# QUANTUM LANGLANDS DUALITY OF REPRESENTATIONS OF $\mathcal{W}$-ALGEBRAS 

TOMOYUKI ARAKAWA AND EDWARD FRENKEL


#### Abstract

We prove duality isomorphisms of certain representations of $\mathcal{W}$ algebras which play an essential role in the quantum geometric Langlands Program and some related results.


## 1. Introduction

Let $G$ be a connected, simply-connected simple algebraic group over $\mathbb{C},{ }^{L} G$ its Langlands dual group, $\mathfrak{g}=\operatorname{Lie}(G),{ }^{L} \mathfrak{g}=\operatorname{Lie}\left({ }^{L} G\right)$. By a level $\kappa$ we will mean a choice of a symmetric invariant bilinear form on $\mathfrak{g}$. We will denote by $\check{\kappa}$ the level for ${ }^{L} \mathfrak{g}$ whose restriction to the Cartan subalgebra ${ }^{L} \mathfrak{h} \subset{ }^{L} \mathfrak{g}$ is dual to the restriction of $\kappa$ to its Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ under the canonical isomorphism $\mathfrak{h}^{*} \cong{ }^{L} \mathfrak{h}$.

Let $X$ be a smooth projective curve over $\mathbb{C}$. Denote by Bun $_{G}$ the moduli stack of principal $G$-bundles on $X$, and by $D_{\kappa}-\bmod (\operatorname{Bun} G)$ the derived category of $\left(\kappa+\kappa_{c}\right)$ twisted $D$-modules on $\operatorname{Bun}_{G}$. Here $\kappa_{c}$ corresponds to the critical level of $G$ (and square root of the canonical line bundle on $\operatorname{Bun}_{G}$ ); that is, $\kappa_{c}=\kappa_{\mathfrak{g}} / 2$, where $\kappa_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$.

In what follows, we call $\kappa$ irrational if $\kappa / \kappa_{\mathfrak{g}} \in \mathbb{C} \backslash \mathbb{Q}$, and generic if $\kappa / \kappa_{\mathfrak{g}} \in \mathbb{C} \backslash S$ for some countable subset $S \subset \mathbb{C}$.

The global quantum geometric Langlands correspondence [Sto, Gai1, FG] states that for irrational $\kappa$ there should be an equivalence of derived categories ${ }^{1}$

$$
\begin{equation*}
\mathbb{L}_{\kappa}: D_{\kappa}-\bmod \left(\operatorname{Bun}_{G}\right) \xrightarrow{\sim} D_{-\check{\kappa}}-\bmod \left(\operatorname{Bun}_{L_{G}}\right) \tag{1.1}
\end{equation*}
$$

In recent works [Gai1, CG, FG, Gai2], various constructions of the equivalence $\mathbb{L}_{\kappa}$ have been proposed that use representations of the $\mathcal{W}$-algebras $\mathcal{W}^{\kappa}(\mathfrak{g})$ and $\mathcal{W}^{\check{\kappa}}\left({ }^{L_{\mathfrak{g}}}\right)$ and the isomorphism [FF91, FF92]

$$
\mathcal{W}^{\kappa}(\mathfrak{g}) \cong \mathcal{W}^{\check{ }}\left({ }^{L} \mathfrak{g}\right)
$$

In particular, in D. Gaitsgory's construction [Gai2] an essential role is played by the duality isomorphisms

$$
\begin{equation*}
T_{\lambda, \check{\mu}}^{\kappa} \cong \check{T}_{\check{\mu}, \lambda}^{\check{\kappa}} . \tag{1.2}
\end{equation*}
$$

Here $\lambda$ (resp., $\check{\mu}$ ) is a dominant integral weight of $\mathfrak{g}$ (resp., ${ }^{L} \mathfrak{g}$ ); $T_{\lambda, \check{\mu}}^{\kappa}$ and $\check{T_{\check{\mu}, \lambda}^{\check{\kappa}} \text { are }}$ certain representations of $\mathcal{W}^{\kappa}(\mathfrak{g})$ and $\mathcal{W}^{\check{ }}\left({ }^{L} \mathfrak{g}\right)$ (see Section 2.2 for the definition). These isomorphisms also appeared in [CG, FG] in a similar context.

[^0]The goal of this paper is to prove the isomorphisms (1.2) for irrational $\kappa$ (see Theorem 2.2 below) and some closely related results.

The paper is organized as follows. In Section 2, we introduce the functor $H_{D S, \check{\mu}}^{\bullet}(?)$ of quantum Drinfeld-Sokolov reduction twisted by $\check{\mu} \in \check{P}_{+}$and the family of modules $T_{\lambda, \check{\mu}}^{\kappa}=H_{D S, \check{\mu}}^{0}\left(\mathbb{V}_{\lambda, \kappa}\right)$, where $\mathbb{V}_{\lambda, \kappa}$ is the Weyl module over $\widehat{\mathfrak{g}}$ of highest weight $\lambda \in P_{+}$and level $\kappa+\kappa_{c}$. We then state our main results: $H_{D S, \check{\mu}}^{j}\left(\mathbb{V}_{\lambda, \kappa}\right)=0$ if $j \neq 0$ for any $\kappa \in \mathbb{C}$ (Theorem 2.1); the isomorphisms (1.2) for irrational $\kappa$ (Theorem 2.2); and irreducibility of $T_{\lambda, \check{\mu}}^{\kappa}$ for irrational $\kappa$ (Theorem 2.3).

In Section 3 we prove Theorem 2.2 using a realization of $T_{\lambda, \check{\mu}}^{\kappa}$ for irrational $\kappa$ as the intersection of the kernels of powers of screening operators. In Section 4.1, using the results of [Ara04, Ara07], we prove that $T_{\lambda, \check{\mu}}^{\kappa}$ is irreducible for irrational $\kappa$ and to identify its highest weight as a $\mathcal{W}$-algebra module. Using this fact, we give another proof of Theorem 2.2. Then we prove the vanishing Theorem 2.1 in Section 4.2 and compute the characters of $T_{\lambda, \check{\mu}}^{\kappa}$ in Section 4.3. In Section 4.4 we show that the statement of Theorem 2.2 with rational $\kappa$ is false already for $\mathfrak{g}=\mathfrak{s l}_{2}$. Finally, in Section 5 we construct a BGG-type resolution of the modules for $T_{\lambda, \check{\mu}}^{\kappa}$ with irrational $\kappa$ and discuss the $\kappa \rightarrow \infty$ limit of this resolution, following [FF96].

Acknowledgements. We thank Thomas Creutzig, Dennis Gaitsgory and Davide Gaiotto for valuable discussions. We are grateful to Sam Raskin for his helpful comments on the first version of this paper; in particular, he pointed out that a conjecture we had made in the first version about the rational levels was incorrect.
T.A. is partially supported by JSPS KAKENHI Grant Number 17H01086, 17K18724.

## 2. Statement of the main results

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank $r,\left\{e_{i}, h_{i}, f_{i}\right\}$ its standard generators, $\mathfrak{g}=\mathfrak{n}_{-} \oplus \mathfrak{h} \oplus \mathfrak{n}_{+}$its triangular decomposition. Let $\Delta$ be the set of roots of $\mathfrak{g}$, $\Delta_{+} \subset \Delta$ the set of positive roots, $\Pi$ the set of simple roots, $P$ the weight lattice, $\check{P}$ the coweight lattice. In what follows, we will use the notation $e_{\alpha}$ (resp., $f_{\alpha}$ ) for a non-zero element of $\mathfrak{n}_{+}$(resp., $\mathfrak{n}_{-}$) corresponding to a root $\alpha \in \Delta_{+}$(resp., $\left.-\alpha \in \Delta_{-}\right)$.

Let $\widehat{\mathfrak{g}}_{\kappa}=\mathfrak{g}((t)) \oplus \mathbb{C} \mathbf{1}$ be the affine Kac-Moody Lie algebra associated with $\mathfrak{g}$ and level $\kappa+\kappa_{c}$, defined by the commutation relation

$$
[x f, y g]=[x, y] f g+\left(\kappa+\kappa_{c}\right)(x, y) \operatorname{Res}_{t=0}(g d f) \mathbf{1}
$$

$[\mathbf{1}, \widehat{\mathfrak{g}}]=0$. Let

$$
V^{\kappa}(\mathfrak{g})=U\left(\widehat{\mathfrak{g}}_{\kappa}\right) \otimes_{U(\mathfrak{g}[[t]] \otimes \mathbb{C} \mathbf{1})} \mathbb{C}
$$

where $\mathbb{C}$ is the regarded as a one-dimensional representation of $\mathfrak{g}[[t]] \otimes \mathbb{C} 1$ on which $\mathfrak{g}[[t]]$ acts trivially and $\mathbf{1}$ acts as the identity. $V^{\kappa}(\mathfrak{g})$ is naturally a vertex algebra, and is called the universal affine vertex algebra associated to $\mathfrak{g}$ at level $\kappa+\kappa_{c}$. A $V^{\kappa}(\mathfrak{g})$-module is the same as a smooth $\widehat{\mathfrak{g}}_{\kappa}$-module.

Let $H_{D S}^{\bullet}(M)$ be the functor of quantum Drinfeld-Sokolov reduction with coefficients in a $\widehat{\mathfrak{g}}$-module $M$ [FF90a, FF92] (see Chapter 15 of [FBZ04] for a survey).

By definition, we have

$$
H_{D S}^{\bullet}(M)=H^{\infty / 2+\bullet}\left(\mathfrak{n}_{+}((t)), M \otimes \mathbb{C}_{\Psi}\right)
$$

where $H^{\infty / 2+\bullet}\left(\mathfrak{n}_{+}((t)), ?\right)$ denotes the functor of Feigin's semi-infinite cohomology of $\mathfrak{n}_{+}((t))$ and $\mathbb{C}_{\Psi}$ is the one-dimensional representation of $\mathfrak{n}_{+}((t))$ corresponding to a non-degenerate character $\Psi: \mathfrak{n}_{+}((t)) \rightarrow \mathbb{C}$. The latter is defined by the formula

$$
\begin{equation*}
\Psi(x f(t))=\psi(x) \cdot \operatorname{Res}_{t=0} f(t) d t, \quad x \in \mathfrak{n}_{+}, \quad f(t) \in \mathbb{C}((t)) \tag{2.1}
\end{equation*}
$$

where $\psi$ is a character of $\mathfrak{n}_{+}$given by the formulas

$$
\psi\left(e_{\alpha}\right)= \begin{cases}1, & \text { if } \alpha \text { is simple }  \tag{2.2}\\ 0, & \text { otherwise }\end{cases}
$$

Formula (2.1) shows that if we identify the dual space to $\mathfrak{n}_{+}((t))$ with $\mathfrak{n}_{+}^{*}((t)) d t$ using the non-degenerate pairing between the latter and $\mathfrak{n}_{+}((t))$ defined by the formula

$$
\langle\varphi g(t), x f\rangle=\langle\varphi, x\rangle \cdot \operatorname{Res}_{t=0} f(t) g(t) d t, \quad \varphi \in \mathfrak{n}_{+}^{*}, \quad x \in \mathfrak{n}_{+},
$$

then $\Psi$ corresponds to the element $\psi d t \in \mathfrak{n}_{+}^{*}((t)) d t$.
Let $\Lambda^{\infty / 2+\bullet}\left(\mathfrak{n}_{+}\right)$be the fermionic ghosts vertex algebra associated with $\mathfrak{n}_{+}$. As a vector space, it is an irreducible module over the Clifford algebra $\mathrm{Cl}_{\mathfrak{n}_{+}}$associated to the vector space

$$
\mathfrak{n}_{+}((t)) \oplus \mathfrak{n}_{+}^{*}((t)) d t
$$

with a non-degenerate bilinear form induced by the above pairing. The algebra $\mathrm{Cl}_{\mathfrak{n}_{+}}$is topologically generated by $\psi_{\alpha, n}=e_{\alpha} t^{n}, \psi_{\alpha, n}^{*}=e_{\alpha}^{*} t^{n-1} d t, \alpha \in \Delta_{+}, n \in \mathbb{Z}$ with the relations

$$
\left[\psi_{\alpha, n}, \psi_{\beta, m}^{*}\right]_{+}=\delta_{\alpha, \beta} \delta_{n,-m}, \quad\left[\psi_{\alpha, n}, \psi_{b, m}\right]_{+}=\left[\psi_{\alpha, n}^{*}, \psi_{\beta, m}^{*}\right]_{+}=0
$$

The module $\Lambda^{\infty / 2+\bullet}\left(\mathfrak{n}_{+}\right)$is generated by a vector $|0\rangle$ such that

$$
\begin{equation*}
\psi_{\alpha, n}|0\rangle=0, \quad n \geqslant 0, \quad \psi_{\alpha, m}^{*}|0\rangle=0, \quad m>0 \tag{2.3}
\end{equation*}
$$

We define a $\mathbb{Z}$-grading on $\bigwedge^{\infty / 2+\bullet}\left(\mathfrak{n}_{+}\right)$by the formulas $\operatorname{deg}|0\rangle=0, \operatorname{deg} \psi_{\alpha, n}=$ $-1, \operatorname{deg} \psi_{\alpha, n}^{*}=1$.

The graded space $H_{D S}^{\bullet}(M)$ is the cohomology of the complex $(C(M), d)$, where

$$
C(M)=M \otimes \bigwedge^{\infty / 2+\bullet\left(\mathfrak{n}_{+}\right)}
$$

with respect to the differential

$$
\begin{equation*}
d=d_{\mathrm{st}}+\widehat{\Psi} \tag{2.4}
\end{equation*}
$$

where $d_{\text {st }}$ is the standard differential computing semi-infinite cohomology

$$
H^{\infty / 2+\bullet}\left(\mathfrak{n}_{+}((t)), M\right)
$$

(see formula (15.1.5) of [FBZ04]) and $\widehat{\Psi}$ stands for the contraction operator on $\bigwedge^{\infty / 2+\bullet}\left(\mathfrak{n}_{+}\right)$corresponding to $\Psi$ viewed as an element of $\mathfrak{n}_{+}^{*}((t)) d t$. In other words,

$$
\widehat{\Psi}=\sum_{i=1}^{r} \psi_{\alpha_{i}, 1}^{*} .
$$

It is known that $H_{D S}^{i}\left(V^{\kappa}(\mathfrak{g})\right)=0$ for $i \neq 0$ (see Theorem 15.1.9 of [FBZ04]). The vertex algebra $H_{D S}^{0}\left(V^{\kappa}(\mathfrak{g})\right)$ is called the (principal) $\mathcal{W}$-algebra associated with $\mathfrak{g}$ at level $\kappa+\kappa_{c}$. We denote it by $\mathcal{W}^{\kappa}(\mathfrak{g})$.

We have the Feigin-Frenkel duality isomorphism [FF91, FF92]

$$
\begin{equation*}
\mathcal{W}^{\kappa}(\mathfrak{g}) \cong \mathcal{W}^{\check{\kappa}}\left({ }^{L} \mathfrak{g}\right) \tag{2.5}
\end{equation*}
$$

where ${ }^{L} \mathfrak{g}$ is the Langlands dual Lie algebra to $\mathfrak{g}$ and $\check{\kappa}$ is the invariant bilinear form on $L_{\mathfrak{g}}$ that is dual to $\kappa$ (see the Introduction).
2.1. Twist. For $\check{\mu} \in \check{P}$, we define a character $\Psi_{\check{\mu}}$ of $\mathfrak{n}_{+}((t))$ by the formula

$$
\begin{equation*}
\Psi_{\check{\mu}}\left(e_{\alpha} f(t)\right)=\psi\left(e_{\alpha}\right) \cdot \operatorname{Res}_{t=0} f(t) t^{\langle\check{\mu}, \alpha\rangle} d t, \quad f(t) \in \mathbb{C}((t)) . \tag{2.6}
\end{equation*}
$$

Given a $V^{\kappa}(\mathfrak{g})$-module $M$, we define a new differential on the complex $C(M)$ :

$$
\begin{equation*}
d_{\check{\mu}}=d_{\mathrm{st}}+\widehat{\Psi}_{\check{\mu}} \tag{2.7}
\end{equation*}
$$

where $d_{\text {st }}$ is the standard differential appearing in (2.4) and $\widehat{\Psi}_{\breve{\mu}}$ is the contraction operator corresponding to the character $\Psi_{\check{\mu}}$, viewed as an element of $\mathfrak{n}_{+}^{*}((t)) d t$ :

$$
\widehat{\Psi}_{\breve{\mu}}=\sum_{i=1}^{r} \psi_{\alpha_{i},\left\langle\check{\mu}, \alpha_{i}\right\rangle+1}^{*} .
$$

We then define the functor $H_{D S, \check{\mu}}^{\bullet}(?)$ by the formula

$$
H_{D S, \check{\mu}}^{\bullet}(M)=H^{\infty / 2+\bullet}\left(\mathfrak{n}_{+}((t)), M \otimes \mathbb{C}_{\Psi_{\tilde{\mu}}}\right)
$$

where $\mathbb{C}_{\Psi_{\tilde{\mu}}}$ is the one-dimensional representation of $\mathfrak{n}((t))$ corresponding to the character $\Psi_{\check{\mu}}$. We note that the functor $H_{D S, \check{\mu}}^{\bullet}(?)$ has been studied in [FG], Sect. 18 (mostly, in the critical level case $\kappa=0$ ).

We define the structure of a $\mathcal{W}^{\kappa}(\mathfrak{g})$-module on $H_{D S, \check{\mu}}^{i}(M), i \in \mathbb{Z}$, as follows.
For every $\check{\mu} \in \check{P}$, let $\sigma_{\check{\mu}}$ be the following "spectral flow" automorphism of $\widehat{\mathfrak{g}}_{\kappa}$ :

$$
\begin{aligned}
e_{i} t^{n} & \mapsto e_{i} t^{n-\check{\mu}_{i}} \\
f_{i} t^{n} & \mapsto f_{i} t^{n+\check{\mu}_{i}} \\
h_{i} t^{n} & \mapsto h_{i} t^{n}-\left(\kappa+\kappa_{c}\right)\left(e_{i}, f_{i}\right) \check{\mu}_{i} \delta_{n, 0}
\end{aligned}
$$

where

$$
\check{\mu}_{i}=\left\langle\check{\mu}, \alpha_{i}\right\rangle .
$$

Note that if $\check{\mu} \in{ }^{L} P=\operatorname{Hom}\left(\mathbb{C}^{\times}, H\right) \subset \check{P}$, then $\sigma_{\check{\mu}}=\operatorname{Ad}_{-\check{\mu}(t)}$, where $-\check{\mu}(t) \in$ $H((t)) \subset G((t))$. For general $\check{\mu} \in \check{P}, \sigma_{\check{\mu}} \in \operatorname{Aut}\left(\widehat{\mathfrak{g}}_{\kappa}\right)$ is a Tits lifting of the element of the extended affine Weyl group corresponding to $\check{\mu}$.

