Bernoulli and Cauchy numbers with level 2 associated with Stirling numbers with level 2

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1 Introduction

In the literature, the Stirling numbers with higher level (level s) seem to have been firstly studied by Tweedie [21] in 1918. Namely, those of the first kind $\left[\!\left[{n \atop k}\right]\!\right]_s$ and the second kind $\left\{\!\!\left\{{n \atop k}\right\}\!\!\right\}_s$ appear as

$$x(x+1^s)(x+2^s)\cdots(x+(n-1)^s)=\sum_{k=0}^n \left[\!\!\left[{n \atop k}\right]\!\!\right]_s x^k$$

and

$$x^n=\sum_{k=0}^n \left\{\!\!\left\{{n \atop k}\right\}\!\!\right\}_s x(x-1^s)(x-2^s)\cdots\bigl(x-(k-1)^s\bigr)\,,$$

respectively. They satisfy the recurrence relations

$$\left[\!\!\left[{n \atop k}\right]\!\!\right]_s=\left[\!\!\left[{n-1 \atop k-1}\right]\!\!\right]_s+(n-1)^s\left[\!\!\left[{n-1 \atop k}\right]\!\!\right]_s$$

and

$$\left\{\!\!\left\{{n \atop k}\right\}\!\!\right\}_s=\left\{\!\!\left\{{n-1 \atop k-1}\right\}\!\!\right\}_s+k\left\{\!\!\left\{{n-1 \atop k}\right\}\!\!\right\}_s \tag{1}$$

with $\left[\!\left[{0 \atop 0}\right]\!\right]_s=\left\{\!\!\left\{{0 \atop 0}\right\}\!\!\right\}_s=1$ and $\left[\!\left[{n \atop 0}\right]\!\right]_s=\left\{\!\!\left\{{n \atop 0}\right\}\!\!\right\}_s=0$ $(n\ge 1)$. Recently, in [15, 16], the Stirling numbers with higher level have been rediscovered and studied more deeply, in particular, from the aspects of combinatorics. When $s=1$, they are the original Stirling numbers of both kinds. When $s=2$, they have been often studided as central factorial numbers of both kinds (see, e.g., [1]).

Some typical values of Stirling numbers of the first kind with higher level are given as

$$\left[\!\!\left[{n \atop 1}\right]\!\!\right]_s=\bigl((n-1)!\bigr)^s\,,$$
$$\left[\!\!\left[{n \atop 2}\right]\!\!\right]_s=\bigl((n-1)!\bigr)^s H_{n-1}^{(s)}\,,$$

$$\left[\!\!\left[{n \atop 3} \right]\!\!\right]_s = \big((n-1)!\big)^s \frac{(H_{n-1}^{(s)})^2 - H_{n-1}^{(2s)}}{2},$$

where $H_n^{(k)}$ are the generalized harmonic numbers of order k defined by $H_n^{(k)} = \sum_{j=1}^n \frac{1}{j^k}$ $(n \ge 0)$ and $H_n = H_n^{(1)}$ are the classical harmonic numbers. More generally,

$$\left[\!\!\left[{n \atop m} \right]\!\!\right]_s = \sum_{1 \le i_1 < \cdots < i_{n-m} \le n-1} (i_1 \cdots i_{n-m})^s .$$

Some typical values of Stirling numbers of the second kind with higher level are given as

$$\left\{\!\!\left\{ {n \atop 1} \right\}\!\!\right\}_s = 1, \quad \left\{\!\!\left\{ {n \atop 2} \right\}\!\!\right\}_s = \sum_{k=0}^{n-2} 2^{sk} = \frac{2^{s(n-1)} - 1}{2^s - 1},$$

$$\left\{\!\!\left\{ {n \atop 3} \right\}\!\!\right\}_s = \sum_{j=0}^{n-3} 3^{n-j-3} \sum_{k=0}^{j} 2^{sk} = \frac{2^s(3^{(n-2)s} - 2^{(n-2)s})}{(2^s-1)(3^s-2^s)} - \frac{3^{(n-2)s} - 1}{(2^s-1)(3^s-1)}$$

$$\left\{\!\!\left\{ {n \atop n-1} \right\}\!\!\right\}_s = \sum_{j=1}^{n-1} j^s .$$

