A CLASS OF HOLOMORPHIC DIRICHLET-HURWITZ-LERCH EISENSTEIN SERIES AND RAMANUJAN'S FORMULA FOR SPECIFIC VALUES OF THE RIEMANN ZETA-FUNCTION

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Abstract. We show in this paper that complete asymptotic expansions exist for a class of holomorphic Dirichlet-Hurwitz-Lerch Eisenstein series (Theorems 1 and 2), which, together with their remainders in exact form (Theorem 3), naturally transfer to several (additive and multiplicative) character analogues of Ramanujan's formula for specific values of the Riemann zeta-function (Theorem 4 and Corollary 4.1), and further to the (quasi) modular relations for similar character analogues of the classical Eisenstein series with integer weights (Corollary 4.2). Prior to the (sketched) derivation of our main formula, we prepare several basic (but new) results on Dirichlet-Hurwitz-Lerch L-functions (Theorems 5, 6 and Lemmas 1–3), which play underlying rôles in all aspects of the proofs; the detailed version of the proofs will appear in a forthcoming article [16].

1. Introduction

Let $\mathfrak{H}^{\pm}$ be the complex upper and lower half-planes, where the argument of each leaf is chosen as

$$\mathfrak{H}^{-} = \{z \in \mathbb{C} \mid -\pi < \arg z < 0\} \qquad \mathfrak{H}^{+} = \{z \in \mathbb{C} \mid 0 < \arg z < \pi\}.$$

Throughout the paper, s is a complex variable, α, β, μ and ν real parameters, $z \in \mathfrak{H}^{\pm}$ a complex parameter, and χ and ψ any primitive Dirichlet characters modulo $f \geq 1$ and $g \geq 1$ respectively. We frequently use the notations $e(s) = e^{2\pi i s}$, $e_h(s) = e(s/h) = e^{2\pi i s/h}$ for $h \geq 1$, $\chi_a(m) = \chi(a+m)$ and $\psi_b(n) = \psi(b+n)$ for any integers a, b, m and n, also $\varepsilon(z) = \operatorname{sgn}(\arg z)$ for $|\arg z| > 0$, and further the parameter $\tau = e^{\mp\varepsilon(z)\pi i/2} z$ for $z \in \mathfrak{H}^{\pm}$, where τ varies within the sector $|\arg \tau| < \pi/2$.

We introduce here the holomorphic Dirichlet-Hurwitz-Lerch Eisenstein series $F^{\pm}_{\chi_a,\psi_b}$ defined by

$$F^{\pm}_{\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z) = \sum_{m,n=-\infty}^{\infty}{}' \frac{\chi_a(m)\psi_b(n)e_f\{(\alpha+m)\mu\}e_f\{(\beta+n)\nu\}}{\{\alpha+m+(\beta+n)z\}^s}, \tag{1.1}$$

converging absolutely for $\sigma > 2$, where (and hereafter) the primed summation symbols indicate omission of the impossible terms of the form $1/0^s$, and the argument of each summand is chosen such that $\arg\{\alpha+m+(\beta+n)z\}$ falls within $[-\pi,\pi[$ in $F^{-}_{\chi_a,\psi_b}$, while within $]-\pi,\pi]$ in $F^{+}_{\chi_a,\psi_b}$. The main object of study is the arithmetical mean

$$F_{\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z) = \frac{1}{2}\bigl\{F^{-}_{\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z) + F^{+}_{\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z)\bigr\}, \tag{1.2}$$

for which we show that complete asymptotic expansions exist as $\tau \to \infty$ (Theorem 1) and $\tau \to 0$ (Theorem 2) both through the sector $|\arg \tau| < \pi/2$, whose proofs, by means of Mellin-Barnes type integrals, lead us to to extract exponentially small order terms from

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the remainder (Theorem 3). The asymptotic series in Theorem 2, or even its first derivative, in fact terminates up to finite terms if s is at any integer point; this combined with Theorems 1 and 3 naturally transfer to several (additive and multiplicative) character analogues of Ramanujan's formula for specific values of the Riemann zeta-function (Theorem 4 and Corollary 4.1), and also of the (quasi) modular relations for similar character analogues of the classical Eisenstein series with integer weights (Corollary 4.2). Crucial rôles in the proofs are played by some basic (but new) results on Dirichlet-Hurwitz-Lerch L-functions, which are prepared in Theorems 5, 6 and Lemmas 1–3, prior to the (sketched) derivation of our main formulae in the final section.

Let $\zeta(s)$ denote the Riemann zeta-function. Berndt [3] first studied, in the present direction of research, the case $(\mu,\nu)=(0,0)$ of (1.2), and further derived certain character analogues of Ramanujan's formula for specific values of $\zeta(s)$, where the results related to [3] have been shown, e.g, in [2][4]; the reader is referred to [15, Sect.1] for more detailed history.

