On universality in penalisation problems with multiplicative weights

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Abstract

We give a general framework for the universality classes of σ-finite measures in penalisation problems with multiplicative weights. We discuss penalisation problems for Brownian motions, Lévy processes and Langevin processes in our framework.

Keywords and phrases: Markov process; martingale; limit theorem; penalisation; conditioning
AMS 2010 subject classifications: 60F05 (60G44 60J57)

1 Introduction

For a measure μ and a non-negative measurable function f, we write $\mu[f]$ for the integral $\int f \mathrm{d}\mu$.

For a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_s)_{s\geq 0}$, and for a non-negative process $\Gamma = (\Gamma_t)_{t\geq 0}$ called a *weight*, we mean by a *penalisation* a problem of finding a limit probability P^Γ on $(\Omega, \mathcal{F})$ called the *penalised probability* such that

$$\frac{P[F_s \Gamma_t]}{P[\Gamma_t]} \underset{t\to\infty}{\longrightarrow} P^\Gamma[F_s] \tag{1.1}$$

is satisfied for all $s \geq 0$ and all bounded $\mathcal{F}_s$-measurable functional F_s. Under the penalised probability P^Γ, the process $(\Gamma_t)_{t\geq 0}$ is prevented from taking small values; this is why Roynette–Vallois–Yor [14] (see also [15]) called this problem the penalisation. Conditioning a process to stay in a domain D may be regarded as a special case of the penalisation, as we take the weight $\Gamma_t = 1_{\{\tau_D > t\}}$ where τ_D denotes the exit time of D.

Although the penalised probability P^Γ depends upon the weight Γ, we can often find a σ-finite measure $\mathscr{P}$ on $(\Omega, \mathcal{F})$ independent of a particular weight such that

$$P^\Gamma(A) = \frac{\mathscr{P}[\Gamma_\infty; A]}{\mathscr{P}[\Gamma_\infty]}, \quad A \in \mathcal{F} \tag{1.2}$$

holds with a suitable limit Γ_∞ of Γ_t in a certain class of weights Γ. In this case we say that Γ belongs to the *universality class* of $\mathscr{P}$. The aim of this paper is to gain a clear

[1]Graduate School of Science, Kyoto University.
[2]The research of Kouji Yano was supported by JSPS KAKENHI grant no.'s JP19H01791, JP19K21834, JP21H01002 and JP18K03441 and by JSPS Open Partnership Joint Research Projects grant no. JPJSBP120209921.

insight into the universality classes in penalisation problems. For this purpose, we confine ourselves to multiplicative weights.

Let $\{B = (B_t)_{t\ge 0}, W_x\}$ denote the canonical representation of the one-dimensional Brownian motion with $W_x(B_0 = x) = 1$ and let $\mathcal{F}_t^B = \sigma(B_s : s \le t)$ denote the natural filtration of the coordinate process B. Let $\tau_D = \inf\{t \ge 0 : B_t = 0\}$ denote the exit time of B from the non-zero real $D = \mathbb{R} \setminus \{0\}$. Let $x \in D$ be fixed. It is then well-known that

$$W_x[F_s|\tau_D > t] \underset{t\to\infty}{\longrightarrow} W_x^{\pm 3\mathrm{B}}[F_s] = \frac{1}{|x|} W_x\left[F_s|B_s|1_{\{\tau_D > s\}}\right] \tag{1.3}$$

for all bounded $\mathcal{F}_s^B$-measurable functional F_s, where $W_x^{\pm 3\mathrm{B}}$ denotes the law of $\pm$ times 3-dimensional Bessel process starting from x. This conditioning to avoid zero may be regarded as a special case of the penalisation with the weight being given by $\Gamma_t = 1_{\{\tau_D > t\}}$. Note that $W_x^{\pm 3\mathrm{B}}$ is locally absolutely continuous with respect to W_x, i.e. $W_x^{\pm 3\mathrm{B}}|_{\mathcal{F}_s^B}$ is absolutely continuous with respect to $W_x|_{\mathcal{F}_s^B}$ for all $s \ge 0$. But $W_x^{\pm 3\mathrm{B}}$ and W_x are mutually singular on $\mathcal{F}_\infty^B := \sigma(B)$, because $W_x^{\pm 3\mathrm{B}}(\tau_D = \infty) = W_x(\tau_D < \infty) = 1$. While the original process $\{B, W_x\}$ is recurrent, the *penalised process* $\{B, W_x^{\pm 3\mathrm{B}}\}$ is transient.

Roynette–Vallois–Yor ([13] and [12]) have studied the penalisation problems for the one-dimensional Brownian motion. They determined the penalised probabilities for $\Gamma_t = f(\overline{X}_t)$, a function of a supremum, $\Gamma_t = f(L_t)$, a function of a local time at 0, and $\Gamma_t = \exp(-\int_0^t v(B_s)\mathrm{d}s)$, a Kac killing weight. For the special case $\Gamma_t = \mathrm{e}^{-L_t}$, we have

$$\frac{W_0[F_s \mathrm{e}^{-L_t}]}{W_0[\mathrm{e}^{-L_t}]} \underset{t\to\infty}{\longrightarrow} W_0^\Gamma[F_s] = \frac{1}{1+|x|} W_0\left[F_s(1+|B_s|)\mathrm{e}^{-L_s}\right] \tag{1.4}$$

for all $s \ge 0$ and all bounded $\mathcal{F}_s^B$-measurable functional F_s. Although W_0^Γ is locally absolutely continuous with respect to W_0, the two measures W_0^Γ and W_0 are mutually singular on $\mathcal{F}_\infty^B$, because $W_0^\Gamma(L_\infty < \infty) = W_0(L_\infty = \infty) = 1$. While the original process $\{B, W_0\}$ is recurrent, the penalised process $\{B, W_0^\Gamma\}$ is transient.

Najnudel–Roynette–Yor ([8]) have introduced the σ-finite measure $\mathscr{W}_0$ defined by

$$\mathscr{W}_0 = \int_0^\infty \frac{\mathrm{d}u}{\sqrt{2\pi u}} \Pi^{(u)} \bullet W_0^{\mathrm{s3B}}, \tag{1.5}$$

where $\Pi^{(u)}$ stands for the law of the Brownian bridge from 0 to 0 of length u, W_0^{s3B} for the law of the symmetrised Bessel process, and $\bullet$ for the law of the concatenated path of two independent paths. They proved that the penalised probability W_0^Γ for any weight Γ in the previous paragraph is absolutely continuous on $\mathcal{F}_\infty^B$ with respect to $\mathscr{W}_0$:

$$W_0^\Gamma[F] = \frac{\mathscr{W}_0[F\Gamma_\infty]}{\mathscr{W}_0[\Gamma_\infty]} \tag{1.6}$$

for all bounded $\mathcal{F}_\infty^B$-measurable functional F. Moreover, if we define $\mathscr{W}_x(\cdot) = \mathscr{W}_0(x + B \in \cdot)$, we have

$$W_x^{\pm 3\mathrm{B}}[F] = \frac{\mathscr{W}_x[F; \tau_D = \infty]}{\mathscr{W}_x(\tau_D = \infty)} \tag{1.7}$$

for all $x > 0$ and all bounded $\mathcal{F}_\infty^B$-measurable functional F. In other words, all the weights belong to the universality class of $\mathscr{W}_x$.

K.Yano–Y.Yano–Yor [20, 21], Y.Yano [22] and recently Takeda–K.Yano [16] studied the penalisation problems for one-dimensional stable Lévy processes and found out that there are two different universality classes. In this paper, we would like to give a general framework to characterise universality classes, where we will give some new results.

Groeneboom–Jongbloed–Wellner [6] studied the conditioning to stay positive for the Langevin process. Profeta [10] studied penalisation problems with several kinds of weights. In this paper, we shall also discuss universality classes for those penalisation problems.

This paper is organized as follows. In Section 2 we develop a general study on penalised probabilities with multiplicative weights. In Section 3 we define the unweighted measures and discuss the subsequent Markov property of them. In Section 4 we state and prove our main theorems on universality classes. In Section 5 we give a general discussion on penalisation problems with multiplicative weights. In Sections 6, 7 and 8, we look at some known results of penalisation problems for Brownian motions, Lévy processes and Langevin processes in our framework. In Section 9 as an appendix, we discuss extension of the transformed probability measures given by local absolute continuity.

2 Penalised probability

For a measure μ and a non-negative measurable function f, we write $f \cdot \mu$ for the transformed measure defined by $(f \cdot \mu)(A) = \int_A f \mathrm{d}\mu$ for all measurable set A. Let $(\mathcal{F}_s)_{s \geq 0}$ be a filtration. For two measures μ and ν, we say that μ is *locally absolutely continuous* with respect to ν if $\mu|_{\mathcal{F}_s}$ is absolutely continuous with respect to $\nu|_{\mathcal{F}_s}$ for all $s \geq 0$. We say the two measures are *locally equivalent* if they are locally absolutely continuous with respect to each other. For a parameterised family $(\mu_\lambda)_\lambda$ of finite measures and a finite measure μ, we say that

$$\lim_\lambda \mu_\lambda = \mu \textit{ along } (\mathcal{F}_s)_{s \geq 0} \tag{2.1}$$

if

$$\lim_\lambda \mu_\lambda[F_s] = \mu[F_s] \tag{2.2}$$

holds for all $s \geq 0$ and all bounded measurable functional F_s.

