RIMS Kôkyûroku Bessatsu
B90 (2022), 1–16

Supersingular abelian varieties and curves, and their moduli spaces, with a remark on the dimension of the moduli of supersingular curves of genus 4

By

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Abstract

We give a survey on fundamentals of supersingular abelian varieties and supersingular curves and their moduli spaces. As an application of the explicit description of supersingular locus in the low dimensional case, we obtain a new result on the dimension of some components of the moduli space of supersingular curves of genus 4.

§ 1. Introduction

In [20], Li and Oort studied the structure of the supersingular locus S_g in the moduli space of principally polarized abelian varieties of dimension g in characteristic $p > 0$. In particular, the dimension of every irreducible component of S_g and the number of irreducible components of S_g were determined. The proof was done by introducing the notion of rigid polarized flag type quotients (PFTQs) and by describing the supersingular locus S_g as a quotient of the union of the moduli space $\mathcal{P}'_{g,\eta}$ of rigid PFTQs, with an explicit description of $\mathcal{P}'_{g,\eta}$. The first aim of this paper is to explain an overview of this theory with reviewing some basic facts on supersingular abelian varities and on supersingular curves. The second aim is to describe the explicit structure of the moduli space of PFTQs for $g \leq 4$, and give an application to a problem of the moduli space of supersingular curves of genus 4. The moduli space is regarded as the intersection

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Received January 2, 2021. Revised June 9, 2021.
2020 Mathematics Subject Classification(s): Primary: 14K10, secondary: 11G10; 14L05
Key Words: *Supersingular abelian varieties, Superspecial abelian varieties, Supersingular curves, Superspecial curves, Moduli space.*
Supported by JSPS Grant-in- Aid for Scientific Research (C) 17K05196 and the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

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of S_g and the Torelli locus $\mathcal{T}_g$ for $g = 4$, where the Torelli locus is the image of the Torelli map

$$\mathcal{M}_g \to \mathcal{A}_g$$

sending C to the Jacobian variety $\mathrm{Jac}(C)$ of C. If $g \leq 3$, then it is known that $\mathcal{T}_g$ is open dense in $\mathcal{A}_g$, and this fact produces some general results on supersingular curves, but for $g \geq 4$, there are many open problems remaining, especially it is still an open problem whether there exists a supersingular curve of genus g in characteristic p for given (g, p) with $g \geq 5$. Remark that quite recently this problem for $g = 4$ was solved affirmatively in [18], but for example we still have few knowledge about the structure of $S_4 \cap \mathcal{T}_4$. This paper will determine the dimension of some components of $S_4 \cap \mathcal{T}_4$ (Corollary 4.4).

The organization of this paper is as follows. In Section 2, we recall the definitions of supersingular abelian varieties and supersingular curves and review their basic facts and the Dieudonné theory used later on. In Section 3, we give an overview of the theory of Li and Oort on the moduli space of principally polarized supersingular abelian varieties. In Section 4, we review the explicit structure of the moduli space of PFTQs in the case of genus $g \leq 4$ and show our main result on the supersingular locus in the moduli space of curves of genus 4. For our purpose, it is important to know the singularity of S_g. For this, we need to investigate the moduli space of PFTQs rather than that of rigid PFTQs.

Acknowledgements

This work was supported by JSPS Grant-in-Aid for Scientific Research (C) 17K05196 and by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

§ 2. Supersingular abelian varieties and curves

We recall the definition of supersingular abelian varieties and curves, and review basic facts and known results on them. We also recall the Dieudonné theory on the classification of p-divisible groups, which will be used in the latter sections.

§ 2.1. Supersingular elliptic curves

Let k be an algebraically closed field in characteristic p. Let E be an elliptic curve over k. We say that E is *supersingular* if the group $E[p](k)$ of k-rational points on the kernel $E[p]$ of p-multiplication $p : E \to E$ consists of only 0. This is equivalent to saying that $B := \mathrm{End}(E) \otimes \mathbb{Q}$ is a quaternion algebra over $\mathbb{Q}$, where $\mathrm{End}(E)$ is the ring of endomorphisms on E. This quaternion algebra is ramified only at p and ∞. There are other criterions on the supersingularity: E is supersingular if and only if the Frobenius F^* on the first cohomology $H^1(E, \mathcal{O}_E)$ is zero or equivalently the Cartier operator $\mathcal{C}$ on the space $H^0(E, \Omega_E)$ of the regular differential forms on E is zero.

