INDEPENDENCE VS RANK

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1. Introduction

This text is based on my talk at the RIMS Model Theory Workshop 2021. The main goal of the talk was to provide a mild introduction to the results from my paper with Jan Dobrowolski [6], where we develop a new family of local ranks, initially aiming for applications in the NSOP_1 context. In this text, I will recall a few basic facts about NSOP_1 theories and then discuss our local ranks.

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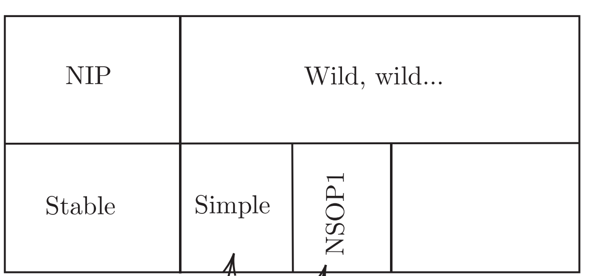
2. Where are we?

There is a very good website presenting the universe of model theory, "forkingand-dividing.com" made by Gabe Conant, where you can find many first order theories categorized among the neo-stability hierarchy. We reproduce the arrangement of the dividing lines from this website in the following picture.



As you can see, the stability was generalized into simplicity (exposed nicely in [3], [11], [14]), and then into NSOP_1 (No Strict Order Property of the first kind). Actually the lower half of the whole rectangular is the class of NSOP theories, and it contains also subclasses of theories of the form NSOP_n, where n is a natural number, but we are mostly interested in transferring geometric ideas and techniques from stable theories into simple theories and then into NSOP_1 theories, so we are not so much interested in other subclasses of the NSOP class here. The difficulties show up just after reaching the border between simple and NSOP_1 sectors.

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At the core of pure model theory, we have stability, and at the core of stability there is geometric stability theory, which was used in several celebrated results and applications outside of the model theory (e.g. [8], [9]). One of the main geometric tools and techniques related to the model-theoretic stability theory is the notion of a ternary independence relation (cf. [1]), which plays a key role in the description of simple theories (cf. [12]). More precisely, a theory is simple if and only if there is a ternary relation satisfying a collection of properties, and, in the case of a simple theory, this ternary relation turns out to be the forking independence relation (denoted by "$\mathop{\smile\hskip-0.9em\raise0.4ex\vert}$").

The second geometric tool in the model theory, which is in the scope of this paper, is the notion of a rank. A notion of rank may be used to characterize dividing lines in the stability hierarchy. A theory is simple if and only if the well-known local rank $D(x = x, \varphi, k)$ (definition will be recalled later) is finite for every choice of a formula φ and every natural number k (cf. Proposition 3.13 in [3]).

These two tools, independence and rank, are nicely related in the case of simple theories: the rank decreases in an extension of types if and only if this extension is a forking extension (Proposition 5.22 in [3]). The aforementioned local rank was also used to develop the theory of generics in simple theories ([13]). Let us see how much of this nice picture from simple theories survives in a more hostile NSOP_1 environment.

Before moving to the definitions, we borrow one more picture. This time it is a table presented by Byunghan Kim in some of his talks on the NSOP_1 theories. The table shows how we pass from a stable theory to its simple and NSOP_1 counterparts. Very elegant. I added two arrows, expressing what is intuitively needed to be added to pass to the next column.

Stable	Simple	NSOP_1
infinite set	random graph	generic binary function
ACF	bounded PAC fields	ω-free PAC fields
vector spaces	vectors spaces with bilinear form over finite field	vector spaces with bilinear form over model of ACF

→ + randomness → + multi-structure

3. Simplicity

In this very short section, we recall the main definitions and facts, which will be basis for their counterparts in the NSOP_1 context.

Consider an $\mathcal{L}$-theory T and its monster model $\mathfrak{C} \models T$. Take an $\mathcal{L}$-formula $\varphi(x, y)$, parameters $a \in \mathfrak{C}^x$ and $b \in \mathfrak{C}^y$, and small sets $A, B \subseteq \mathfrak{C}$.