Given a $V^{\kappa}(\mathfrak{g})$-module (equivalently, a $\widehat{\mathfrak{g}}_{\kappa}$-module) $M$, let $\sigma_{\breve{\mu}}^{*} M$ be the vector space $M$ with the action of $\widehat{\mathfrak{g}}_{\kappa}$ twisted by the automorphism $\sigma_{\check{\mu}}$, i.e. $x \in \widehat{\mathfrak{g}}_{\kappa}$ acts as $\sigma_{\check{\mu}}(x)$. We will use the same notation $\sigma_{\breve{\mu}}^{*} M$ for the corresponding $V^{\kappa}(\mathfrak{g})$-module.

We also define an automorphism similar to $\sigma_{\check{\mu}}$ on the Clifford algebra $\mathrm{Cl}_{\mathfrak{n}_{+}}$:

$$
\begin{aligned}
\psi_{\alpha, n} & \mapsto \psi_{\alpha, n-\check{\mu}_{i}} \\
\psi_{\alpha, n}^{*} & \mapsto \psi_{\alpha, n+\check{\mu}_{i}}^{*}
\end{aligned}
$$

Let $\sigma_{\mu}^{*} \bigwedge^{\infty / 2+\bullet}\left(\mathfrak{n}_{+}\right)$be the twist of $\bigwedge^{\infty / 2+\bullet}\left(\mathfrak{n}_{+}\right)$, considered as a $\mathrm{Cl}_{\mathfrak{n}_{+}-\text {module, by }}$ this automorphism.

Combining these two automorphisms, we obtain an automorphism of $C\left(V^{\kappa}(\mathfrak{g})\right)=$ $C\left(V^{\kappa}(\mathfrak{g})\right) \otimes \bigwedge^{\infty / 2+\bullet}\left(\mathfrak{n}_{+}\right)$which we will also denote by $\sigma_{\check{\mu}}$. For any $V^{\kappa}(\mathfrak{g})$-module $M$, let $\sigma_{\mu}^{*} C\left(V^{\kappa}(\mathfrak{g})\right)$ be the corresponding twist of $C(M)=M \otimes \bigwedge^{\infty / 2+\bullet}\left(\mathfrak{n}_{+}\right)$viewed as a module over the tensor product of the enveloping algebra of $\widehat{\mathfrak{g}}_{\kappa}$ and $\mathrm{Cl}_{\mathfrak{n}_{+}}$, or equivalently, as a module over the vertex algebra $C\left(V^{\kappa}(\mathfrak{g})\right)$.

According to [Li], the action of all fields from $C\left(V^{\kappa}(\mathfrak{g})\right)$ on $\sigma_{\mu}^{*} C\left(V^{\kappa}(\mathfrak{g})\right)$ can be described explicitly by the formula

$$
\begin{equation*}
A \in C\left(V^{\kappa}(\mathfrak{g})\right) \mapsto Y_{C(M)}(\Delta(\check{\mu}, z) A, z), \tag{2.8}
\end{equation*}
$$

where $\Delta(\check{\mu}, z)$ is Li's delta operator (see [Li], Section 3) corresponding to the field

$$
\begin{equation*}
\check{\mu}_{i}(z)+\sum_{\alpha \in \Delta_{+}}\left\langle\alpha_{i}, \check{\mu}\right\rangle: \psi_{\alpha}(z) \psi_{\alpha}^{*}(z): \tag{2.9}
\end{equation*}
$$

in $C\left(V^{\kappa}(\mathfrak{g})\right)$, where $\check{\mu}$ is viewed as an element of $\mathfrak{h}=\check{P} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$.
The $\mathbb{Z}$-grading on $C\left(V^{\kappa}(\mathfrak{g})\right)$ and the differential $d$ given by formula (2.4) endow $\left(C\left(V^{\kappa}(\mathfrak{g})\right), d\right)$ with the structure of a differential graded vertex superalgebra. Its 0 th cohomology is $\mathcal{W}^{\kappa}(\mathfrak{g})$ and all other cohomologies vanish. Furthermore, $\mathcal{W}^{\kappa}(\mathfrak{g})$ can be embedded into the vertex subalgebra of $C\left(V^{\kappa}(\mathfrak{g})\right)$ generated by the fields (2.9) with $\check{\mu} \in \check{P}$ [FF92, FBZ04]. This vertex subalgebra is in fact isomorphic to the Heisenberg vertex algebra $\pi^{\kappa}$ and this embedding is equivalent to the Miura map, see Section 3 below for more details.

For any $\widehat{\mathfrak{g}}_{\kappa}$-module $M$, the complex $C(M)=M \otimes \bigwedge^{\infty / 2+\bullet}\left(\mathfrak{n}_{+}\right)$is naturally a $C\left(V^{\kappa}(\mathfrak{g})\right)$-module. The $\mathcal{W}$-algebra $\mathcal{W}^{\kappa}(\mathfrak{g})$, viewed as a subalgebra of $\pi^{\kappa} \subset$ $C\left(V^{\kappa}(\mathfrak{g})\right)$, acts on $C(M)$ and therefore on the cohomology of $d$ on $C(M)$, which is $H_{D S}^{\bullet}(M)$. This gives us a more explicit description of the action of $\mathcal{W}^{\kappa}(\mathfrak{g})$ on $H_{D S}^{\bullet}(M)$.

Now take the $C\left(V^{\kappa}(\mathfrak{g})\right)$-module $\sigma_{\mu}^{*} C(M)$. As a vector space, it is $C(M)$, but it is equipped with a modified structure of $C\left(V^{\kappa}(\mathfrak{g})\right.$ )-module; namely, the one obtained by twisting the action by $\sigma_{\check{\mu}}$ (see formula (2.8)). Since $\pi^{\kappa}$ is a vertex subalgebra of $C\left(V^{\kappa}(\mathfrak{g})\right)$, we obtain that $\sigma_{\mu}^{*} C(M)$ is a $\pi^{\kappa}$-module, and hence a $\mathcal{W}^{\kappa}(\mathfrak{g})$-module. However, the action of $\mathcal{W}^{\kappa}(\mathfrak{g})$ now commutes not with $d$ but with $\sigma_{\check{\mu}}(d)=d_{\check{\mu}}$. Indeed, we have

$$
\sigma_{\check{\mu}}\left(d_{\mathrm{st}}\right)=d_{\mathrm{st}}, \quad \sigma_{\check{\mu}}(\widehat{\Psi})=\widehat{\Psi}_{\check{\mu}}
$$

Hence, under the $\sigma_{\check{\mu}}$-twisted action, $\mathcal{W}^{\kappa}(\mathfrak{g})$ naturally acts on the cohomologies of the complex $C(M)$ with respect to the differential $d_{\check{\mu}}$. Thus, we obtain the structure of a $\mathcal{W}^{\kappa}(\mathfrak{g})$-module on $H_{D S, \check{\mu}}^{i}(M), i \in \mathbb{Z}$.
2.2. Family of modules. We define a family of modules over $\mathcal{W}^{\kappa}(\mathfrak{g})$ parametrized by $\lambda \in P_{+}, \check{\mu} \in \check{P}_{+}$.

For $\lambda \in \mathfrak{h}^{*}$, let $\mathbb{V}_{\lambda}^{\kappa}$ denote the irreducible highest weight representation of $\widehat{\mathfrak{g}}_{\kappa}$ with highest weight $\lambda$.

If $\lambda \in P_{+}$, we also denote by $\mathbb{V}_{\lambda, \kappa}$ the Weyl module induced from the irreducible finite-dimensional representation $V_{\lambda}$ of $\mathfrak{g}$. If $\kappa$ is irrational and $\lambda \in P_{+}$, then $\mathbb{V}_{\lambda}^{\kappa}=\mathbb{V}_{\lambda, \kappa}$.

In Section 4.2 we will prove the following:
Theorem 2.1. For any $\kappa \in \mathbb{C}$ and any $\lambda \in P_{+}, \check{\mu} \in \check{P}$, we have $H_{D S, \check{\mu}}^{j}\left(\mathbb{V}_{\lambda, \kappa}\right)=0$ for all $j \neq 0$.

It is easy to see that if $\check{\mu} \in \check{P} \backslash \check{P}_{+}$, then $H_{D S, \check{\mu}}^{0}\left(\mathbb{V}_{\lambda}^{\kappa}\right)=0$ as well (see Sect. 18 of [FG]). However, if $\check{\mu} \in \check{P}_{+}$, then the $\mathcal{W}^{\kappa}(\mathfrak{g})$-module $H_{D S, \check{\mu}}^{0}\left(\mathbb{V}_{\lambda}^{\kappa}\right)$ is non-zero.

Now we introduce our main objects of study in this paper, the modules

$$
\begin{equation*}
T_{\lambda, \check{\mu}}^{\kappa}=H_{D S, \check{\mu}}^{0}\left(\mathbb{V}_{\lambda, \kappa}\right), \quad \lambda \in P_{+}, \check{\mu} \in \check{P}_{+} \tag{2.10}
\end{equation*}
$$

Theorem 2.1 implies a character formula for $T_{\lambda, \check{\mu}}^{\kappa}$ which is independent of $\kappa$ (see Section 4.3). Because of that, the modules $T_{\lambda, \mu}^{\kappa}$ may be viewed as specializations to different values of $\kappa$ of a single free $\mathbb{C}[\kappa]$-module.

Switching from $\mathfrak{g}$ to ${ }^{L_{g}}$, we also have the $\mathcal{W}^{\check{\kappa}}\left({ }^{L} \mathfrak{g}\right)$-modules

$$
\check{T}_{\check{\mu}, \lambda}^{\check{\kappa}}=H_{D S, \lambda}^{0}\left(\mathbb{V}_{\check{\mu}, \check{\kappa}}\right)
$$

The following theorem is the main result of this paper:
Theorem 2.2. Let $\kappa$ be irrational. Then for any $\lambda \in P_{+}$and $\check{\mu} \in \check{P}_{+}$there is an isomorphism

$$
\begin{equation*}
T_{\lambda, \check{\mu}}^{\kappa} \cong \check{T}_{\check{\mu}, \lambda}^{\check{\kappa}} \tag{2.11}
\end{equation*}
$$

of modules over $\mathcal{W}^{\kappa}(\mathfrak{g}) \cong \mathcal{W}^{\check{\kappa}}\left({ }^{L} \mathfrak{g}\right)$.
We will also prove the following result:
Theorem 2.3. Let $\kappa$ be irrational. Then $T_{\lambda, \check{\mu}}^{\kappa}$ is irreducible for any $\lambda \in P_{+}$and $\check{\mu} \in \check{P}_{+}$.

A natural extension of the isomorphism (2.11) with $\lambda=0$ and arbitrary $\check{\mu} \in \check{P}_{+}$ to the case of the critical level (i.e. $\kappa=0$ ) has been proved in [FG], Theorem 18.3.1(2), and it is likely to hold for other $\lambda \in P_{+}$as well. For other rational values of $\kappa$, the isomorphism (2.11) does not hold for general $\lambda$ and $\check{\mu}$, even though the modules $T_{\lambda, \check{\mu}}^{\kappa}$ and $\check{T}_{\check{\mu}, \lambda}^{\check{\kappa}}$ have equal characters for all $\kappa$, according to the character formula of Section 4.3. The reason is that for rational values of $\kappa$ these two modules are usually reducible and have different composition series.

Let us comment on the role of Theorem 2.2 in Gaitsgory's work on the quantum geometric Langlands correspondence.

Let $\operatorname{KL}(G)_{\kappa}$ be the category of $\widehat{\mathfrak{g}}_{\kappa}$-modules on which $\mathfrak{g}[[t]]$ acts locally finitely and $t \mathfrak{g}[t]$ acts locally nilpotently, and let Whit $(G)_{\kappa}$ be the category of $\left(\kappa+\kappa_{c}\right)$ twisted Whittaker $D$-modules on the affine Grassmannian $\operatorname{Gr}_{G}=G((t)) / G[[t]]$. The fundamental local equivalence that was conjectured by Gaitsgory and Lurie and proved by Gaitsgory for irrational $\kappa$ states that there is an equivalence

$$
F L E_{\kappa \rightarrow \check{\kappa}}: \operatorname{KL}(G)_{\kappa} \xrightarrow{\sim} \operatorname{Whit}\left({ }^{L} G\right)_{\check{\kappa}}
$$

of chiral categories. It follows that there are two functors

$$
\mathrm{KL}(G)_{\kappa} \otimes \mathrm{KL}\left({ }^{L} G\right)_{\check{\kappa}} \rightarrow \mathcal{W}^{\kappa}(\mathfrak{g})-\bmod
$$

given by

$$
M \otimes N \mapsto H_{D S}^{0}\left(M \star F L E_{\check{\kappa} \rightarrow \kappa}(N)\right), \quad M \otimes N \mapsto H_{D S}^{0}\left(F L E_{\kappa \rightarrow \check{\kappa}}(M) \star N\right),
$$

where $\star$ denotes the convolution product (see e.g. [FG]). Theorem 2.2 implies that these two functors coincide. According to Gaitsgory [Gai2], a Ran space version of this statement can be used to prove the existence of the quantum geometric Langlands correspondence $\mathbb{L}_{\kappa}$ discussed in the Introduction. The isomorphism (2.11) also appeared in a similar context in [CG, FG].

## 3. Proof of Theorem 2.2

Our proof uses a BGG-type resolution of the Weyl module $\mathbb{V}_{\lambda, \kappa}$ with irrational $\kappa$ in terms of the Wakimoto modules. This resolution allows us to express $T_{\lambda, \check{\mu}}^{\kappa}$ with irrational $\kappa$ as the intersection of the kernels of powers of the screening operators acting on particular Fock representations of the Heisenberg vertex algebra $\pi^{\kappa} \cong \check{\pi}^{\check{\kappa}}$. More precisely, we obtain

$$
T_{\lambda, \check{\mu}}^{\kappa}=\bigcap_{i=1}^{r} \operatorname{Ker}_{\pi_{\lambda-\kappa \check{\mu}}^{\kappa}} S_{i}^{W}\left(\lambda_{i}+1\right), \quad \check{T}_{\check{\mu}, \lambda}^{\check{\kappa}}=\bigcap_{i=1}^{r} \operatorname{Ker}_{\check{\pi}_{\tilde{\mu}-\check{\kappa} \lambda}^{\check{\kappa}}} \check{S}_{i}^{W}\left(\check{\mu}_{i}+1\right)
$$

where $r=\operatorname{rank} \mathfrak{g}$ and $S_{i}^{W}\left(\lambda_{i}+1\right)$ and $\check{S}_{i}^{W}\left(\check{\mu}_{i}+1\right)$ are the operators introduced below. We then show that

$$
\operatorname{Ker}_{\pi_{\lambda-\kappa \check{\mu}}^{\kappa}} S_{i}^{W}\left(\lambda_{i}+1\right)=\operatorname{Ker}_{\check{\check{\pi}} \check{\tilde{\mu}-\check{\kappa} \lambda}} \check{S}_{i}^{W}\left(\check{\mu}_{i}+1\right)
$$

for each $i=1, \ldots, r$. The latter statements are independent from each other for different $i$, and each of them reduces to the rank 1 case, i.e. the case of $\mathfrak{g}=\mathfrak{s l}_{2}$. In that case the kernels on both sides are in fact known to be isomorphic to the same irreducible representation of the Virasoro algebra [K, FFu, TK86]. This completes the proof of Theorem 2.2. The details are given in the rest of this section.

In the next two sections we then present some further results. In Section 4, we use the results of [Ara04, Ara07] to prove that $T_{\lambda, \check{\mu}}^{\kappa}$ is irreducible for all irrational $\kappa$ and to identify its highest weight as a $\mathcal{W}$-algebra module. We use this fact to give a different proof of Theorem 2.2, bypassing the information about representations of the Virasoro algebra. Then we prove Theorem 2.1 and compute the characters of $T_{\lambda, \check{\mu}}^{\kappa}$. In Section 5, we construct a BGG-type resolution of the modules for $T_{\lambda, \check{\mu}}^{\kappa}$ with irrational $\kappa$ and discuss the $\kappa \rightarrow \infty$ limit of this resolution, along the lines of [FF96].
3.1. Heisenberg subalgebra. Let $\kappa_{0}$ be the invariant bilinear form normalized so that the square length of the maximal root of $\mathfrak{g}$ is equal to 2 ; that is, $\kappa_{0}=\kappa_{\mathfrak{g}} / 2 h^{\vee}$, where $h^{\vee}$ is the dual Coxeter number of $\mathfrak{g}$. From now on, we will view $\kappa$ as a complex number by identifying it with the ratio $\kappa / \kappa_{0}$. Then the complex numbers $\kappa$ and $\check{\kappa}$ are related by the standard formula:

$$
\check{\kappa}=\frac{1}{m \kappa},
$$

where $m$ is the lacing number of $\mathfrak{g}$, i.e. the maximal number of the edges in the Dynkin diagram of $\mathfrak{g}$.

In what follows, we will use the notation $(\alpha \mid \beta)$ for $\kappa_{0}(\alpha, \beta)$.
Let $\pi^{\kappa}$ be the Heisenberg vertex algebra of level $\kappa$. It is generated by fields $b_{i}(z)$, $i=1, \ldots, r=\operatorname{rank} \mathfrak{g}$, with the OPEs

$$
\begin{equation*}
b_{i}(z) b_{j}(w) \sim \frac{\kappa\left(\alpha_{i} \mid \alpha_{j}\right)}{(z-w)^{2}} \tag{3.1}
\end{equation*}
$$

Let $\pi_{\lambda}^{\kappa}$ be the irreducible highest weight representation of $\pi^{\kappa}$ with highest weight $\lambda \in \mathfrak{h}^{*}$.

Let $\mathbb{W}_{\lambda}^{\kappa}=M_{\mathfrak{g}} \otimes \pi_{\lambda}^{\kappa}$ be the Wakimoto module of highest weight $\lambda$ and level $\kappa+\kappa_{c}$ ([FF90b, Fre05]), where $M_{\mathfrak{g}}$ is the tensor product of $\operatorname{dim} \mathfrak{n}_{+}$copies of the $\beta \gamma$ system.

The vacuum Wakimoto module $\mathbb{W}_{0}^{\kappa}$ is naturally a vertex algebra and there is an injective vertex algebra homomorphism $V^{\kappa}(\mathfrak{g}) \hookrightarrow \mathbb{W}_{0}^{\kappa}$ [Fre05].