2 Stirling numbers with higher level

Given a positive integer s, let $\left[\!\left[{n \atop k} \right]\!\right]_s$ denote the number of ordered s-tuples $(\sigma_1, \sigma_2, \dots, \sigma_s) \in \mathfrak{S}_{(n,k)} \times \mathfrak{S}_{(n,k)} \times \cdots \times \mathfrak{S}_{(n,k)} = \mathfrak{S}_{(n,k)}^s$, such that

$$\min(\sigma_1) = \min(\sigma_2) = \cdots = \min(\sigma_s). \tag{2}$$

For example, $\left[\!\left[{3 \atop 2} \right]\!\right]_3 = 9$, the relevant 3-tuples being

$$((1)(2\,3),(1)(2\,3),(1)(2\,3))), \quad ((1)(2\,3),(1)(2\,3),(1\,3)(2)), \quad ((1)(2\,3),(1\,3)(2),(1)(2\,3)),$$
$$((1)(2\,3),(1\,3)(2),(1\,3)(2))), \quad ((1\,2)(3),(1\,2)(3),(1\,2)(3)), \quad ((1\,3)(2),(1)(2\,3),(1)(2\,3)),$$
$$((1\,3)(2),(1)(2\,3),(1\,3)(2))), \quad ((1\,3)(2),(1\,3)(2),(1)(2\,3)), \quad ((1\,3)(2),(1\,3)(2),(1\,3)(2)).$$

If $n, k \ge 0$, then let $\Pi_{(n,k)}$ denote the set of all partitions of $[n]$ having exactly k non-empty blocks. Given a partition π in Π_n, let $\min(\pi)$ denote the set of the minimal elements in each block of π. Given a positive integer s, let $\left\{\!\left\{ {n \atop k} \right\}\!\right\}_s$ denote the number of ordered s-tuples $(\pi_1, \pi_2, \dots, \pi_s) \in \Pi_{(n,k)} \times \Pi_{(n,k)} \times \cdots \times \Pi_{(n,k)} = \Pi_{(n,k)}^s$, such that

$$\min(\pi_1) = \min(\pi_2) = \cdots = \min(\pi_s). \tag{3}$$

This sequence is called *Stirling numbers of the second kind with higher level*. For example, $\left\{\!\left\{ {3 \atop 2} \right\}\!\right\}_3 = 9$, the relevant 3-tuples being

$$(1/23, 1/23, 1/23), \qquad (1/23, 1/23, 13/2), \qquad (1/23, 13/2, 1/23),$$

$$\begin{array}{lll} (1/23,13/2,13/2), & (12/3,12/3,12/3), & (13/2,1/23,1/23), \\ (13/2,1/23,13/2), & (13/2,13/2,1/23), & (13/2,13/2,13/2). \end{array}$$

The Stirling numbers of the second kind with higher level can be expressed in terms of iterated summations.

Theorem 1. *For $2 \le k \le n$,*

$$\left\{\!\!\left\{ n \atop k \right\}\!\!\right\}_s = k^{s(n-k+1)} \sum_{i_{k-1}=1}^{n-k+1} \left(\frac{k-1}{k}\right)^{si_{k-1}} \sum_{i_{k-2}=1}^{i_{k-1}} \left(\frac{k-2}{k-1}\right)^{si_{k-2}} \cdots \sum_{i_2=1}^{i_3} \left(\frac{2}{3}\right)^{si_2} \sum_{i_1=1}^{i_2} \left(\frac{1}{2}\right)^{si_1} .$$

The (ordinary) generating function of Stirling numbers of the second kind with higher level can be given as follows.

Theorem 2. *For $k \ge 1$,*

$$\sum_{n=k}^{\infty} \left\{\!\!\left\{ n \atop k \right\}\!\!\right\}_s x^n = \frac{x^k}{(1-x)(1-2^s x)\cdots(1-k^s x)} .$$

Corollary 1. *We have the following rational explicit formula*

$$\left\{\!\!\left\{ n \atop k \right\}\!\!\right\}_s = \sum_{j=0}^{k} \frac{j^{sn}}{\prod_{i=0, i\ne j}^{k} (j^s - i^s)} .$$

There exist orthogonality relationships of Stirling numbers of both kinds with higher level.

Theorem 3. *We have the relations*

$$\sum_{k=0}^{\max\{n,m\}} (-1)^{n-k} \left[\!\!\left[n \atop k \right]\!\!\right]_s \left\{\!\!\left\{ k \atop m \right\}\!\!\right\}_s = \delta_{n,m} , \tag{4}$$

$$\sum_{k=0}^{\max\{n,m\}} (-1)^{k-m} \left\{\!\!\left\{ n \atop k \right\}\!\!\right\}_s \left[\!\!\left[k \atop m \right]\!\!\right]_s = \delta_{n,m} , \tag{5}$$

where $\delta_{n,m}$ is the Kronecker delta.

