

TWO INTEGRAL REPRESENTATIONS FOR APÉRY CONSTANT AND ITS APPLICATIONS TO MULTIPLE ZETA VALUES

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ABSTRACT. We generalize the proof of Basel problem by Boo Rim Choe (1987) to obtain two integral representations for Apéry constant. As applications, we also show integral representations for multiple values $\zeta(3, 2, \dots, 2)$ and $t(3, 2, \dots, 2)$.

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1. INTRODUCTION

1.1. **Basel Problem: arcsin and $\zeta(2)$.** Let ζ denote the Riemann zeta function. Boo Rim Choe (1987) [2] gave evaluation of the integral

$$(1) \quad \frac{\pi^2}{8} = \int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}} dx = \frac{3}{4}\zeta(2)$$

which provides another proof of *Basel problem* $\zeta(2) = \frac{\pi^2}{6}$ (dating back to Euler around 1735). Actually, there is a counterpart of this:

$$(2) \quad \frac{2}{\pi} \int_0^1 \frac{\arcsin^2 x}{2!} \frac{dx}{\sqrt{1-x^2}} = \frac{1}{4}\zeta(2)$$

as the even part of $\zeta(2)$; we will explain why this equation involves $2/\pi$ and $\arcsin^2 x/2!$ later on. The aim of this article is to show analogous integral evaluation

$$(3) \quad \int_0^1 \frac{\arcsin x \arccos x}{x} dx = \frac{7}{8}\zeta(3),$$

$$(4) \quad \frac{2}{\pi} \int_0^1 \frac{\arcsin^2 x \arccos x}{2!} \frac{dx}{x} = \frac{1}{8}\zeta(3)$$

and discuss its applications to central binomial series, multiple values shown as Theorems 2.8, 2.13. We also make two Conjectures 2.19 and 3.3.

1.2. **Central binomial sums.** One of important topics in number theory is *central binomial sums*. Informally speaking, it is an infinite series involving $\binom{2n}{n}$. Lehmer [10] discussed two types of such sums

$$\text{I. } \sum_{n=0}^{+\infty} a_n \binom{2n}{n}, \quad \text{II. } \sum_{n=0}^{+\infty} a_n \binom{2n}{n}^{-1}.$$

Some examples are

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{8^n} = \sqrt{2}, \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{10^n} = \sqrt{\frac{5}{3}}.$$

He also presented connection between such series and Maclaurin series of $\arcsin x$ and $\arcsin^2 x$. Other examples are

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}} = \frac{1}{3}\zeta(2), \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}} = \frac{2}{5}\zeta(3),$$

as they arise in the work of Apéry [1] and van der Poorten [14] to prove irrationality of $\zeta(3)$.

Remark 1.1. For a nonnegative integer n , let

$$(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1, \\ (2n)!! = 2n(2n-2)\cdots 4\cdot 2.$$

We understand $(-1)!! = 0!! = 1$. Notice that the following relation holds.

$$\frac{\binom{2n}{n}}{2^{2n}} = \frac{(2n-1)!!}{(2n)!!}.$$

2. MAIN RESULTS

2.1. Arcsin and $\zeta(3)$. Toward the proofs of (3), (4), we first setup some notation for convenience.

Definition 2.1. Let $\mathbf{R}[[x]]$ denote the set of real power series. For $f(x) \in \mathbf{R}[[x]]$, define $W : \mathbf{R}[[x]] \rightarrow \mathbf{R}[[x]]$ by

$$Wf(x) = \int_0^1 f(xu) \frac{du}{\sqrt{1-u^2}}.$$

In particular,

$$Wf(x)|_{x=1} = \int_0^1 f(u) \frac{du}{\sqrt{1-u^2}}.$$

Fact 2.2. Recall from calculus that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ ($a_n \in \mathbf{R}$) is a convergent power series with the radius of convergence R , then so is $\int_0^x f(u) du$ and moreover it is given by termwise integration

$$\int_0^x f(u) du = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

In the sequel, we will use this result without mentioning explicitly.

Lemma 2.3. Let $f(x) \in \mathbf{R}[[x]]$.

