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ARC SPACES AND CHIRAL SYMPLECTIC CORES

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Dedicated to the 70th birthday of Professor Masaki Kashiwara

Abstract. We introduce the notion of chiral symplectic cores in a vertex Poisson variety, which can be viewed as analogs of symplectic leaves in Poisson varieties. As an application we show that any quasi-lisse vertex algebra is a quantization of the arc space of its associated variety, in the sense that its reduced singular support coincides with the reduced arc space of its associated variety. We also show that the coordinate ring of the arc space of Slodowy slices is free over its vertex Poisson center, and the latter coincides with the vertex Poisson center of the coordinate ring of the arc space of the dual of the corresponding simple Lie algebra.

1. Introduction

Any vertex algebra is canonically filtered [Li], and hence can be viewed as a quantization of its associated graded vertex Poisson algebra. Since the structure of a vertex algebra is usually quite complicated, it is often very useful to reduce a problem of a vertex algebra to that of the geometry of the associated vertex Poisson scheme, that is, the spectrum of the associated graded vertex Poisson algebra (see e.g. [Fre, A3, A4]). Since a vertex Poisson scheme can be regarded as a chiral analogue of a Poisson scheme, it is natural to try to upgrade notions in Poisson geometry to the setting of vertex Poisson schemes. We note that the *arc space* $J_\infty X$ of an affine Poisson scheme X is a basic example of vertex Poisson schemes ([A1]).

In [BG] Brown and Gordon introduced the notion of *symplectic cores* in a Poisson variety which is expected to be the finest possible *algebraic* stratification in which the Hamiltonian vector fields are tangent, and showed that the symplectic cores in fact coincide with the symplectic leaves if there is only finitely many numbers of symplectic leaves. In this paper we introduce the notion of *chiral symplectic cores* in a vertex Poisson scheme, which we expect to be the finest possible algebraic stratification in which the *chiral* Hamiltonian vector fields are tangent.

We have two major applications of the notion of chiral symplectic cores.

First, recall that a vertex algebra V is called *quasi-lisse* if its associated variety X_V has finitely many symplectic leaves ([AK]). For instance, a simple affine vertex algebra V associated with a simple Lie algebra $\mathfrak{g}$ is quasi-lisse if and only if X_V is contained in the nilpotent cone of $\mathfrak{g}$. Therefore [A3], simple admissible affine vertex algebras are quasi-lisse. We refer to [AM1, AM2, AM3] for other examples of simple quasi-lisse vertex algebras. Furthermore, all the vertex algebras obtained from *four-dimensional* $\mathcal{N}=2$ superconformal field theories ([BLL$^+$]) are expected

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to be quasi-lisse ([A5, BR]) see e.g. [BPRvR, LP, SXY, BKN, Cre, BMR, A6] for examples of vertex algebras obtained from 4d $\mathcal{N}=2$ SCFTs. It is also believed in physics that there exist Higgs branch vertex algebras and Column branch vertex algebras in *three-dimensional* gauge theories that are expected to be quasi-lisse as well ([CCG]).

We show that any quasi-lisse vertex algebra V is a quantization of the reduced arc space of its associated variety, in the sense that its reduced singular support $\mathrm{Specm}(\mathrm{gr}\, V)$ coincides with $J_\infty X_V$ as topological spaces (Theorem 9.2). Moreover, for a quasi-lisse vertex algebra V, we show that each irreducible component of $J_\infty X_V$ (there are finitely many of them) is a symplectic core closure (Theorem 9.2).

Second, let $\mathfrak{g}$ be a complex simple Lie algebra with adjoint group G. We identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ through the Killing form of $\mathfrak{g}$. Denote by $\mathscr{S}_f$ the *Slodowy slice* $f+\mathfrak{g}^e$ associated with an $\mathfrak{sl}_2$-triple (e,h,f) of $\mathfrak{g}$. The affine variety $\mathscr{S}_f$ has a Poisson structure obtained from that of $\mathfrak{g}^*$ by Hamiltonian reduction [GG]. Consider the adjoint quotient morphism

$$\psi_f\colon \mathscr{S}_f \to \mathfrak{g}^*/\!/G.$$

It is known [Pre1] that any fiber $\psi_f^{-1}(\xi)$ of this morphism is the closure of a symplectic leave, which is irreducible and reduced. We show that any fiber of the induced vertex Poisson algebra morphism

$$J_\infty\psi_f\colon J_\infty\mathscr{S}_f \to J_\infty(\mathfrak{g}^*/\!/G)$$

is an irreducible and reduced chiral Poisson subscheme of $J_\infty\mathscr{S}_f$. This result enables us to show that the morphism $(J_\infty\psi_f)^*$ induces an *isomorphism* of vertex Poisson algebras between $\mathbb{C}[J_\infty\mathfrak{g}^*]^{J_\infty G}$ and the vertex Poisson center of $\mathbb{C}[J_\infty\mathscr{S}_f]$, and that $\mathbb{C}[J_\infty\mathscr{S}_f]$ is free over its vertex Poisson center (Theorem 11.1). As a consequence, we obtain that the center of the *affine W-algebra* $\mathcal{W}^{cri}(\mathfrak{g},f)$ associated with $(\mathfrak{g},f)$ at the critical level is identified with the *Feigin-Frenkel center* $\mathfrak{z}(\widehat{\mathfrak{g}})$, that is, the center of the affine vertex algebra $V^{cri}(\mathfrak{g})$ at the critical level (cf. Theorem 12.1). This later fact was claimed in [A2] but the proof was incomplete. We take the opportunity of this work to clarify this point.

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Notations. The topology is always the Zariski topology. So the term *closure* always refers to the Zariski closure.

2. Vertex algebras

Let V be a vector space over $\mathbb{C}$.

Definition 2.1. The vector space V is called a *vertex algebra* if it is equipped with the following data:

- (*the vacuum vector*) a vector $|0\rangle \in V$,

- (*the vertex operators*) a linear map
$$V \to (\operatorname{End} V)[[z, z^{-1}]], \quad a \mapsto a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1},$$
such that for all $a, b \in V$, $a_{(n)}b = 0$ for n sufficiently large.
- (*the translation operator*) a linear map $T\colon V \to V$.

These data are subject to the following axioms:

- $|0\rangle(z) = \mathrm{id}_V$. Furthermore, for all $a \in V$, $a(z)|0\rangle \in V[[z]]$ and $\lim\limits_{z\to 0} a(z)|0\rangle = a$.
- for any $a \in V$,
$$[T, a(z)] = \partial_z a(z),$$
and $T|0\rangle = 0$.
- for all $a, b \in V$, $(z-w)^{N_{a,b}}[a(z), b(w)] = 0$ for some $N_{a,b} \in \mathbb{Z}_{\geqslant 0}$.

Assume from now that V is a vertex algebra. A consequence of the definition are the following relations, called *Borcherds identities*:

$$[a_{(m)}, b_{(n)}] = \sum_{i\geqslant 0} \binom{m}{i} (a_{(i)}b)_{(m+n-i)}, \tag{1}$$

$$(a_{(m)}b)_{(n)} = \sum_{j\geqslant 0} (-1)^j \binom{m}{j} (a_{(m-j)}b_{(n+j)} - (-1)^m b_{(m+n-j)}a_{(j)}), \tag{2}$$

for $m, n \in \mathbb{Z}$.

A *vertex ideal* I of V is a T-invariant subspace of V such that $a_{(n)}b \in I$ for all $a \in I$, $b \in V$. By the *skew-symmetry property* which says that for all $a, b \in V$, the identity
$$a(z)b = e^{zT} b(-z)a$$
holds in $V((z))$, a vertex ideal I of V is also a T-invariant subspace of V such that $b_{(n)}a \in I$ for all $a \in I$, $b \in V$.

The vertex algebra V is called *commutative* if all vertex operators $a(z)$, $a \in V$, commute each other, that is,
$$[a_{(m)}, b_{(n)}] = 0, \qquad \forall a, b \in \mathbb{Z},\ m, n \in \mathbb{Z}.$$
By (1), V is a commutative vertex algebra if and only if $a(z) \in \operatorname{End} V[[z]]$ for all $a \in V$.

A commutative vertex algebra has a structure of a unital commutative algebra with the product:
$$a \cdot b = a_{(-1)}b,$$
where the unit is given by the vacuum vector $|0\rangle$. The translation operator T of V acts on V as a derivation with respect to this product:
$$T(a \cdot b) = (Ta) \cdot b + a \cdot (Tb).$$
Therefore a commutative vertex algebra has the structure of a *differential algebra*, that is, a unital commutative algebra equipped with a derivation.

Conversely, there is a unique vertex algebra structure on a differential algebra R with derivation ∂ such that:
$$a(z)b = \left(e^{z\partial} a\right) b = \sum_{n\geqslant 0} \frac{z^n}{n!} (\partial^n a) b,$$

for all $a, b \in R$. We take the unit as the vacuum vector. This correspondence gives that the category of commutative vertex algebras is the same as that of differential algebras [Bor].

3. Jet schemes and arc spaces

Our main references about jet schemes and arc spaces are [Mus, EM, Ish2].

Denote by Sch the category of schemes of finite type over $\mathbb{C}$. Let X be an object of this category, and $n \in \mathbb{Z}_{\geqslant 0}$.

Definition 3.1. An *n-jet of X* is a morphism

$$\operatorname{Spec} \mathbb{C}[t]/(t^{n+1}) \longrightarrow X.$$

The set of all n-jets of X carries the structure of a scheme $J_n X$, called the *n-th jet scheme of X*. It is a scheme of finite type over $\mathbb{C}$ characterized by the following functorial property: for every scheme Z over $\mathbb{C}$, we have

$$\operatorname{Hom}_{Sch}(Z, J_n X) = \operatorname{Hom}_{Sch}(Z \times_{\operatorname{Spec}\mathbb{C}} \operatorname{Spec} \mathbb{C}[t]/(t^{n+1}), X).$$

The $\mathbb{C}$-points of $J_n X$ are thus the $\mathbb{C}[t]/(t^{n+1})$-points of X. From Definition 3.1, we have for example that $J_0 X \simeq X$ and that $J_1 X \simeq \mathrm{T}X$, where $\mathrm{T}X$ denotes the total tangent bundle of X.

The canonical projection $\mathbb{C}[t]/(t^{m+1}) \to \mathbb{C}[t]/(t^{n+1})$, $m \geqslant n$, induces a *truncation morphism*

$$\pi^X_{m,n} \colon J_m X \to J_n X.$$

Define the (formal) disc as

$$D := \operatorname{Spec} \mathbb{C}[[t]].$$

The projections $\pi^X_{m,n}$ yield a projective system $\{J_m X, \pi^X_{m,n}\}_{m \geqslant n}$ of schemes.

Definition 3.2. Denote by $J_\infty X$ its projective limit in the category of schemes,

$$J_\infty X = \varprojlim J_n X.$$

It is called the *arc space*, or *the infinite jet scheme*, of X.

Thus elements of $J_\infty X$ are the morphisms

$$\gamma \colon D \to X,$$

and for every scheme Z over $\mathbb{C}$,

$$\operatorname{Hom}_{Sch}(Z, J_\infty X) = \operatorname{Hom}_{Sch}(Z \widehat{\times}_{\operatorname{Spec}\mathbb{C}} D, X),$$

where $Z \widehat{\times}_{\operatorname{Spec}\mathbb{C}} D$ means the formal completion of $Z \times_{\operatorname{Spec}\mathbb{C}} D$ along the subscheme $Z \times_{\operatorname{Spec}\mathbb{C}} \{0\}$. In other words, the contravariant functor

$$Sch \to Set, \quad Z \mapsto \operatorname{Hom}_{Sch}(Z \widehat{\times}_{\operatorname{Spec}\mathbb{C}} D, X)$$

is represented by the scheme $J_\infty X$.

We denote by $\pi^X_{\infty,n}$ the morphism:

$$\pi^X_{\infty,n} \colon J_\infty X \to J_n X.$$

It is surjective if X is smooth. The canonical injection $\mathbb{C} \hookrightarrow \mathbb{C}[[t]]$ induces a morphism $\iota^X_\infty \colon X \to J_\infty X$, and we have $\pi^X_{\infty,0} \circ \iota^X_\infty = \mathrm{id}_X$. Hence ι^X_∞ is injective and $\pi^X_{\infty,0}$ is surjective (for any X). Similarly, the canonical injection $\mathbb{C} \hookrightarrow \mathbb{C}[t]/(t^{n+1})$

induces a morphism $\iota_n^X\colon X \to J_nX$, and we have $\pi_{n,0}^X \circ \iota_n^X = \mathrm{id}_X$. Hence ι_n^X is injective and $\pi_{n,0}^X$ is surjective (for any X).

When the variety X is obvious, we simply write $\pi_{m,n}, \pi_{\infty,n}, \iota_n, \iota_\infty, \dots$ for $\pi_{m,n}^X, \pi_{\infty,n}^X$, $\iota_n^X, \iota_\infty^X, \dots$.

In the case where $X = \operatorname{Spec}\mathbb{C}[x^1, \dots, x^N] \cong \mathbb{A}^N$, $N \in \mathbb{Z}_{>0}$, is an affine space, we have the following explicit description of $J_\infty X$. Giving a morphism $\gamma\colon D \to \mathbb{A}^N$ is equivalent to giving a morphism $\gamma^*\colon \mathbb{C}[x^1, \dots, x^N] \to \mathbb{C}[[t]]$, or to giving

$$\gamma^*(x^i) = \sum_{j \geqslant 0} \gamma^i_{(-j-1)} t^j, \qquad i = 1, \dots, N.$$

Define functions over $J_\infty \mathbb{A}^N$ by setting for $i = 1, \dots, N$:

$$x^i_{(-j-1)}(\gamma) = j! \gamma^i_{(-j-1)}.$$

Then

$$J_\infty \mathbb{A}^N = \operatorname{Spec}\mathbb{C}[x^i_{(-j-1)}\,;\, i = 1, \dots, N,\, j \geqslant 0].$$

Define a derivation T of the algebra $\mathbb{C}[x^i_{(-j-1)}\,;\, i = 1, \dots, N,\, j \geqslant 0]$ by

$$T x^i_{(-j)} = j x^i_{(-j-1)}, \qquad j > 0.$$

Here we identify x^i with $x^i_{(-1)}$.