Definition 3.1. (1) $\varphi(x, b)$ divides over A if there exists an A-indiscernible sequence $(b_i)_{i<\omega}$, with $b_0 = b$, such that the set $\{\varphi(x, b_i) \mid i < \omega\}$ is inconsistent.
(2) $\varphi(x, b)$ forks over A if there are $\psi_1(x), \ldots, \psi_n(x) \in \mathcal{L}(\mathfrak{C})$ such that $\varphi(x, b) \vdash \psi_1(x) \vee \ldots \vee \psi_n(x)$ and each $\psi_i(x)$ divides over A.
(3) A collection of $\mathcal{L}(\mathfrak{C})$-formulas divides [forks] over A if it implies an $\mathcal{L}(\mathfrak{C})$-formula which divides [forks] over A.
(4) (forking independence) $a \mathop{\smile\!\!\!\!\!\!|}_A B$ if $\mathrm{tp}(a/AB)$ does not fork over A.

The above forking independence $\mathop{\smile\!\!\!\!\!\!|}$ is the canonical independence relation in any simple theory. Now, we define the second geometric tool for the class of simple theories, namely the collection of local ranks depending on a finite collection of $\mathcal{L}$-formulas $\Delta = \{\varphi_1(x, y), \ldots, \varphi_n(x, y)\}$ and a natural number $k < \omega$. For any collection of $\mathcal{L}(\mathfrak{C})$-formulas $\pi(x)$, we set:

Definition 3.2.
- $D(\pi, \Delta, k) \geqslant 0$ if $\pi(x)$ is consistent,
- $D(\pi, \Delta, k) \geqslant \alpha + 1$ if for some $j \leqslant n$ there is a sequence $(b_i)_{i<\omega}$ such that the set $\{\varphi_j(x, b_i) \mid i < \omega\}$ is k-inconsistent and
$$D(\pi(x) \cup \{\varphi_j(x, b_i)\}, \Delta, k) \geqslant \alpha$$
for every $i < \omega$.
- for a limit cardinal λ, we set $D(\pi, \Delta, k) \geqslant \lambda$ if $D(\pi, \Delta, k) \geqslant \alpha$ for every $\alpha < \lambda$.

As we already mentioned, notions of independence and rank may be used to characterize dividing lines in the stability hierarchy. The following theorem shows that the above defined notions of independence and rank suit well for this purpose in the case of simple theories.

Theorem 3.3. *The following are equivalent.*
(1) *T is simple.*
(2) *$\mathop{\smile\!\!\!\!\!\!|}$ is symmetric.*
(3) *$D(\{x = x\}, \{\varphi\}, k) < \omega$ for all $\varphi(x, y) \in \mathcal{L}$ and $k < \omega$.*

Also in the case of simple theories, the relation between the independence and rank is very natural as we see in the following fact:

Fact 3.4. *If T is simple, then the following are equivalent:*
(1) *$a \mathop{\smile\!\!\!\!\!\!|}_A B$,*
(2) *$D(\mathrm{tp}(a/AB), \Delta, k) = D(\mathrm{tp}(a/A), \Delta, k)$ for every finite Δ and $k < \omega$.*

Here, the main goal for us is to reproduce Theorem 3.3 and Fact 3.4 in the more general context of NSOP_1 theories by introduction of new notions of independence and rank.

4. NSOP_1

Byunghan Kim introduced a new notion of dividing, where the A-indiscernible sequence from Definition 3.1.(1) is also independent over A with respect to the forking independence. Actually, this will make this sequence a Morley sequence and it turns out that in the context of NSOP_1 theories, passing to Morley sequences is the right move, as in an NSOP_1 theory all the essential data is carried by the

Morley sequences. We will bring some evidence on that in a moment, but we need to clarify a few terms in the first place.