[1] Moreover, suppose it is odd in the form

$$f(x) = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} a_{2k+1} x^{2k+1}, \quad a_{2k+1} \in \mathbf{R}.$$

Then

$$Wf(x) = \sum_{k=0}^{\infty} \frac{a_{2k+1}}{2k+1} x^{2k+1}.$$

[2] Moreover, suppose it is even in the form

$$f(x) = \sum_{k=0}^{\infty} \frac{2^{2k}}{\binom{2k}{k}} a_{2k} x^{2k}, \quad a_{2k} \in \mathbf{R}.$$

Then

$$Wf(x) = \frac{\pi}{2} \sum_{k=0}^{\infty} a_{2k} x^{2k}.$$

Thus, by the operator W we can “kill” $\frac{\binom{2k}{k}}{2^{2k}}$ or $\frac{2^{2k}}{\binom{2k}{k}}$ from coefficients and instead $\frac{1}{2^{2k+1}}$ or $\frac{\pi}{2}$ shows up, respectively.

Proof. To show (1), recall that

$$\int_0^1 \frac{u^{2k+1}}{\sqrt{1-u^2}} du = \frac{(2k)!!}{(2k+1)!!} = \frac{2^{2k}}{\binom{2k}{k}} \frac{1}{2k+1}.$$

Then

$$\begin{aligned} Wf(x) &= \int_0^1 \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} a_{2k+1} x^{2k+1} \frac{u^{2k+1}}{\sqrt{1-u^2}} du \\ &= \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} a_{2k+1} x^{2k+1} \int_0^1 \frac{u^{2k+1}}{\sqrt{1-u^2}} du = \sum_{k=0}^{\infty} \frac{a_{2k+1}}{2k+1} x^{2k+1}. \end{aligned}$$

We can verify (2) likewise with

$$\int_0^1 \frac{u^{2k}}{\sqrt{1-u^2}} du = \frac{\pi}{2} \frac{\binom{2k}{k}}{2^{2k}}.$$

□

Lemma 2.4. Let $f(x) \in \mathbf{R}[[x]]$. Suppose moreover $f(0) = 0$. Then

$$(5) \quad W \left(\int_0^x \frac{f(y)}{y} dy \right) = \int_0^1 \frac{f(xu)}{u} \arccos u du.$$

In particular, for $x = 1$, we have

$$W \left(\int_0^x \frac{f(y)}{y} dy \right) \Big|_{x=1} = \int_0^1 \frac{f(u)}{u} \arccos u du.$$

Proof. If $x = 0$, then both sides in (5) are 0. Suppose $x \neq 0$. Exchanging order of the double integral (Fubini’s Theorem), we have

$$\begin{aligned} W \left(\int_0^x \frac{f(y)}{y} dy \right) &= \int_0^1 \int_0^{xu} \frac{f(y)}{y} dy \frac{du}{\sqrt{1-u^2}} \\ &= \int_0^x \int_{y/x}^1 \frac{f(y)}{y} \frac{1}{\sqrt{1-u^2}} du dy \\ &= \int_0^x \frac{f(y)}{y} \arccos \frac{y}{x} dy \\ &= \int_0^1 \frac{f(xu)}{u} \arccos u du. \end{aligned}$$

□

2.2. **Multiple zeta, t -values.** As a natural generalization of Riemann zeta function, let us introduce the following sums.

Definition 2.5. For positive integers i_1, \dots, i_k such that $i_1 \geq 2$, define the *multiple zeta value* and *multiple t -value* by

$$\zeta(i_1, i_2, \dots, i_k) = \sum_{n_1 > n_2 > \dots > n_k} \frac{1}{n_1^{i_1} n_2^{i_2} \dots n_k^{i_k}},$$

$$t(i_1, i_2, \dots, i_k) = \sum_{\substack{n_1 > n_2 > \dots > n_k \\ n_j \text{ odd}}} \frac{1}{n_1^{i_1} n_2^{i_2} \dots n_k^{i_k}}.$$

Sometimes it is better to interpret $\zeta(i_1, \dots, i_k)$ as

$$2^{i_1 + \dots + i_k} \sum_{\substack{m_1 > m_2 > \dots > m_k \\ m_j \text{ even}}} \frac{1}{(m_1)^{i_1} (m_2)^{i_2} \dots (m_k)^{i_k}}$$

in contrast to t -values.