More generally, if $X \subset \mathbb{A}^N$ is an affine subscheme defined by an ideal $I = (f_1, \dots, f_r)$ of $\mathbb{C}[x^1, \dots, x^N]$, that is, $X = \operatorname{Spec} R$ with

$$R = \mathbb{C}[x^1, x^2, \cdots, x^N]/(f_1, f_2, \cdots, f_r),$$

then its arc space $J_\infty X$ is the affine scheme $\operatorname{Spec}(J_\infty R)$, where

$$J_\infty R := \frac{\mathbb{C}[x^i_{(-j-1)}\,;\, i = 1, 2, \cdots, N,\, j \geqslant 0]}{(T^j f_i\,;\, i = 1, \dots, r,\, j \geqslant 0)}, \tag{3}$$

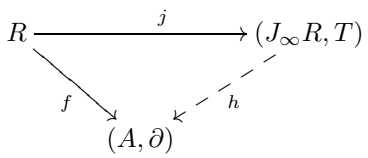
and T is as defined above.

Similarly, we have for any $n \in \mathbb{Z}_{\geqslant 0}$,

$$J_n R := \frac{\mathbb{C}[x^i_{(-j-1)}\,;\, i = 1, 2, \cdots, N,\, j = 0, \dots, n]}{(T^j f_i\,;\, i = 1, \dots, r,\, j = 0, \dots, n)}. \tag{4}$$

The derivation T acts on the quotient ring $J_\infty R$ given by (3). Hence for an affine scheme $X = \operatorname{Spec} R$, the coordinate ring $J_\infty R = \mathbb{C}[J_\infty X]$ of its arc space $J_\infty X$ is a differential algebra, hence is a commutative vertex algebra.

Remark 3.3 ([EM]). The differential algebra $(J_\infty R, T)$ is universal in the following sense. We have a $\mathbb{C}$-algebra homomorphism $j\colon R \to J_\infty R$ such that if (A, ∂) is another differential algebra, and if $f\colon R \to A$ is a $\mathbb{C}$-algebra homomorphism, then there is a unique differential algebra homomorphism $h\colon J_\infty R \to A$ making the following diagram commutative:



The map from a scheme to its n-th jet schemes and arc space is functorial. If $f\colon X \to Y$ is a morphism of schemes, then we naturally obtain a morphism $J_n f\colon J_n X \to J_n Y$ making the following diagram commutative,

$$\begin{array}{ccc} J_n X & \xrightarrow{J_n f} & J_n Y \\ {\scriptstyle \pi^X_{n,0}}\downarrow & & \downarrow{\scriptstyle \pi^Y_{n,0}} \\ X & \xrightarrow[f]{} & Y \end{array}$$

We also have the following for every schemes X, Y,

$$J_n(X \times Y) \cong J_n X \times J_n Y. \tag{5}$$

If A is a group scheme over $\mathbb{C}$, then $J_n A$ is also a group scheme over $\mathbb{C}$. Moreover, by (5), if A acts on X, then $J_n A$ acts on $J_n X$.

From now on, whenever dealing with the schemes $J_n X$ and $J_\infty X$ we will restrict to their $\mathbb{C}$-valued points, unless otherwise specified. Since the ground field is $\mathbb{C}$, $\mathbb{C}$-valued points corresponds to maximal ideals [Ish1, Proposition 2.10].

Denote by X_{red} the reduced scheme of X. Since $\mathbb{C}[[t]]$ is a domain, we have $\mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[[t]], X) = \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[[t]], X_{\mathrm{red}})$. Hence, the natural morphism $X_{\mathrm{red}} \to X$ induces an isomorphism $J_\infty X_{\mathrm{red}} \xrightarrow{\simeq} J_\infty X$ of topological spaces. (Note that the analogous assertion is false for the spaces $J_n X$.) Similarly, if $X = X_1 \cup \ldots \cup X_r$, where all X_i are closed in X, then

$$J_\infty X = J_\infty X_1 \cup \ldots \cup J_\infty X_r.$$

Moreover, we have the following result (which is false for the jet spaces $J_n X$).

Theorem 3.4 (Kolchin [Kol])**.** *The arc space $J_\infty X$ is irreducible if X is irreducible.*

More precisely, we have for any $n \in \mathbb{Z}_{\geqslant 0}$,

$$J_n X = \pi^{-1}_{n,0}(X_{sing}) \cup \overline{\pi^{-1}_{n,0}(X_{reg})}, \tag{6}$$

and $\overline{\pi^{-1}_{n,0}(X_{reg})}$ is an irreducible component of $J_n X$. Here, X_{sing} denotes the singular locus of X, and X_{reg} its open complement in X. Kolchin's theorem says that

$$J_\infty X = \overline{\pi^{-1}_{\infty,0}(X_{reg})}.$$

Let $n \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$. The natural projection $\pi^X_{n,0}\colon J_n X \to X$ corresponds to the embedding $R \hookrightarrow J_n R$, $x^i \to x^i_{(-1)}$ in the case where $X = \mathrm{Spec}\,R$ is affine. If $\mathfrak{m}$ is a maximal ideal of $J_n R$, note that $\pi^X_{n,0}(\mathfrak{m}) = \mathfrak{m} \cap R$.

For I an ideal of R, we denote by $J_n I$ the smallest T-stable ideal of $J_n R$ containing I, that is, $J_n I$ is generated by the elements $T^j a$, $j = 0, \ldots, n$, $a \in I$. Recall that ι^X_n denotes the embedding $X \hookrightarrow J_n X$, and observe that $\iota^X_n(\mathfrak{m}) = J_n(\mathfrak{m})$, for $\mathfrak{m}$ a maximal ideal of R.

4. Vertex Poisson algebras and chiral Poisson ideals

Definition 4.1. A commutative vertex algebra V is called a *vertex Poisson algebra* if it is also equipped with a linear operation,

$$V \to \mathrm{Hom}(V, z^{-1}V[z^{-1}]), \quad a \mapsto a_-(z),$$

such that

$$(Ta)_{(n)} = -na_{(n-1)}, \tag{7}$$

$$a_{(n)}b = \sum_{j\geqslant 0}(-1)^{n+j+1}\frac{1}{j!}T^j(b_{(n+j)}a), \tag{8}$$

$$[a_{(m)}, b_{(n)}] = \sum_{j\geqslant 0}\binom{m}{j}(a_{(j)}b)_{(m+n-j)}, \tag{9}$$

$$a_{(n)}(b\cdot c) = (a_{(n)}b)\cdot c + b\cdot(a_{(n)}c) \tag{10}$$

for $a, b, c \in V$ and $n, m \geqslant 0$. Here, by abuse of notations, we have set

$$a_-(z) = \sum_{n\geqslant 0} a_{(n)}z^{-n-1}$$

so that the $a_{(n)}$, $n \geqslant 0$, are "new" operators, the "old" ones given by the field $a(z)$ being zero for $n \geqslant 0$ since V is commutative.

The equation (10) says that $a_{(n)}$, $n \geqslant 0$, is a derivation of the ring V. Note that (8), (9) and (10) are equivalent to the "skewsymmetry", the "Jacobi identity" and the "left Leibniz rule" in [Kac, §5.1].

It follows from the definition, that we also have the "right Leibniz rule" ([Kac, Exercise 4.2]):

$$(a\cdot b)_{(n)}c = \sum_{i\geqslant 0}(b_{(-i-1)}a_{(n+i)}c + a_{(-i-1)}b_{(n+i)}c), \tag{11}$$

for all $a, b, c \in V$, $n \in \mathbb{Z}_{\geqslant 0}$.

Arc spaces over an affine Poisson scheme naturally give rise to a vertex Poisson algebras, as shows the following result.

Theorem 4.2 ([A1, Proposition 2.3.1])**.** *Let X be an affine Poisson scheme, that is, $X = \operatorname{Spec} R$ for some Poisson algebra R. Then there is a unique vertex Poisson algebra structure on $J_\infty R = \mathbb{C}[J_\infty X]$ such that*

$$a_{(n)}b = \begin{cases} \{a, b\} & \text{if } n = 0 \\ 0 & \text{if } n > 0, \end{cases}$$

for $a, b \in R$.

Let V be a vertex Poisson algebra, and I an ideal of V in the associative sense.

Definition 4.3. We say that I is a *chiral Poisson ideal* of V if $a_{(n)}I \subset I$ for all $a \in V$, $n \in \mathbb{Z}_{\geqslant 0}$.

A *vertex Poisson ideal* of V is a chiral Poisson ideal that is stable under the action of T. The quotient space V/I inherits a vertex Poisson algebra structure from V if I is a vertex Poisson ideal.

Lemma 4.4 ([Dix, 3.3.2])**.** *If I is a vertex (resp. chiral) Poisson ideal of V, then so is its radical $\sqrt{I}$.*

Definition 4.5. Let V be a vertex Poisson algebra. We denote by $\mathcal{Z}(V)$ the *vertex Poisson center* of V:

$$\mathcal{Z}(V) := \{z \in V \mid z_{(n)}a = 0, \forall\, a \in V, n \geqslant 0\}$$

By (8), we also have $\mathcal{Z}(V) = \{z \in V \mid a_{(n)}z = 0, \forall\, a \in V, n \geqslant 0\}$.

The vertex Poisson center $\mathcal{Z}(V)$ is a vertex Poisson ideal of V. Indeed, it is clearly invariant by the derivations $a_{(n)}$, $a \in V$, $n \in \mathbb{Z}_{\geqslant 0}$. Moreover, it is invariant by T by the axiom (7).

We say that a scheme X is a *vertex Poisson scheme* if its structure sheaf $\mathcal{O}_X$ is a sheaf of vertex Poisson algebras. If $X = \operatorname{Spec} V$ is an affine vertex Poisson scheme and I is a chiral Poisson ideal of V, we call the spectrum $\operatorname{Spec}(V/I)$ a *chiral Poisson subscheme* of X. A *chiral Poisson scheme* is a chiral Poisson subscheme of some vertex Poisson scheme.

By Lemma 4.4, the reduced scheme of a vertex Poisson scheme (resp. chiral Poisson scheme) is also a vertex Poisson scheme (resp. chiral Poisson scheme). In this case, we rather call it a *vertex Poisson variety* or a *chiral Poisson variety*.

Lemma 4.6. *Let I be an ideal of $J_\infty R$ in the associative sense. Then I is a chiral Poisson ideal of $J_\infty R$ if and only if $a_{(n)}I \subset I$ for all $a \in R$, $n \in \mathbb{Z}_{\geqslant 0}$.*

Proof. The "only if" part is obvious.

Assume that $a_{(n)}I \subset I$ for all $a \in R$, $n \in \mathbb{Z}_{\geqslant 0}$. We wish to show that $a_{(n)}I \subset I$ for all $a \in J_\infty R$, $n \in \mathbb{Z}_{\geqslant 0}$. Let $u \in I$. First, by (7),

$$(T^j a)_{(n)}u = \begin{cases} (-1)^n \dfrac{n!}{(n-j)!} a_{(n-j)}u & \text{if } 0 \leqslant j \leqslant n, \\ 0 & \text{if } j > n, \end{cases}$$

for all $a \in R$, $n, j \in \mathbb{Z}_{\geqslant 0}$. Hence $(T^j a)_{(n)}u \in I$ for all $a \in R$, $n, j \in \mathbb{Z}_{\geqslant 0}$ by our assumption. Next, by (11) and the above, $(a \cdot b)_{(n)}u$ is in I for all a, b of the form $T^j v$, $v \in R$. Since $J_\infty R$ is generated as a commutative algebra by the elements $T^j v$, $v \in R$, we get the expected statement. □

5. Rank stratification

Let $X = \operatorname{Spec} R$ be a reduced Poisson scheme, and $\{x^1, \dots, x^r\}$ a generating set for R. Let $n \in \mathbb{Z}_{\geqslant 0}$. Then $\{T^j x^i \mid i = 1, \dots, r, j = 0, \dots, n\}$ is a generating set for $J_n R = \mathbb{C}[J_n X]$. We have by the equality (10) of [A1] that for any $x, y \in R$,

$$x_{(k)}(T^l y) = \begin{cases} \frac{l!}{(l-k)!} T^{l-k}\{x, y\} & \text{if } l \geqslant k, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Hence, for $x \in R$, the derivations $x_{(k)}$ of $J_\infty R$ acts on $J_n R$ if $k \in \{0, \dots, n\}$ by the description (4) of $J_n R$.

Consider the $(n+1)r$-size square matrix

$$\mathscr{M}_n = \left(x^i_{(p)}(T^q x^j)\right)_{1 \leqslant i,j \leqslant r,\ 0 \leqslant p,q \leqslant n} \in \operatorname{Mat}_{(n+1)r}(J_n R).$$

For $x \in J_n X$, set

$$\mathscr{M}_n(x) = \left(x^i_{(p)}(T^q x^j) + \mathfrak{m}_x\right)_{1 \leqslant i,j \leqslant r,\ 0 \leqslant p,q \leqslant n} \in \operatorname{Mat}_{(n+1)r}(\mathbb{C}),$$

where $\mathfrak{m}_x$ is the maximal ideal of $J_n R$ corresponding to x.

