \mid \text{ht}(\alpha)\} + \text{rk}(\mathfrak{g})$. \square

The rest of this paragraph is devoted to checking (10) for each classical type.

For $\mathfrak{g} = \mathfrak{sl}_n$ (resp. $\mathfrak{g} = \mathfrak{so}_n$, $\mathfrak{g} = \mathfrak{sp}_n$), we let λ be the the element of $\mathcal{P}(n)$ (resp. $\mathcal{P}_1(n)$, $\mathcal{P}_{-1}(n)$) associated with the nilpotent orbit \mathbb{O}_q , and we choose a nilpotent element $f \in \mathbb{O}_q$.

Case $\mathfrak{g} = \mathfrak{sl}_n$. In this case $\mathfrak{g} = \check{\mathfrak{g}}$ and by [Ar15a, Table 2], $\lambda = (q^m, s)$, so (10) obviously holds.

Case $\mathfrak{g} = \mathfrak{so}_n$, n even. In this case $\mathfrak{g} = \check{\mathfrak{g}}$. Following [Ar15a, Table 2] there are four sub-cases to analyse.

(1) $\lambda = (q^m, s)$ with q, m odd, $0 \leq s \leq q$ odd. Then, clearly,

$$\dim \mathfrak{g}^f = \frac{1}{2}(m^2(q-s) + (m+1)^2s - m - 1) = \dim d_{\mathfrak{g}}(q),$$

whence (10).

(2) $\lambda = (q^m, s, 1)$ with q odd, m even, $0 \leq s \leq q-1$ odd. Then

$$\dim \mathfrak{g}^f = \frac{1}{2}(m^2(q-s) + (m+1)^2(s-1) + (m+2)^2 - m - 2).$$

On the other hand, since $n = mq + s + 1$ with $0 \leq s + 1 \leq q$,

$$d_{\mathfrak{g}}(q) = \frac{1}{2}(m^2(q-s-1) + (m+1)^2(s+1) - m),$$

whence (10).

(3) $\lambda = (q + 1, q^m, s)$ with q, m even, $0 \leq s \leq q - 1$ odd. Then

$$\dim \mathfrak{g}^f = \frac{1}{2}(1 + (m + 1)^2(q - s) + (m + 2)^2s - 2).$$

On the other hand, since $n = (m + 1)q + s + 1$ with $0 \leq s + 1 \leq q$,

$$d_{\mathfrak{g}}(q) = \frac{1}{2}((m + 1)^2(q - s - 1) + (m + 2)^2(s + 1) - 2(m + 2)),$$

whence (10).

(4) $\lambda = (q + 1, q^m, q - 1, s, 1)$ with q, m even, $0 \leq s \leq q - 1$ odd. Then

$$\dim \mathfrak{g}^f = \frac{1}{2}(1 + (m + 1)^2 + (m + 2)^2(q - 1 - s) + (m + 3)^2(s - 1) + (m + 4)^2 - 4).$$

On the other hand, since $n = (m + 2)q + s + 1$ with $0 \geq s + 1 \geq q$,

$$d_{\mathfrak{g}}(q) = \frac{1}{2}((m + 2)^2(q - s - 1) + (m + 3)^2(s + 1) - 2(m + 2)),$$

whence (10).

Case $\mathfrak{g} = \mathfrak{so}_n$, n odd. Following [Ar15a, Table 2], there are two sub-cases to analyse.

(1) $\lambda = (q^m, s)$ with q odd, m even, $0 \leq s \leq q$ odd. Then

$$\dim \mathfrak{g}^f = \frac{1}{2}(m^2(q - s) + (m + 1)^2s - m - 1) = d_{\mathfrak{g}}(q),$$

whence (10).

(2) $\lambda = (q^m, s, 1)$ with q, m odd, $0 \leq s \leq q - 1$ odd. Then

$$\dim \mathfrak{g}^f = \frac{1}{2}(m^2(q - s) + (m + 1)^2(s - 1) + (m + 2)^2 - m - 2).$$

On the other hand, $n - 1 = qm + s$ with $0 \leq s \leq q - 1$. So

$$d_{\mathfrak{g}}(q) = \frac{1}{2}(m^2(q - s) + (m + 1)^2s + m + 1) = d_{\mathfrak{g}}(q),$$

by Lemma 3.1, whence (10).

Case $\mathfrak{g} = \mathfrak{sp}_n$. Following [Ar15a, Table 2], there are two sub-cases to analyse.

(1) $\lambda = (q^m, s)$ with q odd, m even, $0 \leq s \leq q - 1$ even. Then

$$\dim \mathfrak{g}^f = \frac{1}{2}(m^2(q - s) + (m + 1)^2s + m) = d_{\mathfrak{g}}(q),$$

whence (10).