Fact 2.6. Let $\{m\}^n$ denote the sequence $\underbrace{(m, m, \dots, m)}_n$. For a multi-index

$$\mathbf{i} = (a_1 + 1, \{1\}^{b_1-1}, a_2 + 1, \{1\}^{b_2-1}, \dots, a_k + 1, \{1\}^{b_k-1}),$$

with integers $k, a_j, b_j \geq 1$, define its *dual*

$$\mathbf{i}^\dagger = (b_k + 1, \{1\}^{a_k-1}, b_{k-1} + 1, \{1\}^{a_{k-1}-1}, \dots, b_1 + 1, \{1\}^{a_1-1}).$$

Duality formula for multiple zeta values claims that $\zeta(\mathbf{i}) = \zeta(\mathbf{i}^\dagger)$ for all indices such that the first argument is at least 2. Historically, Drinfeld and Kontsevich found *iterated integral expressions* for multiple zeta values and proved the duality in 1990s. Afterward, Kaneko, Hoffman, Zagier and many other researchers developed the theory.

Example 2.7.

[1] the celebrated Euler-Goldbach theorem claims that

$$\zeta(2, 1) = \zeta(3).$$

We can derive this relation from iterated integral expressions

$$\zeta(2, 1) = \int_0^1 \frac{dx_3}{x_3} \int_0^{x_3} \frac{dx_2}{1-x_2} \int_0^{x_2} \frac{dx_1}{1-x_1}$$

and

$$\zeta(3) = \int_0^1 \frac{dy_3}{y_3} \int_0^{y_3} \frac{dy_2}{y_2} \int_0^{y_2} \frac{dy_1}{1-y_1}$$

with changing variables by $y_j = 1 - x_{4-j}$.

[2] Observe that

$$t(2) = \frac{3}{4}\zeta(2), \quad t(3) = \frac{7}{8}\zeta(3)$$

and Hoffman [7, p.4] shows that

$$t(2, 1) = -\frac{1}{2}t(3) + t(2) \log 2 \quad (\neq t(3)).$$

2.3. $\zeta(2, \dots, 2)$ and $t(2, \dots, 2)$.

Theorem 2.8. For $n \geq 1$,

$$\frac{\zeta(\{2\}^n)}{2^{2n}} = \frac{1}{(2n+1)!} \left(\frac{\pi}{2}\right)^{2n}, \quad t(\{2\}^n) = \frac{1}{(2n)!} \left(\frac{\pi}{2}\right)^{2n}.$$

In fact, we can prove these by equating coefficients of x^{2n} in

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(\frac{\pi x}{2}\right)^{2n} = \frac{\sin \frac{\pi x}{2}}{\frac{\pi x}{2}} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(2n)^2}\right)$$

and

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(\frac{\pi x}{2}\right)^{2n} = \cos \frac{\pi x}{2} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(2n-1)^2}\right).$$

However, we give a different proof here because it suggests the application to evaluation of $\zeta(3, 2, \dots, 2)$ and $t(3, 2, \dots, 2)$ in the next subsection. For this purpose, we need a lemma.

Lemma 2.9. For $n \geq 1$, $|x| \leq 1$, we have

$$\frac{\arcsin^{2n} x}{(2n)!} = \sum_{k > m_1 > \dots > m_{n-1} > 0} \frac{2^{2k}}{\binom{2k}{k}} \frac{1}{(2k)^2 (2m_1)^2 \dots (2m_{n-1})^2} x^{2k},$$

$$\frac{\arcsin^{2n-1} x}{(2n-1)!} = \sum_{k > m_1 > \dots > m_{n-1} \geq 0} \frac{\binom{2k}{k}}{2^{2k}} \frac{1}{(2k+1)(2m_1+1)^2 \dots (2m_{n-1}+1)^2} x^{2k+1}.$$

Proof. This is a rephrasing of J.M.Borwein–Chamberland [4, (1.1)-(1.4)]. □

Proof of Theorem 2.8. Lemma 2.9 asserts that

$$\frac{\arcsin^{2n} x}{(2n)!} = \sum_{k > m_1 > \dots > m_{n-1} > 0} \frac{2^{2k}}{\binom{2k}{k}} \frac{1}{(2k)^2 (2m_1)^2 \dots (2m_{n-1})^2} x^{2k}$$

so that

$$W \left(\frac{\arcsin^{2n} x}{(2n)!} \right) = \frac{\pi}{2} \sum_{k > m_1 > \dots > m_{n-1} > 0} \frac{1}{(2k)^2 (2m_1)^2 \dots (2m_{n-1})^2} x^{2k}.$$