For $p > 2$, we consider an elliptic curve of the Legendre form

$$E : y^2 = f(x) := x(x-1)(x-t).$$

Put $m = (p-1)/2$. Then E is supersingular if and only if the x^{p-1}-coefficient of $f(x)^m$ is zero, in other words

$$\sum_{i=0}^{m} \binom{m}{i}^2 t^i = 0$$

cf. [29, Chap. V, Theorem 4.1 (b)]. As any solution of this equation gives a nonsingular elliptic curve, this implies in particular that there exists a supersingular elliptic curve if $p > 2$. Also for $p = 2$, there exists a supersingular elliptic curve $y^2 + y = x^3$. Moreover, for example by looking at isomorphisms between Legendre forms above, one can show that the number of isomorphism classes is equal to

$$\frac{p-1}{12} + \left\{1 - \left(\frac{-3}{p}\right)\right\}/3 + \left\{1 - \left(\frac{-4}{p}\right)\right\}/4,$$

cf. Deuring [3] and Igusa [11], also see [29, Chap. V, Theorem 4.1 (c)]. Remark that this number is equal to the class number of $B = \mathbb{Q}_{\infty,p}$.

§ 2.2. Supersingular abelian varieties

Let us recall the definition in the higher dimensional case. Let k be an algebraically closed field of characteristic $p > 0$. We choose a supersingular elliptic curve E over k.

Definition 2.1. Let X be an abelian variety of dimension $g \geq 2$ over k.

(1) We say that X is *superspecial* if X is isomorphic to E^g

(2) We say that X is *supersingular* if X is isogenous to E^g.

Remark. This definition is independent of the choice of E. In fact, due to Deligne, Ogus and Shioda, for any supersingular elliptic curves $E_1, \ldots, E_{2g}$, we have

$$E_1 \times \cdots \times E_g \simeq E_{g+1} \times \cdots \times E_{2g},$$

cf. [23, Theorem 6.2] and [28, Theorem 3.5], also see [20, 1.6].

Let X be an abelian variety over k and X^t denote its dual abelian variety. A *polarization* on X is an isogeny

$$\eta : X \to X^t$$

obtained as $\eta(x) = \mathcal{L}^{-1} \otimes T_x^*\mathcal{L}$ for an ample line bundle $\mathcal{L}$ on X, where $T_x : X \to X$ is the translation map by x. We say that η is *principal* if η is an isomorphism.

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g. Let S_g be the supersingular locus in $\mathcal{A}_g$, which is the closed subset

$$\{(X, \eta) \in \mathcal{A}_g \mid X \text{ is supersingular}\}$$

of $\mathcal{A}_g$. We consider this as a closed subscheme of $\mathcal{A}_g$ by giving it the induced reduced structure.

§ 2.3. Supersingular curves

In this paper, a curve always means a nonsingular projective variety of dimension one. For a curve C over k, let $\mathrm{Jac}(C)$ denote the Jacobian variety of C, equipped with the canonical principal polarization. Here are the definitions of the supersingularity and the superspeciality of curves.

Definition 2.2. Let C be a curve over k.

(1) C is called *superspecial* if $\mathrm{Jac}(C)$ is superspecial.

(2) C is called *supersingular* if $\mathrm{Jac}(C)$ is supersingular.

Contrary to the case of abelian varieties, the problem asking whether there exists a superspecial curve of genus g in characteristic p for given (p, g) is still open in general. However, for $g \leq 3$, some general results on the existence have been obtained by making use of the fact that the Torelli locus $\mathcal{T}_g$ is open dense in $\mathcal{A}_g$ for $g \leq 3$. For $g = 1$, the existence of a supersingular elliptic curve is due to Deuring [3] (also see Igusa [11]), as we have already explained it in Section 2.1. The existence for $g = 2$ and $p \geq 5$ was proved by Serre [27] and Ibukiyama-Katsura-Oort [10, Proposition 3.1]. For the existence for $g = 3$ and $p \geq 3$, see Oort [24, 5.12] and Ibukiyama [8, Theorem 1], where Oort uses a geometrical approach and Ibukiyama uses an arithmetical approach.

For $g \geq 4$, there is no general answer to the existence of superspecial curve of genus g. However, there are many results on the existence and a few results on the non-existence. If you find a maximal curve over $\mathbb{F}_{p^2}$ for example in the table of site [31], then it is a superspecial curve. As for the non-existence result, here is a celebrated result:

Theorem 2.3 (Ekedahl [4, Theorem 1.1 on p. 165])**.**

(1) *There is no superspecial curve if* $p^2 - p < 2g$.

(2) *There is no superspecial hyperelliptic curve if* $p - 1 < 2g$ *and* $(p, g) \neq (2, 1)$.

