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Abstract

A direct, explicit derivation of the recently discovered time–information uncertainty relation in thermodynamics [S. B. Nicholson *et al* (2020), *Nat. Phys.* **16**, 1211] is presented.

The evolution of entropy and related uncertainty relations are of great importance in nonequilibrium thermodynamics and statistical mechanics [1–3]. Nicholson *et al* have recently discovered a time–information uncertainty relation in thermodynamics [4]:

$$|\dot{S}|/k_B \leqslant \Delta\dot{I}\Delta I. \quad (1)$$

S is the Shannon entropy:

$$S/k_B = -\sum p_x \ln p_x, \quad (2)$$

where k_B is the Boltzmann constant, p_x is the probability of the state x $(= 1, 2, \ldots, N)$, and the simplified symbol of summation in this note denotes:

$$\sum a_x \equiv \sum_{x=1}^{N} a_x, \quad (3)$$

where a_x is a general variable. $\dot{S}$ is the evolution rate or time derivative of the entropy:

$$\dot{S} \equiv \frac{dS}{dt}. \quad (4)$$

ΔI is the standard deviation of the surprisal or information content I_x:

$$I_x = -\ln p_x, \quad (5)$$

and $\Delta\dot{I}$ is the standard deviation of the evolution rate of the surprisal [5–7]. Because p_x is a stochastic variable, other statistical variables such as S, I_x, and their derivatives are also stochastic. The relation of equation (1) provides an upper bound of the entropy evolution rate in a system, and thus is positioned as a milestone in multiple fields including informatics, nonequilibrium thermodynamics, and energy science and engineering. The time–information uncertainty relation and the associated speed limit for flows of heat and entropy were validated with a number of examples [4]. This uncertainty applies to various systems ranging from energy transducers [8–10] to consciousness neuroinformatics [11–13]. In this short note, we present a more explicit derivation of the time–information uncertainty relation of equation (1), solely using the most primitive variable, p_x, as a supplement of the original article by Nicholson *et al* [4], for the convenience of the community.

Let us start with the formulation of the standard deviation of the surprisal and of the surprisal evolution rate. From equation (5),

$$\Delta I = \sqrt{\sum p_x (I_x - \langle I \rangle)^2} = \sqrt{\sum p_x \left(-\ln p_x + \sum p_x \ln p_x\right)^2}, \quad (6)$$

where the symbol of expectation denotes:

$$\langle a \rangle \equiv \sum p_x a_x. \tag{7}$$

$$\begin{aligned} \Delta \dot{I} &= \sqrt{\sum p_x (\dot{I}_x - \langle \dot{I} \rangle)^2} \\ &= \sqrt{\sum p_x \left(-\dot{p}_x/p_x + \sum \dot{p}_x\right)^2}, \end{aligned} \tag{8}$$

since

$$\dot{I}_x = \frac{d}{dt}(-\ln p_x) = -\dot{p}_x/p_x. \tag{9}$$

Next, let us move onto the entropy evolution rate. From equation (2),

$$\begin{aligned} \dot{S}/k_B &= \frac{d}{dt}\left(-\sum p_x \ln p_x\right) \\ &= -\sum \dot{p}_x \ln p_x - \sum p_x \frac{d}{dt}(\ln p_x) \\ &= -\sum \dot{p}_x \ln p_x - \sum \dot{p}_x. \end{aligned} \tag{10}$$

Because $\sum p_x = 1, \sum \dot{p}_x = \frac{d}{dt}\sum p_x = 0$. This fact allows for a trick to multiply the second term of equation (10) by $-\sum p_x \ln p_x\ (= S/k_B)$:

$$\dot{S}/k_B = -\sum \dot{p}_x \ln p_x + \left(\sum \dot{p}_x\right)\left(\sum p_x \ln p_x\right). \tag{11}$$

From equation (11), by further tricky transformations,

$$\begin{aligned} \dot{S}/k_B = &-\sum \dot{p}_x \ln p_x + \left(\sum \dot{p}_x\right)\left(\sum p_x \ln p_x\right) \\ &+\left(\sum \dot{p}_x\right)\left(\sum p_x \ln p_x\right) - \left(\sum \dot{p}_x\right)\left(\sum p_x \ln p_x\right) \\ &= -\sum \dot{p}_x \ln p_x + \left(\sum \dot{p}_x\right)\left(\sum p_x \ln p_x\right) \\ &+\left(\sum p_x \ln p_x\right)\left(\sum \dot{p}_x\right) - \sum p_x \left\{\left(\sum \dot{p}_x\right)\left(\sum p_x \ln p_x\right)\right\} \\ &= -\sum \left\{\dot{p}_x \ln p_x - \dot{p}_x \sum p_x \ln p_x - p_x \ln p_x \sum \dot{p}_x + p_x \left(\sum \dot{p}_x\right)\left(\sum p_x \ln p_x\right)\right\} \\ &= -\sum p_x \left(-\dot{p}_x/p_x + \sum \dot{p}_x\right)\left(-\ln p_x + \sum p_x \ln p_x\right). \end{aligned} \tag{12}$$

For the transformation across the second equality sign in equation (12), a general property $A = \sum p_x A$, where A is a constant, was used. Using the Cauchy–Schwarz inequality $(\sum a_x b_x)^2 \leqslant (\sum a_x{}^2)(\sum b_x{}^2)$, where b_x is a general variable, from equation (12),

$$\begin{aligned} (\dot{S}/k_B)^2 &= \left\{\sum p_x \left(-\dot{p}_x/p_x + \sum \dot{p}_x\right)\left(-\ln p_x + \sum p_x \ln p_x\right)\right\}^2 \\ &\leqslant \left\{\sum p_x \left(-\dot{p}_x/p_x + \sum \dot{p}_x\right)^2\right\}\left\{\sum p_x \left(-\ln p_x + \sum p_x \ln p_x\right)^2\right\} \\ &= (\Delta \dot{I})^2 (\Delta I)^2, \end{aligned} \tag{13}$$

via equations (6) and (8). Hence, $|\dot{S}|/k_B \leqslant \Delta \dot{I} \Delta I$ (equation (1)).

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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References

[1] Seifert U 2005 *Phys. Rev. Lett.* **95** 040602
[2] Parrondo J M R, Horowitz J M and Sagawa T 2015 *Nat. Phys.* **11** 131
[3] Horowitz J M and Gingrich T R 2020 *Nat. Phys.* **16** 15
[4] Nicholson S B, García-Pintos L P, del Campo A and Green J R 2020 *Nat. Phys.* **16** 1211
[5] Shannon C E 1948 *Bell Syst. Tech. J.* **27** 379
[6] Landauer R 1961 *IBM J. Res. Dev.* **5** 183
[7] Sekimoto K 2010 *Stochastic Energetics* (Heidelberg: Springer)

[8] Shiraishi N, Saito K and Tasaki H 2016 *Phys. Rev. Lett.* **117** 190601
[9] Tanabe K 2016 *J. Phys. Soc. Jpn.* **85** 064003
[10] Josefsson M, Svilans A, Burke A M, Hoffmann E A, Fahlvik S, Thelander C, Leijnse M and Linke H 2018 *Nat. Nanotechnol.* **13** 920
[11] Tononi G 2004 *BMC Neurosci.* **5** 42
[12] Oizumi M, Tsuchiya N and Amari S 2016 *Proc. Nat'l. Acad. Sci. USA* **113** 14817
[13] Tanabe K 2020 arXiv2006.16243