Let S be a locally compact separable metric space and let $\mathbb{D}$ denote the space of càdlàg paths from $[0, \infty)$ to S. Let $X = (X_t)_{t \geq 0}$ denote the coordinate process: $X_t(\omega) = \omega(t)$ for $t \geq 0$ and $\omega \in \mathbb{D}$. Let $\mathcal{F}_t^X = \sigma(X_s : s \leq t)$ denote the natural filtration of X and set $\mathcal{F}_t = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}^X$ so that $(\mathcal{F}_t)_{t \geq 0}$ is a right-continuous filtration. We write $\mathcal{F}_\infty = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t) = \sigma(X)$. For $t \geq 0$, let θ_t denote the shift operator of $\mathbb{D}$: $\theta_t \omega(s) = \omega(t + s)$ for $s \geq 0$.

Let $\{X, \mathcal{F}_\infty, (P_x)_{x \in S}\}$ denote the canonical representation of a strong Markov process taking values in S with respect to the augmented filtration $(\mathcal{G}_t)_{t \geq 0}$ of $(\mathcal{F}_t)_{t \geq 0}$. A process

$\Gamma = (\Gamma_t)_{t\ge 0}$ is called a *weight* if it is a non-negative càdlàg process. A weight Γ is called *multiplicative* if Γ is adapted to $(\mathcal{F}_t)_{t\ge 0}$ and

$$\Gamma_t = \Gamma_s \cdot (\Gamma_{t-s} \circ \theta_s),\ P_x\text{-a.s. for all } 0 \le s \le t < \infty \text{ and all } x \in S. \tag{2.3}$$

Let Γ be a multiplicative weight. Since $\Gamma_0 = \Gamma_0 \cdot (\Gamma_0 \circ \theta_0) = \Gamma_0^2$, we note that

$$\text{for any } x \in S \text{ we have either } P_x(\Gamma_0 = 1) = 1 \text{ or } P_x(\Gamma_0 = 0) = 1. \tag{2.4}$$

We set

$$S^\Gamma = \{x \in S : P_x(\Gamma_0 = 1) = 1\}. \tag{2.5}$$

It is easy to see that

$$\tau^\Gamma := \inf\{t \ge 0 : X_t \notin S^\Gamma\} = \inf\{t \ge 0 : \Gamma_t = 0\}\ P_x\text{-a.s. for all } x \in S, \tag{2.6}$$

since $[\Gamma_{t_0} = 0$ implies $\Gamma_t = 0$ for all $t \ge t_0]$ because of the multiplicativity.

We introduce the following assumptions:

(A1) There is a Borel function φ^Γ on S such that $\varphi^\Gamma > 0$ on S^Γ and

$$P_x[\Gamma_t \varphi^\Gamma(X_t)] = \varphi^\Gamma(x) \quad \text{for all } x \in S \text{ and } t \ge 0. \tag{2.7}$$

(A2) It holds that

$$P_x[\Gamma_{\boldsymbol{e}(q)}] \to 0 \text{ as } q \downarrow 0 \text{ for all } x \in S^\Gamma, \tag{2.8}$$

where we abuse P_x for the extended probability measure of P_x supporting a standard exponential variable $\boldsymbol{e}$ independent of $\mathcal{F}_\infty$ and we set $\boldsymbol{e}(q) = \boldsymbol{e}/q$ for $q > 0$.

Note that, by the dominated convergence theorem, the condition **(A2)** follows from the following condition:

(**A2′**) It holds that

$$P_x[\Gamma_t] \to 0 \text{ as } t \to \infty \text{ for all } x \in S^\Gamma. \tag{2.9}$$

By the multiplicativity, the condition (2.7) is equivalent to the condition that

$$(\Gamma_t \varphi^\Gamma(X_t))_{t\ge 0} \text{ is a right-continuous } ((\mathcal{G}_t)_{t\ge 0}, P_x)\text{-martingale for all } x \in S \tag{2.10}$$

(for right-continuity, see, e.g., [5, Theorem 5.8]). Under **(A1)**, for $x \in S^\Gamma$, we may define a probability measure P_x^Γ on $(\mathbb{D}, \mathcal{F}_\infty)$, which we call the *penalised probability* of P_x for Γ, by the following (see Section 9):

$$P_x^\Gamma|_{\mathcal{F}_t} = \frac{\Gamma_t \varphi^\Gamma(X_t)}{\varphi^\Gamma(x)} \cdot P_x|_{\mathcal{F}_t} \quad \text{for all } t \ge 0. \tag{2.11}$$

It is then immediate that the *penalised process* $\{X, \mathcal{F}_\infty, (P_x^\Gamma)_{x\in S}\}$ is a Markov process with respect to $(\mathcal{F}_t)_{t\ge 0}$.

We write $\xrightarrow{P}$ for convergence in probability. In addition to **(A1)** and **(A2)**, we also introduce the following assumptions:

(A3) There is a non-negative finite $\mathcal{F}_\infty$-measurable functional Γ_∞ such that

$$P_x^\Gamma\left(\Gamma_t \underset{t\to\infty}{\longrightarrow} \Gamma_\infty > 0\right) = 1 \text{ for all } x \in S^\Gamma. \tag{2.12}$$

Note that in many examples we have **(A3)** and $P_x(\liminf_{t\to\infty}\Gamma_t = 0) = 1$, which implies that the two measures P_x^Γ and P_x are mutually singular on $\mathcal{F}_\infty$.

The following is a routine argument.

Proposition 2.1. *Let Γ be a multiplicative weight. Then the following hold.*

(i) Under **(A1)**, *it holds that*

$$P_x^\Gamma(\tau^\Gamma = \infty) = 1 \textit{ for all } x \in S^\Gamma. \tag{2.13}$$

(ii) Under **(A1)**, **(A2)** *and* **(A3)**, *it holds that*

$$P_x^\Gamma\left(\varphi^\Gamma(X_t) \underset{t\to\infty}{\longrightarrow} \infty\right) = 1 \quad \textit{for all } x \in S^\Gamma. \tag{2.14}$$

Proof. (i) We apply the optional stopping theorem to the $((\mathcal{G}_t)_{t\ge0}, P_x)$-martingale $M_t := \Gamma_t\varphi^\Gamma(X_t)/\varphi^\Gamma(x)$ (by **(A1)**) to see that

$$\begin{aligned} P_x^\Gamma(\tau^\Gamma > t) =& P_x\left[M_t; \tau^\Gamma > t\right] && (2.15)\\ =& P_x[M_{t\wedge\tau^\Gamma}] - P_x\left[M_{t\wedge\tau^\Gamma}; \tau^\Gamma \le t\right] && (2.16)\\ =& P_x[M_0] - P_x\left[M_{\tau^\Gamma}; \tau^\Gamma \le t\right] = 1, && (2.17)\end{aligned}$$

which implies that $P_x^\Gamma(\tau^\Gamma = \infty) = 1$.

(ii) Let $0 \le s \le t < \infty$ and $A_s \in \mathcal{F}_s$. We then have

$$\begin{aligned} P_x^\Gamma\left[\frac{1}{\Gamma_t\varphi^\Gamma(X_t)}; A_s\right] =& \frac{1}{\varphi^\Gamma(x)}P_x(A_s,\ \tau^\Gamma > t)\\ \le& \frac{1}{\varphi^\Gamma(x)}P_x(A_s,\ \tau^\Gamma > s) = P_x^\Gamma\left[\frac{1}{\Gamma_s\varphi^\Gamma(X_s)}; A_s\right]. \end{aligned} \tag{2.18}$$

This shows that $N_t := 1/\{\Gamma_t\varphi^\Gamma(X_t)\}$ is a non-negative P_x^Γ-supermartingale with respect to the completed filtration $(\overline{\mathcal{F}}_t^{P_x^\Gamma})_{t\ge0}$ of $(\mathcal{F}_t)_{t\ge0}$, and consequently it converges P_x^Γ-a.s. as $t\to\infty$ to some random variable N_∞. By **(A3)**, we see that

$$\frac{1}{\varphi^\Gamma(X_t)} = \Gamma_t N_t \underset{t\to\infty}{\longrightarrow} \Gamma_\infty N_\infty \quad P_x^\Gamma\text{-a.s.}, \tag{2.19}$$

which implies $1/\varphi^\Gamma(X_{\boldsymbol{e}(q)}) \xrightarrow[q\downarrow0]{P_x^\Gamma} \Gamma_\infty N_\infty$. Using Fatou's lemma, we obtain

$$P_x^\Gamma[\Gamma_\infty N_\infty] \le \liminf_{q\downarrow0} P_x^\Gamma\left[\frac{1}{\varphi^\Gamma(X_{\boldsymbol{e}(q)})}\right] = \frac{1}{\varphi^\Gamma(x)}\lim_{q\downarrow0} P_x\left[\Gamma_{\boldsymbol{e}(q)}\right] = 0 \tag{2.20}$$

by **(A2)**. Hence we obtain (2.14). □

3 Subsequent Markov property

Let Γ be a multiplicative weight satisfying **(A1)**, **(A2)** and **(A3)**. For $x \in S^\Gamma$, we may define a measure $\mathscr{P}_x^\Gamma$ on $(\mathbb{D}, \mathcal{F}_\infty)$, which we call the *unweighted measure* of P_x^Γ, by

$$\mathscr{P}_x^\Gamma = \varphi^\Gamma(x)\Gamma_\infty^{-1} \cdot P_x^\Gamma \quad \text{on } \mathcal{F}_\infty. \tag{3.1}$$

Note that $\mathscr{P}_x^\Gamma$ is σ-finite on $\mathcal{F}_\infty$, because $\mathbb{D} = \bigcup_{n\in\mathbb{N}}\{\Gamma_\infty > 1/n\}$, $\mathscr{P}_x^\Gamma$-a.e. and

$$\mathscr{P}_x^\Gamma(\Gamma_\infty > 1/n) \le n\varphi^\Gamma(x) < \infty \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

The family of the unweighted measures satisfies the following property.