Let us summarize known results in the case of $g = 4$ (apology: we do not do so for $g \geq 5$ in this note). First, there is no superspecial curve of genus 4 in characteristic $p = 2, 3$ and 7, see Ekedahl's theorem above for $p = 2, 3$ and Kudo-Harashita [14] for $p = 7$. So far, we have no non-existence result for $p > 7$. There is a superspecial curve of genus 4 in characteristic 5, which is unique (up to isomorphism over an algebraically closed field), see [5] and also [14, Corollary 5.11]. In [4, p. 173] Ekedahl showed the existence for

$$p \equiv -1 \bmod 5, 6, 9, 16 \text{ or } p \equiv -7 \bmod 16 \tag{2.1}$$

(unfortunately there is a typo saying "genus 3" instead of "genus 4" at line 22 on [4, p. 173]), and also the result [1, Corollary 2.16] by Brock can give (other) realizations of superspecial curves of genus 4 in characteristic p for p satisfying (2.1). Next we recall the existence result for superspecial Howe curves, where a Howe curve is a curve of genus 4 obtained by taking the fiber product over $\mathbb{P}^1$ of two genus-1 double covers of the $\mathbb{P}^1$ (cf. [18, Definition 2.1]. In [7] Howe studied such curves to quickly construct curves of genus 4 with many rational points). Note that any Howe curve is nonhyperelliptic ([17, Lemma 2.1]).

Theorem 2.4 (Kudo, Harashita and Howe [17, Theorem 1.1])**.** *For every prime p with $7 < p \leq 20000$ or with $p \equiv 5 \bmod 6$, there exist a superspecial Howe curve of genus 4 in characteristic p.*

There are some enumeration results on superspecial curves of genus 4, cf. [14], [15], [16] and [17]. These results are summarized in Kudo's article [13] of this volume.

Next, let us collect the results on the existence of supersingular curves. For $g \leq 3$, it suffices to look at the case of (p, g), for which there is no superspecial curve. The case is only $(p, g) = (3, 2)$. In the case we have a supersingular (but not superspecial) curve $y^2 = x^5 + 1$. For $g = 4$, please see the recent result by Kudo, Harashita and Senda:

Theorem 2.5 ([18, Corollary 1.2])**.** *There exists a supersingular curve of genus 4 in arbitrary positive characteristic.*

Thus we conclude that there exist supersingular curves of genus g in characteristic p for all (p, g) provided $g \leq 4$. As far as I know, there is no non-existence result even for $g \geq 5$. In [30], van der Geer and van der Vlugt showed the existence for arbitrary g in characteristic 2. The survey paper [26] by Pries would be helpful to find many other known results and questions around this field.

Let $\mathcal{M}_g$ be the moduli space of curves of genus g. Let $\mathcal{T}_g$ be the Torelli locus, that is the image of the morphism

$$\mathcal{M}_g \to \mathcal{A}_g$$

sending a curve C to its Jacobian variety $\mathrm{Jac}(C)$. We are interested in $\mathcal{T}_g \cap S_g$ (the moduli space of supersingular curves), but we know very little about the space if $g \geq 4$, especially about the dimension and about the number of irreducible compoments and so on. In this paper, we shall give a partial result in case $g = 4$.

Theorem 2.6. *Let $\mathcal{W}$ be a component of $\mathcal{T}_4 \cap S_4$. Assume that $\mathcal{W}$ contains a superspecial nonhyperelliptic point. Then the dimension of $\mathcal{W}$ is three.*

Remark that S_4 contains a superspecial nonhyperelliptic point in many characteristics, by Theorem 2.4 and the fact that any Howe curve is nonhyperelliptic ([17, Lemma 2.1]).

§ 2.4. Dieudonné theory

Let k be a perfect field in characteristic $p > 0$. Let X be an abelian variety over k of dimension g. We write $X[n]$ for the n-kernel $\ker(n : X \to X)$, which is a finite group scheme of rank n^{2g}. Consider $X[p^\infty] = \varinjlim_n X[p^n]$, which is called the *$p$-divisible group associated to X.*

Let $W = W(k)$ be the ring of Witt vectors over k and set

$$A = W[F, V]/(Fa - a^\sigma F, Va^\sigma - aV, FV - p, VF - p; a \in W),$$

where σ is the Frobenius on W. A *Dieudonné module* is an A-module which is finitely generated as a W-module.

Theorem 2.7 (Dieudonné theory)**.** *There exists a categorical (anti-)equivalence $\mathbb{D}$ from the category of p-divisible groups over k and that of Dieudonné modules which are free as W-module.*

A polarization η on an abelian variety X over k defines an alternating form on $M = \mathbb{D}(X[p^\infty])$

$$\langle\ ,\ \rangle : M \times M \to W$$

with $\langle Fx, y\rangle = \langle x, Vy\rangle^\sigma$, where σ is the Frobenius map on W, cf. [21, p. 101] and [20, 5.9]. In general, for a Dieudonné module M, a *quasi-polarization* on M is an alternating form $\langle\ ,\ \rangle : M \times M \to \mathrm{frac}\, W$ satisfying the above relation. For a quasi-polarized Dieudonné module $(M, \langle\ ,\ \rangle)$, we have

$$M^t = \mathrm{Hom}_W(M, W) \simeq \{x \in M \otimes \mathrm{frac}\, W \mid \langle x, M\rangle \subset W\}.$$

If the polarization η on X is principal, the induced quasi-polarization on $M = \mathbb{D}(X[p^\infty])$ is perfect (also called *principal*), i.e., it induces an isomorphism $M^t \simeq M$.

