Triplet of Fibonacci Duals

— with or without constraint —

Seiichi Iwamoto
Professor emeritus, Kyushu University

Yutaka Kimura
Department of Management Science and Engineering
Faculty of Systems Science and Technology
Akita Prefectural University

Abstract

We consider a dual relation between minimization (primal) problem and maximization (dual) problem from a view point of complementarity. An identity

$$\text{(CI)} \quad \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_{n-1}-x_n)\mu_n + x_n\mu_n \;=\; x_0\mu_1$$

is called *complementary* [20,22]. We present three types of complementary identities, which take a fundamental role in analyzing respective pairs of primal and dual. Moreover, we show that a primal and its dual satisfy *Fibonacci Complementary Duality* [18, 19, 21, 22].

1 Introduction

A wide class of quadratic optimization problems has been discussed by Bellman and others [1–12, 23]. Dynamic programming has solved its partial class [2, 17, 18, 26]. Further a dual approach has been treated based upon convex-concavity [14, 16, 25].

Recently some new dual approaches — plus-minus method, extended Lagrangean method, ineqlualty method and others — have been derived in [18–22]. In this paper, we propose a complementary duality based upon an identity.

2 Complementary identities

Let $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ be any two sequences of real number with $x_0 = c$. Then an identity

$$\text{(C}_1\text{)} \quad c\mu_1 \;=\; \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_{n-1}-x_n)\mu_n + x_n\mu_n$$

holds true. This identity is called *complementary* [20,22]. Further we assume that $\mu_n = 0$. Then an identity

$$\text{(C}_2\text{)} \quad c\mu_1 \;=\; \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_{n-1}-x_n)\mu_n$$

holds true. This is a *conditional complementarity*.

On the other hand, we assume that $x_n = 0$. Then an identity

$$(\mathrm{C}_3) \quad c\mu_1 \;=\; \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k - \mu_{k+1})] + (x_{n-1}-x_n)\mu_n$$

holds true. This is also a *conditional complementarity*.

3 Three pairs

We consider three pairs of minimization (primal) problems and maximization (dual) problems, which are (P_1) vs (D_1), (P_2) vs (D_2) and (P_3) vs (D_3). It is shown that each pair is dual to each other. It turns out that the duality is based upon the complementary identity and an elementary inequality with equality

$$2xy \;\le\; x^2 + y^2 \quad \text{on} \;\; R^2 \;;\;\; x = y. \tag{1}$$

Both the primal (P_1) and the dual (D_1) are *unconditional*. The primal (P_2) is unconditional, while the dual (D_2) is *conditional* on μ_n. The primal (P_3) is conditional on x_n, while the dual (D_3) is unconditional.

3.1 (P_1) vs (D_1)

Let us consider the first pair:

$$(\mathrm{P}_1) \quad \begin{array}{l} \text{minimize} \;\; \displaystyle\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 + x_n^2 \\ \text{subject to} \quad \text{(i)} \;\; x \in R^n, \quad \text{(ii)} \;\; x_0 = c, \end{array}$$

$$(\mathrm{D}_1) \quad \begin{array}{l} \text{Maximize} \;\; 2c\mu_1 - \displaystyle\sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2 - \mu_n^2 \\ \text{subject to} \quad \text{(i)} \quad \mu \in R^n. \end{array}$$

An identity (C_1) with the elementary inequality (1) yields an inequality

$$\begin{aligned} & 2c\mu_1 - \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2 - \mu_n^2 \\ \le\; & \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 + x_n^2 \end{aligned}$$

for any feasible pair $(x,\ \mu)$. Then it turns out that both are dual to each other. An equality condition is

$$(\mathrm{EC}_1)\qquad \begin{aligned} x_{k-1}-x_k &= \mu_k, \quad x_k = \mu_k-\mu_{k+1} \quad 1\le k\le n-1 \\ x_{n-1}-x_n &= \mu_n, \quad x_n = \mu_n. \end{aligned}$$

The equality condition (EC_1) is a linear system of $2n$-equation on $2n$-variable (x,μ). Let (x,μ) be a solution. Then both sides become a common value with five expressions.

$$(5\mathrm{V}_1)\qquad \begin{aligned} &\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2+x_k^2\right]+(x_{n-1}-x_n)^2+x_n^2 \\ =\ & c(c-x_1) \\ =\ & 2c\mu_1-\sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]-\mu_n^2-\mu_n^2 \\ =\ & \sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]+\mu_n^2+\mu_n^2 \\ =\ & c\mu_1. \end{aligned}$$

Let (x,μ) be a solution of (EC_1). Then the primal (P_1) has a minimum value

$$\begin{aligned} m_1 &= \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2+x_k^2\right]+(x_{n-1}-x_n)^2+x_n^2 \\ &= c(c-x_1) \end{aligned}$$

at x, while the dual (D_1) has a maximum value

$$\begin{aligned} M_1 &= 2c\mu_1-\sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]-\mu_n^2-\mu_n^2 \\ &= \sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]+\mu_n^2+\mu_n^2 \\ &= c\mu_1 \end{aligned}$$

at μ.