Theorem 3.1. *Let Γ be a multiplicative weight satisfying* **(A1)**-**(A3)**. *Then, for any $x \in S^\Gamma$, any non-negative $\mathcal{F}_t$-measurable functional F_t and any non-negative $\mathcal{F}_\infty$-measurable functional G, it holds that*

$$\mathscr{P}_x^\Gamma[F_t(G\circ\theta_t)] = P_x\left[F_t\mathscr{P}_{X_t}^\Gamma[G]; \tau^\Gamma > t\right]. \tag{3.3}$$

Proof. By definition of $\mathscr{P}_x^\Gamma$, we have

$$\begin{aligned}\mathscr{P}_x^\Gamma[(F_t\Gamma_t)((G\Gamma_\infty)\circ\theta_t)] =& \mathscr{P}_x^\Gamma[F_t(G\circ\theta_t)\Gamma_\infty] && (3.4)\\ =& \varphi^\Gamma(x)P_x^\Gamma[F_t(G\circ\theta_t)]. && (3.5)\end{aligned}$$

By the Markov property for X under P_x^Γ, by the local equivalence between P_x^Γ and P_x, and by the global equivalence between P_x^Γ and $\mathscr{P}_x^\Gamma$, we obtain

$$\begin{aligned}(3.5) =& \varphi^\Gamma(x)P_x^\Gamma\left[F_tP_{X_t}^\Gamma[G]\right] && (3.6)\\ =& P_x\left[F_t\varphi^\Gamma(X_t)\Gamma_tP_{X_t}^\Gamma[G]\right] && (3.7)\\ =& P_x[F_t\Gamma_t\mathscr{P}_{X_t}[G\Gamma_\infty]], && (3.8)\end{aligned}$$

where we used the fact obtained from Proposition 2.1 that $X_t \in S^\Gamma$, P_x-a.s. on $\{\Gamma_t > 0\}$. Thus we obtain

$$\mathscr{P}_x^\Gamma[F_t\Gamma_t(G\Gamma_\infty)\circ\theta_t] = P_x\left[F_t\Gamma_t\mathscr{P}_{X_t}^\Gamma[G\Gamma_\infty]\right]. \tag{3.9}$$

Replacing F_t by $F_t\Gamma_t^{-1}1_{\{\tau^\Gamma>t\}}$ and G by $G\Gamma_\infty^{-1}1_{\{\Gamma_\infty>0\}}$, we obtain the desired identity, since $\tau^\Gamma = \infty$ and $\Gamma_\infty > 0$, $\mathscr{P}_x^\Gamma$-a.e. The proof is now complete. □

Theorem 3.1 asserts that, the process under $\mathscr{P}_x^\Gamma$ behaves until a fixed time t as the process under P_x killed upon leaving S^Γ, and it starts afresh at time t to behave as the process under $\mathscr{P}_{X_t}^\Gamma$. In this sense, we may call this property (3.3) the *subsequent Markov property*.

4 Universality class

Let $\mathcal{E}$ be a particular multiplicative weight satisfying **(A1)-(A3)**. We would like to give a sufficient condition for existence of a positive function $c(x)$ such that

$$S^\Gamma \subset S^{\mathcal{E}} \quad \text{and} \quad \mathscr{P}^\Gamma_x = c(x) 1_{\{\Gamma_\infty > 0\}} \cdot \mathscr{P}^{\mathcal{E}}_x \quad \text{for all } x \in S^\Gamma. \tag{4.1}$$

We note that $[\mathscr{P}^\Gamma_x = c(x) 1_{\{\Gamma_\infty > 0\}} \cdot \mathscr{P}^{\mathcal{E}}_x]$ yields [Γ belongs to the universality class of $\mathscr{P}^{\mathcal{E}}_x$] in the sense we mentioned in Introduction.

Theorem 4.1 (Universality theorem). *Let $\mathcal{E}$ and Γ be two multiplicative weights satisfying* **(A1)-(A3)**. *Suppose there exists a positive function $c(x)$ such that*

$$P^{\mathcal{E}}_x\left(\Gamma_t \underset{t\to\infty}{\longrightarrow} \Gamma_\infty\right) = 1, \quad \frac{\varphi^\Gamma(X_t)}{\varphi^{\mathcal{E}}(X_t)} \xrightarrow[t\to\infty]{P^{\mathcal{E}}_x} c(x) \quad \textit{for all } x \in S^\Gamma \tag{4.2}$$

and

$$P^\Gamma_x\left(\mathcal{E}_t \underset{t\to\infty}{\longrightarrow} \mathcal{E}_\infty > 0\right) = 1, \quad \frac{\varphi^\Gamma(X_t)}{\varphi^{\mathcal{E}}(X_t)} \xrightarrow[t\to\infty]{P^\Gamma_x} c(x) \quad \textit{for all } x \in S^\Gamma. \tag{4.3}$$

(Notice that these assumptions do not follow from **(A3)***.) Then* (4.1) *holds.*

Proof. Let $x \in S^\Gamma$ be fixed. Since $P_x = P^\Gamma_x$ on $\mathcal{F}_0$, we have

$$P_x(\mathcal{E}_0 = 1) = P^\Gamma_x(\mathcal{E}_0 = 1) \geq P^\Gamma_x(\mathcal{E}_\infty > 0) = 1, \tag{4.4}$$

which shows $x \in S^{\mathcal{E}}$. By the assumptions, we have

$$R_t \xrightarrow[t\to\infty]{P^{\mathcal{E}}_x} R_\infty \text{ and } R_t \xrightarrow[t\to\infty]{P^\Gamma_x} R_\infty \quad \text{with } R_t = \frac{\Gamma_t \varphi^\Gamma(X_t)}{\varphi^{\mathcal{E}}(X_t)} \text{ and } R_\infty = c(x)\Gamma_\infty. \tag{4.5}$$

Let $s > 0$ and let F_s be a non-negative $\mathcal{F}_s$-measurable functional. For $t > s$, we have

$$P^{\mathcal{E}}_x\left[F_s \cdot \frac{R_t}{1 + R_t + \mathcal{E}_t}\right] = \frac{1}{\varphi^{\mathcal{E}}(x)} P_x\left[F_s \cdot \frac{R_t}{1 + R_t + \mathcal{E}_t} \cdot \mathcal{E}_t \varphi^{\mathcal{E}}(X_t)\right] \tag{4.6}$$

$$= \frac{\varphi^\Gamma(x)}{\varphi^{\mathcal{E}}(x)} P^\Gamma_x\left[F_s \cdot \frac{R_t}{1 + R_t + \mathcal{E}_t} \cdot \frac{\mathcal{E}_t \varphi^{\mathcal{E}}(X_t)}{\Gamma_t \varphi^\Gamma(X_t)}\right] \tag{4.7}$$

$$= \frac{\varphi^\Gamma(x)}{\varphi^{\mathcal{E}}(x)} P^\Gamma_x\left[F_s \cdot \frac{\mathcal{E}_t}{1 + R_t + \mathcal{E}_t}\right]. \tag{4.8}$$

Letting $t \to \infty$ and applying the dominated convergence theorem, we obtain

$$P^{\mathcal{E}}_x\left[F_s \cdot \frac{R_\infty}{1 + R_\infty + \mathcal{E}_\infty}\right] = \frac{\varphi^\Gamma(x)}{\varphi^{\mathcal{E}}(x)} P^\Gamma_x\left[F_s \cdot \frac{\mathcal{E}_\infty}{1 + R_\infty + \mathcal{E}_\infty}\right]. \tag{4.9}$$

Since $s > 0$ and F_s are arbitrary, we obtain

$$c(x)\varphi^{\mathcal{E}}(x)\Gamma_\infty \cdot P^{\mathcal{E}}_x = \varphi^\Gamma(x)\mathcal{E}_\infty \cdot P^\Gamma_x, \tag{4.10}$$

which yields

$$c(x) 1_{\{\Gamma_\infty > 0\}} \cdot \mathscr{P}^{\mathcal{E}}_x = 1_{\{\mathcal{E}_\infty > 0\}} \cdot \mathscr{P}^\Gamma_x = \mathscr{P}^\Gamma_x, \tag{4.11}$$

since $P^\Gamma_x(\mathcal{E}_\infty > 0) = 1$. We thus obtain the desired result. □

5 Penalisation problems

We give two systematic methods of ensuring the conditions **(A1)** and **(A2)** in penalisation problems.

5.1 Constant clock

We give a general framework for penalisation problems with constant clock.