Lemma 5.1. *Let $x \in J_n X$. We have*

$$\operatorname{rank} \mathscr{M}_n(x) = (n+1) \operatorname{rank} \mathscr{M}_0(\pi^X_{n,0}(x)).$$

In particular, $\operatorname{rank} \mathscr{M}_n(x)$ is independent of the choice of generators $\{x^1, \dots, x^r\}$ and depends only on $\pi^X_{n,0}(x) \in X$.

Proof. By definition, $\mathscr{M}_n$ is the following matrix:

$$\begin{pmatrix} x^1_{(0)}x^1 & \cdots & x^1_{(0)}x^r & \cdots\cdots & x^1_{(0)}(T^nx^1) & \cdots & x^1_{(0)}(T^nx^r) \\ \vdots & & & & & & \vdots \\ x^r_{(0)}x^1 & \cdots & x^r_{(0)}x^r & \cdots\cdots & x^r_{(0)}(T^nx^1) & \cdots & x^r_{(0)}(T^nx^r) \\ \vdots & & & & & & \vdots \\ \vdots & & & & & & \vdots \\ x^1_{(n)}x^1 & \cdots & x^1_{(n)}x^r & \cdots\cdots & x^1_{(n)}(T^nx^1) & \cdots & x^1_{(n)}(T^nx^r) \\ \vdots & & & & & & \vdots \\ x^r_{(n)}x^1 & \cdots & x^r_{(n)}x^r & \cdots\cdots & x^r_{(n)}(T^nx^1) & \cdots & x^r_{(n)}(T^nx^r) \end{pmatrix}$$

So by (12) it has the form

$$\begin{pmatrix} \mathscr{M}_0 & * & \cdots & * \\ 0 & 1!\mathscr{M}_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & n!\mathscr{M}_0 \end{pmatrix},$$

whence the first statement. Here we identify the elements $\{x^i, x^j\} = x^i_{(0)}x^j$ of R with elements of J_nR through the embedding $R \hookrightarrow J_nR$. The independence of the choice of generators follows from [Van]. □

For $x \in J_nX$, let $\operatorname{rk} x$ to be $1/(n+1) \times \operatorname{rank} \mathscr{M}_n(x)$. By Lemma 5.1, $\operatorname{rk} x$ is a non-negative integer. Moreover, for $x \in J_\infty X$, $\operatorname{rk} \pi^X_{\infty,n}(x)$ does not depend on n. So we can define $\operatorname{rk} x$ to be this number. Lemma 5.1 says that $\operatorname{rk} x$ is nothing but the rank of the matrix $\mathscr{M}_0$ at $x_0 := \pi^X_{\infty,0}(x)$.

Let $\mathcal{L}$ be a chiral Poisson subscheme of $J_\infty X$. We define the *rank stratification* of $\mathcal{L}$ as follows. For $j \in \mathbb{Z}_{\geqslant 0}$, set:

$$\mathcal{L}^0_j := \{x \in \mathcal{L} \mid \operatorname{rk} x = j\} \subset \mathcal{L}_j := \{x \in \mathcal{L} \mid \operatorname{rk} x \leqslant j\},$$

Also, set $\bar{\mathcal{L}} = \pi^X_{\infty,0}(\mathcal{L}) \subset X$, and put

$$\bar{\mathcal{L}}^0_j := \{x \in \bar{\mathcal{L}} \mid \operatorname{rank} \mathscr{M}_0(x) = j\} \subset \bar{\mathcal{L}}_j := \{x \in \bar{\mathcal{L}} \mid \operatorname{rank} \mathscr{M}_0(x) \leqslant j\}.$$

For $\mathcal{L} = J_\infty X$, we have $\bar{\mathcal{L}} = X$ and

$$X = \bigsqcup_j \bar{\mathcal{L}}^0_j$$

is precisely the rank stratification of X defined by Brown and Gordon [BG]. Note that $\mathcal{L}_j = (\pi^X_{\infty,0})^{-1}(\bar{\mathcal{L}}_j)$ and $\mathcal{L}^0_j = (\pi^X_{\infty,0})^{-1}(\bar{\mathcal{L}}^0_j)$ by definition.

Lemma 5.2. (1) *$\mathcal{L}_j$ is a closed subset of $\mathcal{L}$ with $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \cdots \subseteq \mathcal{L}_d = \mathcal{L}$ for some $d \in \mathbb{Z}_{\geqslant 0}$.*

(2) *$\mathcal{L}_j$ is a chiral Poisson subscheme of $J_\infty X$.*

Proof. Part (1) is clear by Lemma 5.1 and [BG, Lemma 3.1 (1)].

(2) Let $\mathscr{I}_j$ be the defining ideal of $\bar{\mathcal{L}}_j$. By [BG, Lemma 3.1 (2)], $\mathscr{I}_j$ is a Poisson ideal of R. On the other hand, observe that for $n \geqslant 0$,

$$(\pi^X_{n,0})^{-1}(\bar{\mathcal{L}}_j) \cong \bar{\mathcal{L}}_j \times_X J_nX.$$

Hence the defining ideal of $\mathcal{L}_j = (\pi^X_{n,0})^{-1}(\bar{\mathcal{L}}_j)$ is $\mathscr{I}_j \otimes_R J_\infty R$. So it is enough to show that $\mathscr{I}_j \otimes_R J_\infty R$ is chiral Poisson.

Let $u = \sum_i b_i c_i \in \mathscr{I}_j \otimes_R J_\infty R$, with $b_i \in \mathscr{I}_j, c_i \in J_\infty R$, Then by (10) and Theorem 4.2, for all $a \in R$ and $k \geqslant 0$,

$$a_{(k)}u = \sum_i \big((a_{(k)}b_i)\cdot c_i + b_i \cdot (a_{(k)}c_i)\big) = \sum_i \big(\delta_{k,0}\{a, b_i\}\cdot c_i + b_i \cdot (a_{(k)}c_i)\big)$$

is in $\mathscr{I}_j \otimes_R J_\infty R$ since $\mathscr{I}_j$ is a Poisson ideal of R. So $\mathscr{I}_j \otimes_R J_\infty R$ is a chiral Poisson ideal by Lemma 4.6.

□

6. Chiral Poisson cores and chiral symplectic cores

Let $X = \operatorname{Spec} R$ be a reduced Poisson scheme. For I an ideal of R, the *Poisson core* of I is the biggest Poisson ideal contained in I. We denote it by $\mathscr{P}_R(I)$. The *symplectic core* $\mathscr{C}_R(x)$ of a point $x \in X$ is the equivalence class of x for $\sim$, with

$$x \sim y \iff \mathscr{P}_R(\mathfrak{m}_x) = \mathscr{P}_R(\mathfrak{m}_y),$$

where $\mathfrak{m}_x$ denotes the maximal ideal of R corresponding to x. We refer the reader to [BG] for more details about Poisson cores and symplectic cores.

The aim of this section is to define analogue notions in the setting of vertex Poisson algebras.

Let V be a vertex Poisson algebra. By *ideal of* V we mean an ideal of V in the associative sense. We will always specify *vertex Poisson ideal* or *chiral Poisson ideal* (see Section 4) if necessary. An ideal I of V is said to be *prime* if it is prime in the associative sense.

Definition 6.1. The *chiral Poisson core* of an ideal I of V is the biggest chiral Poisson ideal of V contained in I. It exists since the sum of two chiral Poisson ideals is chiral Poisson. We denote the chiral Poisson core of I by $\mathscr{P}_V(I)$.

Lemma 6.2. *Let I be an ideal of V.*

(1) $\mathscr{P}_V(I) = \{x \in I \mid a^1_{(n_1)} \dots a^k_{(n_k)} x \in I \text{ for all } a^i \in V,\ k \geqslant 0,\ n_i \geqslant 0\}$.
(2) ([Dix, Lemma 3.3.2(ii)]) *If I is prime, then $\mathscr{P}_V(I)$ is prime.*
(3) *If I is radical, then $\mathscr{P}_V(I)$ is radical.*

Proof. Set $J = \{x \in I \mid a^1_{(n_1)} \dots a^k_{(n_k)} x \in I \text{ for all } a^i \in V,\ k \geqslant 0,\ n_i \geqslant 0\}$.

(1) By construction, $J \subset I$ and J is chiral Poisson. Hence $J \subset \mathscr{P}_V(I)$. But if K is a chiral Poisson ideal of V contained in I, then for all $x \in K$, $a \in V$ and $n \geq 0$, $a_{(n)}x \in K \subset I$, whence $x \in J$. In conclusion, $J = \mathscr{P}_V(I)$.

(3) Assume that I is radical. Since $\mathscr{P}_V(I)$ is chiral Poisson, $\sqrt{\mathscr{P}_V(I)}$ is chiral Poisson as well by Lemma 4.4, and it is contained in $\sqrt{I}$ since $\mathscr{P}_V(I)$ is contained in I. Hence,

$$\sqrt{\mathscr{P}_V(I)} \subset \mathscr{P}_V(\sqrt{I}) = \mathscr{P}_V(I).$$

But clearly $\mathscr{P}_V(I) \subset \sqrt{\mathscr{P}_V(I)}$, whence the equality $\sqrt{\mathscr{P}_V(I)} = \mathscr{P}_V(I)$ and the statement. □

Corollary 6.3. *Assume that there are finitely many minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ over I, that is, $I = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_r$, and the prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are minimal. If I is chiral Poisson, then so are the prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$.*

Proof. If I is chiral Poisson, then $I \subset \mathscr{P}_V(\mathfrak{p}_i) \subset \mathfrak{p}_i$ for all i. But by Lemma 6.2 (2), the ideals $\mathscr{P}_V(\mathfrak{p}_i)$, $i = 1, \dots, r$, are all prime. By minimality of the prime ideals $\mathfrak{p}_i$ we deduce that $\mathfrak{p}_i = \mathscr{P}_V(\mathfrak{p}_i)$ for all i. In particular, the prime ideals $\mathfrak{p}_i$, $i = 1, \dots, r$, are all chiral Poisson. □

Set

$$\mathcal{L} := \operatorname{Specm}(V).$$

We define a relation $\sim$ on $\mathcal{L}$ by

$$x \sim y \iff \mathscr{P}_V(\mathfrak{m}_x) = \mathscr{P}_V(\mathfrak{m}_y),$$

where $\mathfrak{m}_x$ is the maximal ideal corresponding to $x \in \mathcal{L}$.

Clearly $\sim$ is an equivalence relation. We denote the equivalence class in $\mathcal{L}$ of x by $\mathscr{C}_{\mathcal{L}}(x)$, so that

$$\mathcal{L} = \bigsqcup_x \mathscr{C}_{\mathcal{L}}(x).$$

We call the set $\mathscr{C}_{\mathcal{L}}(x)$ the *chiral symplectic core of x in $\mathcal{L}$*.

For I an ideal of V, we denote by $\mathscr{V}(I)$ the corresponding zero locus in $\operatorname{Specm} V$, that is,

$$\mathscr{V}(I) = \{x \in \mathcal{L} \mid f(x) = 0 \text{ for all } f \in I\} = \{x \in \mathcal{L} \mid \mathfrak{m}_x \supset I\},$$

and by $\widetilde{\mathscr{V}}(I)$ the corresponding closed scheme for the Zariski topology, that is,

$$\widetilde{\mathscr{V}}(I) = \{\mathfrak{p} \in \operatorname{Spec} V \mid \mathfrak{p} \supset I\}.$$

Lemma 6.4. *Let $x \in \mathcal{L}$. Then $\widetilde{\mathscr{V}}(\mathscr{P}_V(\mathfrak{m}_x))$ is the smallest chiral Poisson scheme containing $\overline{\mathscr{C}_{\mathcal{L}}(x)}$. Moreover, it is reduced and irreducible.*

Proof. First of all, since $\mathscr{P}_V(\mathfrak{m}_x)$ is a chiral Poisson, prime and radical ideal of V by Lemma 6.2, $\widetilde{\mathscr{V}}(\mathscr{P}_V(\mathfrak{m}_x))$ is a reduced irreducible (closed) chiral Poisson subscheme of $\mathcal{L}$. For any $y \in \mathscr{C}_{\mathcal{L}}(x)$, we have $\mathfrak{m}_y \supset \mathscr{P}_V(\mathfrak{m}_y) = \mathscr{P}_V(\mathfrak{m}_x)$. Hence $\overline{\mathscr{C}_{\mathcal{L}}(x)} \subset \widetilde{\mathscr{V}}(\mathscr{P}_V(\mathfrak{m}_x))$. Next, if I is a chiral Poisson ideal of V such that $\overline{\mathscr{C}_{\mathcal{L}}(x)} \subset \widetilde{\mathscr{V}}(I)$, then in particular $\mathfrak{m}_x \supset I$. Since I is chiral Poisson, we get that $\mathfrak{m}_x \supset \mathscr{P}_V(\mathfrak{m}_x) \supset I$ by maximality of $\mathscr{P}_V(\mathfrak{m}_x)$. Hence $\overline{\mathscr{C}_{\mathcal{L}}(x)} \subset \widetilde{\mathscr{V}}(\mathscr{P}_V(\mathfrak{m}_x)) \subset \widetilde{\mathscr{V}}(I)$. This proves the statement. □

Lemma 6.5. *Let $\mathcal{L}'$ be a reduced closed vertex Poisson subscheme of $\mathcal{L}$. Then for any $x \in \mathcal{L}'$, we have $\mathscr{C}_{\mathcal{L}'}(x) = \mathscr{C}_{\mathcal{L}}(x)$.*

Proof. Since $\mathcal{L}'$ is a reduced closed vertex Poisson subscheme of $\mathcal{L}$, $\mathcal{L}' = \operatorname{Specm}(V/I)$, where I is a vertex Poisson ideal of V. The maximal ideals of V/I are precisely the quotients $\mathfrak{m}/I$, where $\mathfrak{m}$ is a maximal ideal of V containing I, and $\mathscr{P}_{V/I}(\mathfrak{m}/I) = \mathscr{P}_V(\mathfrak{m})/I$.