For an abelian variety X, its *a-number* is defined to be

$$a(X) = \dim_k \operatorname{Hom}(\alpha_p, X),$$

where α_p is the kernel of the Frobenius map F on the additive group $\mathbb{G}_a$. Let M be the Dieudonné module of $X[p^\infty]$. Then we have

$$a(X) = \dim_k M/(F,V)M$$

(cf. [20, 5.2]). As $a(X)$ depends only on M, we often write $a(M)$ for it. Via a canonical isomorphism $M/VM \simeq H^1(X, \mathcal{O}_X)$ (cf. [21, Section 5]), we get

$$\begin{aligned} a(X) &= \dim \operatorname{coker}(F : M/VM \to M/VM) \\ &= g - \operatorname{rank}(F : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)). \end{aligned}$$

Thus, the a-number is nothing but g subtracted by the Hasse-Witt rank.

The Dieudonné module of a supersingular (resp. superspecial) abelian variety is called *supersingular* (resp. *superspecial*). It is known (cf. [25, Theorem 2]) that a Dieudonné module M is superspecial if and only if $a(M) = g$ (equivalently the Hasse-Witt rank is zero).

§ 3. The moduli space of supersingular abelian varieties

In [20], Li and Oort gave a description of the moduli space of principally polarized supersingular abelian varities. Here we recall their theory. They introduced the notion of (rigid) polarized flag type quotionts in order to describe the supersingular locus. Hence we start with recalling it.

§ 3.1. Polarized flag type quotients

Let k be a perfect field in characteristic $p > 0$. Let M be a supersingular Dieudonné module over k. It is known that there exist superspecial Dieudonné modules (in $M \otimes_W \operatorname{frac}(W)$) containing M, and moreover there exists a smallest one, say N, among them ([19, Lemma 1.3]). Note that N/M is of finite length. Consider the operator

$$\varphi(M) := M + p^{-1}V^2 M$$

in $M \otimes_W \operatorname{frac}(W)$. As N is superspecial, we have $\varphi(N) = N$. Since $\varphi^i(M) \subset N$ for all $i \geq 0$, the ascending filtration $M \subset \varphi(M) \subset \varphi^2(M) \subset \cdots$ is stable. Moreover, $\varphi^{g-1}(M) = N$ holds (cf. [19, Corollary 1.7], also see [32, Lemma 9] for a general result). By $F^{g-1}N \subset M$, putting $M_i = M + F^{g-1-i}N$ we have a filtration

$$M = M_0 \subset M_1 \subset \cdots \subset M_{g-1} = N.$$

For M with $a(M) = 1$, we have $\dim_k M_i/M_{i-1} = i$ (cf. [22, Theorem 2.2]). With this observation, taking account of quasi-polarization and so on, Li and Oort introduced the following notion.

Definition 3.1. *A polarized flag type quotient (PFTQ) of Dieudonné modules* consists of quasi-polarized Dieudonné modules $(M_i, \langle\ ,\ \rangle)$ of rank $2g$ for $i = 0, \dots, g-1$ with isogenies

$$M_0 \subset M_1 \subset \cdots \subset M_{g-2} \subset M_{g-1}$$

such that

(i) $M_{g-1} \simeq A_{1,1}^g$ where $A_{1,1}$ is the Dieudonné module of supersingular elliptic curve. The quasi-polarization $\langle\ ,\ \rangle$ on M_{g-1} induces $M_{g-1}^t = F^{g-1}M_{g-1}$.

(ii) $(F, V)M_i \subset M_{i-1}$ and $\dim M_i/M_{i-1} = i$.

(iii) $F^{i-j}V^jM_i \subset M_i^t$ for $0 \leq j \leq i/2$.

The PFTQ of Dieudonné modules is called *rigid* if in addition it satisfies

(iv) $M_i = M_0 + F^{g-1-i}M_{g-1}$.