Proposition 5.1. *Let Γ be a multiplicative weight. Let $\rho(t)$ be a function such that*

$$\rho(t) \underset{t\to\infty}{\longrightarrow} \infty \quad \text{and} \quad \frac{\rho(t)}{\rho(t-s)} \underset{t\to\infty}{\longrightarrow} 1 \quad \text{for all } s > 0, \tag{5.1}$$

or in other words, $\rho(\log t)$ is divergent and slowly varying at $t = \infty$. Suppose there exists a process $(M_s)_{s\ge 0}$ such that $P_x(M_0 > 0) = P_x(\Gamma_0 = 1)$ for all $x \in S$ and

$$\rho(t)P_x[\Gamma_t|\mathcal{F}_s] \underset{t\to\infty}{\longrightarrow} M_s \quad \text{in } L^1(P_x) \text{ for all } x \in S \text{ and all } s \ge 0. \tag{5.2}$$

Then the weight Γ satisfies **(A1)** *and* **(A2′)** *with*

$$\varphi^\Gamma(x) = \lim_{t\to\infty} \rho(t)P_x[\Gamma_t], \tag{5.3}$$

and the following penalisation limit with constant clock holds:

$$\frac{\Gamma_t \cdot P_x}{P_x[\Gamma_t]} \underset{t\to\infty}{\longrightarrow} P_x^\Gamma \quad \text{along } (\mathcal{F}_s)_{s\ge 0} \text{ for all } x \in S^\Gamma. \tag{5.4}$$

Proof. The convergence (5.2) for $s = 0$ becomes (5.3). By the multiplicativity $\Gamma_t = \Gamma_s \cdot (\Gamma_{t-s} \circ \theta_s)$ and by the Markov property, we have

$$\rho(t)P_x[\Gamma_t|\mathcal{F}_s] = \frac{\rho(t)}{\rho(t-s)}\Gamma_s \cdot \rho(t-s)P_{X_s}[\Gamma_{t-s}] \underset{t\to\infty}{\longrightarrow} \Gamma_s\varphi^\Gamma(X_s) \quad \text{in } P_x\text{-a.s.}. \tag{5.5}$$

which yields $M_s = \Gamma_s\varphi^\Gamma(X_s)$. Hence we have

$$P_x[\Gamma_t\varphi^\Gamma(X_t)] = \lim_{u\to\infty} \rho(u)P_x[P_x[\Gamma_u|\mathcal{F}_t]] = \lim_{u\to\infty} \rho(u)P_x[\Gamma_u] = \varphi^\Gamma(x), \tag{5.6}$$

which shows that **(A1)** is satisfied. As $\rho(t) \to \infty$, we obtain **(A2′)**. For $s > 0$ and for a bounded $\mathcal{F}_s$-measurable functional F_s, we obtain

$$\rho(t)P_x[F_s\Gamma_t] = P_x[F_s\rho(t)P_x[\Gamma_t|\mathcal{F}_s]] \underset{t\to\infty}{\longrightarrow} P_x[F_sM_s] = \varphi^\Gamma(x)P_x^\Gamma[F_s]. \tag{5.7}$$

This shows (5.4). □

5.2 Exponential clock

Conditioning and penalisation problems with exponential clock have been widely studied; see [3], [4], [9], [19] and [11]. We give a general framework for them.

Proposition 5.2. *Let $r(q)$ be a function defined for small $q > 0$ such that $r(q) \to \infty$ as $q \downarrow 0$. We abuse P_x for the extended probability measure of P_x supporting a standard exponential variable $\boldsymbol{e}$ independent of $(\mathcal{F}_t)_{t\geq 0}$ and set $\boldsymbol{e}(q) = \boldsymbol{e}/q$ for $q > 0$. Suppose there exists a process $(M_s)_{s\geq 0}$ such that $P_x(M_0 > 0) = P_x(\Gamma_0 = 1)$ for all $x \in S$ and*

$$\lim_{q\downarrow 0} r(q)P_x[\Gamma_{\boldsymbol{e}(q)}|\mathcal{F}_s] = \lim_{q\downarrow 0} r(q)P_x[\Gamma_{\boldsymbol{e}(q)}1_{\{\boldsymbol{e}(q)>s\}}|\mathcal{F}_s] = M_s \quad \text{in } L^1(P_x) \quad \text{for all } x \in S \text{ and all } s \geq 0. \tag{5.8}$$

Then the weight Γ satisfies **(A1)** *and* **(A2)** *with*

$$\varphi^\Gamma(x) = \lim_{q\downarrow 0} r(q)P_x[\Gamma_{\boldsymbol{e}(q)}], \tag{5.9}$$

and the following penalisation limit with exponential clock holds:

$$\lim_{q\downarrow 0} \frac{\Gamma_{\boldsymbol{e}(q)} \cdot P_x}{P_x[\Gamma_{\boldsymbol{e}(q)}]} = \lim_{q\downarrow 0} \frac{\Gamma_{\boldsymbol{e}(q)}1_{\{\boldsymbol{e}(q)>s\}} \cdot P_x}{P_x[\Gamma_{\boldsymbol{e}(q)}; \boldsymbol{e}(q) > s]} = P_x^\Gamma \quad \text{along } (\mathcal{F}_s)_{s\geq 0} \text{ for all } x \in S^\Gamma. \tag{5.10}$$

Proof. The convergence (5.8) for $s = 0$ becomes (5.9). By the multiplicativity $\Gamma_t = \Gamma_s \cdot (\Gamma_{t-s} \circ \theta_s)$, by the Markov property and by the memoryless property

$$\boldsymbol{e}(q) - s \text{ given } \{\boldsymbol{e}(q) > s\} \stackrel{\text{law}}{=} \boldsymbol{e}(q), \tag{5.11}$$

we have

$$r(q)P_x[\Gamma_{\boldsymbol{e}(q)}1_{\{\boldsymbol{e}(q)>s\}}|\mathcal{F}_s] = \mathrm{e}^{-qs}r(q)P_x[\Gamma_{\boldsymbol{e}(q)+s}|\mathcal{F}_s] \tag{5.12}$$

$$= \mathrm{e}^{-qs}\Gamma_s r(q)P_{X_s}[\Gamma_{\boldsymbol{e}(q)}] \underset{q\downarrow 0}{\longrightarrow} \Gamma_s\varphi^\Gamma(X_s) \quad P_x\text{-a.s.}, \tag{5.13}$$

which yields $M_s = \Gamma_s\varphi^\Gamma(X_s)$. Hence we obtain

$$P_x[\Gamma_t\varphi^\Gamma(X_t)] = P_x[M_t] = \lim_{q\downarrow 0} r(q)P_x[P_x[\Gamma_{\boldsymbol{e}(q)}|\mathcal{F}_t]] = \lim_{q\downarrow 0} r(q)P_x[\Gamma_{\boldsymbol{e}(q)}] = \varphi^\Gamma(x), \tag{5.14}$$

which shows that **(A1)** is satisfied. As $r(q) \to \infty$, we obtain **(A2)**. For $s > 0$ and for a bounded $\mathcal{F}_s$-measurable functional F_s, we obtain

$$r(q)P_x[F_s\Gamma_{\boldsymbol{e}(q)}] = P_x[F_s r(q)P_x[\Gamma_{\boldsymbol{e}(q)}|\mathcal{F}_s]] \underset{q\downarrow 0}{\longrightarrow} P_x[F_sM_s] = \varphi^\Gamma(x)P_x^\Gamma[F_s]. \tag{5.15}$$

This shows (5.10). □

6 Brownian penalisation revisited

Let us look at some results of Roynette–Vallois–Yor [13, 12] and Najnudel–Roynette–Yor [8] in our framework.

Let $\{B = (B_t)_{t\geq 0}, (W_x)_{x\in\mathbb{R}}\}$ denote the canonical representation of the one-dimensional Brownian motion with $W_x(B_0 = x) = 1$. Set $\overline{B}_t = \sup_{s\leq t} B_s$ and let L_t denote the local time of B at 0. For the shift operator on the path space, we have

$$B_{t+s} = B_t \circ \theta_s, \quad \overline{B}_{t+s} = \overline{B}_s \vee (\overline{B}_t \circ \theta_s), \quad L_{t+s} = L_s + (L_t \circ \theta_s). \tag{6.1}$$

For a technical reason, we set

$$S = \{(x, y, l) \in \mathbb{R}^3 : y \geq x,\ l \geq 0\} \tag{6.2}$$

as the state space and consider the coordinate process $X = (X_t)_{t\geq 0} = (X_t^B, X_t^{\mathrm{sup}}, X_t^{\mathrm{lt}})_{t\geq 0}$ on the space of càdlàg paths from $[0, \infty)$ to S. Writing $a \vee b = \max\{a, b\}$, we define $P_{(x,y,l)}$ by the law on $\mathbb{D}$ of $(B, y \vee \overline{B}, l + L)$ under W_x, and adopt the notation of Section 2. By the identities (6.1), we see that the process $\{X, \mathcal{F}_\infty, (P_{(x,y,l)})_{(x,y,l)\in S}\}$ is a strong Markov process with respect to the augmented filtration.