Let $x = 1$. The left hand side becomes

$$\int_0^1 \frac{\arcsin^{2n} u}{(2n)!} \frac{du}{\sqrt{1-u^2}} = \left[\frac{\arcsin^{2n+1} u}{(2n+1)!} \right]_0^1 = \frac{1}{(2n+1)!} \left(\frac{\pi}{2} \right)^{2n+1}$$

while the right hand side turns to be $\frac{\pi}{2} \frac{\zeta(\{2\}^n)}{2^{2n}}$. Hence we proved

$$\frac{\zeta(\{2\}^n)}{2^{2n}} = \frac{1}{(2n+1)!} \left(\frac{\pi}{2} \right)^{2n}.$$

It is quite similar to show $t(\{2\}^n) = \frac{1}{(2n)!} \left(\frac{\pi}{2} \right)^{2n}$ using $\frac{\arcsin^{2n-1} x}{(2n-1)!}$ and the operator W .

2.4. $\zeta(3, 2, \dots, 2)$ and $t(3, 2, \dots, 2)$. We just found $\zeta(2, \dots, 2)$ and $t(2, \dots, 2)$ above. A natural subsequence is to evaluate $\zeta(3, 2, \dots, 2)$ and $t(3, 2, \dots, 2)$ via integrals on powers of arcsin again.

Definition 2.10. For $n \geq 1$, set

$$I(n) = \int_0^1 \frac{\arcsin^n x}{x} dx.$$

Example 2.11. Observe the first several values.

$$(6) \quad I(1) = \frac{\pi}{2} \log(2)$$

$$(7) \quad I(2) = \frac{\pi^2}{4} \log(2) - \frac{7}{8} \zeta(3)$$

$$(8) \quad I(3) = \frac{\pi^3}{8} \log(2) - \frac{9\pi}{16} \zeta(3)$$

$$(9) \quad I(4) = \frac{\pi^4}{16} \log(2) - \frac{9\pi^2}{16} \zeta(3) + \frac{93}{32} \zeta(5)$$

$$(10) \quad I(5) = \frac{\pi^5}{32} \log(2) - \frac{15\pi^3}{32} \zeta(3) + \frac{225\pi}{64} \zeta(5)$$

$$(11) \quad I(6) = \frac{\pi^6}{64} \log(2) - \frac{45\pi^4}{128} \zeta(3) + \frac{675\pi^2}{128} \zeta(5) - \frac{5715}{256} \zeta(7)$$

Remark 2.12.

[1] Indeed, Wolfram alpha [15] returns the algebraic expressions (6)-(11) for integrals

$$I(n) = \int_0^{\pi/2} y^n \cot y dy$$

while she outputs only numerical values for

$$\int_0^1 \frac{\arcsin^n x}{x} dx.$$

In fact, there is a precise formula for $I(n)$ giving a \mathbf{Q} -linear combination of $\log 2$ and single Riemann zeta values. For the sake of completeness, we discuss it here although we do not need it in the sequel. Let η denote the Dirichlet eta function, that is, $\eta(1) = \log 2$ and $\eta(j) = (1 - 2^{1-j}) \zeta(j)$ ($j \geq 2$). Then, there holds

$$(12) \quad I(2n+1) = \frac{(2n+1)!}{2^{2n+1}} \sum_{j=0}^n \frac{(-1)^j \pi^{2n+1-2j}}{(2n+1-2j)!} \eta(2j+1),$$

$$(13) \quad I(2n) = \frac{(2n)!}{2^{2n}} \left(\sum_{j=0}^{n-1} \frac{(-1)^j \pi^{2n-2j}}{(2n-2j)!} \eta(2j+1) + (-1)^n 2(1 - 2^{-2n-1}) \zeta(2n+1) \right).$$