Hence for $x \in \mathcal{L}'$, we have

$$\begin{aligned}\mathscr{C}_{\mathcal{L}'}(x) &= \{y \in \mathcal{L}' \mid \mathscr{P}_{V/I}(\mathfrak{m}_y/I) = \mathscr{P}_{V/I}(\mathfrak{m}_x/I)\} \\ &= \{y \in \mathcal{L}' \mid \mathscr{P}_V(\mathfrak{m}_y)/I = \mathscr{P}_V(\mathfrak{m}_x)/I\} = \{y \in \mathcal{L}' \mid \mathscr{P}_V(\mathfrak{m}_y) = \mathscr{P}_V(\mathfrak{m}_x)\},\end{aligned}$$

where $\mathfrak{m}_x$ is the maximal ideal of V corresponding to x. The maximal ideal $\mathfrak{m}_x$ contains I because $x \in \mathcal{L}'$. On the other hand,

$$\mathscr{C}_{\mathcal{L}}(x) = \{y \in \mathcal{L} \mid \mathscr{P}_V(\mathfrak{m}_y) = \mathscr{P}_V(\mathfrak{m}_x)\}.$$

But if $\mathscr{P}_V(\mathfrak{m}_y) = \mathscr{P}_V(\mathfrak{m}_x)$ for some $y \in \mathcal{L}$, then $\mathfrak{m}_y \supset \mathscr{P}_V(\mathfrak{m}_y) = \mathscr{P}_V(\mathfrak{m}_x) \supset I$ because I is a chiral Poisson ideal of V contained in $\mathfrak{m}_x$. This shows that if $y \in \mathscr{C}_{\mathcal{L}}(x)$, then $y \in \mathcal{L}'$. Therefore

$$\mathscr{C}_{\mathcal{L}}(x) = \{y \in \mathcal{L}' \mid \mathscr{P}_V(\mathfrak{m}_y) = \mathscr{P}_V(\mathfrak{m}_x)\} = \mathscr{C}_{\mathcal{L}'}(x).$$

□

Proposition 6.6. *Let $z \in \mathcal{Z}(V)$ and $x \in \mathcal{L}$. Then z is constant on $\mathscr{V}(\mathscr{P}_V(\mathfrak{m}_x))$ and so on $\overline{\mathscr{C}_{\mathcal{L}}(x)}$.*

Proof. Let $y \in \mathscr{V}(\mathscr{P}_V(\mathfrak{m}_x))$, and let χ_x, χ_y be the homomorphisms $\chi_x \colon V \to \mathbb{C}$, $\chi_y \colon V \to \mathbb{C}$, corresponding to the maximal ideals $\mathfrak{m}_x, \mathfrak{m}_y$. It is enough to show that $\chi_x(z) = \chi_y(z)$. Set $\lambda := \chi_x(z)$. Then $z - \lambda \in \ker \chi_x = \mathfrak{m}_x$. In addition since z is in the center, so is $z - \lambda$, and then $a_{(n)}(z - \lambda) = 0$ for any $a \in V$ and $n \geqslant 0$. Therefore $z - \lambda \in \mathscr{P}_V(\mathfrak{m}_x) \subset \mathfrak{m}_y$, whence $\chi_y(z) = \lambda$. By Lemma 6.4, we conclude that z is constant on $\overline{\mathscr{C}_{\mathcal{L}}(x)}$. □

Lemma 6.7. *Let $X = \operatorname{Specm} R$ be a reduced Poisson scheme, and let $\mathcal{L}$ be a closed vertex Poisson subscheme of $J_\infty X$. Set $\bar{\mathcal{L}} = \pi^X_{\infty,0}(\mathcal{L})$. Let $x \in \mathcal{L}$.*

1. *We have $\pi^X_{\infty,0}(\mathscr{C}_{\mathcal{L}}(x)) \subset \mathscr{C}_{\bar{\mathcal{L}}}(\pi^X_{\infty,0}(x))$.*
2. *If $\operatorname{rk} x = j$, then $\mathscr{C}_{\mathcal{L}}(x) \subset \mathcal{L}^0_j$.*

Proof. (1) We have $\mathcal{L} = \operatorname{Specm} J_\infty R/I$, with I a vertex Poisson ideal of $J_\infty R$.

Let $x \in \mathcal{L}$. Let us first show that:

$$\mathscr{P}_{R/(I\cap R)}(\mathfrak{m}_x \cap R/(I \cap R)) = \mathscr{P}_{J_\infty R/I}(\mathfrak{m}_x/I) \cap R/(I \cap R). \tag{13}$$

The inclusion $\mathscr{P}_{J_\infty R/I}(\mathfrak{m}_x/I) \cap R/(I \cap R) \subset \mathscr{P}_{R/(I\cap R)}(\mathfrak{m}_x \cap R/(I \cap R))$ is clear because the left-hand side is Poisson and is contained in $\mathfrak{m}_x \cap R/(I \cap R)$. For the converse inclusion, let $a \in R$, $k \in \mathbb{Z}_{\geqslant 0}$ and $b \in \mathscr{P}_{R/(I\cap R)}(\mathfrak{m}_x \cap R/(I \cap R))$. Then by Theorem 4.2

$$a_{(k)}(b + I \cap R) = \delta_{k,0}(\{a, b\} + \{a, I \cap R\}) \in \mathscr{P}_{R/(I\cap R)}(\mathfrak{m}_x \cap R/(I \cap R))$$

since $\mathscr{P}_{R/(I\cap R)}(\mathfrak{m}_x \cap R/(I \cap R))$ and $I \cap R$ are Poisson. By Lemma 4.6, this shows that $\mathscr{P}_{R/(I\cap R)}(\mathfrak{m}_x \cap R/(I \cap R))$ is a chiral Poisson ideal of $J_\infty R/I$, contained in $\mathfrak{m}_x/I$, whence the expected equality (13).

Let now $y \in \mathscr{C}_{\mathcal{L}}(x)$. Then $\mathscr{P}_{J_\infty R/I}(\mathfrak{m}_y/I) = \mathscr{P}_{J_\infty R/I}(\mathfrak{m}_x/I)$. From (13), we deduce that

$$\mathscr{P}_{R/(I\cap R)}(\mathfrak{m}_y \cap R/(I \cap R)) = \mathscr{P}_{R/(I\cap R)}(\mathfrak{m}_x \cap R/(I \cap R)),$$

and so

$$\pi^X_{\infty,0}(y) \in \mathscr{C}_{\bar{\mathcal{L}}}(\pi^X_{\infty,0}(x))$$

since $\mathfrak{m}_y \cap R/(I \cap R)$ is the maximal ideal of $R/(I \cap R)$ corresponding to $\pi^X_{\infty,0}(y) \in \bar{\mathcal{L}}$. This proves the statement.

(2) By Lemma 5.1, $\mathcal{L}^0_j = (\pi^X_{\infty,0})^{-1}(\overline{\mathcal{L}}^0_j)$. On the other hand, by [BG, Proposition 3.6], $\mathscr{C}_{\bar{\mathcal{L}}}(\pi^X_{\infty,0}(x)) \subset \overline{\mathcal{L}}^0_j$. Hence by (1),

$$\mathscr{C}_{\mathcal{L}}(x) \subset (\pi^X_{\infty,0})^{-1}(\mathscr{C}_{\bar{\mathcal{L}}}(\pi^X_{\infty,0}(x))) \subset (\pi^X_{\infty,0})^{-1}(\overline{\mathcal{L}}^0_j) = \mathcal{L}^0_j.$$

□

By Lemma 6.7 and Lemma 5.1, note that the stratification by chiral symplectic cores is a refinement of the rank stratification.

7. n-chiral Poisson cores in n-th jet schemes

Let R be a Poisson algebra, $n \in \mathbb{Z}_{\geqslant 0}$. Recall that the derivations $a_{(k)}$, $k \geqslant 0$, of $J_\infty R$ acts on $J_n R$ by (4) and (12). We say that an ideal I of $J_n R$ is *n-chiral Poisson* if $a_{(k)} I \subset I$ for any $a \in R$ and any $k = 0, \dots, n$.

Lemma 7.1. *For a Poisson ideal I of R, $J_n I$ is n-chiral Poisson.*

Proof. This follows from (12) since I is a Poisson ideal of R. □

If I is an ideal of $J_n R$, we define the *n-chiral Poisson core* of I to be the biggest n-chiral Poisson ideal contained in I. We denote it by $\mathscr{P}_{J_n R}(I)$.

Set $X := \operatorname{Specm} R$. We define a relation $\sim$ on $J_n X$ by:

$$x \sim y \iff \mathscr{P}_{J_n R}(\mathfrak{m}_x) = \mathscr{P}_{J_n R}(\mathfrak{m}_y),$$

where $\mathfrak{m}_x$ denotes the maximal ideal of $J_n R$ corresponding to x.

Clearly $\sim$ is an equivalence relation. We denote the equivalence class in $J_n X$ of x by $\mathscr{C}_{J_n X}(x)$, so that

$$J_n X = \bigsqcup_x \mathscr{C}_{J_n X}(x).$$

We call the set $\mathscr{C}_{J_n X}(x)$ the *n-chiral symplectic core of x in $J_n X$*.

Similarly to the case of chiral Poisson cores in a vertex Poisson algebra, we obtain the following facts:

Lemma 7.2.
(1) *Let I be an ideal of $J_n R$. If I is prime (resp. radical) then $\mathscr{P}_{J_n R}(I)$ is prime (resp. radical).*
(2) *Let I be an ideal of $J_n R$. If I is n-chiral Poisson, then so are the minimal prime ideals over I.*
(3) *Let $x \in J_n X$. Then $\widetilde{\mathscr{V}}(\mathscr{P}_{J_n R}(\mathfrak{m}_x))$ is the smallest n-chiral Poisson subscheme of $J_n X$ containing $\overline{\mathscr{C}_{J_n X}(x)}$.*
(4) *Let Y be a reduced Poisson subscheme of X, and let $x \in J_n Y$. Then $\mathscr{C}_{J_n X}(x) = \mathscr{C}_{J_n Y}(x)$ and $\pi^X_{n,0}(\mathscr{C}_{J_n X}(x)) \subset \mathscr{C}_Y(\pi^X_{n,0}(x))$.*

Lemma 7.3. *Let Y be a reduced n-chiral Poisson subscheme of $J_n X$, and $y \in Y$. Let $\{x^1, \dots, x^r\}$ be a generating set for R, and consider the matrix $\mathscr{M}_n(y)$ as in Section 5. Suppose that the matrix $\mathscr{M}_n(y)$ has maximal rank $(n+1)r$. Then the tangent space at y of Y has dimension at least $(n+1)r$. Moreover, Y has dimension at least $(n+1)r$.*

Proof. Since Y is a reduced n-chiral Poisson subscheme of $J_n X$, $Y = \operatorname{Spec} J_n R/I$, where I is a reduced n-chiral Poisson of $J_n R$. Moreover, $I \subset \mathfrak{m}_y$ since $y \in Y$, where $\mathfrak{m}_y$ denotes the maximal ideal of $J_n R$ corresponding to y.

The hypothesis implies that the derivations $x^i_{(k)}$, $i = 1, \dots, r$, $k = 0, \dots, n$, are linearly independent in $\operatorname{Der}(\mathcal{O}_{J_n X, y}, \mathbb{C})$. Indeed, if for some $\lambda^i_{(k)}$, $i = 1, \dots, r$, $k = 0, \dots, n$,

$$\sum_{i=1}^{r} \sum_{k=0}^{n} \lambda^i_{(k)} x^i_{(k)} = 0 \quad \text{in} \quad \operatorname{Der}(\mathcal{O}_{J_n X, y}, \mathbb{C}),$$

then

$$\sum_{i=1}^{r} \sum_{k=0}^{n} \lambda^i_{(k)} (x^i_{(k)} (T^l x^j) + \mathfrak{m}_y) = 0 \quad \text{for all } \ j = 1, \dots, r,\ l = 0, \dots, r,$$

and so $\lambda^i_{(k)} = 0$ for all $i = 1, \dots, r$ and $k = 0, \dots, n$ since the matrix $\mathscr{M}_n(y)$ has rank $(n+1)r$.

Since I is n-chiral Poisson and is contained in $\mathfrak{m}_y$, we get that for all $i = 1, \dots, r$ and $k = 0, \dots, n$, $x^i_{(k)}(I) \subset I \subset \mathfrak{m}_y$. Hence, the derivations $x^i_{(k)}$, $i = 1, \dots, r$, $k = 0, \dots, n$ are also linearly independent in $\mathrm{Der}(\mathcal{O}_{Y,y}, \mathbb{C})$ since $\mathcal{O}_{Y,y} = \mathcal{O}_{J_nX,y}/(\mathcal{O}_{J_nX,y} \cap I)$. This shows that the tangent space at y of Y has dimension at least $(n+1)r$.

The set of points $y \in Y$ such that matrix $\mathscr{M}_n(y)$ has maximal rank $(n+1)r$ is a nonempty open subset of Y. Hence it meets the set of smooth points of Y. By the first step, we deduce that for some smooth point $y \in Y$, the tangent space T_yY has dimension at least $(n+1)r$. Therefore Y has dimension at least $(n+1)r$. □

Recall that ι_n (resp. ι_∞) denotes the canonical embedding from X to J_nX (resp. $J_\infty X$). For x in X, we simply denote by x_n (resp. x_∞) the element $\iota_n(x)$ (resp. $\iota_\infty(x)$).