We show identities which combine Stirling numbers with higher level and Bernoulli polynomials. The Bernoulli polynomials $B_n(x)$ can be defined by the exponential generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} .$$

Theorem 4. *We have the relation*

$$\sum_{\ell=1}^{n-1} \ell(-1)^\ell \left[\!\!\left[\begin{matrix} n \\ n-\ell \end{matrix} \right]\!\!\right]_s \left\{\!\!\left\{ \begin{matrix} n-1+j-\ell \\ n-1 \end{matrix} \right\}\!\!\right\}_s = \frac{B_{sj+1}(0) - B_{sj+1}(n)}{sj+1}\,.$$

Corollary 2. *For $n \ge k \ge 0$, we have*

$$k \left[\!\!\left[\begin{matrix} n \\ n-k \end{matrix} \right]\!\!\right]_s = \sum_{j=1}^{k} (-1)^j \left[\!\!\left[\begin{matrix} n \\ n-k+j \end{matrix} \right]\!\!\right]_s \frac{B_{sj+1}(0) - B_{sj+1}(n)}{sj+1}\,.$$

2.1 Stirling numbers with level 2

When $s = 2$, there is a convenient form to calculate Stirling numbers of the first kind with level 2 from the classical Stirling numbers of the first kind.

Theorem 5.

$$\left[\!\!\left[\begin{matrix} n \\ m \end{matrix} \right]\!\!\right]_2 = \begin{bmatrix} n \\ m \end{bmatrix}^2 - 2 \begin{bmatrix} n \\ m-1 \end{bmatrix} \begin{bmatrix} n \\ m+1 \end{bmatrix} + 2 \begin{bmatrix} n \\ m-2 \end{bmatrix} \begin{bmatrix} n \\ m+2 \end{bmatrix} - \cdots + 2(-1)^{m-1} \begin{bmatrix} n \\ 1 \end{bmatrix} \begin{bmatrix} n \\ 2m-1 \end{bmatrix}\,.$$

When $s = 2$, there is a relation $\left[\!\left[{n \atop m} \right]\!\right]_2 = (-1)^{n-m} t(2n, 2m)$, where $t(n, m)$ are the central factorial numbers of the first kind, defined by

$$x\left(x + \frac{n}{2} - 1\right)\left(x + \frac{n}{2} - 2\right) \cdots \left(x - \frac{n}{2} + 1\right) = \sum_{m=0}^{n} t(n, m) x^m\,.$$

When $s = 2$, we have an convenient identity for the Stirlin numbers of the second kind as

$$\left\{\!\!\left\{ \begin{matrix} n \\ k \end{matrix} \right\}\!\!\right\}_2 = \frac{2}{(2k)!} \sum_{j=1}^{k} (-1)^{k-j} \binom{2k}{k-j} j^{2n}\,.$$

This is an analogous identity for the classical Stirling numbers of the second kind:

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{k-j} j^n\,.$$

However, no convenient form has not been found when $s \ge 3$.

When $s = 2$, there is a relation $\left\{\!\left\{ {n \atop m} \right\}\!\right\}_2 = T(2n, 2m)$, where $T(n, m)$ are the central factorial numbers of the second kind, defined by

$$x^n = \sum_{m=0}^{n} T(n, m) x\left(x + \frac{m}{2} - 1\right)\left(x + \frac{m}{2} - 2\right) \cdots \left(x - \frac{m}{2} + 1\right)\,.$$

3 Poly-Cauchy numbers with level 2

Poly-Cauchy numbers $\mathfrak{C}_n^{(k)}$ *with level* 2 are defined by

$$\mathrm{Lif}_{2,k}(\mathrm{arcsinh}t)=\sum_{n=0}^\infty\mathfrak{C}_n^{(k)}\frac{t^n}{n!}\,, \tag{6}$$

where arcsinh$t$ is the inverse hyperbolic sine function and

$$\mathrm{Lif}_{2,k}(z)=\sum_{m=0}^\infty\frac{z^{2m}}{(2m)!(2m+1)^k}\,.$$

This function is an analogue of *Polylogarithm factorial* or *Polyfactorial* function $\mathrm{Lif}_k(z)$ [7, 8], defined by

$$\mathrm{Lif}_k(z)=\sum_{m=0}^\infty\frac{z^m}{m!(m+1)^k}\,.$$

Several initial values of $\mathfrak{C}_n^{(1)}$ are as follows.

$$\{\mathfrak{C}_{2n}^{(1)}\}_{n\ge 0}=1,\frac{1}{3},-\frac{17}{15},\frac{367}{21},-\frac{27859}{45},\frac{1295803}{33},-\frac{5329242827}{1365},\dots\,.$$