The original idea of Kim was involving Morley sequences understood as in the following definition:

Definition 4.1 (Morley sequence in a type)**.** Let $p(x) \in S(A)$. We say that $(b_i)_{i<\omega}$ is a Morley sequence in p if

(1) $b_i \models p$ for every $i < \omega$,
(2) $(b_i)_{i<\omega}$ is A-indiscernible,
(3) $b_i \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}_A b_{<i}$ for every $i < \omega$.

In the recent papers on Kim's notions of dividing and independence ([4], [7]), we observe that Kim-dividing is defined in accordance with the above definition of Morley sequences. However, in the milestone paper in the subject, [10], the notion of Kim-dividing was defined a bit different. More precisely, the following, stronger, notion of a Morley sequence was involved:

Definition 4.2 (Morley sequence in a global type)**.** Let $q(y) \in S(\mathfrak{C})$ be A-invariant. We say that $(b_i)_{i<\omega}$ is a Morley sequence in q over A if $(b_i)_{i<\omega} \models q^{\otimes\omega}|_A$, i.e. if for every $i < \omega$ we have that $b_i \models q|_{Ab_{<i}}$.

To simplify things, we will use the second definition and just note that both notions of Kim-dividing (depending on the two above notions of a Morley sequence) coincide over models in the NSOP$_1$ context. Let us finally approach the aforementioned definition of dividing relevant for the NSOP$_1$.

Definition 4.3. (1) $\varphi(x, b)$ Kim-divides over A if there exists an A-invariant $q(y) \in S(\mathfrak{C})$ and a sequence $(b_i)_{i<\omega} \models q^{\otimes\omega}|_A$, with $b_0 = b$, such that the set $\{\varphi(x, b_i) \mid i < \omega\}$ is inconsistent.

(2) $\varphi(x, b)$ Kim-forks over A if there are $\psi_1(x), \ldots, \psi_n(x) \in \mathcal{L}(\mathfrak{C})$ such that $\varphi(x, b) \vdash \psi_1(x) \vee \ldots \vee \psi_n(x)$ and each $\psi_i(x)$ Kim-divides over A.
(3) A collection of $\mathcal{L}(\mathfrak{C})$-formulas Kim-divides [Kim-forks] over A if it implies an $\mathcal{L}(\mathfrak{C})$-formula which Kim-divides [Kim-forks] over A.
(4) (Kim-independence) $a \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^K_A B$ if $\mathrm{tp}(a/AB)$ does not Kim-fork over A.

The original definition of the Strict Order Property of the first kind is a bit technical and of a combinatorial nature. Instead of recalling the definition, we provide the following fact, partially a counterpart of Theorem 3.3 above, which describes the NSOP$_1$ in an elegant way:

Theorem 4.4. *The following are equivalent.*

(1) *T is NSOP$_1$ (does not have the Strict Order Property of the first kind).*
(2) *$\mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^K$ is symmetric over models (i.e. for every $a, b \in \mathfrak{C}$ and $M \preceq \mathfrak{C}$ we have that $a \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^K_M b$ implies $b \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^K_M a$).*

As you can see, in Theorem 3.3, there was one more way of describing simple theories. Namely, by finiteness of a family of local ranks. One could ask whether there is a family of local ranks such that finiteness of these ranks corresponds exactly to being an NSOP$_1$ theory? The answer is positive, and such a family of ranks was defined in [4]. More precisely, it was shown in [4] that if the theory is NSOP$_1$ and satisfies the existence axiom for the forking independence, then each rank from the family of local ranks from [4] is finite. The counterpart, i.e. if the theory satisfies the existence axiom for the forking independence and each of the ranks defined in [4] is finite then the theory is NSOP_1, was discussed with Byunghan Kim in a private communication.

Therefore to some extent, we can recover Theorem 3.3 in the NSOP_1 context for the family of ranks from [4]. The next natural question is about translating Fact 3.4 into the NSOP_1 context for the ranks defined in [4]. However, here things are quite open and a counterpart for Fact 3.4 is unknown (cf. Question 4.9 in [4]). One of the reasons for the rank, which will be introduced in a moment, is to approximate Fact 3.4 for NSOP_1 theories. Let us see how it works and what is needed.