Proposition 7.4. *Let $x \in X$, and set $Y := \overline{\mathscr{C}_X(x)}$.*

(1) *For any $n \in \mathbb{Z}_{\geqslant 0}$,*

$$\overline{(\pi^Y_{n,0})^{-1}(Y_{reg})} = \mathscr{V}(\mathscr{P}_{J_nR}(\mathfrak{m}_{x_n})).$$

In particular, if J_nY is irreducible, then

$$J_nY = \mathscr{V}(\mathscr{P}_{J_nR}(\mathfrak{m}_{x_n})).$$

(2) *We have:*

$$J_\infty Y = \mathscr{V}(\mathscr{P}_{J_\infty R}(\mathfrak{m}_{x_\infty}))$$

Proof. (1) By Lemma 7.2 (4), $\mathscr{C}_{J_nX}(x_n) = \mathscr{C}_{J_nY}(x_n)$, and

$$\mathscr{C}_{J_nX}(x_n) \subset (\pi^Y_{n,0})^{-1}(\mathscr{C}_X(x)) \subset (\pi^Y_{n,0})^{-1}(Y_{reg})$$

since by [BG, Lemma 3.3 (2)], $\mathscr{C}_X(x)$ is smooth in its closure. Hence

$$\overline{\mathscr{C}_{J_nX}(x_n)} \subset \overline{(\pi^Y_{n,0})^{-1}(Y_{reg})} \subset J_nY.$$

Since $\overline{(\pi^Y_{n,0})^{-1}(Y_{reg})}$ is an irreducible component of J_nY (cf. Section 3) and since J_nY is n-chiral Poisson, $\overline{(\pi^Y_{n,0})^{-1}(Y_{reg})}$ is an n-chiral Poisson subscheme of J_nX. Hence by Lemma 6.4,

$$\overline{\mathscr{C}_{J_nX}(x_n)} \subset \mathscr{V}(\mathscr{P}_{J_nA}(\mathfrak{m}_{x_n})) \subset \overline{(\pi^Y_{n,0})^{-1}(Y_{reg})},$$

where $A = \mathbb{C}[Y]$. Let $x^1, \dots, x^r$ be generators of A, where $r = \dim Y$. By [BG, Proposition 3.6] (proof of (2)), the matrix $\mathscr{M}_0$ has maximal rank r at $\mathfrak{m}_x$. Hence, by Lemma 5.1, $\mathscr{M}_n$ has rank $(n+1)r$ at $\mathfrak{m}_{x_n}$. Since $\mathscr{P}_{J_nA}(\mathfrak{m}_{x_n})$ is an n-chiral Poisson ideal of J_nA, it results from Lemma 7.3 that $\mathscr{V}(\mathscr{P}_{J_nA}(\mathfrak{m}_{x_n})))$ has dimension at least $(n+1)r$. But

$$\dim \overline{(\pi^Y_{n,0})^{-1}(Y_{reg})} = (n+1)r.$$

Both $\mathscr{V}(\mathscr{P}_{J_nA}(\mathfrak{m}_{x_n})))$ and $\overline{(\pi^Y_{n,0})^{-1}(Y_{reg})}$ are closed and irreducible, whence the first assertion of (1). Indeed note that $\mathscr{V}(\mathscr{P}_{J_nA}(\mathfrak{m}_{x_n}))) = \mathscr{V}(\mathscr{P}_{J_nR}(\mathfrak{m}_{x_n})))$ since $\mathscr{C}_{J_nX}(x_n) = \mathscr{C}_{J_nY}(x_n)$.

The second assertion follows from the fact that $\overline{(\pi^Y_{n,0})^{-1}(Y_{reg})}$ is an irreducible component of J_nY.

Part (2) follows from part (1) and Kolchin's Theorem 3.4. □

8. Partial stratification by chiral symplectic leaves

Recall that there is a well-defined stratification of X by symplectic leaves [BG].

We assume in this section that the Poisson bracket on $X = \operatorname{Specm} R$ is *algebraic*, that is, the symplectic leaves in X are all locally closed. Then [BG, Proposition 3.6] the symplectic leaves coincide with the symplectic cores of X, and the defining ideal of the symplectic core closure of a point $x \in X$ is $\mathscr{P}_R(\mathfrak{m}_x)$.

Let $x_0 \in X$, and $\mathscr{L}_X(x_0)$ the symplectic leaf through x_0 in X. Set

$$Y = \overline{\mathscr{L}_X(x_0)} \quad \text{and} \quad A = \mathbb{C}[Y].$$

Note that $(\pi^Y_{n,0})^{-1}(\mathscr{L}_X(x_0))$ is open in J_nY since $\mathscr{L}_X(x_0)$ is open in its closure Y. Moreover, $(\pi^Y_{n,0})^{-1}(\mathscr{L}_X(x_0)) = J_n\mathscr{L}_X(x_0)$ by [EM, Lemma 2.3]. In particular, one can equip $(\pi^Y_{n,0})^{-1}(\mathscr{L}_X(x_0))$ with the structure of a smooth analytic variety.

Let $x \in J_n\mathscr{L}_X(x_0)$. We denote by $\mathscr{L}_{J_nX}(x)$ the set of all $y \in J_n\mathscr{L}_X(x_0)$ which can be reach from x by traveling along the integral flows of vector fields $a_{(k)}$, $a \in A$, $k = 0, \dots, n$.

We call $\mathscr{L}_{J_nX}(x)$ the *n-chiral symplectic leaf of x in J_nX*. Note that we defined n-chiral symplectic leaves only for elements in $\bigcup_{x' \in X} J_n\mathscr{L}_X(x')$ which is a priori different from J_nX.

Lemma 8.1. *Let $x \in J_n\mathscr{L}_X(x_0)$. Then the defining ideal of the Zariski closure of $\mathscr{L}_{J_nX}(x)$ is $\mathscr{P}_{J_nR}(\mathfrak{m}_x)$.*

Proof. First of all, by construction of $\mathscr{L}_{J_nX}(x)$, we have $\mathscr{L}_{J_nX}(x) \subset J_nY$. So by Lemma 7.2 (4), $\mathscr{P}_{J_nR}(\mathfrak{m}_x) = \mathscr{P}_{J_nA}(\mathfrak{m}_x)$.

We now follow the ideas of the proof of [BG, Lemma 3.5]. Let $\mathscr{K}_x$ be the defining ideal of $\overline{\mathscr{L}_{J_nX}(x)}$.

We first show that $\mathscr{P}_{J_nA}(\mathfrak{m}_x) \subset \mathscr{K}_x$. Let

$$\widetilde{J_nY} := \operatorname{Specm} \widetilde{J_nA}, \quad \text{with} \quad \widetilde{J_nA} = J_nA/\mathscr{P}_{J_nA}(\mathfrak{m}_x),$$

and denote by $\widetilde{a}$ the image of $a \in J_nA$ in $\widetilde{J_nA}$. For $r > 0$, $B(r)$ denotes the open complex analytic disc of radius r. Let $a \in A$ and $k \in \{0, \dots, n\}$. Since $a_{(k)}\mathscr{P}_{J_nA}(\mathfrak{m}_x) \subset \mathscr{P}_{J_nA}(\mathfrak{m}_x)$, $a_{(k)}$ defines a derivation on $\widetilde{J_nA}$, which we denote by $\widetilde{a}_{(k)}$. Consider $\sigma_x \colon B(r) \to J_nY$ and $\widetilde{\sigma}_x \colon B(r) \to \widetilde{J_nY}$ be integral curves of the vector fields $a_{(k)}$ and $\widetilde{a}_{(k)}$ respectively, with $\sigma_x(0) = x$, $\widetilde{\sigma}_x(0) = \widetilde{x}$.

Viewing $\widetilde{J_nY}$ as a subset of J_nY, let us show that $\widetilde{\sigma}_x = \sigma_x$ in a neighborhood of 0. Let $f \in J_nA$. By definition of an integral curve,

$$\frac{d}{dz}(f \circ \sigma_x) = a_{(k)}(f) \circ \sigma_x, \tag{14}$$

$$\frac{d}{dz}(\widetilde{f} \circ \widetilde{\sigma}_x) = \widetilde{a}_{(k)}(\widetilde{f}) \circ \widetilde{\sigma}_x. \tag{15}$$

But the left hand side of (15) is $\dfrac{d}{dz}(f \circ \widetilde{\sigma}_x)$, and the right hand side is $a_{(k)}(f) \circ \widetilde{\sigma}_x$ because $\mathscr{P}_{J_nA}(\mathfrak{m}_x)$ is a chiral Poisson ideal of J_nA. Hence by (14), we conclude by the uniqueness of flows that $\widetilde{\sigma}_x = \sigma_x$ in a neighborhood of 0. Since the n-chiral symplectic leaf $\mathscr{L}_{J_nX}(x)$ is by definition obtained by traveling along integral curves to fields $a_{(k)}$, the chiral symplectic leaf $\mathscr{L}_{J_nX}(x)$, and so its closure, is contained in $\mathscr{V}(\mathscr{P}_{J_nA}(\mathfrak{m}_x))$, whence

$$\mathscr{P}_{J_nA}(\mathfrak{m}_x) \subset \mathscr{K}_x \subset \mathfrak{m}_x.$$

To show the equality $\mathscr{P}_{J_nA}(\mathfrak{m}_x) = \mathscr{K}_x$, it remains to prove that $\mathscr{K}_x$ is an n-chiral Poisson ideal of J_nA.

Let $f \in \mathscr{K}_x$, $a \in A$ and $k \in \{0, \dots, n\}$. Let $\sigma_x \colon B(r) \to J_nY$ be an integral curve to $a_{(k)}$, with $\sigma_x(0) = x$. Then, by definition of an integral curve, (14) holds. On a complex analytic neighborhood of x, $f \circ \sigma_x = 0$ since the image of σ_x is in $\mathscr{L}_{J_nX}(x)$. Hence

$$0 = \frac{d}{dz}(f \circ \sigma_x)(0) = (a_{(k)}(f) \circ \sigma_x)(0) = a_{(k)}(f)(x).$$

As a consequence, $a_{(k)}(\mathscr{K}_x) \subset \mathfrak{m}_{x_n}$ for all $a \in A$ and all $k \in \{0, \dots, n\}$. Repeating this argument with x replaced by each of the members of $\mathscr{L}_{J_nX}(x)$, we conclude that $a_{(k)}(\mathscr{K}_x) \subset \mathscr{K}_x$ for all $a \in A$ and all $k \in \{0, \dots, n\}$, that is, that $\mathscr{K}_x$ is n-chiral Poisson. □

Corollary 8.2. *Let $x \in J_n\mathscr{L}_X(x_0)$. Then the defining ideal of the closure of $\mathscr{C}_{J_nX}(x)$ is $\mathscr{P}_{J_nR}(\mathfrak{m}_x)$.*

Proof. By Lemma 8.1, $\mathscr{L}_{J_nX}(x) \subset \mathscr{C}_{J_nX}(x)$. Indeed, if $y \in \mathscr{L}_{J_nX}(x)$, then $\mathscr{L}_{J_nX}(y) = \mathscr{L}_{J_nX}(x)$ and so $\mathscr{V}(\mathscr{P}_{J_nR}(\mathfrak{m}_y)) = \mathscr{V}(\mathscr{P}_{J_nR}(\mathfrak{m}_x))$ by Lemma 8.1, that is, $\mathscr{P}_{J_nR}(\mathfrak{m}_y) = \mathscr{P}_{J_nR}(\mathfrak{m}_x)$, whence $y \in \mathscr{C}_{J_nX}(x)$. So by Lemma 6.4,

$$\mathscr{V}(\mathscr{P}_{J_nR}(\mathfrak{m}_x)) = \overline{\mathscr{L}_{J_nX}(x)} \subset \overline{\mathscr{C}_{J_nX}(x)} \subset \mathscr{V}(\mathscr{P}_{J_nR}(\mathfrak{m}_x)),$$

whence the statement. □

Corollary 8.3. *Let $x \in X$, and set $Y := \overline{\mathscr{C}_X(x)}$. Then*

$$J_\infty Y = \overline{\mathscr{C}_{J_\infty X}(x_\infty)}.$$

Moreover, for $n \in \mathbb{Z}_{\geqslant 0}$, if J_nY is irreducible, then $J_nY = \overline{\mathscr{C}_{J_nX}(x_n)}$.

Here, recall that x_n (resp. x_∞) stands for $\iota_n(x)$ (resp. $\iota_\infty(x)$) as explained before Proposition 7.4.

Proof. Recall that $\mathscr{L}_X(x) = \mathscr{C}_X(x)$ is contained in the smooth locus of Y, and that $\overline{\pi_{n,0}^{-1}(\mathscr{C}_X(x))}$ has dimension $(n+1)\dim Y$.

Since $\mathscr{P}_R(\mathfrak{m}_x)$ is the defining ideal of $\overline{\mathscr{C}_X(x)}$, it results from Proposition 7.4 that for all $n \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, $\mathscr{V}(\mathscr{P}_{J_nR}(\mathfrak{m}_{x_n})) = \overline{(\pi_{n,0}^Y)^{-1}(Y_{reg})}$, and we have $\mathscr{V}(\mathscr{P}_{J_nR}(\mathfrak{m}_{x_n})) = J_nY$ if J_nY is irreducible. In particular, $J_\infty Y = \mathscr{V}(\mathscr{P}_{J_\infty R}(\mathfrak{m}_{x_\infty}))$.