For simplicity, assume that k is algebraically closed. In order to consider families of PFTQs, we introduce PFTQs of abelian varieties, because they work well over any k-scheme S. Let E be a supersingular elliptic curve over k and set

$$\Lambda_g := \{\text{polarizations } \eta \text{ on } E^g \mid \ker\eta = E^g[F^{g-1}]\}/\text{isom}.$$

Definition 3.2 ([20, 3.6]). Let $\eta \in \Lambda_g$. *A polarized flag type quotient (PFTQ) over S with respect to η* consists of polarized abelian schemes (Y_i, η_i) $(i = 0, \dots, g-1)$ of relative dimension g over S with isogenies

$$Y_{g-1} \xrightarrow{\rho_{g-1}} Y_{g-2} \xrightarrow{\rho_{g-2}} \cdots \xrightarrow{\rho_2} Y_1 \xrightarrow{\rho_1} Y_0$$

such that

(i) $Y_{g-1} = E^g \times S$ and $\eta_{g-1} = \eta \times \mathrm{id}_S$;

(ii) $\ker\rho_i$ is Zariski-locally isomorphic to α_p^i and $\ker\eta_i \subset Y_i[F^{i-j}V^j]$ for all $0 \leq j \leq i/2$;

(iii) $\eta_i = \rho_i^t \circ \eta_{i-1} \circ \rho_i$.

The PFTQ is called *rigid* if in addition it satisfies

(iv) $\ker(Y_{g-1} \to Y_i) = \mathrm{Ker}(Y_{g-1} \to Y_0) \cap Y_{g-1}[F^{g-1-i}]$.

PFTQs (resp. rigid PFTQs) has a fine moduli space:

Lemma 3.3 (Li and Oort [20, 3.7])**.**

(1) *There exists a fine moduli space* $\mathcal{P}_{g,\eta}$ *of PFTQs, which is a projective scheme. Up to isomorphism* $\mathcal{P}_{g,\eta}$ *is independent of the choice of* $\eta \in \Lambda_g$*.*

(2) *There exists a fine moduli space* $\mathcal{P}'_{g,\eta}$ *of rigid PFTQs, which is a quasi-projective scheme. Up to isomorphism* $\mathcal{P}'_{g,\eta}$ *is independent of the choice of* $\eta \in \Lambda_g$*.*

§ 3.2. The result by Li and Oort

Now we review the description of the supersingular locus S_g, obtained by Li and Oort:

Theorem 3.4 (Li and Oort [20, 4.2 and 4.4])**.**

(1) $\mathcal{P}'_{g,\eta}$ *is nonsingular and geometrically integral of dimension* $\left[\frac{g^2}{4}\right]$*. Any generic point of* $\mathcal{P}'_{g,\eta}$ *has* $a(Y_0) = 1$*.*

(2) *The canonical morphism*

$$\Psi : \coprod_{\eta \in \Lambda_g} \mathcal{P}'_{g,\eta} \to S_g$$

sending $\{(Y_i, \eta_i)\}$ *to* (Y_0, η_0) *is surjective and quasi-finite.*

The next corollary follows immediately from this theorem.

Corollary 3.5 (Li and Oort [20, 4.9])**.**

(1) S_g *is equi-dimensional and is of dimension* $\left[\frac{g^2}{4}\right]$*.*

(2) *The number of irreducible components of* S_g *is* $\#\Lambda_g$*.*

Moreover it is known that

$$\#\Lambda_g = \begin{cases} H_g(p,1) & \text{if } g \text{ is odd,} \\ H_g(1,p) & \text{if } g \text{ is even,} \end{cases}$$

where $H_g(p,1)$ (resp. $H_g(1,p)$) is the class number of principal (resp. non-principal) genus of the quaternion unitary group

$$G = \{h \in M_g(B) \mid h\overline{h}^t = rI \text{ for some } r \in \mathbb{Q}^\times\}$$

with $B = \mathrm{End}(E) \otimes \mathbb{Q}$, see [20, 4.6, 4.7 and 4.8] and also [10, Section 2] for the detail.

§ 4. Explicit descriptions in the low dimensional cases

In this section, we review a description of the moduli space of (rigid) PFTQs for $g \leq 4$ and explain a detailed structure of the supersingular locus S_g in the lower dimensional case.

§ 4.1. Genus 2

The references for the structure of S_2 are Katsura-Oort [12] and Li-Oort [20, 9.2]. Let $\eta \in \Lambda_2$. Any PFTQ with respect to η