(1) Supremum penalisation. For an integrable function $f : \mathbb{R} \to [0, \infty)$ such that for some $-\infty < y_0 \leq \infty$ we have $f(y) > 0$ for $y \leq y_0$ and $f(y) = 0$ for $y > y_0$, we set

$$\Gamma_t^{\mathrm{sup},f} = \frac{f(X_t^{\mathrm{sup}})}{f(X_0^{\mathrm{sup}})} 1_{\{X_t^{\mathrm{sup}} \leq y_0\}}, \quad S^{\mathrm{sup},f} = \{(x, y, l) \in S : y \leq y_0\}. \tag{6.3}$$

Then we see that $\Gamma^{\mathrm{sup},f}$ is a multiplicative weight with $S^{\Gamma^{\mathrm{sup},f}} = S^{\mathrm{sup},f}$ (in what follows we will omit similar remarks). By Roynette–Vallois–Yor [12, Theorem 3.6], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = \sqrt{\pi t/2}$ and

$$\varphi^{\mathrm{sup},f}(x, y, l) = y - x + \frac{1}{f(y)} \int_y^{y_0} f(u)\mathrm{d}u, \quad (x, y, l) \in S^{\mathrm{sup},f}, \tag{6.4}$$

so that **(A1)** and **(A2′)** are satisfied. By the discussion of Roynette–Vallois–Yor [12, Subsection 1.4], we can derive that

$$P_{(x,y,l)}^{\mathrm{sup},f}(X_\infty^{\mathrm{sup}} > a) = \frac{\int_a^{y_0} f(u)\mathrm{d}u}{(y - x)f(y) + \int_y^{y_0} f(u)\mathrm{d}u}, \quad y \leq a < \infty, \tag{6.5}$$

and hence that $[X_t^{\mathrm{sup}} = X_\infty^{\mathrm{sup}}$ for large $t]$ and $[\Gamma_t^{\mathrm{sup},f} \to \Gamma_\infty^{\mathrm{sup},f} > 0]$ $P_{(x,y,l)}^{\mathrm{sup},f}$-a.s., which shows **(A3)**. By (ii) of Proposition 2.1, we obtain the following known results:

$$P_{(x,y,l)}^{\mathrm{sup},f}\left(X_t^B \to -\infty,\ \frac{\varphi^{\mathrm{sup},f}(X_t)}{|X_t^B|} \to 1\right) = 1. \tag{6.6}$$

(2) Local time penalisation. For an integrable function $f : [0,\infty) \to [0,\infty)$ such that for some $0 \le l_0 \le \infty$ we have $f(l) > 0$ for $l \le l_0$ and $f(l) = 0$ for $l > l_0$, we set

$$\Gamma_t^{\mathrm{lt},f} = \frac{f(X_t^{\mathrm{lt}})}{f(X_0^{\mathrm{lt}})} 1_{\{X_t^{\mathrm{lt}} \le l_0\}}, \quad S^{\mathrm{lt},f} = \{(x,y,l) \in S : l \le l_0\}. \tag{6.7}$$

By Roynette–Vallois–Yor [12, Theorem 3.13 and Lemma 3.15], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = \sqrt{\pi t/2}$ and

$$\varphi^{\mathrm{lt},f}(x,y,l) = |x| + \frac{1}{f(l)} \int_l^{l_0} f(u)\mathrm{d}u, \quad (x,y,l) \in S^{\mathrm{lt},f}, \tag{6.8}$$

so that **(A1)** and **(A2′)** are satisfied. Moreover, **(A3)** is also satisfied and

$$P_{(x,y,l)}^{\mathrm{lt},f}\big(X_t^B \to \pm\infty\big) = \frac{x^\pm f(l) + \frac{1}{2}\int_l^{l_0} f(u)\mathrm{d}u}{|x| f(l) + \int_l^{l_0} f(u)\mathrm{d}u} \tag{6.9}$$

with $x^\pm = \max\{\pm x, 0\}$. It is then obvious that

$$P_{(x,y,l)}^{\mathrm{lt},f}\left(|X_t^B| \to \infty, \ \frac{\varphi^{\mathrm{lt},f}(X_t)}{|X_t^B|} \to 1 \right) = 1. \tag{6.10}$$

Note that the conditioning to avoid zero, which we have mentioned in Introduction, can be regarded as a special case of the local time penalisation with the weight $1_{\{X_t^{\mathrm{lt}}=0\}} = \Gamma_t^{\mathrm{lt},f}$ for $f(l) = 1_{\{l=0\}}$.

(3) Kac killing penalisation with integrable potential. For an integrable function $v : \mathbb{R} \to [0,\infty)$ satisfying

$$0 < \int_{\mathbb{R}} (1+|x|)v(x)\mathrm{d}x < \infty, \tag{6.11}$$

we set

$$\Gamma_t^{\mathrm{Kac},v} = \exp\left(-\int_0^t v(X_s^B)\mathrm{d}s \right), \quad S^{\mathrm{Kac},v} = S. \tag{6.12}$$

By Roynette–Vallois–Yor [13, Theorem 4.1], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = \sqrt{\pi t/2}$ and $\varphi^{\mathrm{Kac},v}(x,y,l) = \varphi_v(x)$ where φ_v is the unique solution to the Sturm–Liouville equation

$$\frac{1}{2}\frac{\mathrm{d}^2\varphi_v}{\mathrm{d}x^2}(x) = v(x)\varphi_v(x), \quad \lim_{x\to\pm\infty} \frac{\mathrm{d}\varphi_v}{\mathrm{d}x}(x) = \pm 1. \tag{6.13}$$

so that **(A1)** and **(A2′)** are satisfied. Moreover, **(A3)** is also satisfied and

$$P_{(x,y,l)}^{\mathrm{Kac},v}\big(X_t^B \to -\infty\big) = \frac{1}{C_v}\int_x^\infty \frac{\mathrm{d}y}{\varphi_v(y)^2}, \quad P_{(x,y,l)}^{\mathrm{Kac},v}\big(X_t^B \to \infty\big) = \frac{1}{C_v}\int_{-\infty}^x \frac{\mathrm{d}y}{\varphi_v(y)^2} \tag{6.14}$$

with $C_v = \int_{\mathbb{R}} \frac{\mathrm{d}y}{\varphi_v(y)^2}$. By (6.13) it is obvious that

$$P^{\mathrm{Kac},v}_{(x,y,l)}\left(|X^B_t| \to \infty, \ \frac{\varphi^{\mathrm{Kac},v}(X_t)}{|X^B_t|} \to 1\right) = 1. \tag{6.15}$$

(4) Kac killing penalisation with Heviside potential. For $\lambda > 0$, we set

$$\Gamma^{\mathrm{Hev},\lambda}_t = \exp\left(-\lambda \int_0^t 1_{\{X^B_s > 0\}} \mathrm{d}s\right), \quad S^{\mathrm{Kac},v} = S. \tag{6.16}$$

By Roynette–Vallois–Yor [13, Theorem 5.1 and Example 5.4], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = \sqrt{\pi t/2}$ and

$$\varphi^{\mathrm{Hev},\lambda}(x,y,l) = \begin{cases} \frac{1}{\sqrt{2\lambda}} \mathrm{e}^{-\sqrt{2\lambda}x} & (x \geq 0), \\ \frac{1}{\sqrt{2\lambda}} - x & (x < 0), \end{cases} \tag{6.17}$$

so that **(A1)** and **(A2′)** are satisfied. Moreover, **(A3)** is also satisfied and

$$P^{\mathrm{Hev},\lambda}_{(x,y,l)}\left(X^B_t \to -\infty, \ \frac{\varphi^{\mathrm{Hev},\lambda}(X_t)}{|X^B_t|} \to 1\right) = 1. \tag{6.18}$$

($*$) The universality class of Brownian penalisation. Take $\mathcal{E}_t = \exp(-X^{\mathrm{lt}}_t)$ as a special case of (2) with $f(l) = \mathrm{e}^{-l}$. (Note that, by Najnudel–Roynette–Yor [8, Theorem 1.1.2], the corresponding unweighted measure $\mathscr{P}^{\mathcal{E}}_x$ coincides with $\mathscr{W}_x$ given in Introduction.) By the above argument, we see that all the assumptions of Theorem 4.1 are satisfied with $\mathcal{E}$ and $\Gamma = \Gamma^{\mathrm{sup},f}$, $\Gamma^{\mathrm{lt},f}$, $\Gamma^{\mathrm{Kac},v}$ or $\Gamma^{\mathrm{Hev},\lambda}$, so that we obtain the following known result:

$$\mathscr{P}^{\Gamma}_{(x,y,l)} = 1_{\{\Gamma_\infty > 0\}} \cdot \mathscr{P}^{\mathcal{E}}_{(x,y,l)} \quad \text{for all } (x,y,l) \in S^{\Gamma}. \tag{6.19}$$

We remark the following obvious facts: It holds up to $\mathscr{P}^{\mathcal{E}}_{(x,y,l)}$-null sets that

$$\mathbb{D} = \{X^B_t \to \infty \text{ or } X^B_t \to -\infty\}, \tag{6.20}$$

and that the event $\{\Gamma_\infty > 0\}$ becomes

$$\{\Gamma^{\mathrm{sup},f}_\infty > 0\} = \{X^B_t \to -\infty \text{ and } X^{\mathrm{sup}}_\infty \leq y_0\}, \tag{6.21}$$

$$\{\Gamma^{\mathrm{lt},f}_\infty > 0\} = \{[X^B_t \to \infty \text{ or } X^B_t \to -\infty] \text{ and } X^{\mathrm{lt}}_\infty \leq l_0\}, \tag{6.22}$$

$$\{\Gamma^{\mathrm{Kac},v}_\infty > 0\} = \{X^B_t \to \infty \text{ or } X^B_t \to -\infty\}, \tag{6.23}$$

$$\{\Gamma^{\mathrm{Hev},\lambda}_\infty > 0\} = \{X^B_t \to -\infty\}. \tag{6.24}$$

7 Lévy penalisation revisited

Let us look at some results of K.Yano–Y.Yano–Yor [20, 21], Y.Yano [22] and Takeda–K.Yano [16] in our framework.