(2) $\lambda = (q^m, q - 1, s)$ with q odd, m even, $0 \leq s \leq q - 1$ even. Then

$$\dim \mathfrak{g}^f = \frac{1}{2}(m^2 + (m + 1)^2(q - 1 - s) + (m + 2)^2s + m).$$

On the other hand, $n + 1 = q(m + 1) + s$ with $0 \leq s \leq q - 1$. So

$$d_{\mathfrak{g}}(q) = \frac{1}{2}((m + 1)^2(q - s) + (m + 2)^2s) = d_{\mathfrak{g}}(q),$$

by Lemma 3.1, whence (10).

3.1.2. *Second case:* $(q, \check{r}) = \check{r}$. In this case q is necessarily even, $\check{r} = 2$ and either $\mathfrak{g} = \mathfrak{so}_n$ with n odd or else $\mathfrak{g} = \mathfrak{sp}_n$. Note that

$$(11) \quad \begin{aligned} |\Delta(\lambda_q)| &= \#\{\alpha \in \Delta : \langle \check{\rho}, \alpha^\vee \rangle \in q\mathbb{Z}\} \\ &= \#\{\alpha \in \Delta_{\text{long}} : q \mid \text{ht}(\alpha)\} + \#\{\alpha \in \Delta_{\text{short}} : (q/2) \mid \text{ht}(\alpha)\}. \end{aligned}$$

Case $\mathfrak{g} = \mathfrak{so}_n$, n odd. Firstly we rewrite (11) as

$$|\Delta(\lambda_q)| = \#\{\alpha \in \Delta : q | \text{ht}(\alpha)\} \\ + \#\{\alpha \in \Delta_{\text{short}} : (q/2) | \text{ht}(\alpha)\} - \#\{\alpha \in \Delta_{\text{short}} : q | \text{ht}(\alpha)\}.$$

There are exactly $\text{rk}(\mathfrak{g}) = (n-1)/2$ positive short roots, one of height i for each integer $i = 1, 2, \dots, (n-1)/2$. It follows easily that if we write $n = qm_0 + s_0$ where $0 \leq s_0 \leq q-1$, then we have

$$\#\{\alpha \in \Delta_{\text{short}} : (q/2) | \text{ht}(\alpha)\} - \#\{\alpha \in \Delta_{\text{short}} : q | \text{ht}(\alpha)\} = \begin{cases} m_0 & \text{if } m_0 \text{ is even,} \\ m_0 + 1 & \text{if } m_0 \text{ is odd.} \end{cases}$$

We combine equation (9) with the preceding observations to deduce that, in the present case, (8) is equivalent to

$$(12) \quad \dim \mathfrak{g}^f = d_{\mathfrak{g}}(q) + \begin{cases} m_0 & \text{if } m_0 \text{ is even,} \\ m_0 + 1 & \text{if } m_0 \text{ is odd.} \end{cases}$$

We now verify that (12) holds. Following [Ar15a, Table 3], there are two sub-cases to analyse.

(1) $\lambda = (q^m, s)$, with q, m even, $0 \leq s \leq q-1$ odd. In this case

$$\dim \mathfrak{g}^f = \frac{1}{2}(m^2(q-s) + (m+1)^2s - 1)$$

On the other hand $m_0 = m$ is even, and so

$$d_{\mathfrak{g}}(q) + m_0 = \frac{1}{2}(m^2(q-s) + (m+1)^2s - 2m - 1) + m$$

whence (12).

(2) $\lambda = (q^m, q-1, s, 1)$, with q, m even, $0 \leq s \leq q-1$ odd. In this case

$$\dim \mathfrak{g}^f = \frac{1}{2}(m^2 + (m+1)^2(q-1-s) + (m+2)^2(s-1) + (m+3)^2 - 3)$$

On the other hand $m_0 = m+1$ is odd, and so

$$d_{\mathfrak{g}}(q) + m_0 + 1 = \frac{1}{2}((m+1)^2(q-s) + (m+2)^2s - 2(m+1) - 1) + m + 2$$

whence (12).