We can compute $H_{D S}^{\bullet}\left(\mathbb{W}_{\nu}^{\kappa}\right)$ by using a spectral sequence in which the 0th differential is $d_{\text {st }}$. It follows from the construction of $\mathbb{W}_{\nu}^{\kappa}$ that the 0 th cohomology of $d_{\text {st }}$ is isomorphic to $\pi^{\kappa}$ and all other cohomologies vanish. Therefore the spectral sequences collapses and we obtain

$$
H_{D S}^{0}\left(\mathbb{W}_{0}^{\kappa}\right) \cong \pi^{\kappa}
$$

In fact, we can write down explicitly the fields in the complex $\left(C\left(\mathbb{W}_{\nu}^{\kappa}\right), d_{\text {st }}\right)$ corresponding to the generating fields $b_{i}(z)$ of $\pi^{\kappa}$ [FF92] (the factor $\frac{\left(\alpha_{i} \mid \alpha_{i}\right)}{2}$ in front of $h_{i}(z)$ is due to the fact that $b_{i}(z)$ corresponds to the $i$ th simple root rather than coroot):

$$
\begin{equation*}
\mathbf{b}_{i}(z)=\sum_{n \in \mathbb{Z}} \mathbf{b}_{i, n} z^{-n-1}=\frac{\left(\alpha_{i} \mid \alpha_{i}\right)}{2} h_{i}(z)+\sum_{\alpha \in \Delta_{+}}\left(\alpha \mid \alpha_{i}\right): \psi_{\alpha}(z) \psi_{\alpha}^{*}(z): \tag{3.2}
\end{equation*}
$$

The first term contributes $\kappa+\kappa_{c}$ to the level, and the second term contributes $-\kappa_{c}$, so the total level is $\kappa$, which is indeed the level of $\pi^{\kappa}$. Note that the field $\mathbf{b}_{i}(z)$ is nothing but the field $\check{\mu}(z)$ given by formula (2.9) with $\check{\mu}=\alpha_{i}$ (we identify $\mathfrak{h}^{*}$ with $\mathfrak{h}$ using the inner product $\kappa_{0}$ ). We have already mentioned the fact that these fields generate the Heisenberg vertex algebra $\pi^{\kappa}$ in Section 2.1.

By applying the functor $H_{D S}^{0}(?)$ to the embedding $V^{\kappa}(\mathfrak{g}) \hookrightarrow \mathbb{W}_{0}^{\kappa}$, we obtain a vertex algebra homomorphism [FF90a, FF92]

$$
\begin{equation*}
\Upsilon: \mathcal{W}^{\kappa}(\mathfrak{g}) \rightarrow H_{D S}^{0}\left(\mathbb{W}_{0}^{\kappa}\right) \cong \pi^{\kappa} \tag{3.3}
\end{equation*}
$$

called the Miura map, which is injective for all $\kappa$ (see e.g. [Ara17]). In particular, $\mathcal{W}^{\kappa}(\mathfrak{g})$ may be identified with the image of the Miura map inside the Heisenberg vertex algebra $\pi^{\kappa}$. The latter can be described for generic $\kappa$ as the intersection of kernels of the screening operators [FF92]. This fact can actually be taken as a definition of $\mathcal{W}^{\kappa}(\mathfrak{g})$, see [FF96].

The $\mathfrak{n}_{+}((t))$-module $M_{\mathfrak{g}}$ admits a right action $x \mapsto x^{R}$ of $\mathfrak{n}_{+}((t))$ on $M_{\mathfrak{g}}$ that commutes with the left action of $\mathfrak{n}_{+}((t))$ [Fre05]. As a $U\left(\mathfrak{n}_{+}((t))\right)$-bimodule, $M_{\mathfrak{g}}$ is isomorphic to the semi-regular bimodule of $\mathfrak{n}_{+}((t))$ [Vor99, Ara14], and hence we have the following assertion.

Proposition 3.1 ([Ara14, Proposition 2.1]). Let $M$ be a $\mathfrak{n}_{+}((t))$-module that is integrable over $\mathfrak{n}_{+}[[t]]$. There is a linear isomorphism

$$
\Phi: \mathbb{W}_{\lambda}^{\kappa} \otimes M \xrightarrow{\sim} \mathbb{W}_{\lambda}^{\kappa} \otimes M
$$

such that

$$
\Phi \circ \Delta(x)=(x \otimes 1) \circ \Phi, \quad \Phi \circ\left(x^{R} \otimes 1\right)=\left(x^{R} \otimes 1-1 \otimes x\right) \circ \Phi \quad \text { for } x \in \mathfrak{n}_{+}((t)) .
$$

Here $x^{R}$ denotes the right action of $x \in \mathfrak{n}_{+}((t))$ on $\mathbb{W}_{\tilde{\mu}}^{\kappa}$ and $\Delta$ denotes the coproduct: $\Delta(x)=x \otimes 1+1 \otimes x$.

Now we can describe the $\mathcal{W}^{\kappa}(\mathfrak{g})$-modules $H_{D S, \tilde{\mu}}^{i}\left(\mathbb{W}_{\nu}^{\kappa}\right)$.
Lemma 3.2. For any $\nu \in \mathfrak{h}^{*}$, we have

$$
\begin{equation*}
H_{D S, \check{\mu}}^{i}\left(\mathbb{W}_{\nu}^{\kappa}\right) \cong \delta_{i, 0} \pi_{\nu-\kappa \check{\mu}}^{\kappa} \tag{3.4}
\end{equation*}
$$

as $\mathcal{W}^{\kappa}(\mathfrak{g})$-modules.
Proof. By applying Proposition 3.1 for $M=\mathbb{C}_{\Psi_{\tilde{\mu}}}$, we obtain a vector space isomorphism

$$
\begin{aligned}
H^{\infty / 2+i}\left(\mathfrak{n}_{+}((t)), \mathbb{W}_{\nu}^{\kappa} \otimes \mathbb{C}_{\Psi_{\tilde{\mu}}}\right) \xrightarrow{\Phi} H^{\infty / 2+i}\left(\mathfrak{n}_{+}((t)), \mathbb{W}_{\nu}^{\kappa}\right) \otimes \mathbb{C}_{\Psi_{\tilde{\mu}}} \\
\cong H^{\infty / 2+i t}\left(\mathfrak{n}_{+}((t)), \mathbb{W}_{\nu}^{\kappa}\right) \cong \delta_{i, 0} \pi_{\nu}^{\kappa} .
\end{aligned}
$$

According to the definition of the action of $\mathcal{W}^{\kappa}(\mathfrak{g})$ on $H_{D S, \mu}^{i}(?)$ given in Section 2.1, to obtain the structure of a module over $\mathcal{W}^{\kappa}(\mathfrak{g})$ we need to apply the automorphism $\sigma_{\check{\mu}}$ to the fields $\mathbf{b}_{i}(z)$ defined by formula (3.2). Under $\sigma_{\check{\mu}}$, all $\mathbf{b}_{i, n}$ with $n \neq 0$ are invariant but $\mathbf{b}_{i, 0}$ gets shifted by $-\kappa \check{\mu}_{i}$, where $\check{\mu}_{i}=\left\langle\check{\mu}, \alpha_{i}^{\vee}\right\rangle$. Indeed, $h_{i, 0}$ gets shifted by $-\left(\kappa+\kappa_{c}\right) \check{\mu}_{i}$, and the $z^{-1}$-Fourier coefficient of the fermionic term of (3.2) gets shifted by $\kappa_{c} \check{\mu}_{i}$. As the result, we obtain that $H_{D S, \check{\mu}}^{0}\left(\mathbb{W}_{\nu}^{\kappa}\right) \cong \pi_{\nu-\kappa \check{\mu}}^{\kappa}$.
3.2. Screening operators. For each $i=1, \ldots, r$, the screening operator

$$
S_{i}(z): \mathbb{W}_{\nu}^{\kappa} \rightarrow \mathbb{W}_{\nu-\alpha_{i}}^{\kappa}
$$

is defined in [FF99, Fre05] by the formula

$$
\begin{equation*}
S_{i}(z)=e_{i}^{R}(z): e^{\int-\frac{1}{\kappa} b_{i}(z) d z}: \tag{3.5}
\end{equation*}
$$

where

$$
\begin{align*}
& : e^{\int-\frac{1}{\kappa} b_{i}(z) d z}:=  \tag{3.6}\\
& T_{-\alpha_{i}} z^{-\frac{b_{i, 0}}{\kappa}} \exp \left(-\frac{1}{\kappa} \sum_{n<0} \frac{b_{i, n}}{n} z^{-n}\right) \exp \left(-\frac{1}{\kappa} \sum_{n>0} \frac{b_{i, n}}{n} z^{-n}\right)
\end{align*}
$$

Here $z^{-\frac{b_{i, 0}}{\kappa}}=\exp \left(-\frac{b_{i, 0}}{\kappa} \log z\right)$ and $T_{-\alpha_{i}}$ is the translation operator $\pi_{\nu}^{\kappa} \rightarrow \pi_{\nu-\alpha_{i}}^{\kappa}$ sending the highest weight vector to the highest weight vector and commuting with all $b_{j, n}, n \neq 0$.

Let $\nu \in P$ be such that

$$
\begin{equation*}
\left(\nu \mid \alpha_{i}\right)+m \kappa=\frac{\left(\alpha_{i} \mid \alpha_{i}\right)}{2}(n-1) \tag{3.7}
\end{equation*}
$$

for some $n \in \mathbb{Z}_{\geqslant 0}$ and $m \in \mathbb{Z}$. We have

$$
\begin{aligned}
& \left.S_{i}\left(z_{1}\right) S_{i}\left(z_{2}\right) \ldots S_{i}\left(z_{n}\right)\right|_{W_{\nu}^{\kappa}} \\
& =\prod_{i=1}^{n} z_{i}^{-\frac{\left(\nu \mid \alpha_{i}\right)}{\kappa}} \prod_{1 \leqslant i<j \leqslant n}\left(z_{i}-z_{j}\right)^{\frac{\left(\alpha_{i} \mid \alpha_{i}\right)}{\kappa}}: S_{i}\left(z_{1}\right) S_{i}\left(z_{2}\right) \ldots S_{i}\left(z_{n}\right):
\end{aligned}
$$

Let $\mathcal{L}_{n}^{*}(\nu, \kappa)$ be the local system with coefficients in $\mathbb{C}$ associated to the monodromy group of the multi-valued function

$$
\prod_{i=1}^{n} z_{i}^{-\frac{\left(\nu \mid \alpha_{i}\right)}{\kappa}} \prod_{1 \leqslant i<j \leqslant n}\left(z_{i}-z_{j}\right)^{\frac{\left(\alpha_{i} \mid \alpha_{i}\right)}{\kappa}}
$$

on the manifold $Y_{n}=\left\{\left(z_{1}, \ldots, z_{n}\right) \in\left(\mathbb{C}^{*}\right)^{n} \mid z_{i} \neq z_{j}\right\}$, and denote by $\mathcal{L}_{n}(\nu, \kappa)$ the dual local system of $\mathcal{L}^{*}(\nu, \kappa)$ ([AK11]). Then, for an element $\Gamma \in H_{n}\left(Y_{n}, \mathcal{L}_{n}(\nu, \kappa)\right)$,

$$
\begin{equation*}
S_{i}(n, \Gamma):=\int_{\Gamma} S_{i}\left(z_{1}\right) S_{i}\left(z_{2}\right) \ldots S_{i}\left(z_{n}\right) d z_{1} \ldots d z_{n}: \mathbb{W}_{\nu}^{\kappa} \rightarrow \mathbb{W}_{\nu-n \alpha_{i}}^{\kappa} \tag{3.8}
\end{equation*}
$$

defines a $\widehat{\mathfrak{g}}$-module homomorphism.
Theorem 3.3 ([TK86]). Suppose that

$$
\frac{2 d(d+1)}{\kappa\left(\alpha_{i} \mid \alpha_{i}\right)} \notin \mathbb{Z}, \quad \frac{2 d(d-n)}{\kappa\left(\alpha_{i} \mid \alpha_{i}\right)} \notin \mathbb{Z}
$$

for all $1 \leqslant d \leqslant n-1$. Then there exits a cycle $\Gamma \in H_{n}\left(Y_{n}, \mathcal{L}_{n}(\nu, \kappa)\right)$ such that $S_{i}(n, \Gamma)$ is non-zero.

In fact, it follows from more general results in [SV91, Var95] (see [FF96] for a survey) that for irrational $\kappa$ the cohomology group $H_{n}\left(Y_{n}, \mathcal{L}_{n}(\nu, \kappa)\right)$ is one-dimensional. We will choose once and for all its generator $\Gamma$ and will write $S_{i}(n)$ for the corresponding operator $S_{i}(n, \Gamma)$.

The following theorem was proved for $\lambda=0$ in [FF92], and for general $\lambda \in P_{+}$ in [ACL].

Proposition 3.4. Let $\kappa$ be irrational and $\lambda \in P_{+}$. Then there exists a resolution $C_{\lambda}^{\bullet}$ of the Weyl module $\mathbb{V}_{\lambda}^{\kappa}=\mathbb{V}_{\lambda, \kappa}$ of the form

$$
\begin{align*}
& 0 \rightarrow \mathbb{V}_{\lambda}^{\kappa} \rightarrow C_{\lambda}^{0} \xrightarrow{d^{0}} C^{1} \rightarrow \ldots \rightarrow C^{n} \rightarrow 0  \tag{3.9}\\
& C_{\lambda}^{j}=\bigoplus_{\substack{w \in W \\
\ell(w)=j}} \mathbb{W}_{w \circ \lambda}^{\kappa}, \quad w \circ \lambda=w(\lambda+\rho)-\rho
\end{align*}
$$

with the differential $d_{\lambda}^{0}$ given by

$$
\begin{equation*}
d_{\lambda}^{0}=\sum_{i=1}^{r} c_{i} S_{i}\left(\lambda_{i}+1\right) \tag{3.10}
\end{equation*}
$$

for some $c_{i} \in \mathbb{C}$ with $\lambda_{i}=\left\langle\lambda, \alpha_{i}^{\vee}\right\rangle$.
Proof. We recall the proof for completeness. Let $M_{\nu}^{*}$ be the contragradient Verma module over $\mathfrak{g}$ with highest weight $\nu \in \mathfrak{h}^{*}$. Let $M_{\nu}^{* \kappa}$ be the corresponding induced $\widehat{\mathfrak{g}}$ module of level $\kappa$. From the explicit construction of the Wakimoto module $\mathbb{W}_{\nu}^{\kappa}$ (see [Fre05]) it follows that the degree 0 subspace of $\mathbb{W}_{\nu}^{\kappa}$ (with respect to the Sugawara
operator $L_{0}$ shifted by a scalar so that the highest weight vector has degree 0 ), is isomorphic to $M_{\nu}^{*}$ as a $\mathfrak{g}$-module. Therefore we have a canonical homomorphism $M_{\nu}^{* \kappa} \rightarrow \mathbb{W}_{\nu}^{\kappa}$ which is an isomorphism on the degree 0 subspaces.

If this homomorphism were not injective, then its kernel would contain a singular vector of strictly positive degree. Consider then the canonical homomorphism from $M_{\nu}^{* \kappa}$ to the contragradient module of the Verma $\widehat{\mathfrak{g}}$-module $M_{\nu}^{\kappa}$, which induces an isomorphism of degree 0 subspaces. The presence of such a singular vector in $M_{\nu}^{* \kappa}$ implies that the Verma module $M_{\nu}^{\kappa}$ would also contain a singular vector of positive degree. However, if $\nu \in P$ and $\kappa$ is irrational, it is known that there are no such singular vectors in $M_{\nu}^{\kappa}$. Therefore we find that in this case the homomorphism $M_{\nu}^{* \kappa} \rightarrow \mathbb{W}_{\nu}^{\kappa}$ is injective. Since these two $\widehat{\mathfrak{g}}$-modules have the same character, we obtain that $M_{\nu}^{* \kappa} \cong \mathbb{W}_{\nu}^{\kappa}$ if $\nu \in P$ and $\kappa$ is irrational.

Now let $\lambda \in P_{+}$. Then we have the contragradient BGG resolution $C_{\lambda}^{\bullet}(\mathfrak{g})$ of the irreducible $\mathfrak{g}$-module $V_{\lambda}$ with highest weight $\lambda$ such that

$$
C_{\lambda}^{j}(\mathfrak{g})=\bigoplus_{\substack{w \in W \\ \ell(w)=j}} M_{w \circ \lambda}^{*}
$$

Let $C_{\lambda}^{j}(\widehat{\mathfrak{g}})$ be the induced resolution of $\widehat{\mathfrak{g}}$-modules of level $\kappa$. Then for irrational $\kappa$ we have

$$
C_{\lambda}^{j}(\widehat{\mathfrak{g}})=\bigoplus_{\substack{w \in W \\ \ell(w)=j}} M_{w \circ \lambda}^{* \kappa} \simeq \bigoplus_{\substack{w \in W \\ \ell(w)=j}} \mathbb{W}_{w \circ \lambda}^{\kappa}
$$

In particular, the 0th differential $d_{\lambda}^{0}: C_{\lambda}^{0}(\widehat{\mathfrak{g}}) \rightarrow C_{\lambda}^{1}(\widehat{\mathfrak{g}})$ is the sum of non-zero homomorphisms $\phi_{i}: M_{\lambda}^{* \kappa} \rightarrow M_{\lambda-\left(\lambda_{i}+1\right) \alpha_{i}}^{* \kappa}$, or equivalently, $\mathbb{W}_{\lambda}^{\kappa} \rightarrow \mathbb{W}_{-\left(\lambda_{i}+1\right) \alpha_{i}}^{\kappa}$. Since $\operatorname{Hom}_{\widehat{\mathfrak{g}}}\left(M_{\lambda}^{* \kappa}, M_{\lambda-\left(\lambda_{i}+1\right) \alpha_{i}}^{* \kappa}\right) \cong \operatorname{Hom}_{\mathfrak{g}}\left(M_{\lambda}^{* \kappa}, M_{\lambda-\left(\lambda_{i}+1\right) \alpha_{i}}^{* \kappa}\right)$ is one-dimensional, and $S_{i}\left(\lambda_{i}+1\right)$ is a non-zero homomorphism $\mathbb{W}_{\lambda}^{\kappa} \rightarrow \mathbb{W}_{-\left(\lambda_{i}+1\right) \alpha_{i}}^{\kappa}$ by Theorem 3.3, we find that $d_{\lambda}^{0}$ is given by formula (3.10).

The $\widehat{\mathfrak{g}}$-homomorphism $S_{i}\left(\lambda_{i}+1\right): \mathbb{W}_{\lambda}^{\kappa} \rightarrow \mathbb{W}_{\lambda-\left(\lambda_{i}+1\right) \alpha_{i}}^{\kappa}$ induces a linear map

$$
\begin{equation*}
H_{D S, \check{\mu}}^{0}\left(\mathbb{W}_{\lambda}^{\kappa}\right) \rightarrow H_{D S, \check{\mu}}^{0}\left(\mathbb{W}_{\lambda-\left(\lambda_{i}+1\right) \alpha_{i}}^{\kappa}\right) \tag{3.11}
\end{equation*}
$$

for $\lambda \in P_{+}$.
For a positive integer $n$ satisfying (3.7) for some $m \in \mathbb{Z}$, let

$$
\begin{equation*}
S_{i}^{W}(n)=\int_{\Gamma} S_{i}^{W}\left(z_{1}\right) S_{i}^{W}\left(z_{2}\right) \ldots S_{i}^{W}\left(z_{n}\right) d z_{1} \ldots d z_{n}: \pi_{\nu}^{\kappa} \rightarrow \pi_{\nu-n \alpha_{i}}^{\kappa} \tag{3.12}
\end{equation*}
$$

where

$$
\begin{equation*}
S_{i}^{W}(z)=: e^{\int-\frac{1}{\kappa} b_{i}(z) d z}:: \pi_{\nu}^{\kappa} \rightarrow \pi_{\nu-\alpha_{i}}^{\kappa} \tag{3.13}
\end{equation*}
$$

and $\Gamma \in H_{n}\left(Y_{n}, \mathcal{L}_{n}(\nu, \kappa)\right)$.
Next, we find the action of the screening operators on the cohomologies.
Lemma 3.5. Under the isomorphism (3.4), the map (3.11) is identified with the operator

$$
S_{i}^{W}\left(\lambda_{i}+1\right): \pi_{\lambda-\kappa \check{\mu}}^{\kappa} \rightarrow \pi_{\lambda-\kappa \check{\mu}-\left(\lambda_{i}+1\right) \alpha_{i}}^{\kappa} .
$$

Proof. Let $\Phi^{\prime}$ denote the isomorphism (3.4). It follows from Proposition 3.1 that

$$
\Phi^{\prime} \circ S_{i}(z)=\left(S_{i}(z)+: e^{\int-\frac{1}{\kappa} b_{i}(z) d z}:\right) \circ \Phi^{\prime}
$$

This implies that the the operator

$$
S_{i}\left(\lambda_{i}+1\right): \mathbb{W}_{\lambda}^{\kappa} \rightarrow \mathbb{W}_{\lambda-\left(\lambda_{i}+1\right) \alpha_{i}}^{\kappa}
$$

induces on the cohomologies a map

$$
\pi_{\lambda-\kappa \check{\mu}}^{\kappa} \rightarrow \pi_{\lambda-\kappa \check{\mu}-\left(\lambda_{i}+1\right) \alpha_{i}}^{\kappa}
$$

equal to the operator $S_{i}^{W}\left(\lambda_{i}+1\right)$ plus the sum of operators with non-zero weight with respect to the Cartan subalgebra. The latter sum gives rise to the zero map on the cohomologies since both Fock representations $\pi_{\lambda-\kappa \check{\mu}}^{\kappa}$ and $\pi_{\lambda-\kappa \check{\mu}-\left(\lambda_{i}+1\right) \alpha_{i}}^{\kappa}$ have zero weight.