Let $\{Z = (Z_t)_{t\ge 0}, (P^Z_x)_{x\in\mathbb{R}}\}$ denote the canonical representation of one-dimensional strictly α-stable process of index $1 < \alpha < 2$, skewness $-1 \le \beta \le 1$ and scaling parameter $c_\theta > 0$:

$$P^Z_0[\mathrm{e}^{i\lambda Z_t}] = \exp\Big(-c_\theta|\lambda|^\alpha\Big(1 - i\beta\,\mathrm{sgn}(\lambda)\tan\frac{\pi\alpha}{2}\Big)\Big), \quad \lambda\in\mathbb{R}. \tag{7.1}$$

(For the facts in this paragraph, see e.g. [2, Section VIII].) We assume that $1 < \alpha < 2$ so as to exclude the Brownian case and to assure that zero is regular for itself: Writing $T_0 = \inf\{t > 0 : Z_t = 0\}$ for the hitting time of zero, we have

$$P^Z_0(T_0 > 0) = 1. \tag{7.2}$$

Set $\overline{Z}_t = \sup_{s\le t} Z_s$ and let L_t denote the local time of Z at 0. Let

$$\rho := P^Z_0(Z_1 > 0) = \frac{1}{2} + \frac{1}{\pi\alpha}\arctan\Big(\beta\tan\frac{\pi\alpha}{2}\Big) \in [1 - 1/\alpha, 1/\alpha] \tag{7.3}$$

and let k denote the positive constant such that

$$\lim_{y\to\infty} y^\alpha P^Z_0(\overline{Z} > y) = k. \tag{7.4}$$

We set

$$S = \{(x, y, l) \in \mathbb{R}^3 : y \ge x,\ l \ge 0\} \tag{7.5}$$

as the state space and consider the coordinate process $X = (X_t)_{t\ge0} = (X^Z_t, X^{\sup}_t, X^{\mathrm{lt}}_t)_{t\ge0}$ on the space of càdlàg paths from $[0,\infty)$ to S. We define $P_{(x,y,l)}$ by the law on $\mathbb{D}$ of $(Z, y\vee\overline{Z}, l + L)$ under P^Z_x, and adopt the notation of Section 2.

(1) Supremum penalisation. For a non-increasing function $f : \mathbb{R} \to [0,\infty)$ such that for some $-\infty < y_0 \le \infty$ we have $f(y) > 0$ for $y \le y_0$ and $f(y) = 0$ for $y > y_0$, and

$$\int_0^{y_0} x^{\alpha\rho-1} f(y)\mathrm{d}y < \infty, \tag{7.6}$$

we set

$$\Gamma^{\sup,f}_t = \frac{f(X^{\sup}_t)}{f(X^{\sup}_0)} 1_{\{X^{\sup}_t \le y_0\}}, \quad S^{\sup,f} = \{(x, y, l) \in S : y \le y_0\}. \tag{7.7}$$

By K.Yano–Y.Yano–Yor [21, Theorem 5.1], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = t^\rho/k$ and

$$\varphi^{\sup,f}(x, y, l) = (y - x)^{\alpha\rho} + \frac{\alpha\rho}{f(y)}\int_y^{y_0} f(u)(u - x)^{\alpha\rho-1}\mathrm{d}u, \quad (x, y, l) \in S^{\sup,f}, \tag{7.8}$$

so that **(A1)** and **(A2′)** are satisfied. In the same way as that of deducing (6.5), we see that $[X^{\sup}_t = X^{\sup}_\infty$ for large $t]$ and $[\Gamma^{\sup,f}_t \to \Gamma^{\sup,f}_\infty > 0]$ $P^{\sup,f}_{(x,y,l)}$-a.s., which shows **(A3)**. By

(ii) of Proposition 2.1 and by the dominated convergence theorem, we obtain the following known results:

$$P^{\sup,f}_{(x,y,l)}\left(X^Z_t \to -\infty,\ \frac{\varphi^{\sup,f}(X_t)}{(-X^Z_t)^{\alpha\rho}} \to 1 \right) = 1. \tag{7.9}$$

Note that the special case of the supremum penalisation with the weight $1_{\{X^{\sup}_t=0\}} = \Gamma^{\sup,f}_t$ for $f(l) = 1_{\{y=0\}}$ corresponds to the conditioning to stay negative.

(2) Local time penalisation. For an integrable function $f : [0,\infty) \to [0,\infty)$ such that for some $0 \le l_0 \le \infty$ we have $f(l) > 0$ for $l \le l_0$ and $f(l) = 0$ for $l > l_0$, we set

$$\Gamma^{\mathrm{lt},f}_t = \frac{f(X^{\mathrm{lt}}_t)}{f(X^{\mathrm{lt}}_0)} 1_{\{X^{\mathrm{lt}}_t \le l_0\}}, \quad S^{\mathrm{lt},f} = \{(x,y,l) \in S : l \le l_0\}. \tag{7.10}$$

By Takeda–K.Yano [16] and by certain computation in [18, Section 5], we see that all the assumptions of Proposition 5.2 are satisfied with $r(q) = c_r q^{1/\alpha-1}$ for a certain constant $c_r > 0$ and

$$\varphi^{\mathrm{lt},f}(x,y,l) = C_{\alpha,\beta}(1-\beta\,\mathrm{sgn}(x))|x|^{\alpha-1} + \frac{1}{f(l)}\int_l^{l_0} f(u)\mathrm{d}u, \quad (x,y,l) \in S^{\mathrm{lt},f} \tag{7.11}$$

with a certain constant $C_{\alpha,\beta} > 0$, so that **(A1)** and **(A2)** are satisfied. In the same way as that of deducing (6.5), we see that $[X^{\mathrm{lt}}_t = X^{\mathrm{lt}}_\infty$ for large $t]$ and $[\Gamma^{\mathrm{lt},f}_t \to \Gamma^{\mathrm{lt},f}_\infty > 0]$ $P^{\mathrm{lt},f}_{(x,y,l)}$-a.s., which shows **(A3)**. By (ii) of Proposition 2.1, we obtain

$$P^{\mathrm{lt},f}_{(x,y,l)}\left((1-\beta\,\mathrm{sgn}(X^Z_t))|X^Z_t|^{\alpha-1} \to \infty,\ \frac{\varphi^{\mathrm{lt},f}(X_t)}{C_{\alpha,\beta}(1-\beta\,\mathrm{sgn}(X^Z_t))|X^Z_t|^{\alpha-1}} \to 1 \right) = 1; \tag{7.12}$$

in particular,

$$P^{\mathrm{lt},f}_{(x,y,l)}\left(X^Z_t \to -\infty,\ \frac{\varphi^{\mathrm{lt},f}(X_t)}{(-X^Z_t)^{\alpha-1}} \to 2C_{\alpha,1} \right) =1 \quad (\text{if } \beta = 1), \tag{7.13}$$

$$P^{\mathrm{lt},f}_{(x,y,l)}\left(X^Z_t \to \infty,\ \frac{\varphi^{\mathrm{lt},f}(X_t)}{(X^Z_t)^{\alpha-1}} \to 2C_{\alpha,-1} \right) =1 \quad (\text{if } \beta = -1). \tag{7.14}$$

In the case of $-1 < \beta < 1$, we have a stronger convergence result in Takeda–K.Yano [16]:

$$P^{\mathrm{lt},f}_{(x,y,l)}\big(\lim X^Z_t = \limsup X^Z_t = \limsup(-X^Z_t) = \infty\big) = 1 \quad \text{if } -1 < \beta < 1. \tag{7.15}$$

Note that the special case of the local time penalisation with the weight $1_{\{X^{\mathrm{lt}}_t=0\}} = \Gamma^{\mathrm{lt},f}_t$ for $f(l) = 1_{\{l=0\}}$ corresponds to the conditioning to avoid zero. See [17] for comparison of two types of conditionings for Lévy processes.

$(*)$ The universality classes of Lévy penalisation. By (7.9), it holds that

$$\{\Gamma^{\sup,f}_\infty > 0\} =\{X^Z_t \to -\infty \text{ and } X^{\sup}_\infty \le y_0\} \quad \text{up to } \mathscr{P}^{\sup,f}_{(x,y,l)}\text{-null sets} \tag{7.16}$$

in any case of $-1 \le \beta \le 1$.

(∗1) Consider the case of $-1 < \beta < 1$. By (7.15), it holds that

$$\begin{aligned}\{\Gamma^{\mathrm{lt},g}_\infty > 0\} =& \{\lim X^Z_t = \limsup X^Z_t = \limsup(-X^Z_t) = \infty \text{ and } X^{\mathrm{lt}}_\infty \le y_0\} \\ & \text{up to } \mathscr{P}^{\mathrm{lt},g}_{(x,y,l)}\text{-null sets.}\end{aligned} \tag{7.17}$$

This shows that the two σ-finite measures $\mathscr{P}^{\sup,f}_{(x,y,l)}$ and $\mathscr{P}^{\mathrm{lt},g}_{(x,y,l)}$ are singular to each other. Note that (7.9) and (7.15) imply

$$P^{\sup,f}_{(x,y,l)}\left(\frac{\varphi^{\mathrm{lt},g}(X_t)}{\varphi^{\sup,f}(X_t)} \to 0\right) = 1 \tag{7.18}$$

because $\alpha\rho > \alpha - 1$, so that the assumption of Theorem 4.1 is not satisfied.