The paper is organized as follows. Our main formulae (Theorems 1–3) are presented in the next section, while Section 3 is devoted to stating (additive and multiplicative) character analogues of Ramanujan's formula for specific values of $\zeta(s)$ and of (quasi) modular relations for similar character analogues of the classical Eisenstein series of integer weights. The results on Dirichlet-Hurwitz-Lerch L-functions are given in Section 4, while in the final section the proofs of our main formulae are outlined.

2. Statement of results

We prepare several notations before stating of our main results.

Let r be a complex variable, γ and κ real parameters, and c any integer. We introduce the Dirichlet-Hurwitz-Lerch L-function $K_{\chi_c}(r,\gamma,\kappa)$, together with its companion $L_{\chi_c}(r,\gamma,\kappa)$, defined by

$$K_{\chi_c}(r,\gamma,\kappa)=\sum_{-\gamma<k\in\mathbb{Z}}\frac{\chi_c(k)e_f(k\kappa)}{(\gamma+k)^r}\qquad(\operatorname{Re} r>1),\tag{2.1}$$

$$L_{\chi_c}(r,\gamma,\kappa)=\sum_{-\gamma<k\in\mathbb{Z}}\frac{\chi_c(k)e_f\{(\gamma+k)\kappa\}}{(\gamma+k)^r}=e_f(\gamma\kappa)K_{\chi_c}(r,\gamma,\kappa),\tag{2.2}$$

which reduce to the Lerch zeta-function $\phi(r,\gamma,\kappa)$ and to its companion $\psi(r,\gamma,\kappa)$ respectively if $(\chi,f)=(\iota,0)$ with the principal character ι modulo 1, and further to the Dirichlet L-function $L_{\chi_c}(r)$ if $\gamma\in\mathbb{Z}$, $\kappa\in f\mathbb{Z}$ and $c=-\gamma$; the functional equation for (2.1) or (2.2) is given in (4.7) or (4.8) below. We write $q=e(z)=e^{-2\pi\tau}$, and introduce the double q-series $\mathcal{S}_{r,\chi_c,\psi_d}(\gamma,\delta;\kappa,\lambda;q^{1/h})$, defined for any $r\in\mathbb{C}$, any $\gamma,\delta,\kappa,\lambda\in\mathbb{R}$ and any $c,d\in\mathbb{Z}$ by

$$\begin{aligned}\mathcal{S}_{r,\chi_c,\psi_d}(\gamma,\delta;\kappa,\lambda;q^{1/h})&=\sum_{\substack{-\gamma<k\in\mathbb{Z}\\-\delta<l\in\mathbb{Z}}}\frac{\chi_c(k)\psi_d(l)e_f\{(\gamma+k)\kappa\}e_g\{(\delta+l)\lambda\}}{(\delta+l)^r}q^{(\gamma+k)(\delta+l)/h}\\&=g^{-r}\sum_{i=0}^{f-1}\sum_{j=0}^{g-1}\chi_c(i)\psi_d(j)\mathcal{S}_r\Big(\frac{\gamma+i}{f},\frac{\delta+j}{g};\kappa,\lambda;q^{fg/h}\Big),\end{aligned}$$

where $\mathcal{S}_r(\gamma,\delta;\kappa,\lambda;q)$ is the double q-series or generalized Lambert series (defined without characters) of the form

$$\begin{aligned}\mathcal{S}_r(\gamma,\delta;\kappa,\lambda;q) &= \sum_{\substack{-\gamma<k\in\mathbb{Z}\\ -\delta<l\in\mathbb{Z}}} \frac{e\{(\gamma+k)\kappa+(\delta+l)\lambda\}}{(\delta+l)^r} q^{(\gamma+k)(\delta+l)}\\ &= e(\langle\gamma\rangle'\kappa)\sum_{-\delta<l\in\mathbb{Z}} \frac{e\{(\delta+l)\lambda\}q^{\langle\gamma\rangle'(\delta+l)}}{(\delta+l)^r\{1-e(\kappa)q^{\delta+l}\}}.\end{aligned}$$

Let $G_\chi = \sum_{h=0}^{f-1}\chi(h)e_f(h)$ denote Gauß' sum associated with any primitive Dirichlet character χ, $\Gamma(s)$ the gamma function, $(s)_k = \Gamma(s+k)/\Gamma(s)$ for any $k\in\mathbb{Z}$ the rising factorial of s. The convention $\chi_c(x)=0$ for any $x\in\mathbb{R}\setminus\mathbb{Z}$ and for any (shifted) character χ_c modulo $f\geq 1$ is used throughout the following, and hence $\delta(x)=\iota(x)$ for any $x\in\mathbb{R}$ denotes the symbol which equals 1 or 0 according to $x\in\mathbb{Z}$ or otherwise.

We now state our first main result, which gives a transformation formula for (1.2).