5. Rank

The idea for the following rank was in some way motivated by Hans Adler's doctoral dissertation ([1]). Adler defines so called *dividing patterns* and then defines a local rank measuring the length of a maximal dividing pattern. We adapt his idea to the context of Kim-dividing, in other words we propose our own variation on *Kim-dividing patterns*. The main difference is that every sequence (horizontal and vertical in the picture below) behind our notion of rank must be a Morley sequence. As we are working with the stronger notion of a Morley sequence, we need to start with fixing some global types. Let

$$Q := \big((\varphi_0(x;y_0), q_0(y_0)), \ldots, (\varphi_{n-1}(x;y_{n-1}), q_{n-1}(y_{n-1}))\big),$$

where $\varphi_0, \ldots, \varphi_{n-1} \in \mathcal{L}$ and $q_0, \ldots, q_{n-1}$ are global types.

Definition 5.1. We define a local rank, called *Q-rank*,

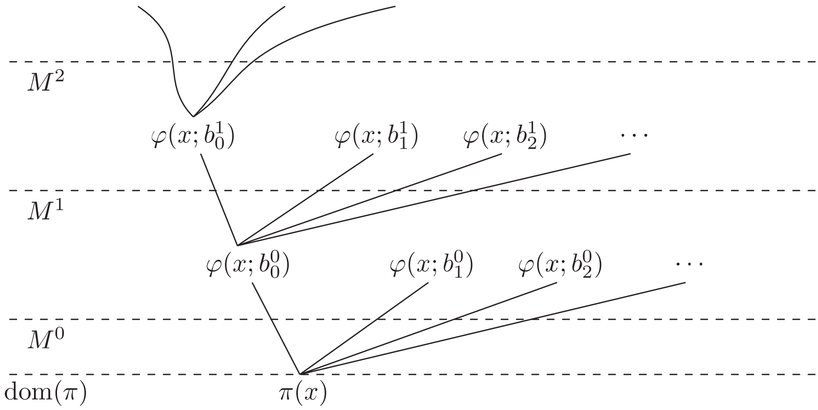
$$D_Q(\,\cdot\,) : \{\text{sets of formulae}\} \to \mathrm{Ord} \cup \{\infty\}.$$

For any set of $\mathcal{L}$-formulae $\pi(x)$ we have $D_Q(\pi(x)) \geqslant \lambda$ if and only if there exists $\eta \in n^\lambda$ and $(b^\alpha, M^\alpha)_{\alpha<\lambda}$ such that

1. $\mathrm{dom}(\pi(x)) \subseteq M^0$,
2. $q_0, \ldots, q_{n-1}$ are M^0-invariant,
3. $M^\alpha \preceq \mathfrak{C}$ for each $\alpha < \lambda$, $(M^\alpha)_{\alpha<\lambda}$ is continuous, and each $M^{\alpha+1}$ is $|M^\alpha|^+$-saturated and strongly $|M^\alpha|^+$-homogeneous.
4. $b^\alpha M^\alpha \subseteq M^{\alpha+1}$ for each $\alpha + 1 < \lambda$,
5. $b^\alpha \models q_{\eta(\alpha)}|_{M^\alpha}$ for each $\alpha < \lambda$,
6. $\pi(x) \cup \{\varphi_{\eta(\alpha)}(x; b^\alpha) \mid \alpha < \lambda\}$ is consistent,
7. each $\varphi_{\eta(\alpha)}(x; b^\alpha)$ Kim-divides over M^α, i.e.: for each $\alpha < \lambda$ there exists an M^α-invariant global type $r_\alpha(y_{\eta(\alpha)})$ extending $\mathrm{tp}(b^\alpha/M^\alpha)$ and $\bar{b}^\alpha = (b^\alpha_i)_{i<\omega} \models r_\alpha^{\otimes\omega}|_{M^\alpha}$ such that $b^\alpha_0 = b^\alpha$ and $\{\varphi_{\eta(\alpha)}(x; b^\alpha_i) \mid i < \omega\}$ is inconsistent.