Note that the numerators of coefficients for numerical integration ([19]) are given as

$$1,17,367,27859,1295803,5329242827,25198857127,11959712166949,\dots$$

([20, A002197]). From higher-order Bernoulli numbers, the denominators of D numbers $D_{2n}(2n)$ ([17, 18]) are given as

$$1,3,15,21,45,33,1365,45,765,1995,3465,1035,20475,189,435,7161,\dots$$

([20, A261274]). Here, the D numbers (or cosecant numbers) $D_{2n}^{(k)}$ may be defined by

$$\left(\frac{t}{\sinh t}\right)^k=\sum_{n=0}^\infty D_{2n}^{(k)}\frac{t^{2n}}{(2n)!}\quad(|t|<\pi)\,.$$

By using the polyfactorial function, poly-Cauchy numbers (of the first kind) $c_n^{(k)}$ are defined as

$$\mathrm{Lif}_k\bigl(\log(1+t)\bigr)=\sum_{n=0}^\infty c_n^{(k)}\frac{t^n}{n!}\,. \tag{7}$$

When $k=1$, by $\mathrm{Lif}_1(z)=(e^z-1)/z$, $c_n=c_n^{(1)}$ are the original Cauchy numbers defined by

$$\frac{t}{\log(1+t)}=\sum_{n=0}^\infty c_n\frac{t^n}{n!}\,.$$

The generating function of poly-Cauchy numbers $c_n^{(k)}$ in (7) can be written in the form of iterated integrals ([7]):

$$\frac{1}{\log(1+x)}\underbrace{\int_0^x \frac{1}{(1+x)\log(1+x)}\cdots\int_0^x \frac{1}{(1+x)\log(1+x)}}_{k-1}\times x\underbrace{dx\cdots dx}_{k-1}=\sum_{n=0}^\infty c_n^{(k)}\frac{x^n}{n!}\,.$$

We can also write the generating function of the poly-Cauchy numbers with level 2 in (6) in the form of iterated integrals.

Theorem 6. *For $k\ge 1$ we have*

$$\frac{1}{\mathrm{arcsinh}x}\underbrace{\int_0^x \frac{1}{\mathrm{arcsinh}x\sqrt{1+x^2}}\cdots\int_0^x \frac{1}{\mathrm{arcsinh}x\sqrt{1+x^2}}}_{k-1}\times x\underbrace{dx\cdots dx}_{k-1}=\sum_{n=0}^\infty \mathfrak{C}_n^{(k)}\frac{x^n}{n!}\,.$$

3.1 Cauchy numbers with level 2

When $k=1$, Cauchy numbers $\mathfrak{C}_{2n}=\mathfrak{C}_{2n}^{(1)}$ with level 2 have a determinant expression.

Theorem 7. *For $n\ge 1$,*

$$\mathfrak{C}_{2n}=(-1)^{n-1}(2n)!\begin{vmatrix}\frac{1}{2^2\cdot 3}\binom{2}{1} & 1 & 0 & & \\ \frac{1}{2^4\cdot 5}\binom{4}{2} & \frac{1}{2^2\cdot 3}\binom{2}{1} & 1 & & \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \frac{1}{2^2\cdot 3}\binom{2}{1} & 1 \\ \frac{1}{2^{2n}(2n+1)}\binom{2n}{n} & \cdots & \cdots & \frac{1}{2^4\cdot 5}\binom{4}{2} & \frac{1}{2^2\cdot 3}\binom{2}{1}\end{vmatrix}.$$

Remark. A determinant expression of the classical Cauchy numbers may be given as

$$c_n=n!\begin{vmatrix}\frac{1}{2} & 1 & 0 & & \\ \frac{1}{3} & \frac{1}{2} & 1 & & \\ \vdots & & \ddots & & 0 \\ \vdots & & & \frac{1}{2} & 1 \\ \frac{1}{n+1} & \cdots & \cdots & \frac{1}{3} & \frac{1}{2}\end{vmatrix}.$$

([2, p.50]).

By the inversion formula below (see, e.g., [14][1]), we also have the following.