$$\rho_1 : \quad E^2 = Y_1 \to Y_0$$

has $\ker \rho \simeq \alpha_p$ and any quotient of E^2 by α_p automatically defines a PFTQ, i.e., by this quotient, η descents to a principal polarization on Y_0. Let us look at this in terms of Dieudonné modules. Put $M_i = \mathbb{D}(Y_i)$. Thanks to the classification [20, 6.1] of quasi-polarizations on superspecial Dieudonné modules, there are elements x, y of M_1 such that $M_1 = Ax \oplus Ay$ with $(F - V)* = 0$ for $* \in \{x, y\}$, and $\langle x, F^2 y\rangle = 1$ and $\langle *_1, F*_2\rangle = 0$ for $*_1, *_2 \in \{x, y\}$. Then M_0 is described as A-span$\langle \tilde{a}x + \tilde{b}y, Fx, Fy\rangle$ for $(\tilde{a}, \tilde{b}) \in W^2 \smallsetminus (pW)^2$, and for any such $(\tilde{a}, \tilde{b})$, the quasi-polarization on M_0 induced by $\langle\ ,\ \rangle$ on M_1 is principal (indeed one can check $\langle v, w\rangle \in W$ for $v, w \in M_0$ and this is a perfect pairing). Thus, since to give a PFTQ $M_0 \subset M_1$ is equivalent to giving a line in $M_1/M_1^t \simeq k^2$, we have $\mathcal{P}_{2,\eta} = \mathcal{P}'_{2,\eta} \simeq \mathbb{P}^1$. The supersingular locus S_g is the union of irreducible varieties W_η ($\eta \in \Lambda_3$)

$$S_2 = \bigcup_{\eta \in \Lambda_2} W_\eta,$$

and the normalization $\tilde{W}_\eta$ of W_η is isomorphic to $\mathcal{P}_{2,\eta}/G_\eta$, where

$$G_\eta = \mathrm{Aut}(E^2, \eta)/\{\pm 1\}.$$

It is known that G_η is isomorphic to one of the following groups

$$\{1\},\ \mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/3\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},\ S_3,\ A_4,\ D_{12},\ S_4,\ A_5.$$

Moreover, an explict formula of $\#\Lambda_2 = H_2(1, p)$ is obtained by Hashimoto and Ibukiyama [6]. For the formula, see Ibukiyama's article [9, Theorem 5.1] of this volume.

§ 4.2. Genus 3

The references of the supersingular locus S_3 and the moduli space of (rigid) PFTQs for $g = 3$ are Oort [24, Section 2] and Li-Oort [20, 9.4], also see Katsura-Oort [12, Sections 5 and 6].

Here is a short review of the structure of the moduli space of PFTQs for $g = 3$. Let $\eta \in \Lambda_3$ and let

$$(E^3, \eta) = (Y_2, \eta_2) \to (Y_1, \eta_1) \to (Y_0, \eta_0)$$

be a PFTQ. Put $M_i := \mathbb{D}(Y_i)$. By [20, 6.1], there are elements x, y of M_2 such that $M_2 = Ax \oplus Ay \oplus Az$ with $(F - V)* = 0$ for $* \in \{x, y, z\}$ and $\langle *, F^3 * \rangle = \varepsilon$ for $* \in \{x, y, z\}$ with $\varepsilon^\sigma = -\varepsilon \in W(\mathbb{F}_{p^2})^\times$. Then M_1 is written as A-span$\langle w := \tilde{a}x + \tilde{b}y + \tilde{c}z, FM_2 \rangle$ with $\tilde{a}, \tilde{b}, \tilde{c} \in W$. Since w has to satisfy $\langle w, Fw \rangle \in W$, to obtain M_2 is equivalent to giving a point on

$$V := V(a^{p+1} + b^{p+1} + c^{p+1}) \subset \mathbf{P}^2,$$

where $a = (\tilde{a} \bmod p)$. To give M_0 is equivalent to giving a line on $M_1/M_1^t \simeq k^2$. The moduli space $\mathcal{P}_{3,\eta}$ is a $\mathbb{P}^1$-bundle over V:

$$\mathcal{P}_{3,\eta} \xrightarrow{\pi} V$$

There exists a section t of π

$$t: \quad V \xrightarrow{\sim} T \subset \mathcal{P}_{3,\eta}$$

with $t(Y_2 \to Y_1) = (Y_2 \to Y_1 \to Y_2/Y_2[F] = Y_0)$. We have

$$\mathcal{P}'_{3,\eta} = \mathcal{P}_{3,\eta} \setminus T.$$

In Theorem 3.4, we consider a morphism from $\mathcal{P}'_{3,\eta}$ to S_3, but it can be extended to $\mathcal{P}_{3,\eta}$:

$$\coprod_{\eta \in \Lambda_3} \mathcal{P}_{3,\eta} \to S_3,$$

which sends $(Y_2, \eta_2) \to (Y_1, \eta_1) \to (Y_0, \eta_0)$ to (Y_0, η_0). Note that this morphisn is not quasi-finite. In fact, this contracts T to a point (superspecial point).