Lemma 1 (EC_1) *has indeed a unique solution:*

$$\begin{aligned} & x = (x_1,\ x_2,\ \ldots,\ x_k,\ \ldots,\ x_{n-1},\ x_n) \\ =\ & \frac{c}{F_{2n+1}}(F_{2n-1},\ F_{2n-3},\ \ldots, F_{2n-2k+1},\ \ldots,\ F_3,\ F_1), \end{aligned} \tag{2}$$

$$\begin{aligned} & \mu = (\mu_1,\ \mu_2,\ \ldots,\ \mu_k,\ \ldots,\ \mu_{n-1},\ \mu_n) \\ =\ & \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-2},\ \ldots, F_{2n-2k+2},\ \ldots,\ F_4,\ F_2). \end{aligned} \tag{3}$$

Proof. From (EC_1), we have a pair of linear systems of n-variable on n-equation:

$$
(\mathrm{EQ}_1) \qquad
\begin{aligned}
c &= 3x_1 - x_2 & c &= 2\mu_1 - \mu_2 \\
x_1 &= 3x_2 - x_3 & \mu_1 &= 3\mu_2 - \mu_3 \\
&\vdots & &\vdots \\
x_{n-2} &= 3x_{n-1} - x_n & \mu_{n-2} &= 3\mu_{n-1} - \mu_n \\
x_{n-1} &= 2x_n & \mu_{n-1} &= 3\mu_n.
\end{aligned}
$$

The left system has a solution x in (2), while the right has a solution μ in (3). □

The primal (P_1) has a minimum value $m_1 = c(c - \hat{x}_1) = \dfrac{F_{2n}}{F_{2n+1}}c^2$ at a path

$$
\begin{aligned}
\hat{x} &= (\hat{x}_1,\ \hat{x}_2,\ \ldots,\ \hat{x}_k,\ \ldots,\ \hat{x}_{n-1},\ \hat{x}_n) \\
&= \frac{c}{F_{2n+1}}(F_{2n-1},\ F_{2n-3},\ \ldots, F_{2n-2k+1},\ \ldots,\ F_3,\ F_1).
\end{aligned}
$$

The dual (D_1) has a maximum value $M_1 = c\mu_1^* = \dfrac{F_{2n}}{F_{2n+1}}c^2$ at a path

$$
\begin{aligned}
\mu^* &= (\mu_1^*,\ \mu_2^*,\ \ldots,\ \mu_k^*,\ \ldots,\ \mu_{n-1}^*,\ \mu_n^*) \\
&= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-2},\ \ldots, F_{2n-2k+2},\ \ldots,\ F_4,\ F_2)
\end{aligned}
$$

where $\{F_n\}$ is the *Fibonacci sequence* [13, 15, 24, 27]. This is defined as the solution to the second-order linear difference equation

$$x_{n+2} - x_{n+1} - x_n = 0, \qquad x_1 = 1,\ x_0 = 0. \tag{4}$$

n	$\cdots$	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
F_n	$\cdots$	-1	1	0	1	1	2	3	5	8	13	21	34	55	89

n	12	13	14	15	16	17	18	19	20	$\cdots$
F_n	144	233	377	610	987	1597	2584	4181	6765	$\cdots$

Table 1 Fibonacci sequence $\{F_n\}$

Hence both optimal values are identical:

$$m_1 = M_1 = \frac{F_{2n}}{F_{2n+1}}c^2.$$

An alternate contexture of both optimal points $\mu^*, \hat{x}$ is Fibonacci backward:

$$
\begin{aligned}
&(\mu_1^*,\ \hat{x}_1,\ \mu_2^*,\ \hat{x}_2,\ \ldots,\ \mu_k^*,\ \hat{x}_k\ \ldots,\ \mu_{n-1}^*,\ \hat{x}_{n-1},\ \mu_n^*,\ \hat{x}_n) \\
= &\frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ F_{2n-2},\ F_{2n-3},\ \ldots, F_{2n-2k+2},\ F_{2n-2k+1},\ \ldots,\ F_4,\ F_3,\ F_2,\ F_1).
\end{aligned}
$$

Thus *Fibonacci Complementary Duality* (FCD) [18, 19, 21, 22] holds between (P_1) and (D_1).