[1]The case where $f_n=0$ for all $n\ge 0$ is considered in [14]

Corollary 3. *For $n \ge 1$,*

$$\frac{1}{2^{2n}(2n+1)}\binom{2n}{n} = \begin{vmatrix} \frac{\mathfrak{C}_2}{2!} & 1 & & 0 \\ -\frac{\mathfrak{C}_4}{4!} & \frac{\mathfrak{C}_2}{2!} & & \\ \vdots & & \ddots & 1 \\ (-1)^{n-1}\frac{\mathfrak{C}_{2n}}{(2n)!} & \cdots & -\frac{\mathfrak{C}_4}{4!} & \frac{\mathfrak{C}_2}{2!} \end{vmatrix}.$$

Lemma 1 (Inversion formula)**.**

$$\sum_{k=0}^{n}(-1)^{n-k}x_{n-k}z_k = f_n \quad \textit{with} \quad x_0 = z_0 = 1$$

$$\iff \quad z_n = \begin{vmatrix} x_1 & 1 & & & 0 \\ x_2 & x_1 & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n-1} & x_{n-2} & \cdots & x_1 & 1 \\ x_n + (-1)^{n-1}f_n & x_{n-1} + (-1)^n f_{n-1} & \cdots & x_2 - f_2 & x_1 + f_1 \end{vmatrix}$$

$$\iff \quad x_n = \begin{vmatrix} z_1 & 1 & & & 0 \\ z_2 & z_1 & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ z_{n-1} & z_{n-2} & \cdots & z_1 & 1 \\ z_n - f_n & z_{n-1} - f_{n-1} & \cdots & z_2 - f_2 & z_1 - f_1 \end{vmatrix}.$$

Poly-Cauchy numbers have an expression of integrals

$$c_n^{(k)} = n! \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{x_1 x_2 \cdots x_k}{n} dx_1 dx_2 \dots dx_k$$

([7]). Poly-Cauchy numbers with level 2 also have a similar expression (or a kind of definition).

Corollary 4. *For $n \ge 0$ and $k \ge 1$, we have*

$$\mathfrak{C}_{2n}^{(k)} = (-4)^n (n!)^2 \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{\frac{x_1 x_2 \cdots x_k}{2}}{n} \binom{-\frac{x_1 x_2 \cdots x_k}{2}}{n} dx_1 dx_2 \dots dx_k\,.$$

4 Poly-Bernoulli numbers with level 2

As poly-Bernoulli numbers [6] and poly-Cauchy numbers are closely connected with each other ([12]), poly-Bernoulli numbers with level 2 can be naturally introduced ([11])

in the connection with poly-Cauchy numbers with level 2. In fact, poly-Bernoulli numbers with level 2 have a good analogy of poly-Bernoulli numbers.

For $k \ge 1$, *poly-Bernoulli numbers* $\mathfrak{B}_n^{(k)}$ *with level* 2 are defined by

$$\frac{\mathrm{Li}_{2,k}\bigl(2\sin(x/2)\bigr)}{2\sin(x/2)} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(k)} \frac{x^n}{n!}\,, \tag{8}$$

where

$$\mathrm{Li}_{2,k}(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^k}$$

is the polylogarithm function with level 2 ([11]). Such a concept is analogous of that of *poly-Bernoulli numbers* $\mathbb{B}_n^{(k)}$, defined by

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(k)} \frac{x^n}{n!}$$

with the polylogarithm function

$$\mathrm{Li}_k(z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)^k}$$

([6]). Then, *Bernoulli numbers* $\mathfrak{B}_n = \mathfrak{B}_n^{(1)}$ *with level* 2 are given by the generating function

$$\frac{1}{4\sin(x/2)} \log \frac{1+2\sin(x/2)}{1-2\sin(x/2)} = \sum_{n=0}^{\infty} \mathfrak{B}_n \frac{x^n}{n!}\,. \tag{9}$$

First several values of Bernoulli numbers with level 2 are given by

$$\{\mathfrak{B}_{2n}\}_{0\le n\le 10} = 1, \frac{2}{3}, \frac{62}{15}, \frac{1670}{21}, \frac{47102}{15}, \frac{6936718}{33}, \frac{29167388522}{1365}, \frac{9208191626}{3}, \frac{150996747969694}{255}, \frac{58943788779804242}{399}, \frac{7637588708954836042}{165}\,.$$

The generating function of the poly-Cauchy numbers with level 2 can be written in the form of iterated integrals ([13, Theorem 2.1]):

$$\frac{1}{\mathrm{arcsinh} x} \underbrace{\int_0^x \frac{1}{\mathrm{arcsinh} x\sqrt{1+x^2}} \cdots \int_0^x \frac{1}{\mathrm{arcsinh} x\sqrt{1+x^2}}}_{k-1} \times x \underbrace{dx \cdots dx}_{k-1} = \sum_{n=0}^{\infty} \mathfrak{C}_n^{(k)} \frac{x^n}{n!} \quad (k \ge 1)\,.$$

We can also write the generating function of the poly-Bernoulli numbers with level 2 in the form of iterated integrals.