So by Corollary 8.2, for all $n \in \mathbb{Z}_{\geqslant 0}$,

$$\overline{\mathscr{C}_{J_nX}(x_n)} = \mathscr{V}(\mathscr{P}_{J_nR}(\mathfrak{m}_{x_n})) = \overline{(\pi_{n,0}^Y)^{-1}(Y_{reg})} \subset J_nY.$$

Taking the limit when n goes to $+\infty$, we obtain:

$$\overline{\mathscr{C}_{J_\infty X}(x_\infty)} = \mathscr{V}(\mathscr{P}_{J_\infty R}(\mathfrak{m}_{x_\infty})) = J_\infty Y.$$

This concludes the proof. □

Now assume further that X has only finitely many symplectic leaves. (Note that the poisson bracket on X is algebraic under this condition by [BG, Proposition 3.6].) If $X_1, \dots, X_r$ are the irreducible components of X, then for some $x_1, \dots, x_r \in X$, we have

$$X_i = \mathscr{V}(\mathscr{P}_R(\mathfrak{m}_i)) = \overline{\mathscr{C}_X(x_i)}, \qquad i = 1, \dots, r,$$

where $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ are the maximal ideals of R corresponding to $x_1, \dots, x_r$, respectively, see [Gin1].

From the decomposition $X = X_1 \cup \ldots \cup X_r$, we get that

$$J_\infty X = J_\infty X_1 \cup \ldots \cup J_\infty X_r$$

since the X_i are closed (see Section 3). Moreover, $J_\infty X_1, \ldots, J_\infty X_r$ are precisely the irreducible components of $J_\infty X$. Indeed, for $i = 1, \ldots, r$, $J_\infty X_i$ is closed in $J_\infty X$ since X_i is closed in X, and for any $i \neq j$, we have $J_\infty X_i \not\subset J_\infty X_j$, otherwise, taking the image by the canonical projection $\pi_{\infty,0} \colon J_\infty X \to X$, we would get $X_i \subset X_j$.

Hence, as a consequence of Corollary 8.3, we obtain the following result.

Theorem 8.4. *Let X be a Poisson scheme. Assume that X has only finitely many symplectic leaves. Then each irreducible components of $J_\infty X$ is the closure of some chiral symplectic core.*

More precisely, if $X_1, \ldots, X_r$ are the irreducible components of X, then for $i = 1, \ldots, r$, $X_i = \overline{\mathscr{C}_X(x_i)}$ for some $x_i \in X_i$, and we have:

$$J_\infty X_i = \overline{J_\infty \mathscr{C}_X(x_i)} = \overline{\mathscr{C}_{J_\infty X}(x_{i,\infty})}.$$

9. Applications to quasi-lisse vertex algebras

In this section we assume that V is a vertex algebra (not necessarily commutative or Poisson).

Recall that V is naturally filtered by the Li filtration ([Li], see also [A1]),

$$V = F^0 V \supset F^1 V \supset \cdots \supset F^p V \supset \cdots,$$

where $F^p V$ is the subspace of V spanned by the vectors

$$a^1_{(-n_1-1)} \cdots a^r_{(-n_r-1)} b,$$

with $a^i \in V$, $b \in V$, $n_i \in \mathbb{Z}_{\geqslant 0}$, $n_1 + \cdots + n_r \geqslant p$. The associated graded vector space $\operatorname{gr} V = \oplus_p F^p V / F^{p+1} V$ is naturally a vertex Poisson algebra [Li]. We have

$$F^1 V = C_2(V) := \operatorname{span}_{\mathbb{C}}\{a_{(-2)} b \mid a, b, \in V\}.$$

Let

$$R_V = V / C_2(V) = F^0 V / F^1 V \subset \operatorname{gr} V$$

be the *Zhu C_2-algebra of V*. It is a Poisson algebra [Zhu], and the Poisson algebra structure can be obtained by restriction to R_V of the vertex Poisson algebra on $\operatorname{gr} V$. Namely,

$$1 = \overline{|0\rangle}, \quad \bar{a} \cdot \bar{b} = \overline{a_{(-1)} b} \quad \text{and} \quad \{\bar{a}, \bar{b}\} = \overline{a_{(0)} b},$$

for $a, b \in V$, where $\bar{a} = a + C_2(V)$.

Let

$$\tilde{X}_V := \operatorname{Spec}(R_V) \quad \text{and} \quad X_V := \operatorname{Specm}(R_V)$$

be the *associated scheme* and the *associated variety* of V, respectively ([A1]).

We assume that the filtration $(F^p V)_p$ is separated, that is, $\bigcap F^p V = \{0\}$ and that V is strongly finitely generated, that is, R_V is finitely generated. Note that the first condition is satisfied if V is positively graded.

Theorem 9.1 ([Li, Lemma 4.2], [A1, Proposition 2.5.1])**.** *The identity map $R_V \to R_V$ induces a surjective vertex Poisson algebra homomorphism*

$$J_\infty R_V = \mathbb{C}[J_\infty(\tilde{X}_V)] \twoheadrightarrow \operatorname{gr} V.$$

The *singular support* of a vertex algebra V is

$$\widetilde{SS}(V) := \operatorname{Spec} \operatorname{gr} V \subset J_\infty(\tilde{X}_V).$$

We set

$$SS(V) := \operatorname{Specm} \operatorname{gr} V \subset J_\infty X_V.$$

In the above inclusion, $J_\infty X_V$ is viewed as a topological space.

Recall from the introduction that the vertex algebra V is called *quasi-lisse* if the Poisson variety X_V has finitely many symplectic leaves ([AK]).

Theorem 9.2. *Assume that V is quasi-lisse. Then $SS(V)$ is a finite union of chiral symplectic cores closures in $\operatorname{gr} V$. Moreover, $SS(V) = J_\infty X_V$ as topological spaces.*

Proof. Set

$$\mathcal{L} = \operatorname{Specm} \operatorname{gr} V = SS(V),$$

and let $X_1, \dots, X_r$ be the irreducible components of X_V. By Theorem 8.4, we have

$$J_\infty X_V = \overline{\mathscr{C}_{J_\infty X_V}(x_{1,\infty})} \cup \ldots \cup \overline{\mathscr{C}_{J_\infty X_V}(x_{r,\infty})}, \tag{16}$$

where $x_i \in (X_i)_{reg}$ for $i = 1, \dots, r$. By Theorem 9.1, $\operatorname{gr} V$ is a vertex Poisson algebra quotient of $J_\infty R_V$, that is, $\operatorname{gr}(V) = J_\infty R_V / I$ with I a vertex Poisson ideal of $J_\infty(R_V)$. Furthermore, the surjective morphisms,

$$J_\infty R_V \twoheadrightarrow \operatorname{gr} V \twoheadrightarrow R_V,$$

induce injective morphisms of varieties,

$$X_V \hookrightarrow \mathcal{L} \hookrightarrow J_\infty X_V,$$

and the composition map is ι_∞. Hence for $x \in X_V$, we get that $\mathfrak{m}_\infty \supset I$, where $\mathfrak{m}_\infty$ denotes the maximal ideal of $J_\infty R_V$ corresponding to x_∞, and so x_∞ is a point of $\mathcal{L}$.

Therefore, by Lemma 6.5, $\mathscr{C}_{J_\infty(X_V)}(x_{i,\infty}) = \mathscr{C}_{\mathcal{L}}(x_{i,\infty})$ for any $i = 1, \dots, r$. Then from (16) and Theorem 9.1, we obtain that

$$\mathcal{L} \subset J_\infty X_V = \overline{\mathscr{C}_{\mathcal{L}}(x_{1,\infty})} \cup \ldots \cup \overline{\mathscr{C}_{\mathcal{L}}(x_{r,\infty})} \subset \mathcal{L},$$

since $\mathcal{L}$ is closed, whence the first statement and the required equality $\mathcal{L} = J_\infty X_V$. □

Corollary 9.3. *Suppose that $\tilde{X}_V$ is smooth, reduced and symplectic. Then $\operatorname{gr} V$ is simple as a vertex Poisson algebra, and hence, V is simple.*

Proof. If X_V is a smooth symplectic variety then $J_\infty X_V$ consists of a single chiral symplectic core. So $J_\infty X_V = \mathscr{C}_{J_\infty X_V}(x)$ for any $x \in J_\infty X_V$. It follows that there is no nonzero proper chiral Poisson subscheme in $J_\infty X_V$. So by Theorem 9.2, there is no nonzero proper chiral Poisson subscheme in $\operatorname{Spec} \operatorname{gr} V$, too. Hence $\operatorname{gr} V$ is simple as a vertex Poisson algebra. And so V is simple since any vertex ideal $I \subset V$ defined a vertex Poisson, and so chiral Poisson, ideal $\operatorname{gr} I$ in $\operatorname{gr} V$. □

For example, if X is a smooth affine variety, then the global section of the chiral differential operators $\mathcal{D}_X^{ch}$ ([MSV, GMS2, BD2]) is simple, because its associated scheme is canonically isomorphic to T^*X. In particular, the global section of the chiral differential operators $\mathcal{D}_{G,k}^{ch}$ on a reductive group G ([GMS1, AG]) is simple at any level k.

10. Adjoint quotient and arc space of Slodowy slice

Recall that $\mathfrak{g}$ is a complex simple Lie algebra. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form $(\ |\ )$ of $\mathfrak{g}$. Let (e,h,f) be an $\mathfrak{sl}_2$-triple, and $\mathscr{S}_f = f + \mathfrak{g}^e$ the corresponding Slodowy slice, with $\mathfrak{g}^e$ the centralizer of e in $\mathfrak{g}$. Recall that $\mathfrak{g}^* \cong \mathfrak{g}$ is a Poisson variety and that the symplectic leaves of $\mathfrak{g}^* \cong \mathfrak{g}$ are the (co)adjoint orbits. The algebra $R_f := \mathbb{C}[\mathscr{S}_f]$ inherits a Poisson structure from $\mathbb{C}[\mathfrak{g}]$ by Hamiltonian reduction [GG]. The Hamiltonian reduction can also be described in terms of the BRST cohomology, essentially following Kostant and Sternberg [KS]. The symplectic leaves of $\mathscr{S}_f$ are precisely the intersections of the adjoint orbits of $\mathfrak{g}$ with $\mathscr{S}_f$.

Let $p_1, \dots, p_\ell$ be homogeneous generators of $\mathbb{C}[\mathfrak{g}^*]^G \cong \mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$.

Consider the adjoint quotient map

$$\psi\colon \mathfrak{g} \to \mathfrak{g}//G \cong \mathbb{C}^\ell, \quad x \mapsto (p_1(x), \dots, p_\ell(x)),$$

and its restriction ψ_f to $\mathscr{S}_f$,

$$\psi_f\colon \mathscr{S}_f \to \mathfrak{g}//G \cong \mathbb{C}^\ell.$$

We first recall some facts about ψ_f and its fibers ([Pre1]).

The morphism ψ_f is faithfully flat. As a consequence, ψ_f is surjective and all fibers have the dimension $r - \ell$, where $r = \dim \mathfrak{g}^e$. Furthermore the fibers of ψ_f are generically smooth, that is, contain a smooth open dense subset of dimension $r - \ell$, and they are irreducible.

Lemma 10.1. *Let $\xi \in \mathfrak{g}//G$. Then $\psi_f^{-1}(\xi)$ is a finite union of symplectic leaves. Hence it is the closure of some symplectic leaf closure.*

Proof. The proof is standard, we recall it for the convenience of the reader.

We first prove the statement for the morphism ψ. Let $x \in \psi^{-1}(\xi)$, and write $x = x_s + x_n$ its Jordan decomposition. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} + \mathfrak{n}_+$ be a triangular decomposition of $\mathfrak{g}$. One can assume that $x_s \in \mathfrak{h}$, and that $x_n \in \mathfrak{n}_+$ since $x_n \in \mathfrak{g}^{x_s}$. Let now $y \in \psi^{-1}(\xi)$. Then $p_i(y) = p_i(x)$ for $i = 1, \dots, \ell$, and so y_s is conjugate to x_s by an element s of the Weyl group $W(\mathfrak{g}, \mathfrak{h})$ of $(\mathfrak{g}, \mathfrak{h})$. Let $\tilde{s}$ be a Tits lifting of s in G, so that $y = \tilde{s}(x_s + \tilde{s}^{-1} y_n)$. Since $W(\mathfrak{g}, \mathfrak{h})$ is finite, it is sufficient to show that there are only finitely many possible choices of y_n up to conjugation by the centralizer G^{y_s} of y_s in G. However, since $y_n \in \mathfrak{g}^{y_s} \cap \mathcal{N}$ and the set $\mathfrak{g}^{y_s} \cap \mathcal{N}$ consists of a finite union of $(G^{y_s})^\circ$-orbits, where $(G^{y_s})^\circ$ is the identity connected component subgroup of G^{y_s}, the assertion follows. In conclusion, $\psi^{-1}(\xi)$ is a finite union of G-orbits of $\mathfrak{g}$, that is, a finite union of symplectic leaves of $\mathfrak{g}$. Since $\psi^{-1}(\xi)$ is irreducible and G-invariant, we deduce that $\psi^{-1}(\xi)$ is the closure of some symplectic leaf.

Next, $\psi_f^{-1}(\xi) = \psi^{-1}(\xi) \cap \mathscr{S}_f$ and the symplectic leaves of $\mathscr{S}_f$ are the intersections of adjoint orbits of $\mathfrak{g}$ with $\mathscr{S}_f$. So $\psi_f^{-1}(\xi)$ is a finite union of symplectic leaves, too. As $\psi_f^{-1}(\xi)$ is irreducible [Gin1], $\psi_f^{-1}(\xi)$ is a symplectic leaf closure. □

By Kostant [Kos],

$$\psi_f^{-1}(0) = \mathscr{S}_f \cap \mathcal{N},$$

with $\mathcal{N}$ the nilpotent cone of $\mathfrak{g}$.