Now we are ready to prove Theorem 2.2.
3.3. Completion of the proof of Theorem 2.2. By Lemmas 3.4 and 3.5, Wakimoto modules are acyclic with respect to the cohomology functor $H_{D S, \check{\mu}}^{i}(?)$ and $T_{\lambda, \check{\mu}}^{\kappa}$ is identified with the 0th cohomology of a complex $\bar{C}_{\lambda}^{\bullet}$ which starts as follows:

$$
\begin{equation*}
0 \rightarrow \pi_{\lambda-\kappa \check{\mu}}^{\kappa} \stackrel{\bar{d}_{\rightarrow}^{0}}{\rightarrow} \bigoplus_{i=1}^{r} \pi_{\lambda-\kappa \check{\mu}-\left(\lambda_{i}+1\right) \alpha_{i}}^{\kappa} \rightarrow \ldots \tag{3.14}
\end{equation*}
$$

with

$$
\begin{equation*}
\bar{d}_{\lambda}^{0}=\sum_{i=1}^{r} c_{i} S_{i}^{W}\left(\lambda_{i}+1\right) \tag{3.15}
\end{equation*}
$$

obtained by applying the functor $H_{D S, \check{\mu}}^{0}(?)$ to each term of the resolution of Proposition 4.5 and using Lemma 3.5. In Section 5 we will prove that the higher cohomologies of the complex (3.14) vanish for irrational $\kappa$. For now, we just focus on its 0th cohomology:

$$
\begin{equation*}
T_{\lambda, \check{\mu}}^{\kappa} \cong \bigcap_{i=1}^{r} \operatorname{Ker}_{\pi_{\lambda-\kappa \check{\mu}}^{\kappa}} S_{i}^{W}\left(\lambda_{i}+1\right) \tag{3.16}
\end{equation*}
$$

Let $\check{\pi}^{\check{\kappa}}$ be the Heisenberg vertex algebra of ${ }^{L} \mathfrak{h}$ of level $\check{\kappa}$. It is generated by the fields ${ }^{L} b_{i}(z), i=1, \ldots, \operatorname{rank}^{L} \mathfrak{g}$, with the OPEs

$$
\begin{equation*}
{ }^{L} b_{i}(z)^{L} b_{j}(w) \sim \frac{\check{\kappa}\left(\left.{ }^{L} \alpha_{i}\right|^{L} \alpha_{j}\right)}{(z-w)^{2}} \tag{3.17}
\end{equation*}
$$

where ${ }^{L_{\alpha}} \alpha_{i}$ is the $i$ th simple root of ${ }^{L} \mathfrak{g}$ and $\check{\kappa}=1 / m \kappa$. Note that $(\cdot \mid \cdot)$ now stands for the inner product on $\left({ }^{L} \mathfrak{h}\right)^{*}$ such that the square length of its maximal root is equal to 2 .

According to [FF92] (see also [Fre05, FBZ04]), the duality (2.5) is induced by the vertex algebra isomorphism

$$
\begin{align*}
\pi^{\kappa} & \xrightarrow{\sim} \check{\pi}^{\check{\kappa}}, \\
b_{i}(z) & \mapsto-m \frac{\kappa\left(\alpha_{i} \mid \alpha_{i}\right)}{2}{ }^{2} b_{i}(z),  \tag{3.18}\\
\kappa & \mapsto \check{\kappa}=\frac{1}{m \kappa},
\end{align*}
$$

where $m$ is the lacing number of $\mathfrak{g}$, that is, the maximal number of the edges in the Dynkin diagram of $\mathfrak{g}$.

In the same way as above, we obtain in the case of ${ }^{L_{\mathfrak{g}}}$ that

$$
\check{T}_{\check{\mu}, \lambda}^{\kappa} \cong \bigcap_{i=1}^{r} \operatorname{Ker}_{\pi_{\tilde{\mu}-\tilde{\kappa} \lambda}^{\tilde{\kappa}}} \check{S}_{i}^{W}\left(\check{\mu}_{i}+1\right) .
$$

Therefore in order to prove Theorem 2.2 it is sufficient to establish the isomorphisms

$$
\begin{equation*}
\operatorname{Ker}_{\pi_{\lambda-\kappa \check{\mu}}^{\kappa}} S_{i}^{W}\left(\lambda_{i}+1\right) \cong \operatorname{Ker}_{\check{\pi_{\tilde{\mu}}^{\tilde{\mu}}-\check{\kappa} \lambda}} \check{S}_{i}^{W}\left(\check{\mu}_{i}+1\right), \quad i=1, \ldots, r \tag{3.19}
\end{equation*}
$$

(for irrational $\kappa$ ).
To prove the latter, observe that we have tensor product decompositions

$$
\pi^{\kappa}=\pi_{i}^{\kappa} \otimes \pi_{i}^{\kappa \perp}, \quad \pi_{\nu}^{\kappa}=\pi_{i, \nu_{i}}^{\kappa} \otimes \pi_{i, \nu^{\perp}}^{\kappa \perp}
$$

where $\pi_{i}^{\kappa}$ is the Heisenberg vertex subalgebra generated by the field $b_{i}(z)$ and $\pi_{i}^{\kappa \perp}$ is its centralizer, which is a Heisenberg vertex algebra generated by the fields orthogonal to $b_{i}(z)$. We denote by $\pi_{i, \nu_{i}}^{\kappa}$ and $\pi_{i, \nu^{\perp}}^{\kappa \perp}$ the corresponding modules. By construction, the operator $S_{i}^{W}\left(\lambda_{i}+1\right)$ commutes with $\pi_{i}^{\kappa \perp} \subset \pi^{\kappa}$. Therefore

$$
\operatorname{Ker}_{\pi_{\lambda-\kappa \check{\mu}}^{\kappa}} S_{i}^{W}\left(\lambda_{i}+1\right)=\pi_{i,(\lambda-\kappa \check{\mu})^{\perp}}^{\kappa \perp} \otimes \operatorname{Ker}_{\pi_{i, \lambda_{i}-\kappa \tilde{\mu}_{i}}^{\kappa}} S_{i}^{W}\left(\lambda_{i}+1\right)
$$

We have a similar decomposition in the case of $L_{\mathfrak{g}}$. Furthermore, under the identification of the Heisenberg vertex algebras $\pi^{\kappa}$ and $\check{\pi}^{\check{\kappa}}$, the subalgebras $\pi_{i}^{\kappa}$ and $\pi_{i}^{\kappa \perp}$ are identified with the corresponding subalgebras $\check{\pi}_{i}^{\check{\kappa}}$ and $\check{\pi}_{i}^{\check{\kappa}} \perp$ of $\check{\pi}^{\check{\kappa}}$. We also have

$$
\operatorname{Ker}_{\check{\pi}_{\tilde{\mu}-\check{\kappa} \lambda}^{\check{\kappa}}} \check{S}_{i}^{W}\left(\check{\mu}_{i}+1\right)=\check{\pi}_{i,(\check{\mu}-\check{\kappa} \lambda)^{\check{\kappa}} \perp}^{\perp} \otimes \operatorname{Ker}_{\pi_{i, \tilde{\mu}_{i}-\check{\kappa} \lambda_{i}}} \check{S}_{i}^{W}\left(\check{\mu}_{i}+1\right)
$$

Since $\pi_{i,(\lambda-\kappa \check{\mu})^{\perp}}^{\perp \perp} \cong \check{\pi}_{i,(\check{\mu}-\check{\kappa} \lambda)^{\perp}}^{\check{L}}$, the isomorphism (3.19) is equivalent to the isomorphism

$$
\begin{equation*}
\operatorname{Ker}_{\pi_{i, \lambda_{i}-\kappa \check{\mu}}^{i}} S_{i}^{W}\left(\lambda_{i}+1\right) \cong \operatorname{Ker}_{\pi_{i, \tilde{\mu}_{i}-\check{\kappa} \lambda_{i}}} \check{S}_{i}^{W}\left(\check{\mu}_{i}+1\right) \tag{3.20}
\end{equation*}
$$

The left hand side of (3.20) is the kernel of the map

$$
\begin{equation*}
S_{i}^{W}\left(\lambda_{i}+1\right): \pi_{i, \lambda_{i}-\kappa \check{\mu}_{i}}^{\kappa} \rightarrow \pi_{i,-2-\lambda_{i}-\kappa \check{\mu}_{i}}^{\kappa} \tag{3.21}
\end{equation*}
$$

As shown in [FF92] (see the proof of Proposition 5, where the notation $\nu$ corresponds to our $\kappa^{1 / 2}$ ), it commutes with the Virasoro algebra $\operatorname{Vir}_{i}^{\kappa}$ generated by the field

$$
\begin{equation*}
T_{i}(z)=\frac{1}{2 \kappa\left(\alpha_{i} \mid \alpha_{i}\right)}: b_{i}(z)^{2}:+\left(\frac{1}{\left(\alpha_{i} \mid \alpha_{i}\right)}-\frac{1}{2 \kappa}\right) \partial_{z} b_{i}(z) \tag{3.22}
\end{equation*}
$$

with central charge $c=13-6 \gamma-6 \gamma^{-1}$ where $\gamma=\frac{2 \kappa}{\left(\alpha_{i} \mid \alpha_{i}\right)}$.

According to the results of $[\mathrm{K}, \mathrm{FFu}, \mathrm{TK} 86]$, for irrational $\kappa$ (and hence $\gamma$ ), the kernel of the operator (3.21) is isomorphic to the irreducible module over the Virasoro algebra (3.22) with lowest weight (lowest eigenvalue of $L_{0}$ )

$$
\begin{equation*}
\Delta_{\lambda_{i}, \check{\mu}_{i}}^{\gamma}=\gamma^{-1} \frac{\lambda_{i}\left(\lambda_{i}+2\right)}{4}+\gamma \frac{\check{\mu}_{i}\left(\check{\mu}_{i}+2\right)}{4}-\frac{\lambda_{i} \check{\mu}_{i}+\lambda_{i}+\check{\mu}_{i}}{2}, \tag{3.23}
\end{equation*}
$$

and the same is true for the kernel on the right hand side of (3.20).
Thus, for irrational $\kappa$ the isomorphisms (3.20) hold for all $i=1, \ldots, r$, and hence so do the isomorphisms (3.19). This completes the proof.

## 4. Irreducibility and vanishing of higher cohomologies

The Miura map $\Upsilon$ induces an injective homomorphism

$$
\begin{equation*}
\Upsilon_{Z h u}: \operatorname{Zhu}\left(\mathcal{W}^{\kappa}(\mathfrak{g})\right) \hookrightarrow \operatorname{Zhu}(\pi)=S(\mathfrak{h}) \tag{4.1}
\end{equation*}
$$

where $\mathrm{Zhu}(V)$ is Zhu's algebra of $V([A C L])$. For $\lambda \in \mathfrak{h}^{*}$

$$
\chi(\lambda):(\text { evaluation at } \lambda) \circ \Upsilon_{Z h u}: \operatorname{Zhu}\left(\mathcal{W}^{\kappa}(\mathfrak{g})\right) \rightarrow \mathbb{C} .
$$

Then

$$
\begin{equation*}
\chi(\lambda)=\chi(\check{\mu}) \Longleftrightarrow \lambda+\rho-\kappa \check{\rho} \in W(\check{\mu}+\rho-\kappa \check{\rho}) . \tag{4.2}
\end{equation*}
$$

Here, $\rho$ and $\check{\rho}$ are the half sum of the positive roots and the positive coroots of $\mathfrak{g}$, respectively.
4.1. Irreducibility and vanishing for irrational $\kappa$. Let $\mathbf{L}_{\chi(\lambda)}^{\kappa}$ be the irreducible representation of $\mathcal{W}^{\kappa}(\mathfrak{g})$ with highest weight $\chi(\lambda)$. Recall that $\mathbb{V}_{\nu}^{\kappa}$ denotes the irreducible highest weight representation of $\widehat{\mathfrak{g}}_{\kappa}$ with highest weight $\nu$.

The following assertion follows from [Ara04] and [Ara07, Theorem 9.14].
Proposition 4.1. Let $\kappa$ be irrational, $\lambda \in P_{+}, \check{\mu} \in P_{+}^{\vee}$. Then $H_{D S}^{i}\left(\mathbb{V}_{\lambda-\kappa \check{\mu}}^{\kappa}\right)=0$ for $i \neq 0$ and $H_{D S}^{0}\left(\mathbb{V}_{\lambda-\kappa \check{\mu}}^{\kappa}\right) \cong \mathbf{L}_{\chi(\lambda-\kappa \check{\mu})}^{\kappa}$.

Recall the isomorphism (3.18) between the Heisenberg algebras $\pi$ and $\check{\pi}$ which induces the duality isomorphism (2.5). It implies the following statement.

Lemma 4.2. Let $\lambda \in P, \check{\mu} \in P^{\vee}$. Under the duality isomorphism (2.5), we have

$$
\mathbf{L}_{\chi(\lambda-\kappa \check{\mu})}^{\kappa} \cong \mathbf{L}_{\chi(\check{\mu}-\check{\kappa} \lambda)}^{\check{\kappa}}
$$

The following assertion was conjectured by Creutzig and Gaiotto [CG].
Theorem 4.3. Let $\kappa$ be irrational. For any $\lambda \in P_{+}, \check{\mu} \in \check{P}_{+}$, we have

$$
T_{\lambda, \check{\mu}}^{\kappa} \cong \mathbf{L}_{\chi(\lambda-\kappa \check{\mu})}^{\kappa}
$$

Corollary 4.4. The modules $T_{\lambda, \check{\mu}}^{\kappa}, \lambda \in P_{+}, \check{\mu} \in \check{P}_{+}$are irreducible for irrational $\kappa$.
This is the statement of Theorem 2.3.
In order to prove Theorem 4.3, we will need the following generalization of Proposition 3.4 which has been proved in [ACL].

Proposition 4.5. Let $\kappa$ be irrational, $\lambda \in P_{+}, \check{\mu} \in \check{P}_{+}$. There exists a resolution $C_{\lambda-\kappa \check{\mu}}^{\bullet}$ of the $\widehat{\mathfrak{g}}$-module $\mathbb{V}_{\lambda-\kappa \check{\mu}}^{\kappa}$ of the form

$$
\begin{aligned}
& 0 \rightarrow \mathbb{V}_{\lambda-\kappa \check{\mu}}^{\kappa} \rightarrow C_{\lambda-\kappa \check{\mu}}^{0} \xrightarrow{d_{\lambda-\kappa \check{\mu}}^{0}} C_{\lambda-\kappa \check{\mu}}^{1} \rightarrow \ldots \rightarrow C_{\lambda-\kappa \check{\mu}}^{n} \rightarrow 0, \\
& C_{\lambda-\kappa \check{\mu}}^{i}=\bigoplus_{\substack{w \in W \\
\ell(w)=i}} \mathbb{W}_{w \circ \lambda-\kappa \check{\mu}}^{\kappa} .
\end{aligned}
$$

The differential $d_{\lambda-\kappa \check{\mu}}^{0}$ is given by

$$
d_{\lambda-\kappa \check{\mu}}^{0}=\sum_{i=1}^{r} c_{i} S_{i}\left(\lambda_{i}+1\right)
$$

for some $c_{i} \in \mathbb{C}$, with $\lambda_{i}=\left\langle\lambda, \alpha_{i}^{\vee}\right\rangle$.
Proof of Theorem 4.3. Let us apply the quantum Drinfeld-Sokolov reduction functor (without twist by $\breve{\mu}$ ) to each term of the resolution of Proposition 4.5. Then we find that $H_{D S}^{0}\left(\mathbb{V}_{\lambda-\kappa \check{\mu}}^{\kappa}\right)$ is the 0 th cohomology of the complex obtained by applying the functor $H_{D S}^{i}(?)$ to the resolution in Proposition 4.5. In the same way as in the proof of Lemma 3.5 we then obtain that

$$
\begin{equation*}
H_{D S}^{0}\left(\mathbb{V}_{\lambda-\kappa \check{\mu}}^{\kappa}\right) \cong \bigcap_{i=1}^{r} \operatorname{Ker}_{\pi_{\lambda-\kappa \check{\mu}}} S_{i}^{W}\left(\lambda_{i}+1\right) \tag{4.3}
\end{equation*}
$$

Combining the isomorphisms (3.16) and (4.3), we obtain an isomorphism

$$
\begin{equation*}
T_{\lambda, \check{\mu}}^{\kappa} \cong H_{D S}^{0}\left(\mathbb{V}_{\lambda-\kappa \check{\mu}}^{\kappa}\right) \tag{4.4}
\end{equation*}
$$

According to Proposition 4.1, $\mathbf{L}_{\chi(\lambda-\kappa \check{\mu})}^{\kappa} \cong H_{D S}^{0}\left(\mathbb{V}_{\lambda-\kappa \check{\mu}}^{\kappa}\right)$. Together with (4.4), this completes the proof of Theorem 4.3.

Note that Theorem 2.2 also follows from Lemma 4.2 and Theorem 4.3. Thus, we obtain an alternative proof of Theorem 2.2. Both proofs rely on resolutions of irreducible $\widehat{\mathfrak{g}}$-modules in terms of Wakimoto modules. The proof given in the previous section uses in addition to that an isomorphism of kernels of screening operators in the rank 1 case, which boils down to some properties of representations of the Virasoro algebras. The proof presented in this section does not use representations of the Virasoro algebra, but uses instead Proposition 4.1 stating that $H_{D S}^{0}\left(\mathbb{V}_{\lambda-\kappa \check{\mu}}^{\kappa}\right)$ is irreducible.
4.2. Cohomology vanishing for arbitrary $\kappa$. In this subsection we prove Theorem 2.1 by generalizing the proof in the case $\lambda=\check{\mu}=0$ given in Sect. 15.2 of [FBZ04] (which followed [dBT]).