If $D_Q(\pi) \geqslant \lambda$ for each $\lambda \in \mathrm{Ord}$, then we set $D_Q(\pi) = \infty$. Otherwise $D_Q(\pi)$ is the maximal $\lambda \in \mathrm{Ord}$ such that $D_Q(\pi) \geqslant \lambda$.

The above, quite complicated, definition can explained as constructing a tree build with instances of formulas φ_i's along proper Morley sequences. Let us explain this concept. For simplicity we assume that $Q = ((\varphi, q))$. Then the witnesses from the definition of $D_Q(\pi) \geqslant \lambda$, $(M^\alpha, b^\alpha_i)_{\alpha<\lambda, i<\omega}$, from the last point of this definition, may be used to draw the following tree ([6]):



Each horizontal sequence of b_i^α's is a Morley sequence in some global M^α-invariant type and so witnesses Kim-dividing of $\varphi(x; b_0^\alpha)$ over M^α. Such an approach was also present in the local ranks from [4] and this is standard.

The first new ingredient in our rank is that we require that also the leftmost branch in our tree forms a Morley sequence (and then via automorphisms we can extend all branches of the tree and notice that they will be also based on Morley sequences), this time in the previously fixed global type q (which is M^0-invariant). In other words, we focus only on Morley sequences - and that is in accordance with the intuition that all the essential data in a NSOP$_1$ theory is coded by Morley sequences. This intuition is based on the following fundamental theorem:

Theorem 5.2 (Theorem 8.1 in [10]). *The following are equivalent.*

(1) *T is NSOP$_1$.*
(2) Kim's lemma for Kim-dividing*: For any $M \preceq \mathfrak{C}$ and any $\varphi(x; b)$, if $\varphi(x; b)$ Kim-divides over M with respect to* ***some*** *M-invariant $q(y) \in S(\mathfrak{C})$ with* $\mathrm{tp}(b/M) \subseteq q(y)$*, then $\varphi(x; b)$ Kim-divides over M with respect to* ***any*** *M-invariant $q(y) \in S(\mathfrak{C})$ with* $\mathrm{tp}(b/M) \subseteq q(y)$.

The second new ingredient in our rank is that we allow "jumps" in the extension of the base parameters between levels. More precisely, instead of the sequence $\mathrm{dom}(\pi) \subseteq M^0 \preceq M^1 \preceq \ldots$, we could consider a more standard sequence $\mathrm{dom}(\pi) \subseteq \mathrm{dom}(\pi)b_0^0 \subseteq \mathrm{dom}(\pi)b_0^0 b_0^1 \subseteq \ldots$. However, let us recall that $\mathop{\smile\!\!\!\!\!|}^K$ does not necessarily satisfy the base monotonicity axiom (which is one of the main obstacles showing up after passing from simple to NSOP$_1$ theories), thus we allow in our rank some freedom in choosing the parameters over which each next level Kim-divides.

Coming back to the general NSOP$_1$ situation, the above rank has the standard properties expected from a rank. It is also finite for any NSOP$_1$ theory, but it might be also finite outside of the NSOP$_1$ context. This is somehow unpredicted, but we obtained that a smooth modification of the above defined rank (cf. [6]) is bounded by the inp-rank (i.e. the burden) and the dp-rank. This makes it more interesting. Recall that the inp-rank is well-defined if and only if the theory is NTP$_2$, and the dp-rank is well-defined if and only if the theory is NIP. Thus we wonder what is the class of theories corresponding to well-definedeness of the aforementioned slight modification of our rank. One could conjecture that it is related to the class of NATP theories appearing in [2].

Now, let us evoke two lemmas, which are in the spirit of Fact 3.4, this time for NSOP_1 theories and our notion of rank.