3.2 (P_2) vs (D_2)

Let us consider the second:

$$(\mathrm{P}_2)\qquad \begin{aligned} &\text{minimize} \quad \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2+x_k^2\right]+(x_{n-1}-x_n)^2 \\ &\text{subject to} \quad \text{(i)}\ \ x\in R^n, \quad \text{(ii)}\ \ x_0=c \end{aligned}$$

$$(\mathrm{D}_2)\qquad \begin{aligned} &\text{Maximize}\quad 2c\mu_1-\sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]-\mu_n^2 \\ &\text{subject to} \quad \text{(i)}\ \ \mu\in R^n, \quad \text{(ii)}\ \ \underline{\mu_n=0}. \end{aligned}$$

An identity (C_2) with the elementary inequality (1) yields an inequality

$$\begin{aligned} &2c\mu_1-\sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]-\mu_n^2 \\ &\le \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2+x_k^2\right]+(x_{n-1}-x_n)^2 \end{aligned}$$

for any feasible pair $(x,\,\mu)$. Then both are dual to each other. An equality condition is

$$(\mathrm{EC}_2)\qquad \begin{aligned} &x_{k-1}-x_k\ =\ \mu_k,\quad x_k\ =\ \mu_k-\mu_{k+1}\quad 1\le k\le n-1 \\ &x_{n-1}-x_n\ =\ \mu_n. \end{aligned}$$

The equality condition (EC_2) is a linear system of $(2n-1)$-equation on $2n$-variable (x,μ). Let (EC_2') be an augmentation of the system (EC_2) with the additional constraint (ii) $\underline{\mu_n=0}$:

$$(\mathrm{EC}_2')\qquad \begin{aligned} &x_{k-1}-x_k\ =\ \mu_k,\quad x_k\ =\ \mu_k-\mu_{k+1}\quad 1\le k\le n-1 \\ &x_{n-1}-x_n\ =\ \mu_n,\quad \underline{\mu_n=0}. \end{aligned}$$

Then (EC_2') is of $2n$-equation on $2n$-variable.

Let (x,μ) be a solution of (EC_2'). Then both sides become a common value with five expressions.

$$(5\mathrm{V}_2)\qquad \begin{aligned} &\sum_{k=1}^{n}\left[(x_{k-1}-x_k)^2+x_k^2\right]+(x_{n-1}-x_n)^2 \\ &=\ c(c-x_1) \\ &=\ 2c\mu_1-\sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]-\mu_n^2 \\ &=\ \sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]+\mu_n^2 \\ &=\ c\mu_1. \end{aligned}$$

The primal (P_2) has a minimum value

$$
\begin{aligned}
m_2 &= \sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 + x_k^2\right] + (x_{n-1} - x_n)^2 \\
&= c(c - x_1)
\end{aligned}
$$

at x, while the dual (D_2) has a maximum value

$$
\begin{aligned}
M_2 &= 2c\mu_1 - \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2 \\
&= \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] + \mu_n^2 \\
&= c\mu_1
\end{aligned}
$$

at μ.

Lemma 2 *The system* (EC_2') *has indeed a unique solution:*

$$
\begin{aligned}
& x = (x_1,\ x_2,\ \ldots,\ x_k,\ \ldots,\ x_{n-1},\ x_n) \\
&= \frac{c}{F_{2n-1}}(F_{2n-3},\ F_{2n-5},\ \ldots, F_{2n-2k-1},\ \ldots,\ F_1,\ F_{-1}), \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
& \mu = (\mu_1,\ \mu_2,\ \ldots,\ \mu_k,\ \ldots,\ \mu_{n-1},\ \mu_n) \\
&= \frac{c}{F_{2n-1}}(F_{2n-2},\ F_{2n-4},\ \ldots, F_{2n-2k},\ \ldots,\ F_2,\ F_0). \qquad (6)
\end{aligned}
$$

Proof. From (EC_2'), we have a pair of linear systems of n-variable on n-equation:

$$
(\mathrm{EQ}_2) \quad
\begin{aligned}
c &= 3x_1 - x_2 & \qquad c &= 2\mu_1 - \mu_2 \\
x_1 &= 3x_2 - x_3 & \mu_1 &= 3\mu_2 - \mu_3 \\
&\ \vdots & &\ \vdots \\
x_{n-3} &= 3x_{n-2} - x_{n-1} & \mu_{n-3} &= 3\mu_{n-2} - \mu_{n-1} \\
x_{n-2} &= 3x_{n-1} - x_n & \mu_{n-2} &= 3\mu_{n-1} - \mu_n \\
x_{n-1} &= x_n & \underline{\mu_n} &\ \underline{= 0}.
\end{aligned}
$$