To see this, we remark that Buhler-Crandall [5, p.280] stated

$$\begin{aligned} & \int_0^{1/2} x^n \cot(\pi x) dx \\ &= \frac{n!}{2^n} \sum_{\substack{1 \leq k \leq n \\ k \text{ odd}}} \frac{(-1)^{(k-1)/2}}{\pi^k} \frac{\eta(k)}{(n-k+1)!} + \frac{1}{2} ((-1)^n + 1) \frac{4n!(1 - 2^{-n-1})}{(2\pi)^{n+1}} \zeta(n+1). \end{aligned}$$

However, the sign $\frac{1}{2}((-1)^n + 1)$ must be $\cos \frac{n\pi}{2}$ (for $n = 2$, the coefficient of $\zeta(3)$ is negative; see (7)). To correct this, set

$$\begin{aligned} J(n) &:= \int_0^{1/2} x^n \cot(\pi x) dx \\ &= \frac{n!}{2^n} \left(\sum_{\substack{1 \leq k \leq n \\ k \text{ odd}}} \frac{(-1)^{(k-1)/2}}{\pi^k} \frac{\eta(k)}{(n-k+1)!} \right) + \cos \frac{n\pi}{2} \frac{4n!(1 - 2^{-n-1})}{(2\pi)^{n+1}} \zeta(n+1). \end{aligned}$$

Then, with $y = \sin(\pi x)$, we find

$$J(n) = \frac{1}{\pi^{n+1}} \int_0^1 \frac{\arcsin^n(y)}{y} dy = \frac{1}{\pi^{n+1}} I(n).$$

Thus, $I(n) = \pi^{n+1} J(n)$. Writing down the cases for the index even and odd with $k = 2j + 1$, we get (12), (13) and hence justified (6)-(11).

[2] We can also view $I(n)$ as a *log-sine integral*:

$$I(n) = \int_0^1 \frac{\arcsin^n x}{x} dx = \underbrace{[\log x(\arcsin^n x)]_0^1}_0 - n \int_0^1 \log x \frac{\arcsin^{n-1} x}{\sqrt{1-x^2}} dx$$

$$= -n \int_0^{\pi/2} y^{n-1} \log(\sin y) dy.$$

See J.M.Borwein-Broadhurst-Kamnitzer [3] for relation of such integrals and central binomial series, for example.

Theorem 2.13. For $n \geq 0$,

$$\frac{\zeta(3, \{2\}^n)}{2^{2n+3}} = \frac{2}{\pi} \int_0^1 \frac{\arcsin^{2n+2} x \arccos x}{(2n+2)! x} dx,$$

$$t(3, \{2\}^n) = \int_0^1 \frac{\arcsin^{2n+1} x \arccos x}{(2n+1)! x} dx.$$

The proof is quite same to the one for Theorem 2.8. Hence we omit it.

Remark 2.14. Recently, Lupu [11], Murakami [12] and Zagier [18] obtained similar results. Each of their proofs is different from ours.

Example 2.15. For $n = 0$, we have

$$(14) \quad \int_0^1 \frac{\arcsin x \arccos x}{x} dx = \frac{7}{8} \zeta(3),$$

$$(15) \quad \frac{2}{\pi} \int_0^1 \frac{\arcsin x \arccos x}{2! x} dx = \frac{1}{8} \zeta(3)$$

as we mentioned in Introduction. For $n \geq 1$, with $\arccos x = \frac{\pi}{2} - \arcsin x$, we see that

$$t(3, 2) = \int_0^1 \frac{\arcsin^3 x \arccos x}{3! x} dx = \frac{1}{3!} \left(\frac{\pi}{2} I(3) - I(4) \right) = \frac{1}{64} (3\pi^2 \zeta(3) - 31\zeta(5)),$$

$$\frac{\zeta(3, 2)}{2^5} = \frac{2}{\pi} \frac{1}{4!} \left(\frac{\pi}{2} I(4) - I(5) \right) = \frac{1}{64} (\pi^2 \zeta(3) - 11\zeta(5)),$$

$$t(3, 2, 2) = \frac{1}{5!} \left(\frac{\pi}{2} I(5) - I(6) \right) = \frac{1}{2048} (2\pi^4 \zeta(3) - 60\pi^2 \zeta(5) + 381\zeta(7))$$

and so on.