Assume $n \geq 3$ with $(n, p) = 1$. Let $S_{3,n}$ be the supersingular locus in the moduli space $\mathcal{A}_{3,n}$ of principally polarized abelian threefolds with level n-structure. Even with level n-structure, we have a similar description

$$\coprod_{\eta \in \Lambda_{g,n}} \mathcal{P}_{3,\eta} \to S_{3,n}.$$

As an application of this description, in [24] Oort studied the singularity of $S_{3,n}$ at a superspecial point (considered as the image of T):

Theorem 4.1 ([24, 2.9]). *For any irreducible component Z of $S_{3,n}$ and for any superspecial point $x \in Z$, the tangent space at x of Z is of dimension* 6 $(= \dim \mathcal{A}_{3,n})$.

Let $B_{g,n}$ be the locus in $\mathcal{A}_{g,n}$ consisting of Jacobians of hyperelliptic good curves, where "good" means "stable and its Jacobian variety is an abelian variety". Note $\dim B_{g,n} = 2g-1$. Oort showed a discrepancy between the dimension of the tangent space of Z at a superspecial point and that of components of the formal completion of $B_{g,n}$ at a superspecial point and conclude

Theorem 4.2 ([24, 1.10]). *Any component of $S_{3,n}$ is not contained in $B_{3,n}$. In particular, there exists a nonhyperelliptic supersingular curve of genus* 3.

In the next section, we use this idea in the case of genus 4 and obtain a result on the dimension of some components of the moduli space of supersingular curves of genus 4.

§ 4.3. Genus 4

In this subsection, we review the structure of the moduli space of PFTQs for $g = 4$ (cf. [20, 9.6]), and as an application of it we shall show the next theorem, which enables us to determine the dimension of some components of the moduli space of supersingular curves of genus 4 (Corollary 4.4).

Theorem 4.3. *Assume $p > 2$ and $n \geq 3$. For any irreducible component Z of $S_{4,n}$ and for any superspecial point $s \in Z$, the tangent space at s of Z is of dimension* 10 $(= \dim \mathcal{A}_{4,n})$.

Let $\eta \in \Lambda_4$ and let

$$(E^4, \eta) = (Y_3, \eta_3) \to (Y_2, \eta_2) \to (Y_1, \eta_1) \to (Y_0, \eta_0)$$

be a PFTQ with respect to η. Put $M_i := \mathbb{D}(Y_i)$. By [20, 6.1], there are elements x, y, z, u of M_3 such that

$$M_3 = Ax \oplus Ay \oplus Az \oplus Au$$

with $(F-V)* = 0$ for $* = x, y, z, u$, and $\langle x, F^4 y\rangle = 1$, $\langle z, F^4 u\rangle = 1$ and $\langle *_1, F^4 *_2\rangle = 0$ for $*_1 = x, y$ and $*_2 = z, u$. We call such x, y, z, u a *standard basis* of M_3.

To obtain Y_2 is equivalent to getting a submodule

$$M_2 = A\text{-span}\langle v := \tilde{a}x + \tilde{b}y + \tilde{c}z + \tilde{d}u, Fx, Fy, Fz, Fu\rangle$$

of M_3 for $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in W$ such that $\langle v, F^2 v\rangle \in W$. In other words, if we put $a = (\tilde{a} \bmod p)$ and so on, then (a, b, c, d) satisfies

$$ab^{p^2} - a^{p^2}b + cd^{p^2} - c^{p^2}d = 0.$$

Consider the generic part where $(a:c) \notin \mathbb{P}^1(\mathbb{F}_{p^2})$. Then M_1 is generated by an elemenet w of the form $\tilde{r}v + \tilde{s}Fy + \tilde{t}Fu$ and $(F,V)M_2$:

$$M_1 = A\text{-span}\langle w = \tilde{r}v + \tilde{s}Fy + \tilde{t}Fu, Fv, Vv, F^2x, F^2y, F^2z, F^2u\rangle.$$

If this comes from a PFTQ, we require $\langle w, Fw\rangle \in W$, explicitly

$$r(as^p - ct^p - sr^{p-1}a^p - tr^{p-1}c^p) = 0.$$

This equation says that $\mathcal{P}_{4,\eta}$ has two components. The component defined by $r = 0$ does not contain $\mathcal{P}'_{4,\eta}$ and therefore is called a *garbage component*. We consider only the other component (*non-garbage component*). Finally to get M_0 is equivalent to get a line in M_1/M_1^t. Let $\mathcal{Z}_\eta$ be this $\mathbb{P}^1$-bundle over the non-garbage component. We have a (surjective and generically quasi-finite) morphism

$$\coprod_{\eta\in\Lambda_4} \mathcal{Z}_\eta \to S_4,$$

defined by sending $(Y_3,\eta_3) \to (Y_2,\eta_2) \to (Y_1,\eta_1) \to (Y_0,\eta_0)$ to (Y_0,η_0).