We start by representing the complex $C\left(\mathbb{V}_{\lambda, \kappa}\right)$ as a tensor product of two subcomplexes. Let $\left\{J^{a}\right\}$ be a basis of $\mathfrak{g}$ which is the union of the basis $\left\{J^{\alpha}\right\}_{\alpha \in \Delta_{+}}$of $\mathfrak{n}_{+}\left(\right.$where $\left.J^{\alpha}=e^{\alpha}\right)$ and a basis $\left\{J^{\bar{a}}\right\}_{\bar{a} \in \Delta_{-} \cup I}$ of $\mathfrak{b}_{-}=\mathfrak{n}_{-} \oplus \mathfrak{h}$ consisting of root vectors $f^{\alpha}, \alpha \in \Delta_{+}$, in $\mathfrak{n}_{-}$and vectors $h^{i}, i \in I=\{1, \ldots, \ell\}$, in $\mathfrak{h}$. Thus, we use Latin upper indices to denote arbitrary basis elements, Latin indices with a bar to denote elements of $\mathfrak{b}_{-}$, and Greek indices to denote basis elements of $\mathfrak{n}_{+}$.

Denote by $c_{d}^{a b}$ the structure constants of $\mathfrak{g}$ with respect to the basis $\left\{J^{a}\right\}$.

Define the following currents:

$$
\begin{equation*}
\widehat{J}^{a}(z)=\sum_{n \in \mathbb{Z}} \widehat{J}_{n}^{a} z^{-n-1}=J^{a}(z)+\sum_{\beta, \gamma \in \Delta_{+}} c_{\gamma}^{a \beta}: \psi_{\gamma}(z) \psi_{\beta}^{*}(z): \tag{4.5}
\end{equation*}
$$

Now, the first complex, denoted by $C\left(\mathbb{V}_{\lambda, \kappa}\right)_{0}$, is spanned by all monomials of the form

$$
\begin{equation*}
\widehat{J}_{n_{1}}^{\bar{a}(1)} \ldots \widehat{J}_{n_{r}}^{\bar{a}(r)} \psi_{\alpha(1), m_{1}}^{*} \ldots \psi_{\alpha(s), m_{s}}^{*} v, \quad v \in V_{\lambda} \tag{4.6}
\end{equation*}
$$

(recall that $J^{\bar{a}} \in \mathfrak{b}_{-}$). The second complex, denoted by $C\left(\mathbb{V}_{\lambda, \kappa}\right)^{\prime}$, is spanned by all monomials of the form

$$
\widehat{J}_{n_{1}}^{\alpha(1)} \ldots \widehat{J}_{n_{r}}^{\alpha(r)} \psi_{\alpha(1), m_{1}} \ldots \psi_{\alpha(s), m_{s}}
$$

(recall that $J^{\alpha} \in \mathfrak{n}_{+}$). We have an analogue of formula (15.2.3) of [FBZ04]: the natural map

$$
\begin{equation*}
C\left(\mathbb{V}_{\lambda, \kappa}\right)^{\prime} \otimes C\left(\mathbb{V}_{\lambda, \kappa}\right)_{0} \xrightarrow{\sim} C\left(\mathbb{V}_{\lambda, \kappa}\right) \tag{4.7}
\end{equation*}
$$

sending $A \otimes B$ to $A \cdot B$ is an isomorphism of graded vector spaces.
We then have an analogue of Lemma 15.2.5 of [FBZ04]: the cohomology of $\left(C\left(\mathbb{V}_{\lambda, \kappa}\right), d_{\check{\mu}}\right)$ is isomorphic to the tensor product of the cohomologies of the two complexes in (4.7): $\left(C\left(\mathbb{V}_{\lambda, \kappa}\right)_{0}, d_{\check{\mu}}\right)$ and $\left(C\left(\mathbb{V}_{\lambda, \kappa}\right)^{\prime}, d_{\check{\mu}}\right)$. This is proved in the same way as in [FBZ04], using the commutation relations established in Sect. 15.2.4, in which we set $\chi=\Psi_{\check{\mu}}$.

In the same way as in Sect. 15.2.6, we prove that the cohomology of the complex $\left(C\left(\mathbb{V}_{\lambda, \kappa}\right)^{\prime}, d_{\check{\mu}}\right)$ is one-dimensional, in cohomological degree 0 . Thus, we have an analogue of Lemma 15.2.7: the cohomology of $\left(C\left(\mathbb{V}_{\lambda, \kappa}\right), d_{\check{\mu}}\right)$ is isomorphic to the cohomology of its subcomplex $\left(C\left(\mathbb{V}_{\lambda, \kappa}\right)_{0}, d_{\check{\mu}}\right)$.

To compute $H^{\bullet}\left(C\left(\mathbb{V}_{\lambda, \kappa}\right)_{0}, d_{\check{\mu}}\right)$, we introduce a double complex as in Sect. 15.2.8 of [FBZ04]. The convergence of the resulting spectral sequence is guaranteed by the fact that $\left(C\left(\mathbb{V}_{\lambda, \kappa}\right)_{0}, d_{\breve{\mu}}\right)$ is a direct sum of finite-dimensional subcomplexes obtained via the $\mathbb{Z}$-grading introduced below in Section 4.3. The 0th differential is $\hat{\Psi}_{\check{\mu}}$. We have an analogue of formula (15.2.4) from [FBZ04]:

$$
\begin{equation*}
\left[\hat{\Psi}_{\breve{\mu}}, \widehat{J}_{n}^{\bar{a}}\right]=\sum_{\substack{\beta \in \Delta_{+} \\ k \in \mathbb{Z}^{-}}}\left(\left[\sigma_{\check{\mu}}\left(p_{-}\right), J_{n}^{\bar{a}}\right] \mid J_{-k}^{\beta}\right) \psi_{\beta, k+1}^{*} \tag{4.8}
\end{equation*}
$$

where

$$
p_{-}=\sum_{i=1}^{\ell} \frac{\left(\alpha_{i}, \alpha_{i}\right)}{2} f_{i}
$$

(here $f_{i}=f_{i} \cdot 1$ ), $\sigma_{\check{\mu}}$ is the automorphism introduced in Section 2.1, and we use the notation

$$
\left(A t^{n} \mid B t^{m}\right)=\kappa_{0}(A, B) \delta_{n,-m}
$$

In [FBZ04], formula (15.2.4) (to which our formula (4.8) specializes when $\check{\mu}=0$ ) was used to show that $\mathfrak{b}_{-} t^{-1} \mathbb{C}\left[t^{-1}\right]$ has a basis consisting of the elements $P_{i}^{(n)}, n<$ $0, i=1, \ldots, r$, forming a basis of the Lie subalgebra

$$
\widehat{\mathfrak{a}}_{-}=\operatorname{Ker} \operatorname{ad}\left(p_{-}\right) \subset \mathfrak{b}_{-} t^{-1} \mathbb{C}\left[t^{-1}\right]
$$

and elements $I_{n}^{\alpha}, \alpha \in \Delta_{+}, n<0$, such that $\operatorname{ad}\left(p_{-}\right) \cdot I_{n}^{\alpha}=f_{\alpha} t^{n}$ (here $f_{\alpha}$ is a generator of the one-dimensional subspace of $\mathfrak{n}_{-}$corresponding to the root $-\alpha$ ).

The existence of this basis is equivalent to the surjectivity of the map

$$
\begin{equation*}
\operatorname{ad} p_{-}: \mathfrak{b}_{-} t^{-1} \mathbb{C}\left[t^{-1}\right] \rightarrow \mathfrak{n}_{-}\left[t^{-1}\right] \tag{4.9}
\end{equation*}
$$

which implies the following direct sum decomposition (as a vector space)

$$
\begin{equation*}
\mathfrak{b}_{-} t^{-1} \mathbb{C}\left[t^{-1}\right]=\widehat{\mathfrak{a}}_{-} \oplus\left(\operatorname{ad} p_{-}\right)^{-1}\left(\mathfrak{n}_{-}\left[t^{-1}\right]\right) \tag{4.10}
\end{equation*}
$$

where the second vector space on the right hand side denotes a particular choice of a subspace of $\mathfrak{b}_{-} t^{-1} \mathbb{C}\left[t^{-1}\right]$ that isomorphically maps onto $\mathfrak{n}_{-}\left[t^{-1}\right]$ under the map ad $p_{-}$. This decomposition, in turn, implies that the complex $C\left(\mathbb{V}_{0}^{\kappa}\right)_{0}$ is isomorphic, as a vector space, to the tensor product

$$
\begin{equation*}
C\left(\mathbb{V}_{0}^{\kappa}\right)_{0}=U\left(\widehat{\mathfrak{a}}_{-}\right) \otimes \mathbb{C}\left[\widehat{I}_{n}^{\alpha}\right]_{\alpha \in \Delta_{+}, n<0} \otimes \bigwedge\left(\psi_{\alpha, n}^{*}\right)_{\alpha \in \Delta_{+}, n<0} \tag{4.11}
\end{equation*}
$$

where $\mathbb{C}\left[\widehat{I}_{n}^{\alpha}\right]_{\alpha \in \Delta_{+}, n<0}$ stands for the linear span of lexicographically ordered monomials in the $\widehat{I}_{n}^{\alpha}$. The differential $\chi=\widehat{\Psi}_{0}$ acts as follows:

$$
\begin{equation*}
\left[\widehat{\Psi}_{0}, \widehat{P}_{n}^{(i)}\right]=0, \quad\left[\widehat{\Psi}_{0}, \widehat{I}_{n}^{\alpha}\right]=\psi_{\alpha, n+1}^{*}, \quad\left[\widehat{\Psi}_{0}, \psi_{\alpha, n}^{*}\right]_{+}=0 \tag{4.12}
\end{equation*}
$$

In Sect. 15.2.9 of [FBZ04], the decomposition (4.11) and formulas (4.12) were used to show that the higher cohomologies of the complex $C\left(\mathbb{V}_{0}^{\kappa}\right)_{0}$ vanish and the 0th cohomology is isomorphic to $U\left(\widehat{\mathfrak{a}}_{-}\right)$. This proves the vanishing of $H_{D S, 0}^{j}\left(\mathbb{V}_{0}^{\kappa}\right)$ for all $j \neq 0$.

We want to apply this argument for arbitrary $\lambda \in P_{+}, \check{\mu} \in \check{P}_{+}$. In order to do that, we need to prove that the linear map

$$
\begin{equation*}
\operatorname{ad}\left(\sigma_{\check{\mu}}\left(p_{-}\right)\right): \mathfrak{b}_{-} t^{-1} \mathbb{C}\left[t^{-1}\right] \rightarrow \mathfrak{n}_{-}\left[t, t^{-1}\right] \rightarrow \mathfrak{n}_{-}\left[t, t^{-1}\right] / \mathfrak{n}_{-}[t] \cong \mathfrak{n}_{-} t^{-1} \mathbb{C}\left[t^{-1}\right] \tag{4.13}
\end{equation*}
$$

(which is the analogue of the map (4.9) for general $\check{\mu}$ ) is surjective. To see that, let

$$
\begin{equation*}
I_{n, \check{\mu}}^{\alpha}=\sigma_{\check{\mu}}\left(I_{n-\langle\alpha, \check{\mu}\rangle}^{\alpha}\right), \quad n<0 . \tag{4.14}
\end{equation*}
$$

Then the formula $\operatorname{ad}\left(p_{-}\right) \cdot I_{n, \check{\mu}}^{\alpha}=f_{\alpha} t^{n}$ implies that

$$
\operatorname{ad}\left(\sigma_{\check{\mu}}\left(p_{-}\right)\right) \cdot I_{n, \check{\mu}}^{\alpha}=\sigma_{\check{\mu}}\left(f_{\alpha} t^{n-\langle\alpha, \check{\mu}\rangle}\right)=f_{\alpha} t^{n} .
$$

Moreover, $I_{m}^{\alpha}$ has the form

$$
I_{m}^{\alpha}=\sum_{i=1}^{r} b_{i} f_{\alpha-\alpha_{i}} t^{m}, \quad b_{i} \in \mathbb{C}
$$

(in this formula, if $\alpha=\alpha_{i}$, then $f_{\alpha-\alpha_{i}}$ stands for the Cartan generator $h_{i}$ ). Therefore

$$
\sigma_{\check{\mu}}\left(I_{m}^{\alpha}\right)=\sum_{i=1}^{r} b_{i} f_{\alpha-\alpha_{i}} t^{m+\left\langle\alpha-\alpha_{i}, \check{\mu}\right\rangle} .
$$

Since $\check{\mu} \in \check{P}_{+}$, it follows that the elements

$$
I_{n, \check{\mu}}^{\alpha}=\sigma_{\check{\mu}}\left(I_{n-\langle\alpha, \check{\mu}\rangle}^{\alpha}\right)=\sum_{i=1}^{r} b_{i} f_{\alpha-\alpha_{i}} t^{n-\left\langle\alpha_{i}, \check{\mu}\right\rangle}
$$

with $n<0$ belong to $\mathfrak{b}_{-} t^{-1} \mathbb{C}\left[t^{-1}\right]$, and so the map (4.13) is indeed surjective.

Therefore, we have the following analogue of the decomposition (4.10)

$$
\begin{equation*}
\mathfrak{b}_{-} t^{-1} \mathbb{C}\left[t^{-1}\right]=\widehat{\mathfrak{a}}_{-}^{\check{\mu}} \oplus\left(\operatorname{ad} \sigma_{\check{\mu}}\left(p_{-}\right)\right)^{-1}\left(\mathfrak{n}_{-} t^{-1} \mathbb{C}\left[t^{-1}\right]\right), \tag{4.15}
\end{equation*}
$$

where $\widehat{\mathfrak{a}}_{-}^{\check{\mu}}$ is the kernel of the map (4.13). This implies an analogue of the tensor product decomposition (4.11):

$$
\begin{equation*}
C\left(\mathbb{V}_{\lambda, \kappa}\right)_{0}=U\left(\widehat{\mathfrak{a}}_{-}^{\check{\mu}}\right) \otimes \mathbb{C}\left[\widehat{I}_{n, \check{\mu}}^{\alpha}\right]_{\alpha \in \Delta_{+}, n<0} \otimes \bigwedge\left(\psi_{\alpha, n}^{*}\right)_{\alpha \in \Delta_{+}, n<0} \otimes V_{\lambda} \tag{4.16}
\end{equation*}
$$

where $I_{n, \check{\mu}}^{\alpha}, \alpha \in \Delta_{+}, n<0$, is defined by the formula (4.14) and $\mathbb{C}\left[\hat{I}_{n, \check{\mu}}^{\alpha}\right]_{\alpha \in \Delta_{+}, n<0}$ stands for the linear span of lexicographically ordered monomials in the $\widehat{I}_{n, \check{\mu}}^{\alpha}$. The differential $\widehat{\Psi}_{\check{\mu}}$ acts as follows:

$$
\begin{gather*}
{\left[\widehat{\Psi}_{\breve{\mu}}, \widehat{P}\right]=0, \quad \forall P \in \widehat{\mathfrak{a}}_{-}^{\mu_{-}}, \quad\left[\widehat{\Psi}_{\breve{\mu}}, \widehat{I}_{n, \breve{\mu}}^{\alpha}\right]=\psi_{\alpha, n+1}^{*},}  \tag{4.17}\\
{\left[\widehat{\Psi}_{\breve{\mu}}, \psi_{\alpha, n}^{*}\right]_{+}=0, \quad \widehat{\Psi}_{\breve{\mu}} \cdot v=0, \quad \forall v \in V_{\lambda} .} \tag{4.18}
\end{gather*}
$$

In the same way as in Sect. 15.2.9 of [FBZ04], we then use the decomposition (4.16) and formulas (4.17), (4.18) to show that the higher cohomologies of the complex $\left(C\left(\mathbb{V}_{\lambda, \kappa}\right)_{0}, \widehat{\Psi}_{\breve{\mu}}\right)$ vanish and the 0th cohomology is isomorphic to $U\left(\widehat{\mathfrak{a}}_{-}^{\check{\mu}}\right) \otimes V_{\lambda}$. This implies the statement of Theorem 2.1.
4.3. Character formula. We define a $\mathbb{Z}_{+}$-grading on the complex $C\left(\mathbb{V}_{\lambda, \kappa}\right)$ as follows: $\operatorname{deg} v_{\lambda, \kappa}=0$, where $v_{\lambda}$ is the highest weight vector of $\mathbb{V}_{\lambda, \kappa}$,
$\operatorname{deg} e_{\alpha} t^{n}=\operatorname{deg} \psi_{\alpha, n}=-n-\langle\alpha, \check{\mu}+\check{\rho}\rangle, \quad \operatorname{deg} f_{\alpha} t^{n}=\operatorname{deg} \psi_{\alpha, n}^{*}=-n+\langle\alpha, \check{\mu}+\check{\rho}\rangle$,
$\operatorname{deg} h_{i, n}=-n$.
We find that $\operatorname{deg} d_{\mathrm{st}}=\operatorname{deg} \widehat{\Psi}_{\breve{\mu}}=0$, so the differential $d_{\check{\mu}}$ preserves the grading and the complex $\left(C\left(\mathbb{V}_{\lambda, \kappa}\right), d_{\breve{\mu}}\right)$ decomposes into a direct sum of homogeneous subcomplexes corresponding to all non-negative degrees. The same is true for the subcomplex $\left(C\left(\mathbb{V}_{\lambda, \kappa}\right)_{0}, d_{\check{\mu}}\right)$.

It is easy to see that the homogeneous subcomplexes of $\left(C\left(\mathbb{V}_{\lambda, \kappa}\right)_{0}, d_{\breve{\mu}}\right)$ are finitedimensional. Hence we can use this $\mathbb{Z}_{+- \text {-grading and the vanishing }}$ Theorem 2.1 to find the character of $T_{\lambda, \tilde{\mu}}^{\kappa}$, which appears as the 0th cohomology of $\left(C\left(\mathbb{V}_{\lambda, \kappa}\right)_{0}, d_{\tilde{\mu}}\right)$, by the taking the alternating sum of characters of the $j$ th terms of $C\left(\mathbb{V}_{\lambda, \kappa}\right)_{0}$ :

$$
\begin{aligned}
\operatorname{char} T_{\lambda, \tilde{\mu}}^{\kappa} & =\sum_{j \geqslant 0}(-1)^{j} \operatorname{char} C^{j}\left(\mathbb{V}_{\lambda, \kappa}\right)_{0} \\
& =\operatorname{char}_{\check{\mu}} V_{\lambda} \cdot \prod_{\substack{\alpha \in \Delta_{+} \\
n \geqslant\langle\alpha, \tilde{\mu}+\overline{+}\rangle}}\left(1-q^{n}\right) \prod_{\substack{\alpha \in \Delta_{+} \\
n>\langle\alpha, \tilde{\mu})}}\left(1-q^{n}\right)^{-1} \prod_{n>0}\left(1-q^{n}\right)^{-r} \\
& =\operatorname{char}_{\tilde{\mu}} V_{\lambda} \cdot \prod_{\alpha \in \Delta_{+}}\left(1-q^{\langle\alpha, \check{\mu}+\check{\rho}\rangle}\right) \prod_{n>0}\left(1-q^{n}\right)^{-r} .
\end{aligned}
$$

Here char ${ }_{\mu} V_{\lambda}$ is the character of the finite-dimensional representation $V_{\lambda}$ with respect to the $\mathbb{Z}_{+}$-grading defined by the formulas $\operatorname{deg} v_{\lambda}=0$, where $v_{\lambda}$ is the highest weight vector of $V_{\lambda}$, and $\operatorname{deg} f_{\alpha}=\langle\alpha, \check{\mu}+\check{\rho}\rangle$.