Lemma 5.3. *Assume that T is $NSOP_1$ and $M \preceq N \preceq \mathfrak{C}$. If for every $Q = ((\varphi, q))$ with q being M-invariant we have that $D_Q(\mathrm{tp}(a/M)) = D_Q(\mathrm{tp}(a/N))$, then $a \mathop{\smile\!\!\!\!\!\!|}\,^K_M N$.*

Lemma 5.4. *Now, assume that T is $NSOP_1$ with existence and that $M \preceq N \preceq \mathfrak{C}$. Consider $Q = ((\varphi, q))$ with q being M-invariant and such that $q^{\otimes\omega}|_M$ is Kim-stationary (i.e. has a unique Kim-nonforking extension over every overset). If $a \mathop{\smile\!\!\!\!\!\!|}\,^K_M N$ then $D_Q(\mathrm{tp}(a/M)) = D_Q(\mathrm{tp}(a/N))$.*

Moreover, we know that the assumption about Kim-stationarity in the above lemma is crucial and can not be relaxed.

6. Generics

The idea of generic elements in model theory has its origins in the algebraic geometry. Roughly speaking, we try to find one point in a given affine variety, which locus is equal to the variety itself. Proving existence of such elements usually involves an argument on the maximality of the dimension. In model theory, generics are mostly considered if there is a definable group. Recall that T is an L-theory with monster model $\mathfrak{C}$, let $A \subseteq B \subseteq \mathfrak{C}$ be small subsets, and let G be an A-definable group in $\mathfrak{C}$. As we already mentioned, the natural way of defining generics may involve dimension. However, this is not always possible due to the lackness of the canonical notion of dimension in some first order theories. Another notion measuring objects in model theory is the notion of dividing and related notion of the forking independence, or in general any canonical notion of independence.

Definition 6.1. Assume that $\mathop{\smile\!\!\!\!\!\!|}\,^*$ is an invariant ternary relation. We say that $g \in G(\mathfrak{C})$ is (left) $\mathop{\smile\!\!\!\!\!\!|}\,^*$-generic over B if for each $h \in G(\mathfrak{C})$, $g \mathop{\smile\!\!\!\!\!\!|}\,^*_B h$ implies that $h \cdot g \mathop{\smile\!\!\!\!\!\!|}\,^*_A B \cup \{h\}$. We say that $p(x) \in S(B)$ is (left) $\mathop{\smile\!\!\!\!\!\!|}\,^*$-generic if there exists $g \models p$, $g \in G(\mathfrak{C})$, such that g is (left) $\mathop{\smile\!\!\!\!\!\!|}\,^*$-generic over B.

The main issue with generics related to a notion of an independence is that they may not exist for a given first order theory. That is the case for $\mathop{\smile\!\!\!\!\!\!|}\,^K$-generics in the theory of infinitely dimensional vector space with bilinear form over an algebraically closed field (cf. [5]). On the other hand, in this theory there is a good notion of dimension and arising from it Γ-independence, denoted by $\mathop{\smile\!\!\!\!\!\!|}\,^\Gamma$, and related $\mathop{\smile\!\!\!\!\!\!|}\,^\Gamma$-generics exist. Following this intuition, we would like to have - in a general NSOP_1 theory - a notion of rank (a substitute of the notion of dimension) which is related to $\mathop{\smile\!\!\!\!\!\!|}\,^K$. Then we could reprove existence of $\mathop{\smile\!\!\!\!\!\!|}\,^K$-generics under extra assumption required by the rank. In this way, existence of $\mathop{\smile\!\!\!\!\!\!|}$-generics was proved for arbitrary simple theory in [13]. More precisely, Fact 3.7 there lists three crucial properties of a rank, which are enough to proceed with the proof: the local ranks need to be finite, related to the notion of independence and in some sense invariant under the shifts by elements of G. In the case of the local ranks from [4] and the theory of infinite dimensional vectors spaces with bilinear form over an algebraically closed field, the first two properties hold (cf. [6]), but this is still not enough to obtain generics. We hope that the rank introduced above will put more light on how to modify the proof of existence of generics for the simple theories and obtain a proof for NSOP_1 theories enjoying additional assumptions obtained by studying our rank.

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