Corollary 2.16. Let $\mathbf{Q}[\pi, \zeta(3), \zeta(5), \dots, \zeta(2n+3)]_{2n+3}$ denote the set of all elements of degree $2n+3$ in the rational polynomial ring in $\pi, \zeta(3), \zeta(5), \dots, \zeta(2n+3)$ with grading $\deg \pi = 1$ and $\deg \zeta(2j+1) = 2j+1$. Then

$$\frac{\zeta(3, \{2\}^n)}{2^{2n+3}}, t(3, \{2\}^n) \in \mathbf{Q}[\pi, \zeta(3), \zeta(5), \dots, \zeta(2n+3)]_{2n+3}.$$

Remark 2.17. Not all multiple values satisfy such a property. For example, as mentioned before,

$$t(2, 1) = \frac{\pi^2}{8} \log 2 - \frac{7}{16} \zeta(3)$$

involves a rational multiple of $\pi^2 \log 2$.

2.5. Conjecture on central binomial sums. Keeping iterated integrals in mind, now we discuss powers of $\log x$ as certain operator; at the end, we make one conjecture.

For $f(x) \in \mathbf{R}[[x]]$ such that $f(0) = 0$, it is the technique to consider

$$\int_0^y \frac{f(x)}{x} dx$$

to construct another series as we often encountered.

We can generalize this little more by changing the part “ $\int \frac{1}{x}$ ” with the iterated integral

$$\underbrace{\int \frac{dx}{x} \int \frac{dx}{x} \cdots \int \frac{dx}{x}}_{r+1} = \int \frac{\log^r x}{r!x}.$$

Fact 2.18 ([13, p.1, 57-58]). For integers $n \geq 1, r \geq 0$, we have

$$\int_0^1 x^n \frac{\log^r x}{r!x} dx = \frac{(-1)^r}{n^{r+1}}.$$

Let us see what if we apply this idea to arcsin integrals. Again, recall that

$$\arcsin x = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} \frac{x^{2k+1}}{2k+1}.$$

We now see another central binomial series

$$- \int_0^1 \arcsin x \frac{\log x}{x} dx = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} \frac{1}{2k+1} \int_0^1 x^{2k+1} \frac{\log x}{x} dx = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} \frac{1}{(2k+1)^3}$$

and similarly

$$\int_0^1 \arcsin x \frac{\log^2 x}{2!x} dx = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} \frac{1}{(2k+1)^4}.$$

Wolfram alpha [15] says that

$$- \int_0^1 \arcsin x \frac{\log x}{x} dx = \frac{1}{48}(\pi^3 + 12\pi \log^2 2).$$

$$\int_0^1 \arcsin x \frac{\log^2 x}{2!x} dx = \frac{1}{48}(6\pi\zeta(3) + 4\pi \log^3 2 + \pi^3 \log 2).$$

This is only computer calculation. Hence let us state it as a conjecture.

Conjecture 2.19.

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} \frac{1}{(2k+1)^3} = \frac{1}{48}(\pi^3 + 12\pi \log^2 2),$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} \frac{1}{(2k+1)^4} = \frac{1}{48} (6\pi\zeta(3) + 4\pi \log^3 2 + \pi^3 \log 2).$$

Remark 2.20. Jianqiang Zhao pointed out (e-mail) to the author that we can prove these with Xu-Zhao [17, Theorem 8.12] (for $m = 4, k = \emptyset$) without much trouble.

3. OTHER MULTIPLE SUMS

3.1. Multiple T, S -values. Following Xu-Zhao [16], let us introduce more general multiple sums here.

Definition 3.1. For a multi-index (i_k, \dots, i_1) with $i_k \geq 2$, define a *multiple T -value* (MTV) and a *multiple S -value* (MSV) by

$$T(i_k, \dots, i_1) = 2^k \sum_{\substack{n_k > n_{k-1} > \dots > n_1 \\ n_j \equiv j \pmod{2}}} \frac{1}{n_k^{i_k} n_{k-1}^{i_{k-1}} \dots n_1^{i_1}},$$

$$S(i_k, \dots, i_1) = 2^k \sum_{\substack{n_k > n_{k-1} > \dots > n_1 \\ n_j \equiv j+1 \pmod{2}}} \frac{1}{n_k^{i_k} n_{k-1}^{i_{k-1}} \dots n_1^{i_1}}.$$

We understand that the empty index \emptyset has weight 0 and

$$\zeta(\emptyset) = t(\emptyset) = T(\emptyset) = S(\emptyset) = 1.$$

TABLE 1. Conjectural dim sequences

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
z_n	1	0	1	1	1	2	2	3	4	5	7	9	12	16	...
t_n	1	0	1	2	3	5	8	13	21	34	55	89	144	233	...
T_n	1	0	1	1	2	2	4	5	9	10	19	23	42	49	...
S_n	1	0	1	2	3	4	6	10	15	22	32	52	76	?	...