(∗2) Consider the case of $\beta = 1$, the spectrally positive case. Take $\mathcal{E}_t = \exp(X^{\sup}_0 - X^{\sup}_t)$ as a special case of (1) with $f(y) = \mathrm{e}^{-y}$. Then, since $\alpha\rho = \alpha - 1$, all the assumptions of Theorem 4.1 are satisfied with $\mathcal{E}$ and $\Gamma = \Gamma^{\sup,f}$ or $\Gamma^{\mathrm{lt},g}$, so that we conclude as a new result that

$$\mathscr{P}^{\Gamma}_{(x,y,l)} = 1_{\{\Gamma_\infty > 0\}} \cdot \mathscr{P}^{\mathcal{E}}_{(x,y,l)} \quad \text{for all } (x,y,l) \in S^{\Gamma}. \tag{7.19}$$

It holds up to $\mathscr{P}^{\mathcal{E}}_{(x,y,l)}$-null sets that

$$\mathbb{D} = \{X^Z_t \to -\infty\}, \tag{7.20}$$

and that the event $\{\Gamma_\infty > 0\}$ becomes

$$\{\Gamma^{\mathrm{lt},g}_\infty > 0\} = \{X^Z_t \to -\infty \text{ and } X^{\mathrm{lt}}_\infty \le l_0\}. \tag{7.21}$$

(∗3) Consider the case of $\beta = -1$, the spectrally negative case. Then

$$\{\Gamma^{\mathrm{lt},g}_\infty > 0\} = \{X^Z_t \to \infty \text{ and } X^{\mathrm{lt}}_\infty \le l_0\} \quad \text{up to } \mathscr{P}^{\mathrm{lt},g}_{(x,y,l)}\text{-null sets,} \tag{7.22}$$

which shows that $\mathscr{P}^{\sup,f}_{(x,y,l)}$ and $\mathscr{P}^{\mathrm{lt},g}_{(x,y,l)}$ are singular to each other.

8 Langevin penalisation revisited

Let us look at some results of Profeta [10] in our framework.

Let $\{(B,A),(W_{(b,a)})_{(b,a)\in\mathbb{R}^2}\}$ denote the canonical representation of the two-dimensional diffusion $(B,A) = (B_t, A_t)_{t\ge 0}$ where B is a Brownian motion starting from b and

$$A_t = a + \int_0^t B_u \mathrm{d}u. \tag{8.1}$$

This two-dimensional diffusion is a special case of the *Langevin process* and the process A is called the *integrated Brownian motion*. Set $\overline{A}_t := \sup_{s \le t} A_s$.

We set

$$S = \{(b, a, y) \in \mathbb{R}^3 : y \ge a\} \tag{8.2}$$

as the state space and consider the coordinate process

$$X = (X_t)_{t \ge 0} = (X_t^B, X_t^A, X_t^{\sup})_{t \ge 0} \tag{8.3}$$

on the space of càdlàg paths from $[0, \infty)$ to S. We define $P_{(b,a,y)}$ by the law on $\mathbb{D}$ of $(B, A, y \vee \overline{A})$ under $W_{(b,a)}$, and adopt the notation of Section 2.

We recall the confluent hypergeometric function (see [1, Chapter 13]):

$$U(\alpha, \beta, z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \mathrm{e}^{-zu} u^{\alpha-1} (1+u)^{\beta-\alpha-1} \mathrm{d}u, \quad \alpha > 0, \ \beta \in \mathbb{R}, \ z > 0. \tag{8.4}$$

It is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^\alpha U(\alpha, \beta, z)) = -\alpha(\beta - \alpha - 1) z^{\alpha-1} U(\alpha+1, \beta, z). \tag{8.5}$$

The following asymptotics are taken from [1, Formulae 13.5.2 and 13.5.8]:

$$\lim_{z \to \infty} z^\alpha U(\alpha, \beta, z) = 1 \ (\beta \in \mathbb{R}), \quad \lim_{z \downarrow 0} z^{\beta-1} U(\alpha, \beta, z) = \frac{\Gamma(\beta - 1)}{\Gamma(\alpha)} \ (1 < \beta < 2). \tag{8.6}$$

(1) Conditioning to stay negative. We write $\tau^A = \inf\{t > 0 : X_t^A \ge 0\}$ for the exit time from $(-\infty, 0)$ for the process X^A and set

$$\Gamma_t^A = 1_{\{\tau^A > t\}}, \quad S^A = \{(b, a, y) \in S : y < 0\} = \{(b, a, y) \in \mathbb{R}^3 : a \le y < 0\}. \tag{8.7}$$

By modifying Profeta [10, Theorem 5], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = c_1 t^{1/4}$ for a certain constant $c_1 > 0$ and

$$\varphi^A(b, a, y) = h(-a, -b), \quad (b, a, y) \in S^A, \tag{8.8}$$

with a continuous function $h : (0, \infty) \times \mathbb{R} \to (0, \infty)$ given as

$$h(x, y) = \begin{cases} (\frac{9}{2}x)^{1/6} z^{1/3} U(\frac{1}{6}, \frac{4}{3}, z) = y^{1/2} z^{1/6} U(\frac{1}{6}, \frac{4}{3}, z) & (y > 0), \\ \frac{1}{6}(\frac{9}{2}x)^{1/6} z^{1/3} U(\frac{7}{6}, \frac{4}{3}, z) \mathrm{e}^{-z} = \frac{1}{6}|y|^{1/2} z^{1/6} U(\frac{7}{6}, \frac{4}{3}, z) \mathrm{e}^{-z} & (y < 0), \end{cases} \tag{8.9}$$

for $x > 0$ and $z = \frac{2}{9}\frac{|y|^3}{x}$, so that **(A1)** and **(A2′)** are satisfied. Moreover, **(A3)** is also satisfied and

$$P_{(b,a,y)}^A \big(X_t^B \to -\infty \text{ and } X_t^A \to -\infty \big) = 1. \tag{8.10}$$

Let us prove this fact, as the part $[X_t^B \to -\infty]$ was not mentioned in [10]. By the formulae (8.6), we see that both $z^{1/6}U(\frac{1}{6}, \frac{4}{3}, z)$ and $z^{1/6}U(\frac{7}{6}, \frac{4}{3}, z)\mathrm{e}^{-z}$ are bounded in $z > 0$, we obtain $h(x, y) \le c_2|y|^{1/2}$ for some constant $c_2 > 0$. It holds $P^A_{(b,a,y)}$-a.s. that, by (ii) of Proposition 2.1,

$$\varphi^A(X_t) = h(-X_t^A, -X_t^B) \to \infty, \tag{8.11}$$

which yields $[|X_t^B| \to \infty]$. But $[P^A_{(b,a,y)}(X_t^B \to \infty) = 0]$, since $[X_t^B \to \infty]$ implies $[X_t^A = a + \int_0^t X_s^B \mathrm{d}s \to \infty]$, which contradicts the fact that $X_0^A = a < 0$ and $\tau^A = \infty$ by (i) of Proposition 2.1. Hence we obtain (8.10).

(2) Supremum penalisation. Let $f : \mathbb{R} \to [0, \infty)$ be a continuous function such that for some $-\infty < y_0 \le 0$, we have $f(y) > 0$ for $y \le y_0$ and $f(y) = 0$ for $y > y_0$. Set

$$\Gamma_t^{\sup,f} = \frac{f(X_t^A)}{f(X_0^A)} 1_{\{X_t^A \le y_0\}}, \qquad \begin{aligned} S^{\sup,f} &= \{(b, a, y) \in S : y \le y_0\} \\ &= \{(b, a, y) \in \mathbb{R}^3 : a \le y < y_0\}. \end{aligned} \tag{8.12}$$

By Profeta [10, Proposition 18 and Theorem 19], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = c_1 t^{1/4}$ and

$$\varphi^{\sup,f}(b, a, y) = h(y - a, -b) + \frac{1}{f(y)} \int_y^{y_0} f(w) \frac{\partial}{\partial w} h(w - a, -b) \mathrm{d}w, \quad (b, a, y) \in S^{\sup,f}, \tag{8.13}$$

so that **(A1)** and **(A2′)** are satisfied. By a similar argument to that deducing (6.5), we see that $[X_t^{\sup} = X_\infty^{\sup}$ for large $t]$ $P^{\sup,f}_{(b,a,y)}$-a.s., and that $[\Gamma_t^{\sup,f} \to \Gamma_\infty^{\sup,f} > 0]$ $P^{\sup,f}_{(b,a,y)}$-a.s., which shows **(A3)**. By the fact that $\frac{\partial h}{\partial w} \ge 0$, we have

$$\varphi^{\sup,f}(b, a, y) \le \left(\sup_{y \le w \le y_0} f(w) \right) h(y_0 - a, -b). \tag{8.14}$$