By the Weyl character formula,

$$
\operatorname{char}_{\check{\mu}} V_{\lambda}=q^{\langle\lambda+\rho, \check{\mu}+\check{\rho}\rangle} \sum_{w \in W}(-1)^{\ell(w)} q^{-\langle w(\lambda+\rho), \check{\mu}+\check{\rho}\rangle} \prod_{\alpha \in \Delta_{+}}\left(1-q^{\langle\alpha, \check{\mu}+\check{\rho}\rangle}\right)^{-1} .
$$

Therefore we obtain the following character formula for $T_{\lambda, \check{\mu}}^{\kappa}$ (for any $\kappa \in \mathbb{C}$ ):

$$
\begin{equation*}
\operatorname{char} T_{\lambda, \breve{\mu}}^{\kappa}=q^{\langle\lambda+\rho, \check{\mu}+\breve{\rho}\rangle} \sum_{w \in W}(-1)^{\ell(w)} q^{-\langle w(\lambda+\rho), \check{\mu}+\check{\rho}\rangle} \prod_{n>0}\left(1-q^{n}\right)^{-r} \tag{4.19}
\end{equation*}
$$

It is independent of $\kappa$ and clearly symmetrical under the exchange of $\lambda$ and $\check{\mu}$ (as well as $\rho$ and $\check{\rho}$ ).
4.4. Failure of Theorem 2.2 for rational $\kappa$. In this subsection we show that the statement of Theorem 2.2 with rational $\kappa$ is false already for $\mathfrak{g}=\mathfrak{s l}_{2}$. In this case, we will use the parameter $\gamma=\kappa / \kappa_{0} \in \mathbb{C}$ (then $\check{\kappa}$ corresponds to $\gamma^{-1}$, $\kappa_{c}$ to $\gamma=-2$ and $\kappa_{\mathfrak{s l}_{2}}$ to $\gamma=4$ ), and will identity weights $\lambda \in P$ with the integers $\langle\check{\alpha}, \lambda\rangle \in \mathbb{Z}$, coweights $\check{\mu} \in \check{P}$ with the integers $\langle\check{\mu}, \alpha\rangle \in \mathbb{Z}$. It is proved in [Ara05] that for any complex $\gamma \neq-2$, the cohomology $H_{D S}^{0}(?)$ defines an exact functor from the category $\mathcal{O}_{\kappa}$ of $\widehat{\mathfrak{g}}_{\kappa}$-modules to the category $\mathcal{O}$ of modules over the Virasoro algebra with the central charge $13-6 \gamma-6 \gamma^{-1}$. It sends the Verma module $\mathbb{M}_{\lambda}^{\kappa}$ over $\widehat{\mathfrak{g}}_{\kappa}$ with highest weight $\lambda$ (resp. the contragradient dual $D\left(\mathbb{M}_{\lambda}^{\kappa}\right)$ of $\mathbb{M}_{\lambda}^{\kappa}$; resp. the unique simple quotient $\mathbb{L}_{\lambda}^{\kappa}$ of $\mathbb{M}_{\lambda}^{\kappa}$ ) to the Verma module (resp. the contragradient dual of the Verma module; resp. a simple module or zero module) over the Virasoro algebra with lowest weight (i.e. the lowest eigenvalue of the element $L_{0}$ ) $\Delta_{\lambda, 0}^{\gamma}=\lambda(\lambda+2) / 4 \kappa-\lambda / 2$ (compare with formula (3.23)).

In particular, $T_{\lambda, 0}^{\kappa}=H_{D S}\left(\mathbb{V}_{\lambda}^{\kappa}\right)$ is a quotient of the Verma module $H_{D S}^{0}\left(\mathbb{M}_{\lambda}^{\kappa}\right)$ and hence is a cyclic module over the Virasoro algebra, generated by its lowest weight vector.

Now suppose that $\gamma<0$. It is proved in [Fre92b] that in this case the Wakimoto module $\mathbb{W}_{\lambda}^{\kappa}$ with $\lambda \in \mathbb{Z}_{+}$is isomorphic to the contradradient dual $D\left(\mathbb{M}_{\lambda}^{\kappa}\right)$ of the Verma module $\mathbb{M}_{\lambda}^{\kappa}$ over $\widehat{\mathfrak{g}}_{\kappa}$ with highest weight $\lambda$, and that $H_{D S, 0}^{0}\left(\mathbb{W}_{\lambda}^{\kappa}\right) \cong \pi_{\lambda}^{\kappa}$ is isomorphic to the contradradient dual of the corresponding Verma module over the Virasoro algebra. Thus, $\pi_{\lambda}^{\kappa}$ is a cocyclic module over the Virasoro algebra for any $\lambda \in \mathbb{Z}_{+}$.

In our counterexample, we will set $\gamma=-2, \lambda=2, \check{\mu}=0$. (Similar counterexamples can also be obtained for any negative integer $\gamma \leqslant-2$ and $\lambda$ from an infinite subset of $\mathbb{Z}_{+}$depending on $\gamma$.) Then we have $\mathbb{V}_{0}^{\kappa} \cong \mathbb{L}_{0}^{\kappa}$ and there is an exact sequence

$$
0 \rightarrow \mathbb{L}_{0}^{\kappa} \rightarrow \mathbb{V}_{2}^{\kappa} \rightarrow \mathbb{L}_{2}^{\kappa} \rightarrow 0
$$

(see e.g. [M, KT]). Applying the functor $H_{D S}^{0}(?)$, we get an exact sequence

$$
\begin{equation*}
0 \rightarrow \mathbf{L}_{\chi(0)}^{\kappa} \rightarrow T_{2,0}^{\kappa} \rightarrow \mathbf{L}_{\chi(2)}^{\kappa} \rightarrow 0 \tag{4.20}
\end{equation*}
$$

The $L_{0}$-lowest weights of $\mathbf{L}_{\chi(0)}^{\kappa}$ and $\mathbf{L}_{\chi(2)}^{\kappa}$ are 0 and -2 , respectively. Therefore the image of $\mathbf{L}_{\chi(0)}^{\kappa}$ in $T_{2,0}^{\kappa}$ is generated by a singular vector of weight 2. Thus, the module $T_{2,0}^{\kappa}$ is a cyclic module over the Virasoro algebra, generated by its lowest weight vector, which is an extension of the irreducible module $\mathbf{L}_{\chi(2)}^{\kappa}$ by the irreducible module $\mathbf{L}_{\chi(0)}^{\kappa}$.

Next, consider $T_{0,2}^{\check{\kappa}}=H_{D S, 2}^{0}\left(\mathbb{V}_{0}^{\check{\kappa}}\right)$. Our character formula (4.19) shows that $T_{0,2}^{\check{\kappa}}$ and $T_{2,0}^{\kappa}$ have the same characters. Therefore, their irreducible subquotients are also
the same. However, we will now show that these two modules are not isomorphic to each other.

The embedding $\mathbb{V}_{0}^{\check{\kappa}} \hookrightarrow \mathbb{W}_{\lambda}^{\check{\kappa}}$ induces a map

$$
\begin{equation*}
T_{0,2}^{\check{\kappa}}=H_{D S, 2}^{0}\left(\mathbb{V}_{0}^{\check{\kappa}}\right) \rightarrow H_{D S, 2}^{0}\left(\mathbb{W}_{\lambda}^{\check{\kappa}}\right)=\pi_{-2 \check{\kappa}}^{\check{\kappa}} \tag{4.21}
\end{equation*}
$$

With our choice of $\kappa$, it follows from [Fre92b] that $\pi_{-2 \check{\kappa}}^{\check{\kappa}}$ is a cocyclic module over the Virasoro algebra, generated by its lowest weight vector. Its character coincides with the character of $\pi_{2}^{\kappa}$, and hence it is isomorphic to $\pi_{2}^{\kappa}$.

We claim that the map (4.21) is injective. This does not follow immediately since we don't know whether $H_{D S, 2}^{0}(?)$ is an exact functor. However, we know from the character formula (4.19) that the weight 2 subspace of $T_{0,2}^{\check{\kappa}}$ is 2 -dimensional. Furthermore, it is clear that the images of $\hat{h}_{-2} v$ and $\hat{h}_{-1}^{2} v$ (where $\{e, h, f\}$ is the standard basis of $\mathfrak{s l}_{2}$ and $v$ is the highest weight vector of $\left.C\left(\mathbb{V}_{0}^{\breve{\kappa}}\right)\right)$ in $T_{0,2}^{\breve{\kappa}}=H_{D S, 2}^{0}\left(\mathbb{V}_{0}^{\breve{\kappa}}\right)$ are linearly independent. Hence they form a basis of this weight 2 subspace. But the map (4.21) sends these vectors to non-zero scalar multiples of the vectors $b_{-2} v_{-2 \check{\kappa}}$ and $b_{-1}^{2} v_{-2 \check{\kappa}}$, which form a basis in the weight 2 subspace of $\pi_{-2 \check{\kappa}}^{\check{\kappa}}$. Therefore, the map (4.21) is injective on the weight 2 subspaces. But $T_{0,2}^{\check{\kappa}_{2}}$ has the same irreducible subquotients as $T_{2,0}^{\kappa}$, i.e. the ones with lowest weights 0 and 2 (see the exact sequence (4.20)). From the injectivity on the weight 2 subspaces, it then follows that the map (4.21) itself is injective.

Recalling that $\pi_{-2 \check{\kappa}}^{\check{\kappa}}$ is a cocyclic module over the Virasoro algebra, we then find that $T_{0,2}^{\check{\kappa}}$ is cocyclic as well. Therefore, we have a non-trivial extension

$$
\begin{equation*}
0 \rightarrow \mathbf{L}_{\chi(2)}^{\kappa} \rightarrow T_{0,2}^{\check{\kappa}} \rightarrow \mathbf{L}_{\chi(0)}^{\kappa} \rightarrow 0 \tag{4.22}
\end{equation*}
$$

Comparing the extensions (4.20) and (4.22), we conclude that $T_{0,2}^{\check{\kappa}}$ is not isomorphic to $T_{2,0}^{\kappa}$. Rather, $T_{0,2}^{\check{\kappa}}$ is isomorphic to a different module: the contragradient dual of $T_{2,0}^{\kappa}$. Thus, we obtain a counterexample to the statement of Theorem 2.2 with rational $\kappa$.

## 5. Resolutions and vanishing

In this section, we will give a more detailed description of the complexes obtained by applying the $\check{\mu}$-twisted Drinfeld-Sokolov reduction functor $H_{D S, \check{\mu}}^{\bullet}(?)$ to the resolution of the Weyl module $\mathbb{V}_{\lambda}^{\kappa}$ described in Proposition 4.5. In our proof of Theorem 2.2, we focused on the 0th differential and the 0th cohomology of this complex, which is the module $T_{\lambda, \check{\mu}}^{\kappa}$. Here, we will give formulas for the higher differentials and will explain the connection to the BGG resolutions of irreducible finite-dimensional representations of the corresponding quantum groups, following [FF96, FF99]. This works for all irrational values of $\kappa$.

Theorem 2.1 then implies that for irrational $\kappa$ this complex is a resolution of the $\mathcal{W}^{\kappa}(\mathfrak{g})$-module $T_{\lambda, \check{\mu}}^{\kappa}$ by Fock representations. As an application, we will write in Section 5.4 the character of the module $T_{\lambda, \check{\mu}}^{\kappa}$ as an alternating sum of characters of the Fock representations appearing in the resolution. This reproduces the character formula from Section 4.3.

Finally, in Section 5.5 we will give an alternative proof, for generic $\kappa$, that the higher cohomologies of this complex (and hence $H_{D S, \check{\mu}}^{j}\left(\mathbb{V}_{\lambda}^{\kappa}\right)$ with $j \neq 0$ ) vanish. It relies on the vanishing of the higher cohomologies in the classical limit $\kappa \rightarrow \infty$. In this limit, the screening operators satisfy the Serre relations of the Lie algebra $\mathfrak{g}$, i.e. they generate an action of the Lie subalgebra $\mathfrak{n}_{-} \subset \mathfrak{g}$. The cohomologies of our complex in the limit $\kappa \rightarrow \infty$ are therefore the cohomologies of $\mathfrak{n}_{-}$acting on the $\kappa \rightarrow \infty$ limit of the Fock representation $\pi_{\lambda}^{\kappa}$. It is easy to show that this action is cofree, so that higher cohomologies vanish. The vanishing of higher cohomologies in the limit $\kappa \rightarrow \infty$ implies the vanishing for generic $\kappa$ as well. This is a generalization of the argument that was used in [FF96], which corresponds to the case $\lambda=0, \breve{\mu}=0$.
5.1. Recollections from [FF96]. Using the results of the earlier works [BMP, SV91, Var95], Feigin and one of the authors showed in [FF96] how to associate linear operators between Fock representations to singular vectors in Verma modules over the quantum group. Let us briefly recall this construction.

Let $q=e^{\pi i / \kappa}$ and $U_{q}(\mathfrak{g})$ the Drinfeld-Jimbo quantum group with generators $e_{i}, K_{i}, f_{i}, i=1, \ldots, r$ and standard relations (see, e.g., [FF96], Sect. 4.5.1). Let $U_{q}\left(\mathfrak{n}_{-}\right)$(resp., $\left.U_{q}\left(\mathfrak{b}_{+}\right)\right)$be the lower nilpotent (resp., upper Borel) subalgebra of $U_{q}(\mathfrak{g})$, generated by $f_{i}\left(\right.$ resp $\left., K_{i}, e_{i}\right)$ where $i=1, \ldots, r$. The generators $f_{i}$ satisfy the $q$-Serre relations

$$
\begin{equation*}
\left(\operatorname{ad}_{q} f_{i}\right)^{-a_{i j}+1} \cdot f_{j}=0 \tag{5.1}
\end{equation*}
$$

where $\left(a_{i j}\right)$ is the Cartan matrix of $\mathfrak{g}$. The notation $\operatorname{ad}_{q} f_{i}$ means the following: introduce a grading on the free algebra with generators $e_{i}, i=1, \ldots, l$, with respect to the root lattice $Q$ of $\mathfrak{g}$, by putting $\operatorname{deg} f_{i}=-\alpha_{i}$. If $x$ is a homogeneous element of this algebra of weight $\gamma \in Q$, put

$$
\operatorname{ad}_{q} f_{i} \cdot x=f_{i} x-q^{\left(\alpha_{i} \mid \gamma\right)} x f_{i} .
$$

Next, we define Verma modules over $U_{q}(\mathfrak{g})$ as follows. Let $\mathbb{C}_{\lambda}$ be the onedimensional representation of $U_{q}\left(\mathfrak{b}_{+}\right)$, which is spanned by a vector $\mathbf{1}_{\lambda}$, such that

$$
e_{i} \cdot \mathbf{1}_{\lambda}=0, \quad K_{i} \cdot \mathbf{1}_{\lambda}=q^{\left(\lambda \mid \alpha_{i}\right)} \mathbf{1}_{\lambda}, \quad i=1, \ldots, r .
$$

The Verma module $M_{\lambda}^{q}$ over $U_{q}(\mathfrak{g})$ of highest weight $\lambda$ is the module induced from the $U_{q}\left(\mathfrak{b}_{+}\right)$-module $\mathbb{C}_{\lambda}$ :

$$
M_{\lambda}^{q}=U_{q}(\mathfrak{g}) \otimes_{U_{q}\left(\mathfrak{b}_{+}\right)} \mathbb{C}_{\lambda}
$$

It is canonically isomorphic to $U_{q}\left(\mathfrak{n}_{-}\right) \mathbf{1}_{\lambda}$, and hence to $U_{q}\left(\mathfrak{n}_{-}\right)$.
Roughly speaking, the screening operators $Q_{i}=\int S_{i}^{W}(z) d z$, where $S_{i}^{W}(z)$ is given by formula (3.13), satisfy the $q$-Serre relations (5.1) and hence generate $U_{q}\left(\mathfrak{n}_{-}\right)$. However, because of the multivalued nature of the OPEs between the fields $S_{i}^{W}(z)$ :

$$
S_{i}^{W}(z) S_{j}^{W}(w)=(z-w)^{\left(\alpha_{i} \mid \alpha_{j}\right) / \kappa}: S_{i}^{W}(z) S_{j}^{W}(w):
$$

and the factor $z^{\left(\lambda \mid \alpha_{i}\right) / \kappa}$ appearing in the expansion of $S_{i}^{W}(z)$ acting from $\pi_{\lambda}^{\kappa}$ to $\pi_{\lambda-\alpha_{i}}^{\kappa}$, a general element of $U_{q}\left(\mathfrak{n}_{-}\right)$, when expressed in terms of the screening operators $Q_{i}$, is not well-defined as a linear operator between Fock representations.

Only those elements are well-defined for which there is a non-trivial integration cycle on the corresponding configuration space (of the variables of the currents $S_{i}^{W}(z)$ that have to be integrated) with values in a one-dimensional local system. Such an integration cycle, in turn, exists if and only if the element of $U_{q}\left(\mathfrak{n}_{-}\right)$, when viewed as a vector in $M_{\lambda}^{q}$ (where $\lambda$ is the highest weight of the Fock representation from which we want our operator to act), is a singular vector, i.e. is annihilated by the generators $e_{i}, i=1, \ldots, r$.

Remark on notation: Our Heisenberg algebra generators $b_{i, n}$ correspond to $\beta^{-2} b_{i, n}$ of [FF96], and our $\kappa$ corresponds to $\beta^{-2}$. However, we have a different sign in the definition of the screening currents $S_{i}^{W}(z)$ (see formula (3.13)) compared to [FF96], and for this reason our $\pi_{\nu}^{\kappa}$ corresponds to $\pi_{-\nu \beta}$ of [FF96]. In addition, our $U_{q}\left(\mathfrak{n}_{-}\right)$corresponds to $U_{q}\left(\mathfrak{n}_{+}\right)$of [FF96], for the same reason. Apart from this sign change, our notation is compatible with that of [FF96].

According to Lemma 4.6 .6 of [FF96], we have the following result.
Lemma 5.1. Let $P \in U_{q}\left(\mathfrak{n}_{-}\right)$be such that $P \cdot \mathbf{1}_{\nu}$ is a singular vector of $M_{\nu}^{q}$ of weight $\nu-\gamma$. Then for irrational $\kappa$ the operator $V_{P}^{\kappa}$ defined by formula (4.6.1) of [FF96] (with $\beta=\kappa^{-1 / 2}$ ) is a well-defined homogeneous linear operator $\pi_{\nu}^{\kappa} \rightarrow \pi_{\nu-\gamma}^{\kappa}$.