For $n \geq 0$, let $\text{MZV}_n, \text{MtV}_n, \text{MTV}_n, \text{MSV}_n$ denote the set of \mathbf{Q} -span of all MZVs, MtVs, MTVs, MSVs of weight n , respectively. Moreover, let

$$z_n = \dim_{\mathbf{Q}} \text{MZV}_n, \quad t_n = \dim_{\mathbf{Q}} \text{MtV}_n,$$

$$T_n = \dim_{\mathbf{Q}} \text{MTV}_n, \quad S_n = \dim_{\mathbf{Q}} \text{MSV}_n.$$

By $(d_n)_{n \geq 0}$ and $(f_n)_{n \geq 0}$, we mean *Padovan* and *Fibonacci* sequences; to be more precise, they are ones satisfying

$$d_0 = 1, d_1 = 0, d_2 = 1 \text{ and } d_n = d_{n-2} + d_{n-3} \text{ for } n \geq 3,$$

$$f_0 = 0, f_1 = 1 \text{ and } f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 2.$$

Conjecture 3.2.

- [1] (Zagier) $z_n = d_n$ for $n \geq 1$.
- [2] (Hoffman) $t_n = f_n$ for $n \geq 2$.

What about T_n, S_n ? Kaneko-Tsumura [8], Xu-Zhao [16] observed that the conjectural sequence $(T_n)_{0 \leq n \leq 13}$ satisfies *restricted* Fibonacci relation

$$T_{2k} = T_{2k-1} + T_{2k-2}, \quad 1 \leq k \leq 6.$$

Here we also observe that the conjectural $(S_n)_{0 \leq n \leq 13}$ satisfies *restricted* Padovan-like relation

$$S_{2k+1} = S_{2k-1} + 2S_{2k-2}, \quad 1 \leq k \leq 5.$$

Thus, S_{13} is conjecturally equal to

$$52 * 2 \times 32 = 116.$$

There might exist such relations for all $(T_n)_{n \geq 0}, (S_n)_{n \geq 0}$ throughout. We are planning to pursue these details at another opportunity.

3.2. Conjecture on $t(\{2\}^{n+1}, 1)$. We evaluated $\zeta(3, \{2\}^n), t(3, \{2\}^n)$ together in Theorem 2.13. Notice that

$$(3, \{2\}^n)^\dagger = (\{2\}^{n+1}, 1)$$

so that MZV duality implies

$$\zeta(3, \{2\}^n)^\dagger = \zeta(\{2\}^{n+1}, 1).$$

It is now natural to ask what $t(\{2\}^{n+1}, 1)$ is.

With (7), (9), (11) and [7, Appendix], we observe that

$$t(2, 1) = \frac{1}{2} \left(\frac{\pi^2}{4} \log 2 - \frac{7}{8} \zeta(3) \right) = \frac{I(2)}{2!},$$

$$t(2, 2, 1) = \frac{1}{24} \left(\frac{\pi^4}{16} \log 2 - \frac{9}{16} \pi^2 \zeta(3) + \frac{93}{32} \zeta(5) \right) = \frac{I(4)}{4!},$$

$$t(2, 2, 2, 1) = \frac{1}{720} \left(\frac{\pi^6}{64} \log(2) - \frac{45\pi^4}{128} \zeta(3) + \frac{675\pi^2}{128} \zeta(5) - \frac{5715}{256} \zeta(7) \right) = \frac{I(6)}{6!}.$$

It would be nice if we can generalize this.

Conjecture 3.3 ([6]). For $n \geq 0$,

$$t(\{2\}^{n+1}, 1) = \int_0^1 \frac{\arcsin^{2n+2} x}{(2n+2)!} \frac{dx}{x}.$$

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