By a similar argument after (8.11), and by (ii) of Proposition 2.1, we can deduce

$$P^{\sup,f}_{(b,a,y)}\big(X_t^B \to -\infty \text{ and } X_t^A \to -\infty\big) = 1. \tag{8.15}$$

(∗) The universality class of Langevin penalisation. We would like to compare the three unweighted measures $\mathscr{P}^A_{(b,a,y)}$, $\mathscr{P}^{\sup,f}_{(b,a,y)}$ and $\mathscr{P}^B_{(b,a,y)}$. Here we write $\tau^B = \inf\{t > 0 : X_t^B \ge 0\}$ for the exit time from $(-\infty, 0)$ for the Brownian motion X^B and set

$$\Gamma_t^B = 1_{\{\tau^B > t\}}, \quad S^B = \{(b, a, y) \in S : b < 0\}. \tag{8.16}$$

The penalisation for the weight Γ^B is nothing else but the conditioning to stay negative for the Brownian motion, so that we obtain $\varphi^B(b, a, y) = -b$. The penalized probability $P^B_{(b,a,y)}$ is the minus times 3-dimensional Bessel process and the corresponding unweighted measure is given as $\mathscr{P}^B_{(b,a,y)} = (-b)P^B_{(b,a,y)}$. Since $X_t^A = a + \int_0^t X_u^B \mathrm{d}u$, we obtain

$$P^B_{(b,a,y)}\big(X_t^B \to -\infty \text{ and } X_t^A \to -\infty\big) = 1. \tag{8.17}$$

We prove the following proposition with conjectured assumptions.

Proposition 8.1. *Set* $Z_t = \dfrac{(-X_t^B)^3}{(-X_t^A)}$. *Then the following assertions hold:*

(i) Suppose the following conjecture is true:

$$Z_t \xrightarrow[t\to\infty]{P^A_{(b,a,y)}} \infty \text{ and } Z_t \xrightarrow[t\to\infty]{P^{\sup,f}_{(b,a,y)}} \infty \text{ for } (b,a,y) \in S^{\sup,f}. \tag{8.18}$$

Then $\mathscr{P}^{\sup,f}_{(b,a,y)}$ *and* $\mathscr{P}^A_{(b,a,y)}$ *coincide for* $(b,a,y) \in S^{\sup,f} (\subset S^A)$.

(ii) Suppose the following conjecture is true:

$$Z_t \xrightarrow[t\to\infty]{P^A_{(b,a,y)}} \infty \text{ and } Z_t \xrightarrow[t\to\infty]{P^B_{(b,a,y)}} \infty \text{ for } (b,a,y) \in S^A \cap S^B. \tag{8.19}$$

Then $\mathscr{P}^A_{(b,a,y)}$ *and* $\mathscr{P}^B_{(b,a,y)}$ *are singular to each other for* $(b,a,y) \in S^A \cap S^B$.

Proof. (i) Set $Z_t^{\sup} = \dfrac{(-X_t^B)^3}{(X_t^{\sup} - X_t^A)}$. Then $Z_t \xrightarrow[t\to\infty]{P} \infty$ both for $P = P^A_{(b,a,y)}$ and for $P = P^{\sup,f}_{(b,a,y)}$. Since $X_t^B < 0$ for large t, we have

$$\frac{h(X_t^{\sup} - X_t^A, -X_t^B)}{h(-X_t^A, -X_t^B)} = \frac{(Z_t^{\sup})^{1/6} U(\frac{1}{6}, \frac{4}{3}, Z_t^{\sup})}{(Z_t)^{1/6} U(\frac{1}{6}, \frac{4}{3}, Z_t)} \xrightarrow[t\to\infty]{P} 1 \tag{8.20}$$

by the assumption. Noting that (8.5) implies

$$\frac{\partial}{\partial x} h(x,y) = c_3 x^{-5/6} \cdot z^{7/6} U(\tfrac{7}{6}, \tfrac{4}{3}, z) \le c_4 x^{-5/6}, \quad x, y > 0, \; z = \frac{2}{9}\frac{|y|^3}{x} \tag{8.21}$$

for some constants $c_3, c_4 > 0$, we obtain

$$\frac{\varphi^{\sup,f}(X_t)}{\varphi^A(X_t)} \xrightarrow[t\to\infty]{P} 1 \tag{8.22}$$

both for $P = P^A_{(b,a,y)}$ and for $P = P^{\sup,f}_{(b,a,y)}$. We may now apply Theorem 4.1 for $\mathcal{E} = \Gamma^A$ and $\Gamma = \Gamma^{\sup,f}$, and thus we obtain the desired result.

(ii) By the assumption, we have

$$R_t := \frac{\Gamma_t^A \varphi^A(X_t)}{\varphi^B(X_t)} = \frac{\Gamma_t^A \cdot (-X_t^B)^{1/2} \cdot (Z_t)^{1/6} U(\frac{1}{6}, \frac{4}{3}, Z_t)}{(-X_t^B)} \xrightarrow[t\to\infty]{P} 0 \tag{8.23}$$

both for $P = P^A_{(b,a,y)}$ and for $P = P^B_{(b,a,y)}$. By the same argument of Theorem 4.1 with $\mathcal{E} = \Gamma^B$ and $\Gamma = \Gamma^A$, we obtain

$$P^B_{(b,a,y)}\left[F_s \cdot \frac{R_t}{1 + R_t + \Gamma_t^B}\right] = \frac{\varphi^A(b,a,y)}{\varphi^B(b,a,y)} P^A_{(b,a,y)}\left[F_s \cdot \frac{\Gamma_t^B}{1 + R_t + \Gamma_t^B}\right]. \tag{8.24}$$

Letting $t \to \infty$, we obtain $P^A_{(b,a,y)}(\Gamma^B_\infty > 0) = 0$. Since $P^B_{(b,a,y)}(\Gamma^B_\infty > 0) = 1$, we obtain the desired result. ∎

9 Appendix: Extension of transformed probability measures

We discuss in general extension of the transformed probability measures given by local absolute continuity like (2.11). Recall that $\mathbb{D}$ is the space of càdlàg paths from $[0, \infty)$ to a locally compact separable metric space S and X is the coordinate process on $\mathbb{D}$.

Theorem 9.1. *Let P be a probability measure on $(\mathbb{D}, \sigma(X))$ and let $(M_t)_{t \geq 0}$ be a non-negative martingale such that $P[M_t] = 1$ for all $t \geq 0$. Then there exists a unique probability measure Q on $(\mathbb{D}, \sigma(X))$ such that*

$$Q|_{\mathcal{F}_t^X} = M_t \cdot P|_{\mathcal{F}_t^X}, \quad t \geq 0, \tag{9.1}$$

where $\mathcal{F}_t^X = \sigma(X_s : s \leq t)$ is the natural filtration of X.

Proof. Since $\bigcup_{t \geq 0} \mathcal{F}_t^X$ is a π-system generating $\sigma(X)$, uniqueness of Q follows immediately from Dynkin's π-λ theorem.

Let us prove existence of Q. For $n \in \mathbb{N}$, let $\mathbb{D}_n$ denote the space of càdlàg paths from $[n-1, n)$ to S, equipped with the σ-field $\mathcal{B}_n$ generated by the coordinate process on $\mathbb{D}_n$. We thus see that $\mathbb{D}$ is the product space of $\{\mathbb{D}_n\}$:

$$\mathbb{D} = \prod_{n=1}^{\infty} \mathbb{D}_n, \quad \sigma(X) = \sigma\left(\prod_{k=1}^{n} B_k \times \prod_{k=n+1}^{\infty} \mathbb{D}_k : B_1 \in \mathcal{B}_1, \ldots, B_n \in \mathcal{B}_n; \ n \in \mathbb{N} \right). \tag{9.2}$$

Let μ_n denote the law on $\mathbb{D}_1 \times \cdots \times \mathbb{D}_n$, the space of càdlàg paths from $[0, n)$ to S, of $(X_t)_{0 \leq t < n}$ under $M_n \cdot P|_{\mathcal{F}_n^X}$. We then see that $\{\mu_n\}$ is a projective sequence:

$$\mu_{n+1}(\cdot \times \mathbb{D}_{n+1}) = \mu_n, \quad n \in \mathbb{N}. \tag{9.3}$$

We may apply Daniell's extension theorem (cf. [7, Theorem 6.14]) to see that there exists a sequence of random variables $\{\xi_n\}$ defined on a probability space $(\Omega', \mathcal{F}', P')$ such that ξ_n for each n takes values in $\mathbb{D}_n$ and the joint distribution of $(\xi_1, \ldots, \xi_n)$ under P' for each n coincides with μ_n.

We now define Q by the law on $\mathbb{D}$ of $(\xi_1, \xi_2, \ldots)$ under P'. For any $A \in \mathcal{F}_n^X$ for each $n \in \mathbb{N}$, we can find $B \subset \mathbb{D}_1 \times \cdots \times \mathbb{D}_n$ which belongs to $\sigma(\prod_{k=1}^{n} B_k : B_1 \in \mathcal{B}_1, \ldots, B_n \in \mathcal{B}_n)$ such that $A = \{(X_t)_{0 \leq t < n} \in B\}$, so that we obtain

$$Q(A) = P'((\xi_1, \ldots, \xi_n) \in B) = \mu_n(B) = P[M_n; (X_t)_{0 \leq t < n} \in B] = P[M_n; A]. \tag{9.4}$$

We thus conclude that Q is as desired. □

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