Theorem 8. *For $k \ge 1$, we have*

$$\frac{1}{2\sin\frac{x}{2}}\underbrace{\int_0^x \frac{1}{2\tan\frac{x}{2}}\cdots\int_0^x \frac{1}{2\tan\frac{x}{2}}}_{k-1}\times\frac{1}{2}\log\frac{1+2\sin\frac{x}{2}}{1-2\sin\frac{x}{2}}\underbrace{dx\cdots dx}_{k-1}=\sum_{n=0}^\infty \mathfrak{B}_n^{(k)}\frac{x^n}{n!}\,.$$

Poly-Cauchy numbers with level 2 can be expressed explicitly in terms of the Stirling numbers of the second kind with level 2. Poly-Bernoulli numbers with level 2 can be expressed explicitly in terms of the Stirling numbers of the second kind with level 2.

Theorem 9. *For $n \ge 0$,*

$$\mathfrak{C}_{2n}^{(k)}=\sum_{m=0}^n \left[\!\!\left[{n \atop m} \right]\!\!\right]_2 \frac{(-4)^{n-m}}{(2m+1)^k}\,,$$

$$\mathfrak{B}_{2n}^{(k)}=\sum_{m=0}^n \left\{\!\!\left\{ {n \atop m} \right\}\!\!\right\}_2 \frac{(-1)^{n-m}(2m)!}{(2m+1)^k}\,.$$

4.1 Relations with poly-Cauchy numbers with level 2

Poly-Cauchy numbers with level 2 can be expressed in terms of poly-Bernoulli numbers with level 2.

Theorem 10. *For integers n and k with $n \ge 1$,*

$$\mathfrak{C}_{2n}^{(k)}=\sum_{m=1}^n\sum_{l=1}^m \frac{(-4)^{n-m}}{(2m)!}\left[\!\!\left[{n \atop m} \right]\!\!\right]_2 \left[\!\!\left[{m \atop l} \right]\!\!\right]_2 \mathfrak{B}_{2l}^{(k)}\,.$$

Remark. Poly-Cauchy numbers can be expressed in terms of poly-Bernoulli numbers ([12, Theorem 2.2]):

$$c_n^{(k)}=\sum_{m=1}^n\sum_{l=1}^m \frac{(-1)^{n-m}}{m!}\left[{n \atop m} \right]\left[{m \atop l} \right] B_l^{(k)}\,.$$

On the contrary, poly-Bernoulli numbers can be expressed in terms of poly-Cauchy numbers:

$$B_n^{(k)}=\sum_{m=1}^n\sum_{l=1}^m (-1)^{n-m}m!\left\{ {n \atop m} \right\}\left\{ {m \atop l} \right\} c_l^{(k)}\,.$$

Similarly, poly-Bernoulli numbers with level 2 can be expressed in terms of poly-Cauchy numbers with level 2.

Theorem 11. *For integers n and k with $n \ge 1$,*

$$\mathfrak{B}_{2n}^{(k)} = \sum_{m=1}^{n} \sum_{l=1}^{m} (-1)^{n-m} 4^{m-l} (2m)! \left\{\!\!\left\{ {n \atop m} \right\}\!\!\right\}_2 \left\{\!\!\left\{ {m \atop l} \right\}\!\!\right\}_2 \mathfrak{C}_{2l}^{(k)} .$$

Other relations with Stirling numbers with level 2 are given as follows.

Theorem 12. *For $n \ge 1$,*

$$\frac{1}{(2n)!} \sum_{m=0}^{n} \left[\!\!\left[{n \atop m} \right]\!\!\right]_2 \mathfrak{B}_{2m}^{(k)} = \frac{1}{(2n+1)^k} , \tag{10}$$

$$\sum_{m=0}^{n} \left\{\!\!\left\{ {n \atop m} \right\}\!\!\right\}_2 4^{n-m} \mathfrak{C}_{2m}^{(k)} = \frac{1}{(2n+1)^k} . \tag{11}$$

Remark. For poly-Bernoulli and poly-Cauchy numbers ([7, Theorem 3]), we have

$$\frac{1}{n!} \sum_{m=0}^{n} \left[{n \atop m} \right] B_m^{(k)} = \frac{1}{(n+1)^k} ,$$

$$\sum_{m=0}^{n} \left\{ {n \atop m} \right\} c_m^{(k)} = \frac{1}{(n+1)^k} .$$

Since the Stirling numbers with level 2 have an explicit expression ([1, Proposition 2.4 (xiii)],[11, (7)]):

$$\left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_2 = \frac{2}{(2k)!} \sum_{j=0}^{k} (-1)^{k-j} \binom{2k}{k-j} j^{2n} , \tag{12}$$

, we have an explicit expression of poly-Bernoulli numbers with level 2.