Proposition 10.2. *Let $n \in \mathbb{Z}_{\geqslant 0}$. Then $(J_n\psi_f)^{-1}(0) = J_n(\psi_f^{-1}(0))$ is a reduced complete intersection, and it is irreducible. Moreover, $(J_\infty\psi_f)^{-1}(0) = J_\infty(\psi_f^{-1}(0))$ is irreducible and reduced.*

Proof. The variety $\mathscr{S}_f \cap \mathcal{N}$ is normal, reduced and is a complete intersection [Gin2], with rational singularities [AKM, Lemma 3.1.2.1]. As $\psi_f^{-1}(0) = \mathscr{S}_f \cap \mathcal{N}$, it follows from the main results of [Mus] and its consequences that $J_n(\psi_f^{-1}(0))$ is also reduced, irreducible and a complete intersection for any $n \geqslant 0$. Now observe that $(J_n\psi_f)^{-1}(0) = J_n(\psi_f^{-1}(0))$ by the properties of jet schemes (cf. Section 3). This proves the first part of the statement.

Since $\psi_f^{-1}(0)$ is irreducible, $J_\infty(\psi_f^{-1}(0))$ is irreducible by Theorem 3.4 and, from the above, we get that $(J_\infty\psi_f)^{-1}(0) = J_\infty(\psi_f^{-1}(0))$ is irreducible. Because all $J_n(\psi_f^{-1}(0))$ are reduced, $J_\infty(\psi_f^{-1}(0))$ is reduced, too. □

Next, we wish to prove that the other fibers of $J_\infty\psi_f$ are also reduced and irreducible. To this end, we use ideas of [Pre1, §§5.3 and 5.4].

The Slodowy slice has a contracting $\mathbb{C}^*$-action. Recall briefly the construction. The embedding $\mathrm{span}_{\mathbb{C}}\{e,h,f\} \cong \mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ exponentiates to a homomorphism $SL_2 \to G$. By restriction to the one-dimensional torus consisting of diagonal matrices, we obtain a one-parameter subgroup $\tilde{\rho}\colon \mathbb{C}^* \to G$. Thus $\tilde{\rho}(t)x = t^{2j}x$ for any $x \in \mathfrak{g}_j = \{y \in \mathfrak{g} \mid [h,y] = 2jy\}$. For $t \in \mathbb{C}^*$ and $x \in \mathfrak{g}$, set

$$\rho(t)x := t^2\tilde{\rho}(t)x. \tag{17}$$

So, for any $x \in \mathfrak{g}_j$, $\rho(t)x = t^{2+2j}x$. In particular, $\rho(t)f = f$ and the $\mathbb{C}^*$-action of ρ stabilizes $\mathscr{S}_f$. Moreover, it is contracting to f on $\mathscr{S}_f$, that is,

$$\lim_{t\to 0}\rho(t)(f+x) = f$$

for any $x \in \mathfrak{g}^e$. The $\mathbb{C}^*$-action ρ induces a positive grading on $\mathscr{S}_f$, and so on $R_f = \mathbb{C}[\mathscr{S}_f] \cong \mathbb{C}[\mathfrak{g}^e]$.

Let $n \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$. Similarly, we define a contracting $\mathbb{C}^*$-action on $J_n\mathscr{S}_f$ and a positive grading on J_nR_f as follows.

Let $x^1, \dots, x^r$ be a basis of $\mathfrak{g}^e$ so that

$$J_nR_f \cong J_n\mathbb{C}[\mathfrak{g}^e] \cong \operatorname{Spec}\mathbb{C}[x^i_{(-j-1)}\,;\, i = 1,\dots,r,\ j = 0,\dots,n].$$

One can assume that the x^i's are Slodowy homogeneous. One defines a grading on J_nR_f by setting

$$\deg x^i_{(-j-1)} = \deg x^i + j.$$

Since the grading is positive, it gives a contracting $\mathbb{C}^*$-action on $J_n\mathscr{S}_f$. Indeed, consider the morphism $\mathbb{C}[J_n\mathscr{S}_f] \to \mathbb{C}[J_n\mathscr{S}_f] \otimes \mathbb{C}[t,t^{-1}]$, $f \mapsto f \otimes t^{\deg f}$, for homogeneous f. Its comomorphism induces a $\mathbb{C}^*$-action

$$\mu_n\colon \mathbb{C}^* \times J_n\mathscr{S}_f \to J_n\mathscr{S}_f$$

which is contracting since $\deg f \geqslant 0$ for any homogeneous f.

The above grading gives an increasing filtration on J_nR_f in an obvious way:

$$\mathscr{F}_p(J_nR_f) := \oplus_{j\leqslant p}(J_nR_f)_j, \qquad p \geqslant 0.$$

Given a quotient M of J_nR_f, we define a filtration $(\mathscr{F}_pM)_p$ of M by setting

$$\mathscr{F}_pM := \tau_n(\mathscr{F}_p(J_nR_f)),$$

where τ_n is the canonical quotient morphism $\tau_n\colon J_nR_f \to M$. We denote by $\mathrm{gr}M$ the corresponding graded space.

For M a subspace of J_nR_f denote by $\mathrm{gr}M$ the homogeneous subspace of J_nR_f with the property that $g \in \mathrm{gr}M \cap (J_nR_f)_p$ if and only if there is $\tilde{g} \in M$ such that

$\tilde{g} - g \in \mathscr{F}_{p-1}(J_n R_f)$. Obviously the subspace $\mathrm{gr}M$ is invariant for the $\mathbb{C}^*$-action μ_n. If M is an ideal of $J_n R_f$ then $\mathrm{gr}M$ is an ideal of $\mathrm{gr}\, J_n R_f$.

For $f \in \mathbb{C}[J_n\mathfrak{g}]$ we denote by $\overline{f}$ its restriction to $J_n\mathscr{S}_f$. Then for $\xi = (\xi_i^{(j)} \mid i = 1, \dots, \ell, j = 0, \dots, n) \in J_n\mathbb{C}^\ell$, $(J_n\psi_f)^{-1}(\xi)$ is the set of common zeroes of the ideal

$$\mathscr{I}_{n,\xi} := \left(\overline{T^j p_i} - \xi_i^{(j)} \mid i = 1, \dots, \ell, j = 0, \dots, n\right).$$

Here, note that $J_n\psi_f$ is the morphism:

$$J_n\psi_f \colon J_\infty\mathscr{S}_f \to J_n\mathbb{C}^\ell, \quad x \mapsto (T^j p_i(x), i = 1, \dots, \ell, j = 0, \dots, n).$$

Lemma 10.3. *Let $n \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ and $\xi \in J_n\mathbb{C}^\ell$. Then the fiber $(J_n\psi_f)^{-1}(\xi)$ is reduced and irreducible.*

Proof. Clearly $\mathrm{gr}\mathscr{I}_{n,\xi} = \mathscr{I}_{n,0}$.

Let $a \in J_n R_f/\mathscr{I}_{n,\xi}$ and suppose that $a^k = 0$ for some $k \in \mathbb{Z}_{\geqslant 0}$. Then $\sigma_n(a)^k = 0$, where σ_n is the symbol of a in $\mathrm{gr}(J_n R_f/\mathscr{I}_{n,\xi}) = J_n R_f/\mathscr{I}_{n,0}$. As $\mathscr{I}_{n,0}$ is radical, $\sigma_n(a) = 0$, and hence $a = 0$. This proves that $\mathscr{I}_{n,\xi}$ is radical.

Similarly, $J_n R_f/\mathscr{I}_{n,\xi}$ is a domain since $\mathrm{gr}(J_n R_f/\mathscr{I}_{n,\xi}) = J_n R_f/\mathscr{I}_{n,0}$ is. Hence $\mathscr{I}_{n,\xi}$ is prime. □

The Poisson bracket on $\mathfrak{g}$ is algebraic since all adjoint orbits are open in their closure. From the inclusion $\overline{G.x} \cap \mathscr{S}_f \subset \overline{G.x} \cap \mathscr{S}_f$, for $x \in \mathscr{S}_f$, we deduce that the symplectic leaf $G.x \cap \mathscr{S}_f$ of $\mathscr{S}_f$ is locally closed, and hence the Poisson bracket on $\mathscr{S}_f$ is algebraic. Indeed, $\overline{G.x} \cap \mathscr{S}_f$ is a finite union of symplectic leaves of $\mathscr{S}_f$ and so [BG, Proposition 3.7] applies. As a result, the hypothesis of Section 8 are satisfied.

Proposition 10.4. *For any $n \in \mathbb{Z}_{\geqslant 0}$, the fiber $(J_n\psi_f)^{-1}(0)$ is the closure of some n-chiral Poisson core in $J_n\mathscr{S}_f$, and $(J_\infty\psi_f)^{-1}(0)$ is the closure of some chiral Poisson core in $J_\infty\mathscr{S}_f$.*

Proof. By Proposition 10.2, $J_n(\psi_f^{-1}(0)) \cong (J_n\psi_f)^{-1}(0)$ is irreducible for any $n \in \mathbb{Z}_{\geqslant 0}$. The statement follows from Corollary 8.3 because $\psi_f^{-1}(0)$ is the closure of some symplectic leaf. □

Theorem 10.5. *Let z be in the vertex Poisson center of $J_\infty R_f$, and $\xi \in J_\infty(\mathfrak{g}/\!/G)$. Then z is constant on $(J_\infty\psi_f)^{-1}(\xi)$.*

Proof. By Proposition 6.6 and Proposition 10.4, any element z in the vertex Poisson center of $J_\infty R_f$ is constant on $(J_\infty\psi_f)^{-1}(0)$. Let now $\xi \in J_\infty(\mathfrak{g}/\!/G)$. Then the symbol $\sigma_\infty(z) \in \mathrm{gr}(J_\infty R_f/\mathscr{I}_{\infty,\xi})$ belongs to the center $\mathcal{Z}(J_\infty R_f/\mathscr{I}_{\infty,0})$ of $\mathrm{gr}(J_\infty R_f/\mathscr{I}_{\infty,0})$. However, $\mathcal{Z}(J_\infty R_f/\mathscr{I}_{\infty,0}) \cong \mathbb{C}$ by the $\xi = 0$ case. Therefore $\sigma_\infty(z)$ is constant, and this happens only if z itself is constant in $J_\infty R_f/\mathscr{I}_{\infty,\xi}$, that is, z is constant on $(J_\infty\psi_f)^{-1}(\xi)$. □

11. Vertex Poisson center and arc space of Slodowy slices

The vertex Poisson algebra structure on $\mathbb{C}[J_\infty R_f]$ can be described using cohomology of some dg-vertex Poisson algebras, which is a tensor product of functions

over $J_\infty\mathfrak{g}$ with *fermionic-ghost* vertex Poisson super-algebra $\wedge^{\frac{\infty}{2}}(\mathfrak{m})$, where $\mathfrak{m}$ is a certain nilpotent algebra $\mathfrak{m}$ ([A3, Theorem 4.6]):

$$\mathbb{C}[J_\infty R_f] \cong H^0(\mathbb{C}[J_\infty\mathfrak{g}] \otimes \wedge^{\frac{\infty}{2}}(\mathfrak{m}), Q_{(0)}).$$

The canonical embedding

$$\mathbb{C}[J_\infty\mathfrak{g}] \longrightarrow \mathbb{C}[J_\infty\mathfrak{g}] \otimes \wedge^{\frac{\infty}{2}}(\mathfrak{m}), \quad f \longmapsto f \otimes 1,$$

induces morphisms of vertex Poisson algebras,

$$\mathcal{Z}(\mathbb{C}[J_\infty\mathfrak{g}]) \longrightarrow \mathcal{Z}(\mathbb{C}[J_\infty\mathfrak{g}] \otimes \wedge^{\frac{\infty}{2}}(\mathfrak{m})) \longrightarrow \mathcal{Z}(H^0(\mathbb{C}[J_\infty\mathfrak{g}] \otimes \wedge^{\frac{\infty}{2}}(\mathfrak{m}), Q_{(0)})).$$

Hence we get a morphism of vertex Poisson algebras,

$$\mathcal{Z}(\mathbb{C}[J_\infty\mathfrak{g}]) \longrightarrow \mathcal{Z}(\mathbb{C}[J_\infty R_f]) \subset \mathbb{C}[J_\infty R_f].$$

Note that this morphism corresponds to the restriction map.

On the other hand, we have an isomorphism [RsT, BD1, EF]: $J_\infty(\mathfrak{g}/\!/G) \cong J_\infty\mathfrak{g}/\!/J_\infty G$, where $J_\infty\mathfrak{g}/\!/J_\infty G = \operatorname{Spec}\mathbb{C}[J_\infty\mathfrak{g}]^{J_\infty G}$. In other words, the infinite jet scheme $J_\infty G$ acts on $J_\infty\mathfrak{g}$ and $\mathbb{C}[J_\infty\mathfrak{g}]^{J_\infty G}$ is the polynomial ring $\mathbb{C}[J_\infty(\mathfrak{g}/\!/G)] = \mathbb{C}[T^j p_i,\, i = 1, \dots, \ell, j \geqslant 0]$. Therefore, we get the following isomorphisms:

$$\mathcal{Z}(\mathbb{C}[J_\infty\mathfrak{g}]) = \mathbb{C}[J_\infty\mathfrak{g}/\!/J_\infty G] \cong \mathbb{C}[J_\infty(\mathfrak{g}/\!/G)].$$

So the above morphism from $\mathcal{Z}(\mathbb{C}[J_\infty\mathfrak{g}])$ to $\mathbb{C}[J_\infty R_f]$ is nothing but the comorphism,

$$(J_\infty\psi_f)^*\colon J_\infty(\mathfrak{g}/\!/G) \to J_\infty R_f,$$

of $J_\infty\psi_f$.