For example, let $P=f_{i}^{n}$, where $n \in \mathbb{Z}_{+}$. Then $P \mathbf{1}_{\nu}$ is a singular vector in $M_{\nu}^{q}$ if $\nu$ satisfies equation (3.7) for some $m \in \mathbb{Z}$. The corresponding operator $V_{P}^{\kappa}: \pi_{\nu}^{\kappa} \rightarrow \pi_{\nu-n \alpha_{i}}^{\kappa}$ is the operator $S_{i}^{W}(n)$ given by formula (3.12).

Denote by $F_{\kappa}^{\bullet}(\mathfrak{g})$ the complex $F_{\beta}^{*}(\mathfrak{g})$ constructed in Sect. 4.6 of [FF96], where $\beta=\kappa^{-1 / 2}$. It consists of Fock representations and its differentials are constructed using the BGG resolution of the trivial representation of $U_{q}(\mathfrak{g})$ and Lemma 5.1. It was proved in Theorem 4.6 .9 of [FF96] that the 0th cohomology of $F_{\kappa}^{\bullet}(\mathfrak{g})$ is the $\mathcal{W}$-algebra $\mathcal{W}^{\kappa}(\mathfrak{g})$ and all other cohomologies vanish for generic $\kappa$.

This will be our complex corresponding to $\lambda=0, \check{\mu}=0$. And now we construct a similar complex $F_{\lambda, \check{\mu}, \kappa}^{\bullet}(\mathfrak{g})$ for all $\lambda \in P_{+}, \check{\mu} \in \check{P}_{+}$. We will show that the 0th cohomology of $F_{\lambda, \check{\mu}, \kappa}^{\bullet}(\mathfrak{g})$ is the $\mathcal{W}^{\kappa}(\mathfrak{g})$-module $T_{\lambda, \check{\mu}}^{\kappa}$ and all other cohomologies vanish for irrational $\kappa$.
5.2. Generalization to non-zero $\lambda$ and $\check{\mu}$. First, we generalize the complex to an arbitrary $\lambda \in P_{+}$and $\check{\mu}=0$. Consider the BGG resolution $B_{\bullet}^{q, \lambda}(\mathfrak{g})$ of the irreducible finite-dimensional representation $L_{\lambda}^{q}$ of $U_{q}(\mathfrak{g})$ with highest weight $\lambda \in P_{+}$(see Remark 4.5.7 of [FF96]). Its degree $j$ part is

$$
B_{j}^{q, \lambda}(\mathfrak{g})=\bigoplus_{\ell(w)=j} M_{w \circ \lambda}^{q}, \quad w \circ \lambda=w(\lambda+\rho)-\rho .
$$

The differential is constructed in the same way as in Sect. 4.5.6 of [FF96] in the case $\lambda=0$ : For any pair $w, w^{\prime}$ of elements of the Weyl group of $\mathfrak{g}$, such that $w \prec w^{\prime \prime}$, we have the embeddings $i_{w^{\prime}, w}^{q}: M_{w^{\prime} \circ \lambda}^{q} \rightarrow M_{w \circ \lambda}^{q}$ satisfying $i_{w_{1}^{\prime}, w}^{q} \circ i_{w^{\prime \prime}, w_{1}^{\prime}}^{q}=i_{w_{2}^{\prime}, s}^{q} \circ i_{w^{\prime \prime}, w_{2}^{\prime}}^{q}$. The differential $d_{j}^{q, \lambda}: B_{j}^{q, \lambda}(\mathfrak{g}) \rightarrow B_{j-1}^{q, \lambda}(\mathfrak{g})$ is given by the formula

$$
\begin{equation*}
d_{j}^{q, \lambda}=\sum_{\ell(w)=j-1, \ell\left(w^{\prime}\right)=j, w \prec w^{\prime}} \epsilon_{w^{\prime}, w} \cdot i_{w^{\prime}, w}^{q} \tag{5.2}
\end{equation*}
$$

The embedding $i_{w^{\prime}, w}^{q}$ is given by the formula $u \mathbf{1}_{w^{\prime} \circ \lambda} \rightarrow u P_{w^{\prime}, w}^{q} \mathbf{1}_{w \circ \lambda}^{q}, \forall u \in U_{q}\left(\mathfrak{n}_{-}\right)$, where $P_{w^{\prime}, w}^{q} \mathbf{1}_{w \circ \lambda}^{q}$ is a singular vector in $M_{w \circ \lambda}^{q}$ of weight $w^{\prime} \circ \lambda$.

Now we use this BGG resolution to construct a complex $F_{\lambda, 0, \kappa}^{\bullet}(\mathfrak{g})$ as in Sect. 4.6 .8 of [FF96]. Namely, we set

$$
F_{\lambda, 0, \kappa}^{j}(\mathfrak{g})=\bigoplus_{\ell(w)=j} \pi_{w \circ \lambda}^{\kappa}
$$

and define the differential of this complex using the differential of $B_{\bullet}^{q, \lambda}(\mathfrak{g})$ by formulas analogous to formula (4.6.5) of [FF96]:

$$
\begin{equation*}
d_{\lambda}^{j}=\sum_{\ell(w)=j-1, \ell\left(w^{\prime}\right)=j, w \prec w^{\prime}} \epsilon_{w^{\prime}, w} \cdot V_{P_{w^{\prime}, w}^{q}}^{\kappa} . \tag{5.3}
\end{equation*}
$$

The nilpotency of this differential follows in the same way as in the case $\lambda=0$ [FF96]. Furthermore, it follows from the construction that the 0th differential of the complex $F_{\lambda, \kappa}^{\bullet}\left(\right.$ recall that $\left.s_{i} \circ \lambda=\lambda-\left(\lambda_{i}+1\right) \alpha_{i}\right)$

$$
\begin{equation*}
d_{\lambda}^{0}: \pi_{\lambda}^{\kappa} \rightarrow \bigoplus_{i=1}^{r} \pi_{\lambda-\left(\lambda_{i}+1\right) \alpha_{i}}^{\kappa} \tag{5.4}
\end{equation*}
$$

is equal to

$$
\begin{equation*}
d_{\lambda}^{0}=\sum_{i=1}^{r} v_{i} S_{i}^{W}\left(\lambda_{i}+1\right) \tag{5.5}
\end{equation*}
$$

where $v_{i} \in \mathbb{C}^{\times}$(compare with formulas (3.14) and (3.15)). The factors $v_{i}$ occur because our choice of integration cycle $\Gamma$ in Theorem 3.3 is a priori different from that of [FF96]. Since the corresponding cohomology group is one-dimensional, the resulting integrals are proportional to each other, and $v_{i}$ is the corresponding proportionality factor.

Finally, we consider arbitrary $\check{\mu} \in \check{P}_{+}$. We define the complex $F_{\lambda, \check{\mu}, \kappa}^{\bullet}(\mathfrak{g})$ as follows:

$$
F_{\lambda, \check{\mu}, \kappa}^{j}(\mathfrak{g})=\bigoplus_{\ell(w)=j} \pi_{w \circ \lambda-\kappa \check{\mu}}^{\kappa}
$$

and define the differentials by the same formula as for the complex $F_{\lambda, \kappa}^{\bullet}(\mathfrak{g})$.
In particular, the 0th differential $d_{\lambda, \check{\mu}}^{0}$ equals the differential (3.15) (up to the inessential scalar multiples in front of the summands), and therefore we find that the 0 th cohomology of our complex $F_{\lambda, \check{\mu}, \kappa}^{\bullet}(\mathfrak{g})$ is the $\mathcal{W}^{\kappa}(\mathfrak{g})$-module $T_{\lambda, \check{\mu}}^{\kappa}$.

Theorem 5.2. Let $\kappa$ be irrational. Then we have
(1) The $j$ th cohomology of the complex $F_{\lambda, \check{\mu}, \kappa}^{\bullet}(\mathfrak{g})$ is isomorphic to $H_{D S, \check{\mu}}^{j}\left(\mathbb{V}_{\lambda}^{\kappa}\right)$.
(2) The $j$ th cohomology of $F_{\lambda, \check{\mu}, \kappa}^{\bullet}(\mathfrak{g})$ is $T_{\lambda, \check{\mu}}^{\kappa}$ if $j=0$ and 0 if $j>0$.
5.3. Proof of Theorem 5.2. We will construct explicitly the higher differentials of the complex (3.9), which is a resolution of the Weyl module $\mathbb{V}_{\lambda}^{\kappa}$ in terms of the Wakimoto modules. This has already been done in [Fre92a, FF99] in the case $\lambda=0$ and the construction generalizes in a straightforward fashion to arbitrary $\lambda \in P_{+}$.

Recall that

$$
C_{\lambda}^{j}=\bigoplus_{\substack{w \in W \\ \ell(w)=j}} \mathbb{W}_{w \circ \lambda}^{\kappa}
$$

Thus, the weights of the Wakimoto modules appearing in $C_{\lambda}^{j}$ are the same as those of the Verma modules appearing in the $j$ th term $B_{j}^{q, \lambda}(\mathfrak{g})$ of the BGG resolution of $L_{\lambda}^{q}$. We define the differentials of the complex $C_{\lambda}^{\bullet}$ by the above formula (5.3), in which we however use a different definition of $V_{P}^{\kappa}$. While in the definition of [FF96], which is used above in formula (5.3), $V_{P}^{\kappa}$ is constructed using the $\mathcal{W}$ algebra screening currents $S_{i}^{W}(z)$, now we use in their place the affine Kac-Moody screening currents $S_{i}(z)$ given by formula (3.5). Let us denote the corresponding operator by $\widetilde{V}_{P}^{\kappa}$.

The fact that an analogue of Lemma 5.1 holds for these screening currents was established in Sect. 3 of [FF99]. This implies that with this definition, we indeed obtain a complex. Furthermore, for irrational $\kappa$ we have $\mathbb{W}_{w \circ \lambda}^{\kappa} \cong M_{w \circ \lambda}^{* \kappa}$, as shown in the proof of Proposition 3.4. Therefore we find that the space of intertwining operators between $\mathbb{W}_{w \circ \lambda}^{\kappa} \rightarrow \mathbb{W}_{w^{\prime} \circ \lambda}^{\kappa}$ with $\ell(w)=j-1, \ell\left(w^{\prime}\right)=j, w \prec w^{\prime}$ is onedimensional. We also know that each operator $\widetilde{V}_{P}^{\kappa}$ is non-zero because this is so in the limit $\kappa \rightarrow \infty$, as explained in Sect. 4 of [FF99]. Therefore the complex constructed this way is indeed isomorphic to the complex from Proposition 3.4.

Now we apply to this complex the functor $H_{D S, \check{\mu}}^{\bullet}(?)$. According to Lemma 3.4, we have $H_{D S, \check{\mu}}^{\bullet}\left(\mathbb{W}_{w \circ \lambda}^{\kappa}\right) \cong \pi_{w \circ \lambda-\kappa \check{\mu}}^{\kappa}$, so as a graded vector space, the complex we obtain is precisely the complex $F_{\lambda, \check{\mu}, \kappa}^{\bullet}(\mathfrak{g})$. Furthermore, in the same way as in the proof of Lemma 3.5 we obtain that the corresponding differentials are given by the same formulas as the differentials of the complex $C_{\lambda}^{\bullet}$ but we have to replace the Kac-Moody screening currents $S_{i}(z)$ by the $\mathcal{W}$-algebra screening currents $S_{i}^{W}(z)$. Thus, we obtain precisely the differentials (5.3) of the complex $F_{\lambda, \check{\mu}, \kappa}^{\bullet}(\mathfrak{g})$.

This proves part (1) of Theorem 5.2. Part (2) now follows from Theorem 2.1 and the definition of $T_{\lambda, \check{\mu}}^{\kappa}$.

It is worth noting that the complex $F_{\lambda, \check{\mu}, \kappa}^{\bullet}(\mathfrak{g})$ can be obtained in two ways: by applying the functor $H_{D S, \tilde{\mu}}^{\bullet}(?)$ to the resolution $C_{\lambda}^{\bullet}$ of $V_{\lambda}^{\kappa}$ (as above), and by applying the functor $H_{D S}^{\bullet}(?)$ to the resolution $C_{\lambda-\kappa \check{\mu}}^{\bullet}$ of $V_{\lambda-\kappa \check{\mu}}^{\kappa}$ from Proposition 4.5. The second way implies that its higher cohomologies vanish because of Proposition 4.1. Hence we obtain another proof of part (2) of Theorem 5.2.
5.4. Character formula. By definition, the character of a $\mathcal{W}^{\kappa}(\mathfrak{g})$-module $M$ is $\operatorname{ch}(M)=\operatorname{Tr}_{M} q^{L_{0}}$, where $L_{0}$ is the grading operator obtained from the Virasoro generator $T(z)$ of $\mathcal{W}^{\kappa}(\mathfrak{g})$. Theorem 5.2 implies that

$$
\operatorname{ch}\left(T_{\lambda, \check{\mu}}^{\kappa}\right)=\sum_{w \in W}(-1)^{\ell(w)} \operatorname{ch}\left(\pi_{w \circ \lambda-\kappa \check{\mu}}^{\kappa}\right)
$$

Next, according to the formula for $T(z)$ given in Sect. 4 of [FF92],

$$
\operatorname{ch}\left(\pi_{\nu-\kappa \check{\mu}}^{\kappa}\right)=q^{\Delta_{\nu, \tilde{\mu}}^{\kappa}} \prod_{n>0}\left(1-q^{n}\right)^{-r}
$$

where

$$
\begin{equation*}
\Delta_{\nu, \check{\mu}}^{\kappa}=\frac{1}{2 \kappa}(\nu \mid \nu+2 \rho)+\frac{\kappa}{2}(\check{\mu} \mid \check{\mu}+2 \check{\rho})-\langle\nu+\rho, \check{\mu}+\check{\rho}\rangle+\langle\rho, \check{\rho}\rangle . \tag{5.6}
\end{equation*}
$$

We also find that for every $w \in W$,

$$
\Delta_{w \circ \lambda, \check{\mu}}^{\kappa}=\widetilde{\Delta}_{\lambda, \check{\mu}}^{\kappa}-\langle w(\lambda+\rho), \check{\mu}+\check{\rho}\rangle,
$$

where

$$
\widetilde{\Delta}_{\lambda, \check{\mu}}^{\kappa}=\frac{1}{2 \kappa}(\lambda \mid \lambda+2 \rho)+\frac{\kappa}{2}(\check{\mu} \mid \check{\mu}+2 \check{\rho})+\langle\rho, \check{\rho}\rangle .
$$

Therefore

$$
\begin{equation*}
\operatorname{ch}\left(T_{\lambda, \check{\mu}}^{\kappa}\right)=q^{\widetilde{\Delta}_{\lambda, \check{\mu}}^{\kappa}} \prod_{n>0}\left(1-q^{n}\right)^{-r} \sum_{w \in W}(-1)^{\ell(w)} q^{-\langle w(\lambda+\rho), \check{\mu}+\check{\rho}\rangle} . \tag{5.7}
\end{equation*}
$$

Note that the eigenvalues of $L_{0}$ coincide with the $\mathbb{Z}_{+}$-grading introduced in Section 4.3 up to a shift by $\Delta_{\lambda, \check{\mu}}^{\kappa}$ given by formula (5.6). Hence formula (5.7) is equivalent to formula (4.19).
5.5. The limit $\kappa \rightarrow \infty$. In order to pass to the limit $\kappa \rightarrow \infty$, we redefine the complex $F_{\lambda, \check{\mu}, \kappa}^{\bullet}(\mathfrak{g})$ slightly. Define the complex $\widetilde{F}_{\lambda, \check{\mu}, \kappa}^{\bullet}(\mathfrak{g})$ by the formula

$$
\widetilde{F}_{\lambda, \kappa}^{j}(\mathfrak{g})=\bigoplus_{\ell(w)=j} \pi_{w \circ \lambda}^{\kappa} .
$$

Let us identify $\pi_{w \circ \lambda-\kappa \check{\mu}}^{\kappa} \cong \pi_{w \circ \lambda}^{\kappa}$ as free modules with one generator over the negative part of the Heisenberg Lie algebra. Then we identify $\widetilde{F}_{\lambda, \kappa}^{\bullet}(\mathfrak{g})$ and $F_{\lambda, \check{\mu}, \kappa}^{\bullet}(\mathfrak{g})$ as vector spaces. The differential on $F_{\lambda, \check{\mu}, \kappa}^{\bullet}(\mathfrak{g})$, given by formula (5.3), gives rise to the following differential on $\widetilde{F}_{\lambda, \tilde{\mu}, \kappa}^{\bullet}(\mathfrak{g})$. Note that the screening current $S_{i}^{W}(z)$ acting on $\pi_{\nu-\kappa \check{\mu}}^{\kappa}$ becomes, under the isomorphism $\pi_{\nu-\kappa \check{\mu}}^{\kappa} \cong \pi_{\nu}^{\kappa}$ the operator $z^{-\check{\mu}_{i}} S_{i}^{W}(z)$, where as before $\check{\mu}_{i}=\left\langle\check{\mu}, \alpha_{i}\right\rangle$. Thus, the differential

$$
\widetilde{d}_{\lambda, \check{\mu}}^{j}: \widetilde{F}_{\lambda, \tilde{\mu}, \kappa}^{j}(\mathfrak{g}) \rightarrow \widetilde{F}_{\lambda, \tilde{\mu}, \kappa}^{j+1}(\mathfrak{g})
$$

is given by the same formula (5.3) in which we replace each $S_{i}^{W}(z)$ by $z^{-\breve{\mu}_{i}} S_{i}^{W}(z)$. For instance, the 0th differential

$$
\begin{equation*}
\widetilde{d}_{\lambda, \check{\mu}}^{0}: \pi_{\lambda}^{\kappa} \rightarrow \bigoplus_{i=1}^{r} \pi_{\lambda-\left(\lambda_{i}+1\right) \alpha_{i}}^{\kappa} \tag{5.8}
\end{equation*}
$$

is equal to

$$
\begin{equation*}
\widetilde{d}_{\lambda, \check{\mu}}^{0}=\sum_{i=1}^{r} v_{i} S_{i,\left(\check{\mu}_{i}\right)}^{W}\left(\lambda_{i}+1\right) \tag{5.9}
\end{equation*}
$$

where

$$
\begin{align*}
& S_{i,(m)}^{W}(n)=\int_{\Gamma} S_{i}^{W}\left(z_{1}\right) S_{i}^{W}\left(z_{2}\right) \ldots S_{i}^{W}\left(z_{n}\right) z_{1}^{-m} \ldots z_{n}^{-m} d z_{1} \ldots d z_{n}  \tag{5.10}\\
& \pi_{\nu}^{\kappa} \rightarrow \pi_{\nu-n \alpha_{i}}^{\kappa}
\end{align*}
$$

(compare with formula (3.12)).