Case $\mathfrak{g} = \mathfrak{sp}_n$. There are exactly $\text{rk}(\mathfrak{g}) = n/2$ positive long roots, one of height $2i+1$ for each integer $i = 0, 1, \dots, n/2-1$. In particular

$$\#\{\alpha \in \Delta_{\text{long}} : q | \text{ht}(\alpha)\} = 0$$

since q is even. Hence (11) becomes

$$|\Delta(\lambda_q)| = \#\{\alpha \in \Delta : (q/2) | \text{ht}(\alpha)\} - \#\{\alpha \in \Delta_{\text{long}} : (q/2) | \text{ht}(\alpha)\}.$$

If $q/2$ is even, then by the above remarks the set $\{\alpha \in \Delta_{\text{long}} : (q/2) | \text{ht}(\alpha)\}$ is empty. If $q/2$ is odd, and we write $n = (q/2)m_1 + s_1$ where $0 \leq s_1 \leq \frac{q}{2}-1$, then it follows easily that

$$\#\{\alpha \in \Delta_{\text{long}} : (q/2) | \text{ht}(\alpha)\} = \begin{cases} m_1 & \text{if } m_1 \text{ is even,} \\ m_1 + 1 & \text{if } m_1 \text{ is odd.} \end{cases}$$

By combining these facts with (9) we deduce that (8) is equivalent, in the present case, to

$$(13) \quad \dim \mathfrak{g}^f = d_{\mathfrak{g}}(q/2) - \begin{cases} 0 & \text{if } q/2 \text{ is even,} \\ m_1 & \text{if } q/2 \text{ is odd and } m_1 \text{ is even,} \\ m_1 + 1 & \text{if } q/2 \text{ is odd and } m_1 \text{ is odd.} \end{cases}$$

We now verify that (13) holds. Following [Ar15a, Table 3], there are three sub-cases to analyse.

- (1) $\lambda = ((q/2)^m, s)$, with $q/2, m$ even, $0 \leq s \leq q/2 - 1$ even. In this case

$$\dim \mathfrak{g}^f = \frac{1}{2}(m^2(q/2 - s) + (m + 1)^2s) = d_{\mathfrak{g}}(q/2) - 0,$$

whence (13).

- (2) $\lambda = (q/2 + 1, (q/2)^m, s)$, with $q/2$ odd, m even, $0 \leq s \leq q/2 - 1$ even. In this case

$$\dim \mathfrak{g}^f = \frac{1}{2}(1 + (m + 1)^2(q/2 - s) + (m + 2)^2s + m).$$

On the other hand, $n = (m + 1)(q/2) + s + 1$, with $m_1 = m + 1$ odd, and $0 \leq s + 1 \leq q/2$.

- If $s < q/2 - 1$, then $0 \leq s + 1 < q/2$, and

$$d_{\mathfrak{g}}(q/2) - m_1 = \frac{1}{2}((m + 1)^2(q/2 - s - 1) + (m + 2)^2(s + 1) + m + 2) - m - 2,$$

whence (13).

- If $s = q/2 - 1$, then $n = (m + 2)(q/2)$ with $m_1 = m + 2$ even, and

$$d_{\mathfrak{g}}(q/2) - m_1 = \frac{1}{2}((m + 1)^2(q/2) + (m + 2)^2 + m + 2) - m - 2,$$

whence (13).

- (3) $\lambda = (q/2 + 1, (q/2)^m, q/2 - 1, s)$, with $q/2$ odd, m even, $0 \leq s \leq q/2 - 1$ even. In this case

$$\dim \mathfrak{g}^f = \frac{1}{2}(1 + (m + 1)^2 + (m + 2)^2(q/2 - 1 - s) + (m + 2)^2s + m).$$

On the other hand, $n = (m + 2)(q/2) + s$, with $m_1 = m + 2$ even, $0 \leq s \leq q/2 - 1$, and

$$d_{\mathfrak{g}}(q/2) - m_1 = \frac{1}{2}((m + 2)^2(q/2 - s) + (m + 3)^2(s + 1) - m - 2) - (m + 2),$$

whence (13).

3.2. Proof of (8) for the simple exceptional Lie algebras. In this paragraph, we show that (8) holds for all simple Lie algebra \mathfrak{g} of exceptional type.

As in §3.1.1 above, when $(q, \check{r}) = 1$ we have using Lemma 3.1,

$$\begin{aligned} |\Delta(\lambda_q)| &= \#\{\alpha \in \Delta: \langle \rho, \alpha^\vee \rangle \in q\mathbb{Z}\} \\ &= \#\{\alpha \in \Delta: q|\text{ht}(\alpha^\vee)\} = d_{\mathfrak{g}}(q) - \text{rk}(\mathfrak{g}) = d_{\mathfrak{g}}(q) - \text{rk}(\mathfrak{g}), \end{aligned}$$

and hence, by (9), it is enough to verify that the equality (10) holds for any $f \in \mathbb{O}_q$. Note that the last three equalities hold even if $(q, \check{r}) \neq 1$. Using Tables 1, 3, 5, 6 and 7 below, we easily verify that (10) holds for \mathfrak{g} any of the simple Lie algebras of type G_2, F_4, E_6, E_7, E_8 , and for $(q, \check{r}) = 1$. In Tables 1, 3, we put in parentheses

the values of q which are not coprime with \check{r} . As we can see, the equality (10) still holds in these cases.