Let us prove Theorem 4.3. We choose $\eta \in \Lambda_4$ and fix it throughout the following argument. We look at only the single component $\mathcal{Z}_\eta$. On the non-garbage component, consider the part defined by $(a,b,c,d) = (1,0,c,0)$, and take the limits $c \to 0$ and $(r,s,t) \to (1,\zeta,0)$ with $\zeta^p = \zeta$. Then we have a PFTQ belonging to $\mathcal{Z}_\eta$ with

$$\begin{aligned}
M_1 &= A\text{-span}\langle x + \zeta Fy, Fx, F^2y, Fz, F^2u\rangle,\\
M_0 &= M_{0\alpha} := A\text{-span}\langle Fx + \alpha(\zeta x + \zeta^2 Fy), F^2y + \alpha(-x - \zeta Fy), Fz, F^2u\rangle,\\
M_1^t &= A\text{-span}\langle Fx + \zeta F^2y, F^2x, F^3y, Fz, F^2u\rangle.
\end{aligned}$$

Consider the deformation space around $\alpha = 0$, where M_{00} (that is, $M_{0\alpha}$ at the point $\alpha = 0$) is $AFx + AF^2y + AFz + AF^2u$. If $p > 2$, then the lines made by moving α for all ζ generate three dimensional subspace of the tangent space. This is proved in the samy way as in [24, 2.8] (the basis here is different from that of [24, 2.8]): the Hasse-Witt matrix of the deformation, obtained by moving α, of M_{00} with respect to the basis Fx, F^2y, Fz, F^2u is a scalar multiple of the 4×4 matrix $\begin{pmatrix} Q & 0_2 \\ 0_2 & 0_2 \end{pmatrix}$ with $Q = \begin{pmatrix} \zeta & -1 \\ \zeta^2 & -\zeta \end{pmatrix}$ for each $\zeta \in \mathbb{F}_p$. If $p > 2$, the Q's for all $\zeta \in \mathbb{F}_p$ generate the 3-dimensional space generated by $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Q_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Q_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Instead of decomposition $M_{00} = (AFx + AF^2y) \oplus (AFz + AF^2u)$, we have other orthogonal decompositions with principaly quasi-polarized direct summands of genus two, for example

$$\begin{aligned}&(AFz + AF^2u) \oplus (AFx + AF^2y),\\ &(AF(x+z) + AF^2y) \oplus (AFz + AF^2(u-y)),\\ &(AFx + AF^2(y+u)) \oplus (AF(z-x) + AF^2u).\end{aligned}$$

In the above argument, instead of $AFx+AF^2y$ we use the first direct summand for each decomposition above, we have a 3-dimensional subspace of the tangent space. A tedious computation shows that the twelve 4×4-matrices: $P_j \begin{pmatrix} Q_i & 0_2 \\ 0_2 & 0_2 \end{pmatrix} P_j^{-1}$ for $i = 1, 2, 3$ and for $j = 0, 1, 2, 3, 4$ with $P_0 = 1_4$ (the identity matrix) and

$$P_1 = \begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1&0&0&0\\ 0&1&0&-1\\ 1&0&1&0\\ 0&0&0&1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1&0&-1&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&1&0&1 \end{pmatrix}$$

span a 10-dimensional space. Thus, Theorem 4.3 was proven.

Although we should struggle for the case of $p = 2$, the next corollary is vacant in the case, because there is no superspecial curve of genus 4 in $p = 2$ (cf. Theorem 2.3).

Corollary 4.4. *Assume $n \geq 3$. Let $\mathcal{W}$ be an irreducible component of $\mathcal{T}_{4,n} \cap S_{4,n}$, where $\mathcal{T}_{4,n}$ is the Torelli locus. If $\mathcal{W}$ contains a superspecial nonhyperelliptic point, then* $\dim \mathcal{W} = 3$.

Proof. By the purity result [2, 4.1] of de Jong and Oort, we have $\dim \mathcal{W} \geq 3$, since it is not empty. If $\dim \mathcal{W} > 3$, then a component Z of $S_{4,n}$ is contained in $\overline{\mathcal{W}}$, since $\dim S_{4,n} = [g^2/4] = 4$. Let s be a superspecial nonhyperelliptic point on $\mathcal{W}$. Then $T_sZ \subset T_s\mathcal{W}$ must hold. Since the Torelli map $\mathcal{M}_{4,n} \to \mathcal{T}_{4,n} \subset \mathcal{A}_{4,n}$ is an immersion at nonhyperelliptic point, we have $\dim T_s\mathcal{W} \leq 3g - 3 = 9$. This contradicts with the theorem above: $\dim T_sZ = 10$. □

Remark. By Theorem 2.4, there exists a superspecial nonhyperelliptic curve C of genus 4 if $7 < p < 20000$ or $p \equiv 5 \pmod 6$.

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