The left system has a solution x in (5), while the right has a solution μ in (6). □

The primal (P_2) has a minimum value $m_2 = c(c - \hat{x}_1) = \dfrac{F_{2n-2}}{F_{2n-1}}c^2$ at a path

$$
\begin{aligned}
& \hat{x} = (\hat{x}_1,\ \hat{x}_2,\ \ldots,\ \hat{x}_k,\ \ldots,\ \hat{x}_{n-1},\ \hat{x}_n) \\
&= \frac{c}{F_{2n-1}}(F_{2n-3},\ F_{2n-5},\ \ldots, F_{2n-2k-1},\ \ldots,\ F_1,\ F_{-1}).
\end{aligned}
$$

The dual (D_2) has a maximum value $M_2 = c\mu_1^* = \dfrac{F_{2n-2}}{F_{2n-1}}c^2$ at a path

$$\begin{aligned}\mu^* &= (\mu_1^*,\ \mu_2^*,\ \ldots,\ \mu_k^*,\ \ldots,\ \mu_{n-1}^*,\ \mu_n^*)\\ &= \frac{c}{F_{2n-1}}(F_{2n-2},\ F_{2n-4},\ \ldots, F_{2n-2k},\ \ldots,\ F_2,\ F_0).\end{aligned}$$

Hence both optimal values are identical:

$$m_2 = M_2 = \frac{F_{2n-2}}{F_{2n-1}}c^2.$$

An alternate contexture of both optimal points $\mu^*, \hat{x}$ is Fibonacci backward:

$$\begin{aligned}&(\mu_1^*,\ \hat{x}_1,\ \mu_2^*,\ \hat{x}_2,\ \ldots,\ \mu_k^*,\ \hat{x}_k\ \ldots,\ \mu_{n-1}^*,\ \hat{x}_{n-1},\ \mu_n^*,\ \hat{x}_n)\\ &= \frac{c}{F_{2n-1}}(F_{2n-2},\ F_{2n-3},\ F_{2n-4},\ F_{2n-5},\ \ldots, F_{2n-2k},\ F_{2n-2k-1},\ \ldots,\ F_2,\ F_1,\ F_0,\ F_{-1}).\end{aligned}$$

Thus FCD holds between (P_2) and (D_2).

3.3 (P_3) vs (D_3)

Let us consider the third:

$$(\mathrm{P}_3)\qquad \begin{array}{ll}\text{minimize} & \displaystyle\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2\\ \text{subject to} & \text{(i)}\ \ x \in R^n, \quad \text{(ii)}\ \ x_0 = c,\ \underline{x_n = 0}\end{array}$$

$$(\mathrm{D}_3)\qquad \begin{array}{ll}\text{Maximize} & \displaystyle 2c\mu_1 - \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2\\ \text{subject to} & \text{(i)}\ \ \mu \in R^n.\end{array}$$

An identity (C_3) with the elementary inequality (1) yields an inequality

$$\begin{aligned}&2c\mu_1 - \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2\\ &\le \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2\end{aligned}$$

for any feasible pair (x, μ). Then both are dual to each other. An equality condition is

$$(\mathrm{EC}_3)\qquad \begin{array}{l} x_{k-1} - x_k = \mu_k, \quad x_k = \mu_k - \mu_{k+1} \quad 1 \le k \le n-1\\ x_{n-1} - x_n = \mu_n.\end{array}$$

The equality condition (EC_3) is a linear system of $(2n-1)$-equation on $2n$-variable (x,μ). Let (EC_3') be an augmentation of the system (EC_3) with the additional constraint (ii) $\underline{x_n = 0}$:

$$(EC_3') \qquad \begin{aligned} x_{k-1} - x_k &= \mu_k, \quad x_k = \mu_k - \mu_{k+1} \quad 1 \le k \le n-1 \\ x_{n-1} - x_n &= \mu_n, \quad \underline{x_n = 0}. \end{aligned}$$

Then (EC_3') is of $2n$-equation on $2n$-variable.