As in §3.1.2 above, when $(q, \check{r}) = \check{r}$ we have

$$\begin{aligned} |\Delta(\lambda_q)| &= |\Delta(\check{\rho}/q - \rho)| \\ &= \#\{\alpha \in \Delta_{\text{long}} : q | \text{ht}(\alpha)\} + \#\{\alpha \in \Delta_{\text{short}} : (q/\check{r}) | \text{ht}(\alpha)\}. \end{aligned}$$

Using this, we compute $|\Delta(\lambda_q)|$ for the simple Lie algebras of type G_2 and F_4 with $(q, \check{r}) = \check{r}$. From Tables 2 and 4, we conclude that (10) holds.

q	\mathcal{N}_q	$\dim \mathcal{N}_q$	$ \Delta(\lambda_q) $
2	A_1	8	4
(3), 4, 5	$G_2(a_1)$	10	2
(6), ≥ 7	G_2	12	0

TABLE 1. Data for G_2 ,
 $(q, \check{r}) = 1$

q	${}^L\mathcal{N}_{q/\check{r}}$	$\dim {}^L\mathcal{N}_{q/\check{r}}$	$ \Delta(\lambda_q) $
3	A_1	6	6
6, 9	$G_2(a_1)$	10	2
≥ 12	G_2	12	0

TABLE 2. Data for G_2 ,
 $(q, \check{r}) = \check{r}$

q	\mathcal{N}_q	$\dim \mathcal{N}_q$	$ \Delta(\lambda_q) $
(2)	$A_1 + A_1$	28	20
3	$A_2 + A_1$	36	12
(4), 5	$F_4(a_3)$	40	8
(6), 7	$F_4(a_2)$	44	4
(8), 9, (10), 11	$F_4(a_1)$	46	2
(12), ≥ 13	F_4	48	0

TABLE 3. Data for F_4 ,
 $(q, \check{r}) = 1$

q	${}^L\mathcal{N}_{q/\check{r}}$	$\dim {}^L\mathcal{N}_{q/\check{r}}$	$ \Delta(\lambda_q) $
2	A_1	16	32
4	$A_2 + A_1$	34	14
6	$F_4(a_3)$	40	8
8	B_3	42	6
10	$F_4(a_2)$	44	4
12, 14, 16	$F_4(a_1)$	46	2
≥ 18	F_4	48	0

TABLE 4. Data for F_4 ,
 $(q, \check{r}) = \check{r}$

q	\mathcal{N}_q	$\dim \mathcal{N}_q$	$ \Delta(\lambda_q) $
2	$3A_1$	40	32
3	$2A_2 + A_1$	54	18
4	$D_4(a_1)$	58	14
5	$A_4 + A_1$	62	10
6, 7	$E_6(a_3)$	66	6
8	D_5	68	4
9, 10, 11	$E_6(a_1)$	70	2
≥ 12	E_6	72	0

TABLE 5. Data for E_6

q	\mathcal{N}_q	$\dim \mathcal{N}_q$	$ \Delta(\lambda_q) $
2	$4A_1$	70	56
3	$2A_2 + A_1$	90	36
4	$A_3 + A_2 + A_1$	100	26
5	$A_4 + A_2$	106	20
6	$E_7(a_5)$	112	14
7	A_6	114	12
8	$E_7(a_4)$	116	10
9	$E_6(a_1)$	118	8
10, 11	$E_7(a_3)$	120	6
12, 13	$E_7(a_2)$	122	4
14, \dots , 17	$E_7(a_1)$	124	2
≥ 18	E_7	126	0

TABLE 6. Data for E_7

q	\mathcal{N}_q	$\dim \mathcal{N}_q$	$ \Delta(\lambda_q) $
2	$4A_1$	128	112
3	$2A_2 + A_1$	168	72
4	$2A_3$	188	52
5	$A_4 + A_3$	200	40
6	$E_8(a_7)$	208	32
7	$A_6 + A_1$	212	28
8	A_7	218	22
9	$E_8(b_6)$	220	20
10, 11	$E_8(a_6)$	224	16
12, 13	$E_8(a_5)$	228	12
14	$E_8(b_4)$	230	10
15, 16, 17	$E_8(a_4)$	232	8
18, 19	$E_8(a_3)$	234	6
20, \dots , 23	$E_8(a_2)$	236	4
24, \dots , 29	$E_8(a_1)$	238	2
≥ 30	E_8	240	0

TABLE 7. Data for E_8

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