Let us now rescale the generators of the Heisenberg Lie algebra as follows:

$$
b_{n}^{i} \mapsto x_{n}^{i}=\frac{b_{n}^{i}}{\kappa}
$$

The OPEs (3.1) imply the commutation relations

$$
\left[x_{n}^{i}, x_{m}^{j}\right]=\frac{1}{\kappa}\left(\alpha_{i} \mid \alpha_{j}\right) n \delta_{n,-m}
$$

We will consider the Heisenberg algebra and its modules with respect to these new generators $x_{n}^{i}, n \in \mathbb{Z}, i=1, \ldots, r$. Then in the limit $\kappa \rightarrow \infty$ the Heisenberg algebra becomes commutative, with generators $x_{n}^{i}$. Let us fix once and for all the highest weight vector in in the Fock representation $\pi_{\nu}^{\kappa}, \nu \in \mathfrak{h}^{*}$. Then we can identify $\pi_{\nu}^{\kappa}$ with $\mathbb{C}\left[x_{n}^{i}\right]_{n<0}$ (this corresponds to choosing a particular $\mathbb{C}\left[\kappa^{-1}\right]$-lattice in $\pi_{\nu}^{\kappa} \otimes \mathbb{C}\left[\kappa, \kappa^{-1}\right] ;$ namely, the one generated by monomials in the $x_{n}^{i}$ applied to the highest weight vector). In the limit $\kappa \rightarrow \infty$, we obtain a module on which the negative subalgebra $\mathbb{C}\left[x_{n}^{i}\right]$ acts freely and all other generators $x_{n}^{i}, n \geqslant 0$ act by 0 . Thus, the $\kappa \rightarrow \infty$ limit of $\pi_{\nu}^{\kappa}$ defined in this way does not depend on $\nu$. We will denote it simply by $\pi^{\infty}$.

According to Lemma 4.3.4 of [FF96], the screening operator $Q_{i}^{\kappa}=\int S_{i}^{W}(z) d z$ : $\pi_{0}^{\kappa} \rightarrow \pi_{-\alpha_{i}}^{\kappa}$ has the following expansion in $\kappa^{-1}=\beta^{2}$ :

$$
Q_{i}^{\kappa}=\kappa^{-1} Q_{i}+\kappa^{-2}(\ldots)
$$

where bracketed dots represent a power series in non-negative powers in $\kappa^{-1}$ (the difference in sign is due to our choice of sign in the definition of the screening currents; see Remark on notation in Section 5.1). The leading term $Q_{i}$ is given by formula (2.2.4) of [FF96] (note that $Q_{i}=T_{i}^{-1} \widetilde{Q}_{i}$ ):

$$
\begin{equation*}
Q_{i}=\sum_{n<0} S_{n+1}^{i} \partial_{n}^{(i)} \tag{5.11}
\end{equation*}
$$

where the $S_{n}^{i}$ are the Schur polynomials given by the generating function

$$
\begin{equation*}
\sum_{n \leqslant 0} S_{n}^{i} z^{n}=\exp \left(\sum_{m<0}-\frac{x_{m}^{i}}{m} z^{m}\right) \tag{5.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\partial_{n}^{(i)}=\sum_{j=1}^{r}\left(\alpha_{i} \mid \alpha_{j}\right) \frac{\partial}{\partial x_{n}^{j}} \tag{5.13}
\end{equation*}
$$

In the same way, we obtain an analogous formula for

$$
Q_{i,\left(\check{\mu}_{i}\right)}^{\kappa}=\int S_{i}^{W}(z) z^{-\check{\mu}_{i}} d z: \pi_{0}^{\kappa} \rightarrow \pi_{-\alpha_{i}}^{\kappa}, \quad \check{\mu}_{i} \geqslant 0
$$

Namely,

$$
Q_{i,\left(\breve{\mu}_{i}\right)}^{\kappa}=\kappa^{-1} Q_{i,\left(\check{\mu}_{i}\right)}+\kappa^{-2}(\ldots),
$$

where

$$
\begin{equation*}
Q_{i,\left(\check{\mu}_{i}\right)}=\sum_{n<-\tilde{\mu}_{i}} S_{n+\check{\mu}_{i}+1}^{i} \partial_{n}^{(i)} \tag{5.14}
\end{equation*}
$$

Thus, for each $\check{\mu} \in \check{P}_{+}$we obtain an $r$-tuple of operators $Q_{i,\left(\check{\mu}_{i}\right)}$ on the space $\pi^{\infty}$. These are actually derivatives of the ring $\pi^{\infty}=\mathbb{C}\left[x_{n}^{i}\right]_{n<0}$.

Lemma 5.3. The operators $Q_{i,\left(\breve{\mu}_{i}\right)}$ satisfy the Serre relations of $\mathfrak{n}_{-} \subset \mathfrak{g}$ :

$$
\begin{equation*}
\left(\operatorname{ad} f_{i}\right)^{-a_{i j}+1} \cdot f_{j}=0 \tag{5.15}
\end{equation*}
$$

Proof. The proof is essentially the same as the proof of Proposition 2.2.8 of [FF96], which corresponds to the case $\check{\mu}=0$. The crucial formula in that proof is the commutation relation

$$
\begin{align*}
& \left(\operatorname{ad} Q_{i}\right)^{m} \cdot Q_{j}=  \tag{5.16}\\
& C_{m}\left(-a_{i j}-m+1\right) \sum_{n_{1}, \ldots, n_{m+1}<0} S_{n_{1}+1}^{i} \ldots S_{n_{m}+1}^{i} S_{n_{m+1}+1}^{j} \frac{1}{n_{1} \ldots n_{m}} \\
& \quad \cdot\left(\sum_{l=1}^{m} \frac{n_{l}}{n_{1}+\ldots \widehat{n}_{l} \ldots+n_{m+1}} \partial_{n_{1}+\cdots+n_{m+1}}^{(i)}-\partial_{n_{1}+\cdots+n_{m+1}}^{(j)}\right)
\end{align*}
$$

where $C_{m}$ is a constant (note that there is a typo in this formula in [FF96]; namely, $S_{n_{1}}^{i} \ldots S_{n_{m}}^{i} S_{n_{m+1}}^{j}$ should be replaced with $S_{n_{1}+1}^{i} \ldots S_{n_{m}+1}^{i} S_{n_{m+1}+1}^{j}$ ). This formula is proved by induction, using the relations

$$
\left[\partial_{n}^{(i)}, S_{m}^{j}\right]=-\left(\alpha_{i} \mid \alpha_{j}\right) \frac{1}{n} S_{m-n}^{j}
$$

(where we set $S_{m}^{j}=0$, if $m>0$ ) and the identity

$$
\frac{1}{a(a+b)}+\frac{1}{b(a+b)}=\frac{1}{a b} .
$$

The following formula is a straightforward generalization of formula (5.16):

$$
\begin{equation*}
\left(\operatorname{ad} Q_{i}\right)^{m} \cdot Q_{j}= \tag{5.17}
\end{equation*}
$$

$$
\begin{array}{r}
C_{m}\left(-a_{i j}-m+1\right) \sum_{n_{1}, \ldots, n_{m}<-\check{\mu}_{i} ; n_{m+1}<-\check{\mu}_{j}} S_{n_{1}+\check{\mu}_{i}+1}^{i} \ldots S_{n_{m}+\check{\mu}_{i}+1}^{i} S_{n_{m+1}+\check{\mu}_{j}+1}^{j} \frac{1}{n_{1} \ldots n_{m}} \\
\cdot\left(\sum_{l=1}^{m} \frac{n_{l}}{n_{1}+\ldots \widehat{n_{l}} \ldots+n_{m+1}} \partial_{n_{1}+\cdots+n_{m+1}}^{(i)}-\partial_{n_{1}+\cdots+n_{m+1}}^{(j)}\right) .
\end{array}
$$

This proves our Lemma.
According to Proposition 2.4.6 of [FF96], in the case of $\check{\mu}=0$ the action of $\mathfrak{n}_{-}$ generated by the operators $Q_{i}, i=1, \ldots, r$, on $\pi^{\infty}$ is "cofree", i.e. $\pi^{\infty} \cong U\left(\mathfrak{n}_{-}\right)^{\vee} \otimes V$ for some graded vector space $V$ with a trivial action of $\mathfrak{n}_{-}$. Here $U\left(\mathfrak{n}_{-}\right)^{\vee}$ is the restricted dual of the free $\mathfrak{n}_{-}$-module $U\left(\mathfrak{n}_{-}\right): U\left(\mathfrak{n}_{-}\right)^{\vee}=\oplus_{\gamma} U\left(\mathfrak{n}_{-}\right)_{\gamma}^{*}$, where for each element $\gamma$ in the root lattice of $\mathfrak{g}, U\left(\mathfrak{n}_{-}\right)_{\gamma}$ stands for the corresponding component in $U\left(\mathfrak{n}_{-}\right)$, which is finite-dimensional. In the same way, one can show that the action of $\mathfrak{n}_{-}$generated by $Q_{i,\left(\check{\mu}_{i}\right)}, i=1, \ldots, r$, on $\pi^{\infty}$ is cofree for all $\check{\mu} \in \check{P}_{+}$as well.

Now we are ready to study the limit of the complex $\widetilde{F}_{\lambda, \tilde{\mu}, \kappa}(\mathfrak{g})$ as $\kappa \rightarrow \infty$. We identify each Fock representation appearing in it with $\mathbb{C}\left[x_{n}^{i}\right]$ as above, and in the formula for the differential rescale the screening current $S_{i}^{W}(z) \mapsto \kappa S_{i}^{W}(z)$. As
explained in Sect. 4.6 of [FF96], the complex defined this way has a well-defined limit as $\kappa \rightarrow \infty$.

Let's first look at the limiting complex $\widetilde{F}_{\lambda, \check{\mu}, \infty}^{\bullet}(\mathfrak{g})$ in the case $\lambda=0, \check{\mu}=0$ considered in [FF96]. It is shown in the proof of Proposition 4.3.5 of [FF96] that the complex $\widetilde{F}_{0,0, \infty}^{\bullet}(\mathfrak{g})$ computes the cohomology of the complex $\operatorname{Hom}_{\mathfrak{n}_{-}}\left(B \boldsymbol{\bullet}(\mathfrak{g}), \pi^{\infty}\right)$, where $B \bullet(\mathfrak{g})$ is the BGG resolution of the trivial representation $L_{0}$ of $\mathfrak{g}$ (this resolution is the $q \rightarrow 1$ limit of the resolution $B_{\bullet}^{q, 0}(\mathfrak{g})$ discussed in Section 5.1 above). Since $\pi^{\infty}$ is a cofree $\mathfrak{n}_{-}$-module, we find that the 0th cohomology is $\operatorname{Hom}_{\mathfrak{n}_{-}}\left(L_{0}, \pi^{\infty}\right)=\left(\pi^{\infty}\right)^{\mathfrak{n}_{-}}$ and all higher cohomologies vanish.

In the same way, we show that for general $\lambda \in P_{+}, \check{\mu} \in \check{P}_{+}$we have

$$
\widetilde{F}_{\lambda, \check{\mu}, \infty}^{\bullet}(\mathfrak{g}) \simeq \operatorname{Hom}_{\mathfrak{n}_{-}}\left(B_{\bullet}^{\lambda}(\mathfrak{g}), \pi^{\infty}\right)
$$

where $B_{\bullet}^{\lambda}(\mathfrak{g})$ is the BGG resolution of the irreducible finite-dimensional representation $L_{\lambda}$ of $\mathfrak{g}$ (the $q \rightarrow 1$ limit of the resolution $B_{\bullet}^{q, \lambda}(\mathfrak{g})$ discussed in Section 5.1) and we consider the action of $\mathfrak{n}_{-}$on $\pi^{\infty}$ generated by the operators $Q_{i,\left(\check{\mu}_{i}\right)}, i=1, \ldots, r$. Since $\pi^{\infty}$ is cofree with respect to this action, we obtain

Proposition 5.4. The 0 th cohomology of the complex $\widetilde{F}_{\lambda, \tilde{\mu}, \infty}^{\bullet}(\mathfrak{g})$ is isomorphic to $\operatorname{Hom}_{\mathfrak{n}_{-}}\left(L_{\lambda}, \pi^{\infty}\right)$ and all higher cohomologies vanish.

Corollary 5.5. For generic $\kappa$, all higher cohomologies of the complex $F_{\lambda, \check{\mu}, \infty}^{\bullet}(\mathfrak{g})$ vanish.

Note that in Theorem 5.2,(2) we have proved (by relying on Theorem 2.1) a slightly stronger statement: All higher cohomologies of the complex $F_{\lambda, \check{\mu}, \infty}^{\bullet}(\mathfrak{g})$ vanish for irrational $\kappa$.

## References

[AK11] Kazuhiko Aomoto and Michitake Kita. Theory of hypergeometric functions. Springer Monographs in Mathematics. Springer-Verlag, Tokyo, 2011. With an appendix by Toshitake Kohno, Translated from the Japanese by Kenji Iohara.
[Ara04] Tomoyuki Arakawa. Vanishing of cohomology associated to quantized Drinfeld-Sokolov reduction. Int. Math. Res. Not., (15):730-767, 2004.
[Ara05] Tomoyuki Arakawa. Representation Theory of Superconformal Algebras and the Kac-Roan-Wakimoto Conjecture, Duke Math. J., 130(3):435-478, 2005.
[Ara07] Tomoyuki Arakawa. Representation theory of $W$-algebras. Invent. Math., 169(2):219320, 2007.
[Ara14] Tomoyuki Arakawa. Two-sided BGG resolution of admissible representations. Represent. Theory, 18(3):183-222, 2014.
[Ara17] Tomoyuki Arakawa. Introduction to $W$-algebras and their representation theory, volume 19 of Springer INdAM Series. Springer, 2017.
[ACL] Tomoyuki Arakawa, Thomas Creutzig, and Andrew R. Linshaw. W-algebras as coset vertex algebras. Invent. math. (2019). https://doi.org/10.1007/s00222-019-00884-3 (arXiv:1801.03822 [math.QA]).
[BMP] Peter Bouwknegt, Jim McCarthy, and Krzysztof Pilch. Quantum Group Structure in the Fock Space Resolutions of $\widehat{s l}(n)$ Representations. Comm. Math. Phys. 131 (1990) 125-155.
[CG] Thomas Creutzig and Davide Gaiotto. Vertex algebras for S-duality. arXiv:1708.00875 [hep-th].
[dBT] Jan de Boer and Tjark Tjin. The relation between quantum $W$-algebras and Lie algebras, Comm. Math. Phys. 160 (1994) 317-332.
[FBZ04] Edward Frenkel and David Ben-Zvi. Vertex algebras and algebraic curves, Second Edition, Mathematical Surveys and Monographs 88. American Mathematical Society, Providence, RI, second edition, 2004.
[FF90a] Boris Feigin and Edward Frenkel. Quantization of the Drinfeld-Sokolov reduction. Phys. Lett. B, 246(1-2):75-81, 1990.
[FF90b] Boris L. Feigin and Edward V. Frenkel. Affine Kac-Moody algebras and semi-infinite flag manifolds. Comm. Math. Phys., 128(1):161-189, 1990.
[FF91] Boris Feigin and Edward Frenkel. Duality in $W$-algebras. Internat. Math. Res. Notices, (6):75-82, 1991.
[FF92] Boris Feigin and Edward Frenkel. Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras. Int. J. Mod. Phys. A7 Suppl. 1A (1992) 197-215.
[FF96] Boris Feigin and Edward Frenkel. Integrals of motion and quantum groups. in Proceedings of the C.I.M.E. School Integrable Systems and Quantum Groups, Italy, June 1993, Lect. Notes in Math. 1620 Springer, Berlin, 1996 (arXiv:arXiv:hep-th/9310022).
[FF99] Boris Feigin and Edward Frenkel. Integrable hierarchies and Wakimoto modules. in Differential Topology, Infinite-Dimensional Lie Algebras, and Applications, D. B. Fuchs 60th Anniversary Collection, A. Astashkevich and S. Tabachnikov (eds.), pp. 27-60, AMS 1999 (arXiv:math/9902103).
[FFu] Boris Feigin and Dmitry Fuchs. in Representations of Lie groups and related topics, eds. A.M. Vershik and D.P. Zhelobenko, Gordon and Breach, London, 1990, pp. 465-554.
[Fre92a] Edward Frenkel. $\mathcal{W}$-algebras and Langlands-Drinfeld correspondence. in New Symmetry Principles in Quantum Field Theory, Eds. J. Fröhlich e. a., pp. 433-447, Plenum Press, New York, 1992.
[Fre92b] Edward Frenkel. Determinant formulas for the free field representations of the Virasoro and Kac-Moody algebras. Phys. Lett. B 286 (1992) 71-77.
[Fre05] Edward Frenkel. Wakimoto modules, opers and the center at the critical level. Adv. Math., 195(2):297-404, 2005.
[FG] Edward Frenkel and Davide Gaiotto. Quantum Langlands dualities of boundary conditions, $D$-modules, and conformal blocks. arXiv:1805.00203.
[FG] Edward Frenkel and Dennis Gaitsgory. Affine Kac-Moody algebras and local geometric Langlands correspondence. in Algebraic Geometry and Number Theory, Progress in Math. 253, pp. 69-260, Birkhäuser Boston, 2006 (math.RT/0508382).
[Gai1] Dennis Gaitsgory. Quantum Langlands Correspondence. arXiv:1601.05279 [math.AG].
[Gai2] Dennis Gaitsgory. The master chiral algebra. Gauge theory, geometric Langlands and vertex operator algebras, Perimeter Institute for Theoretical Physics. March 2018, https://www.perimeterinstitute.ca/videos/master-chiral-algebra.
[K] Victor Kac. Contravariant form for infinite dimensional Lie algebras and superalgebras. Lecture Notes in Physics 94 (1979) 441-445.
[KT] Masaki Kashiwara and Toshiyuki Tanisaki. Kazhdan-Lusztig conjecture for affine Lie algebras with negative level. Duke Math. J., $\mathbf{7 7}(1): 21-62,1995$.
[Li] Haisheng Li. The physics superselection principle in vertex operator algebra theory. $J$. Algebra 196(2):436-457, 1997.
[M] Fedor Malikov. Verma modules over Kac-Moody algebras of rank 2. Algebra i Analiz, 2(2):65-84, 1990.
[SV91] Vadim Schechtman and Alexander Varchenko. Quantum groups and homology of local systems. in Algebraic Geometry and Analytic Geometry, ICM-90 Satellite Conference Proceedings, pp. 182-197, Springer, 1991.
[Sto] Alexander V. Stoyanovsky. Quantum Langlands duality and conformal field theory. arXiv:math/0610974 [math.AG].
[TK86] Akihiro Tsuchiya and Yukihiro Kanie. Fock space representations of the Virasoro algebra. Intertwining operators. Publ. Res. Inst. Math. Sci. 22(2):259-327, 1986.
[Var95] Alexander Varchenko. Multidimensional Hypergeometric Functions and Representation Theory of Lie Algebras and Quantum Groups, World Scientific, 1995
[Vor99] Alexander A. Voronov. Semi-infinite induction and Wakimoto modules. Amer. J. Math., 121(5):1079-1094, 1999.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, JAPAN

E-mail address: arakawa@kurims.kyoto-u.ac.jp
Department of Mathematics, University of California, Berkeley, CA 94720, USA


[^0]:    ${ }^{1} \mathrm{~A}$ similar, but more subtle, equivalence is expected for rational values of $\kappa$ as well.