The map $(J_\infty\psi_f)^*$ is an embedding. Indeed, let $\mathfrak{g}_{reg}$ be the set of regular elements of $\mathfrak{g}$, that is, those elements whose centralizer has minimal dimension ℓ. Since the restriction of the morphism ψ_f to $\mathscr{S}_f \cap \mathfrak{g}_{reg}$ is smooth and surjective (see [Kos], [Pre1, Section 5]), the restriction of $J_n\psi_f$ to $J_n(\mathscr{S}_f \cap \mathfrak{g}_{reg})$ is smooth and surjective for any n as well ([EM, Remark 2.10] or [Fre, §3.4.3]). Therefore the morphism $(J_n\psi_f)^*\colon \mathbb{C}[J_n\mathfrak{g}]^{J_n G} \to J_n R_f$ is an embedding for any n. Moreover, the restriction of $J_\infty\psi_f$ to $J_\infty(\mathscr{S}_f \cap \mathfrak{g}_{reg})$ is (formally) smooth and surjective, whence the morphism $(J_\infty\psi_f)^*\colon \mathbb{C}[J_\infty\mathfrak{g}]^{J_\infty G} \to J_\infty R_f$ is an embedding of vertex Poisson algebras.

The aim of this section is to proof the following result.

Theorem 11.1. *The morphism $(J_\infty\psi_f)^*$ induces an isomorphism of vertex Poisson algebras between $\mathbb{C}[J_\infty\mathfrak{g}]^{J_\infty G}$ and the vertex Poisson center of $\mathbb{C}[J_\infty\mathscr{S}_f]$. Moreover, $\mathbb{C}[J_\infty\mathscr{S}_f]$ is free over its vertex Poisson center.*

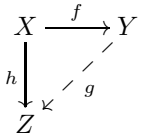
Note that $\mathbb{C}[J_\infty\mathfrak{g}]^{J_\infty G} \cap \mathbb{C}[\mathfrak{g}] = \mathbb{C}[\mathfrak{g}]^G$ and $\mathcal{Z}(\mathbb{C}[J_\infty\mathscr{S}_f]) \cap \mathbb{C}[\mathscr{S}_f] = \mathcal{Z}(\mathbb{C}[\mathscr{S}_f])$, the Poisson center of $\mathbb{C}[\mathscr{S}_f]$. Hence from the above theorem we recover the well-known result of Ginzburg-Premet [Pre2, Question 5.1] which states that

$$\mathcal{Z}(\mathbb{C}[\mathscr{S}_f]) \cong \mathbb{C}[\mathfrak{g}]^G. \tag{18}$$

To prove Theorem 11.1 we first state some preliminary results.

Lemma 11.2 ([GW, Theorem A.2.9])**.** *Let X, Y, Z be irreducible affine varieties. Assume that $f\colon X \to Y$ and $h\colon X \to Z$ are dominant morphisms such that h is*

constant on the fibers of f. Then there exists a rational map $g\colon Y \to Z$ making the following diagram commutative:



Lemma 11.3. *Let X and Y be two normal irreducible affine varieties, and $f\colon X \to Y$ a flat morphism. Then $\mathbb{C}(Y)\cap\mathbb{C}[X] = \mathbb{C}[Y]$. Here, we view $\mathbb{C}[Y]$ as a subalgebra of $\mathbb{C}[X]$ using $f^*\colon \mathbb{C}[Y] \to \mathbb{C}[X]$.*

Proof. Since X is normal and the fibers of f are all of dimension $\dim X - \dim Y$, the image of the set X' of smooth points of X is an open subset Y' of Y such that $Y \setminus Y'$ has codimension at least 2.

Let y in Y' and $x \in f^{-1}(y) \subset X'$. Then we have a flat extension of the local rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Since $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$ are regular local rings, they are factorial. For $a \in \mathbb{C}(Y)\cap\mathbb{C}[X]$, write $a = p/q$ with p, q relatively prime elements of $\mathbb{C}[Y]$. Since p, q are relatively prime, the multiplication by p induces an injective homomorphism

$$\mathcal{O}_{Y,y}/q\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/q\mathcal{O}_{Y,y}.$$

Since $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$, the base change $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} -$ yields an injective homomorphism

$$\mathcal{O}_{X,x}/q\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/q\mathcal{O}_{X,x}.$$

Hence p and q are relatively prime in $\mathcal{O}_{X,x}$. In addition, the image of 1 is 0 because $a = p/q$ is regular in X. As a result, q is invertible in $\mathcal{O}_{X,x}$.

Since the maximal ideal of $\mathcal{O}_{Y,y}$ is the intersection of $\mathcal{O}_{Y,y}$ with the maximal ideal of $\mathcal{O}_{X,x}$, q is invertible in $\mathcal{O}_{Y,y}$, and so a is in $\mathcal{O}_{Y,y}$. As a result, a is regular on Y' and then extends to a regular function on Y since Y is normal. □

Proof of Theorem 11.1. Let us prove the first assertion of the theorem. We view the algebra $J_n(\mathbb{C}[\mathfrak{g}]^G)$ as a subalgebra of J_nR_f for any n. We have already noticed that the inclusion $J_\infty(\mathbb{C}[\mathfrak{g}]^G) \subset \mathcal{Z}(J_\infty R_f)$ holds, with $\mathcal{Z}(J_\infty R_f)$ the vertex Poisson center of $J_\infty R_f$. Conversely, we have to prove that any element z in the vertex Poisson center $\mathcal{Z}(J_\infty R_f)$ can be lifted to a an element of $J_\infty(\mathbb{C}[\mathfrak{g}]^G)$.

Since $J_\infty\mathscr{S}_f$ is the projective limit of the projective system $(J_n\mathscr{S}_f, \pi_{m,n})$, the algebra $J_\infty R_f$ is the inductive limit of the algebras J_nR_f. The injection j_n from J_nR_f to $J_\infty R_f$ is defined by

$$j_n(\mu_n)(\gamma) := \mu_n(\pi_{\infty,n}(\gamma)), \qquad \mu_n \in J_nR_f,\ \gamma \in J_\infty\mathscr{S}_f.$$

Let $z \in \mathcal{Z}(J_\infty R_f) \subset J_\infty R_f$. As $z \in J_\infty R_f$, $z = z_n \in J_nR_f$ for n big enough, where z_n is such that

$$z_n(\gamma_n) = z(j_n(\gamma_n)), \quad z(\gamma) = z_n(\pi_{\infty,n}(\gamma)), \qquad \gamma_n \in J_n\mathscr{S}_f,\ \gamma \in J_\infty\mathscr{S}_f.$$

By Theorem 10.5, z is constant on each fibers of $J_\infty\psi_f$. As a consequence, z_n is constant on each fibers of $J_n\psi_f$. Indeed, let $\xi_n \in J_nY$, and $\gamma_n, \gamma'_n \in J_n\psi_f^{-1}(\xi_n)$. Then $j_n(\gamma_n), j_n(\gamma'_n)$ are in $J_\infty\psi_f^{-1}(j_n(\xi_n))$ since

$$J_\infty\psi_f(j_n(\gamma_n)) = j_n(J_n\psi_f(\gamma_n)) = j_n(\xi_n) = j_n(J_n\psi_f(\gamma'_n)) = J_\infty\psi_f(j_n(\gamma'_n)).$$

Hence,

$$z_n(\gamma_n) = z(j_n(\gamma_n)) = z(j_n(\gamma'_n)) = z_n(\gamma'_n)$$

since z is constant on $J_\infty\psi_f^{-1}(j_n(\xi_n))$.

If z is a constant function, that is, $z \in \mathbb{C}$, then clearly z lies in the vertex Poisson center of $J_\infty\mathbb{C}[\mathfrak{g}^*]$. Hence, one can assume that z is not constant. Furthermore, there is no loss of generality in assuming that z is homogeneous for the Slodowy grading on $J_\infty R_f$ because $\mathcal{Z}(J_\infty R_f)$ is Slodowy invariant. Thus for any $t \in \mathbb{C}^*$, $t.z = t^k z$ for some $k \in \mathbb{Z}_{\geqslant 0}$. So one can assume that the morphisms $z\colon J_\infty\mathscr{S}_f \to \mathbb{C}$ and $z_n\colon J_n\mathscr{S}_f \to \mathbb{C}$ are dominant.

Hence by Lemma 11.2, $z_n \in \mathbb{C}[J_n\mathscr{S}_f]$ induces a rational morphism $\tilde{z}_n$ on $J_n(\mathfrak{g}/\!/G)$ since z_n is constant on the fibers of the dominant morphism $J_n\psi_f$.

As $\mathscr{S}_f$ and $\mathfrak{g}/\!/G$ are affine spaces, $J_n\mathscr{S}_f$ and $J_n(\mathfrak{g}/\!/G)$ are affine spaces for any n. In particular, $J_n\mathscr{S}_f$ and $J_n(\mathfrak{g}/\!/G)$ are normal and irreducible for any n. Therefore Lemma 11.3 can be applied because the morphism $J_n\psi_f\colon J_n\mathscr{S}_f \to J_n(\mathfrak{g}/\!/G)$ is flat for any n. So, $\tilde{z}_n \in \mathbb{C}[J_n(\mathfrak{g}/\!/G)]$. This holds for any n such that $z = z_n$. Since $z = z_n$ for n big enough, we deduce that z can be lifted to an element of $\mathbb{C}[J_\infty(\mathfrak{g}/\!/G)] = \mathbb{C}[J_\infty\mathfrak{g}]^{J_\infty G}$, whence the first part of the theorem.

It remains to prove the freeness. Since $\mathscr{S}_f \cap \mathcal{N}$ enjoys the same geometrical properties as $\mathcal{N}$, that is, $\mathscr{S}_f \cap \mathcal{N}$ is reduced, irreducible and is a complete intersection with rational singularities, the arguments of [EF, Theorem A.4] can be applied in order to get that $\mathbb{C}[J_\infty\mathscr{S}_f]$ is free over its vertex Poisson center (see also [CM, Proposition 2.5 (ii)]). This concludes the proof of the theorem. □

12. Center of W-algebras

Let $V^k(\mathfrak{g})$ be the universal affine vertex algebra associated with $\mathfrak{g}$ at level k, and let $\mathcal{W}^k(\mathfrak{g}, f)$ be the (affine) W-algebraassociated with $(\mathfrak{g}, f)$ at level $k \in \mathbb{C}$. The W-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is defined by the quantized Drinfeld-Sokolov reduction associated with f ([FF1, KRW]).

The embedding $Z(V^k(\mathfrak{g})) \hookrightarrow V^k(\mathfrak{g})$ induces a vertex algebra homomorphism

$$Z(V^k(\mathfrak{g})) \longrightarrow Z(\mathcal{W}^k(\mathfrak{g}, f))$$

for any $k \in \mathbb{C}$. Here, for V a vertex algebra V, $Z(V)$ denotes the vertex center of V, that is,

$$Z(V) = \{z \in V \mid a_{(n)}z = 0 \text{ for all } a \in V, n \geqslant 0\}.$$

Both $Z(V^k(\mathfrak{g}))$ and $Z(\mathcal{W}^k(\mathfrak{g}, f))$ are trivial unless $k = cri$ is the critical level $cri = -h^\vee$ with $h^\vee$ the dual Coxeter number of $\mathfrak{g}$. For $k = cri$, $\mathfrak{z}(\widehat{\mathfrak{g}}) := Z(V^{cri}(\mathfrak{g}))$ is known as the Feigin-Frenkel center [FF2].

Theorem 12.1. *The embedding $\mathfrak{z}(\widehat{\mathfrak{g}}) \hookrightarrow V^{cri}(\mathfrak{g})$ induces an isomorphism*

$$\mathfrak{z}(\widehat{\mathfrak{g}}) \xrightarrow{\sim} Z(\mathcal{W}^{cri}(\mathfrak{g}, f))$$

and we have $\operatorname{gr} Z(\mathcal{W}^{cri}(\mathfrak{g}, f)) \cong \mathcal{Z}(\mathbb{C}[J_\infty\mathscr{S}_f])$.

Proof. Recall that there is an obvious vertex algebra homomorphism $\mathfrak{z}(\widehat{\mathfrak{g}}) \to Z(\mathcal{W}^{cri}(\mathfrak{g}, f))$, see [A2]. Hence it is sufficient to show that the induced homomorphism $\operatorname{gr}\mathfrak{z}(\widehat{\mathfrak{g}}) \to \operatorname{gr} Z(\mathcal{W}^{cri}(\mathfrak{g}, f))$ is an isomorphism.

First, we have ([FF2])

$$\operatorname{gr}\mathfrak{z}(\widehat{\mathfrak{g}}) \cong \mathbb{C}[J_\infty\mathfrak{g}]^{J_\infty G}.$$

On the other hand, we have ([A3, Theorem 4.17])

$$\operatorname{gr}\mathcal{W}^{cri}(\mathfrak{g}, f) \cong \mathbb{C}[J_\infty\mathscr{S}_f],$$

and so

$$\mathcal{Z}(\mathrm{gr}\,\mathcal{W}^{cri}(\mathfrak{g},f)) \cong \mathcal{Z}(\mathbb{C}[J_\infty \mathscr{S}_f]).$$

By Theorem 11.1, $\mathcal{Z}(\mathbb{C}[J_\infty \mathscr{S}_f]) \cong \mathbb{C}[J_\infty \mathfrak{g}]^{J_\infty G}$, which forces the compound map

$$\mathcal{Z}(\mathbb{C}[J_\infty \mathscr{S}_f]) \cong \mathrm{gr}\,\mathfrak{z}(\widehat{\mathfrak{g}}) \longrightarrow \mathrm{gr}\,Z(\mathcal{W}^{cri}(\mathfrak{g},f)) \hookrightarrow \mathcal{Z}(\mathrm{gr}\,\mathcal{W}^{cri}(\mathfrak{g},f)) \cong \mathcal{Z}(\mathbb{C}[J_\infty \mathscr{S}_f])$$

to be an isomorphism. This completes the proof. □

Theorem 12.1 was stated in [A2], but the proof of the surjectivity was incomplete.

Note that the similar argument as above using (18) recovers Premet's result [Pre2] stating that the center of the *finite W-algebra* $U(\mathfrak{g},f)$ associated with $(\mathfrak{g},f)$ is isomorphic to the center of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

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