Proposition 1. *For $n \ge 1$,*

$$\mathfrak{B}_{2n}^{(k)} = \sum_{m=0}^{n} \sum_{j=0}^{m} \frac{2(-1)^{n-j} j^{2n}}{(2m+1)^k} \binom{2m}{m-j} .$$

In particular, Bernoulli numbers with level 2 can be expressed explicitly as

$$\mathfrak{B}_{2n} = \sum_{m=0}^{n} \sum_{j=0}^{m} \frac{2(-1)^{n-j} j^{2n}}{2m+1} \binom{2m}{m-j} .$$

Remark. Note that poly-Bernoulli numbers $\mathbb{B}_n$ can be expressed as

$$\mathbb{B}_n^{(k)} = \sum_{m=0}^{n} \sum_{j=0}^{m} \frac{(-1)^{n-j} j^n}{(m+1)^k} \binom{m}{m-j}$$

and the classical Bernoulli numbers B_n with $B_1 = -1/2$ can be expressed as

$$B_n = \sum_{m=0}^{n} \sum_{j=0}^{m} \frac{(-1)^j j^n}{(m+1)^k} \binom{m}{m-j} .$$

5 Bernoulli numbers with level 2

Bernoulli numbers $\mathfrak{B}_n = \mathfrak{B}_n^{(1)}$ *with level* 2 are given by the generating function

$$\frac{1}{4\sin(x/2)} \log \frac{1+2\sin(x/2)}{1-2\sin(x/2)} = \sum_{n=0}^{\infty} \mathfrak{B}_n \frac{x^n}{n!} . \tag{13}$$

First several values of Bernoulli numbers with level 2 are given by

$$\{\mathfrak{B}_{2n}\}_{0\le n\le 10} = 1, \frac{2}{3}, \frac{62}{15}, \frac{1670}{21}, \frac{47102}{15}, \frac{6936718}{33}, \frac{29167388522}{1365}, \frac{9208191626}{3}, \frac{150996747969694}{255}, \frac{58943788779804242}{399}, \frac{7637588708954836042}{165} .$$

Though this definition may be strange, we shall show some meaningful relations with some classical numbers.

For Bernoulli numbers, the von Staudt-Clausen theorem holds. That is, for every $n > 0$,

$$B_{2n} + \sum_{(p-1)|2n} \frac{1}{p}$$

is an integer. The sum extends over all primes p for which $p-1$ divides $2n$. For Bernoulli numbers with level 2, a similar formula holds ([11, Theorem 14]): for every $n > 0$,

$$\mathfrak{B}_{2n} + \sum_{(p-1)|2n} \frac{(-1)^{n-\frac{p-1}{2}}}{p}$$

is an integer. The sum extends over all odd primes p for which $p-1$ divides $2n$.

5.1 Glaisher's R numbers

In 1898, Glaisher introduced and studied several numbers related to Bernoulli numbers. In order to get several relations about Bernoulli numbers with level 2, first we use the numbers R_n, studied in [3, §132–138] and [4, p.51].The generating functions ([3, p.71]) of R numbers are given by

$$\frac{\cosh x}{2\cosh 2x - 1} = \frac{1+\cosh 2x}{2\cosh 3x} = \sum_{n=0}^{\infty} (-1)^n R_n \frac{x^{2n}}{(2n)!} . \tag{14}$$

The first several values of numbers R_n are given by

$$\{R_n\}_{n\ge 0} = 1, 7, 305, 33367, 6815585, 2237423527, 1077270776465,$$
$$715153093789687, 626055764653322945, 698774745485355051847, \dots$$

([20, A002437,A000364]). In [3, p.71], it is shown that

$$R_n = \frac{3^{2n+1}+1}{4}(-1)^n E_{2n}\,, \tag{15}$$

where Euler numbers E_n are defined by

$$\frac{1}{\cosh x} = \sum_{n=0}^\infty E_n \frac{x^n}{n!}\,.$$

Theorem 13. *For $n \ge 0$, we have*

$$\sum_{k=0}^n \binom{2n+1}{2k}(-4)^k \mathfrak{B}_{2k} = (-1)^n R_n\,.$$

From Theorem 13, we have a determinant expression of Bernoulli numbers with level 2.