Let (x,μ) be a solution of (EC_3'). Then both sides become a common value with five expressions.

$$(5V_3) \qquad \begin{aligned} &\sum_{k=1}^{n} \left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 \\ &= c(c-x_1) \\ &= 2c\mu_1 - \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2 \\ &= \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] + \mu_n^2 \\ &= c\mu_1. \end{aligned}$$

The primal (P_3) has a minimum value

$$\begin{aligned} m_3 &= \sum_{k=1}^{n} \left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 \\ &= c(c-x_1) \end{aligned}$$

at x, while the dual (D_3) has a maximum value

$$\begin{aligned} M_3 &= 2c\mu_1 - \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2 \\ &= \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] + \mu_n^2 \\ &= c\mu_1 \end{aligned}$$

at μ.

Lemma 3 *The system* (EC_3') *has indeed a unique solution:*

$$\begin{aligned} x &= (x_1,\ x_2,\ \ldots,\ x_k,\ \ldots,\ x_{n-1},\ x_n) \\ &= \frac{c}{F_{2n}}(F_{2n-2},\ F_{2n-4},\ \ldots, F_{2n-2k},\ \ldots,\ F_2,\ F_0), \end{aligned} \tag{7}$$

$$\begin{aligned} \mu &= (\mu_1,\ \mu_2,\ \ldots,\ \mu_k,\ \ldots,\ \mu_{n-1},\ \mu_n) \\ &= \frac{c}{F_{2n}}(F_{2n-1},\ F_{2n-3},\ \ldots, F_{2n-2k+1},\ \ldots,\ F_3,\ F_1). \end{aligned} \tag{8}$$

Proof. From (EC'_3), we have a pair of linear systems of n-variable on n-equation:

$$
(EQ_3) \quad
\begin{aligned}
c &= 3x_1 - x_2 \\
x_1 &= 3x_2 - x_3 \\
&\vdots \\
x_{n-3} &= 3x_{n-2} - x_{n-1} \\
x_{n-2} &= 3x_{n-1} - x_n \\
\underline{x_n} &= \underline{0}
\end{aligned}
\qquad
\begin{aligned}
c &= 2\mu_1 - \mu_2 \\
\mu_1 &= 3\mu_2 - \mu_3 \\
&\vdots \\
\mu_{n-3} &= 3\mu_{n-2} - \mu_{n-1} \\
\mu_{n-2} &= 3\mu_{n-1} - \mu_n \\
\mu_{n-1} &= 2\mu_n.
\end{aligned}
$$

The left system has a solution x in (7), while the right has a solution μ in (8). □

The primal (P_3) has a minimum value $m_3 = c(c - \hat{x}_1) = \dfrac{F_{2n-1}}{F_{2n}}c^2$ at a path

$$
\begin{aligned}
\hat{x} &= (\hat{x}_1,\ \hat{x}_2,\ \ldots,\ \hat{x}_k,\ \ldots,\ \hat{x}_{n-1},\ \hat{x}_n) \\
&= \frac{c}{F_{2n}}(F_{2n-2},\ F_{2n-4},\ \ldots, F_{2n-2k},\ \ldots,\ F_2,\ F_0).
\end{aligned}
$$

The dual (D_3) has a maximum value $M_3 = c\mu_1^* = \dfrac{F_{2n-1}}{F_{2n}}c^2$ at a path

$$
\begin{aligned}
\mu^* &= (\mu_1^*,\ \mu_2^*,\ \ldots,\ \mu_k^*,\ \ldots,\ \mu_{n-1}^*,\ \mu_n^*) \\
&= \frac{c}{F_{2n}}(F_{2n-1},\ F_{2n-3},\ \ldots, F_{2n-2k+1},\ \ldots,\ F_3,\ F_1).
\end{aligned}
$$

Hence both optimal values are identical:

$$
m_3 = M_3 = \frac{F_{2n-1}}{F_{2n}}c^2.
$$

An alternate contexture of both optimal points $\mu^*, \hat{x}$ is Fibonacci backward:

$$
\begin{aligned}
&(\mu_1^*,\ \hat{x}_1,\ \mu_2^*,\ \hat{x}_2,\ \ldots,\ \mu_k^*,\ \hat{x}_k\ \ldots,\ \mu_{n-1}^*,\ \hat{x}_{n-1},\ \mu_n^*,\ \hat{x}_n) \\
= &\frac{c}{F_{2n}}(F_{2n-1},\ F_{2n-2},\ F_{2n-3},\ F_{2n-4},\ \ldots, F_{2n-2k+1},\ F_{2n-2k},\ \ldots,\ F_3,\ F_2,\ F_1,\ F_0).
\end{aligned}
$$

Thus FCD holds between (P_3) and (D_3).

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