Theorem 14. *For $n \ge 1$, we have*

$$\mathfrak{B}_{2n} = \frac{(2n)!}{4^n}\begin{vmatrix} \frac{1}{3!} & 1 & & & 0 \\ \frac{1}{5!} & \frac{1}{3!} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ \frac{1}{(2n-1)!} & \frac{1}{(2n-3)!} & \cdots & \frac{1}{3!} & 1 \\ \frac{1+(-1)^{n-1}R_n}{(2n+1)!} & \frac{1+(-1)^n R_{n-1}}{(2n-1)!} & \cdots & \frac{1-R_2}{5!} & \frac{1+R_1}{3!} \end{vmatrix},$$

where R_n are Glaisher's R numbers, given in (15).

Remark. Euler numbers of the second kind $\widehat{E}_n$, defined by

$$\frac{x}{\sinh x} = \sum_{n=0}^\infty \widehat{E}_n \frac{x^n}{n!}\,,$$

have a similar determinant expression ([9, Corollary 2.2],[10, (1.7)]).

$$\widehat{E}_{2n} = (-1)^n (2n)!\begin{vmatrix} \frac{1}{3!} & 1 & & & 0 \\ \frac{1}{5!} & \frac{1}{3!} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ \frac{1}{(2n-1)!} & \frac{1}{(2n-3)!} & \cdots & \frac{1}{3!} & 1 \\ \frac{1}{(2n+1)!} & \frac{1}{(2n-1)!} & \cdots & \frac{1}{5!} & \frac{1}{3!} \end{vmatrix}.$$

By using the inversion formula, we have the determinant expression of $1/(2n+1)!$ in terms of $\mathfrak{B}_n$.

Corollary 5. *For $n \ge 1$, we have*

$$\frac{1}{(2n+1)!} = \begin{vmatrix} \frac{4\mathfrak{B}_2}{2!} & 1 & & & 0 \\ \frac{4^2\mathfrak{B}_4}{4!} & \frac{4\mathfrak{B}_2}{2!} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ \frac{4^{n-1}\mathfrak{B}_{2n-2}}{(2n-2)!} & \frac{4^{n-2}\mathfrak{B}_{2n-4}}{(2n-4)!} & \cdots & \frac{4\mathfrak{B}_2}{2!} & 1 \\ \frac{4^n\mathfrak{B}_{2n}}{(2n)!} - \frac{R_n}{(2n+1)!} & \frac{4^{n-1}\mathfrak{B}_{2n-2}}{(2n-2)!} - \frac{R_{n-1}}{(2n-1)!} & \cdots & \frac{4^2\mathfrak{B}_4}{4!} - \frac{R_2}{5!} & \frac{4\mathfrak{B}_2}{2!} - \frac{R_1}{3!} \end{vmatrix}$$

6 Glaisher's H' numbers

Glaisher's H' numbers $\mathcal{H}_n$ ([5, §34])[2] are defined by

$$\frac{1}{2\cos x - 1} = 1 + \sum_{n=1}^{\infty} 2\mathcal{H}_n \frac{x^{2n}}{(2n)!} \tag{16}$$

and given by

$$\mathcal{H}_n = \sum_{k=1}^{2n}\sum_{j=0}^{k}\sum_{i=0}^{\lfloor (j-1)/2 \rfloor} \binom{k}{j}\binom{j}{i}(-1)^{n-j}2^{k-j}(j-2i)^{2n} \quad (n \ge 1)\,. \tag{17}$$

(*Cf.* [20, A002114]). The first several values of $\mathcal{H}_n$ are

$$\{\mathcal{H}_n\}_{n\ge 1} = 1, 11, 301, 15371, 1261501, 151846331, \\ 25201039501, 5515342166891, 1538993024478301, \dots .$$

Notice that the value for $n = 0$ may be recognized as $H_0 = 1/2$. In the next section, we shall see a nice relation with poly-Bernoulli numbers with level 2 for index 0, yielding a simper expression than the known identity (17). In fact, Glaisher's H' numbers are closely related to poly-Bernoulli numbers with level 2 with index 0.

By Proposition 1, when the index is 0, we can find a simpler relation about Glaisher's H' numbers.

Theorem 15. *For $n \ge 1$,*

$$\begin{aligned} \mathcal{H}_n &= \frac{1}{2}\mathfrak{B}_{2n}^{(0)} \\ &= \sum_{m=0}^{n}\sum_{j=0}^{m}(-1)^{n-j} j^{2n}\binom{2m}{m-j}\,. \end{aligned}$$

[2] Here we use the notation $\mathcal{H}_n$ to avoid confusion with differentiation. In fact, $\mathcal{H}_n = H_n/3$, where H_n are Glaisher's H numbers ([5, §25],[20, A002114]).

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