Theorem 1. *Set*

$$\begin{aligned}(2.3)\qquad \mathcal{A}_{\chi_a}(s,\alpha,\mu) &= \chi(-1)\cos(\pi s)L_{\chi_{-a}}(s,-\alpha,-\mu)+L_{\chi_a}(s,\alpha,\mu)\\ &= e_f\{(\alpha-a)\mu\}G_\chi\frac{(2\pi/f)^s}{2\Gamma(s)}\big\{\chi(-1)e^{\pi is/2}L_{\overline{\chi}}\big(1-s,\mu,-(\alpha-a)\big)\\ &\quad + e^{-\pi is/2}L_{\overline{\chi}}(1-s,-\mu,\alpha-a)\big\},\end{aligned}$$

where the second equality is derived by the functional equation for $L_{\chi_c}(r,\gamma,\kappa)$. Then for any $z\in\mathfrak{H}^+$, any $\alpha,\beta,\mu,\nu\in\mathbb{R}$ and any $a\in\mathbb{Z}$, on the whole s-plane we have

$$\begin{aligned}(2.4)\quad F_{\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z) &= \psi_b(-\beta)\mathcal{A}_{\chi_a}(s,\alpha,\mu)+e_f\{(\alpha-a)\mu\}G_\chi\frac{(2\pi/f)^s}{\Gamma(s)}\\ &\quad\times\big\{e^{-\pi is/2}\mathcal{S}_{1-s,\psi_b,\overline{\chi}}(\beta,-\mu;\nu,\alpha-a;q^{1/f})\\ &\quad+\chi(-1)\psi(-1)e^{\pi is/2}\mathcal{S}_{1-s,\psi_{-b},\overline{\chi}}(-\beta,\mu;-\nu,-(\alpha-a);q^{1/f})\big\},\end{aligned}$$

the right side of which provides the holomorphic continuation of the left side to the whole s-plane.

Remark. The q-series $\mathcal{S}_{1-s,\psi_{\pm b},\overline{\chi}}(\pm\beta,\mp\mu;\pm\nu,\pm(\alpha-a);q^{1/f})$ on the right side of (2.4) give the (convergent) asymptotic expansion as $\tau\to\infty$ through $|\arg\tau|<\pi/2$, since each term of the series is of order $O[\exp\{-2\pi\langle\mp\beta\rangle'(\mp\mu+m)\tau/f\}]$ as $\tau\to\infty$ (for $m>\pm\mu$).

Next let $\widetilde{\mathbb{C}^\times}$ denote the universal covering of the punctured complex plane $\mathbb{C}^\times=\mathbb{C}\setminus\{0\}$, where the mapping $\widetilde{\mathbb{C}^\times}: \widetilde{X}\mapsto\log\widetilde{Y}=\log|\widetilde{Y}|+i\arg\widetilde{Y}\in\mathbb{C}$ is bijective (with the range of $\arg\widetilde{Y}$ being extended over $\mathbb{R}$). We define for any $X\in\mathbb{C}$ and $\widetilde{Y}\in\widetilde{\mathbb{C}^\times}$ the operation

$$\begin{aligned}(2.5)\qquad \widetilde{\mathbb{C}^\times}\ni\widetilde{Y}\longmapsto\widetilde{Y}^X &= \exp(X\log\widetilde{Y})=\exp\{X(\log|\widetilde{Y}|+i\arg\widetilde{Y})\}\\ &= |\widetilde{Y}|^X\exp(X\arg\widetilde{Y})\in\mathbb{C}.\end{aligned}$$

Let $\widetilde{e}(\kappa)$ for any $\kappa\in\mathbb{R}$ denote the point defined by $\log\widetilde{e}(\kappa)=2\pi i\kappa$, and write $\widetilde{e}(0)=\widetilde{1}$. Then $\widetilde{e}(\kappa)^\gamma=e(\gamma\kappa)$ holds for all $\gamma\in\mathbb{R}$ by (2.5).

We introduce here the generating function $F_{\chi_c}(X,\widetilde{Y};Z)$ for $(X,\widetilde{Y})\in\mathbb{C}\times\widetilde{\mathbb{C}^\times}$ with the variable $Z\in\mathbb{C}$, defined by

$$F_{\chi_c}(X,\widetilde{Y};Z)=\sum_{h=0}^{f-1}\chi_c(h)\frac{Z\widetilde{Y}^{(X+h)/f}e^{(X+h)Z}}{\widetilde{Y}^1e^{fZ}-1}. \tag{2.6}$$

It is convenient for describing specific values of $L_{\chi_c}(r,\gamma,\kappa)$ at integer points to use the sequence of functions $\mathcal{C}_{k,\chi_c}:\mathbb{C}\times\widetilde{\mathbb{C}^\times}\ni(X,\widetilde{Y})\mapsto\mathcal{C}_{k,\chi_c}(X,\widetilde{Y})\in\mathbb{C}$ $(k=0,1,\dots)$ defined by the Taylor series expansion

$$F_{\chi_c}(X,\widetilde{Y};Z)=\sum_{k=0}^{\infty}\frac{\mathcal{C}_{k,\chi_c}(X,\widetilde{Y})}{k!}Z^k, \tag{2.7}$$

centered at $Z=0$; this reduces to the usual Bernoulli polynomial $B_k(X)$ if $(\chi,f)=(\iota,1)$ and $\widetilde{Y}=\widetilde{1}$, and to the Bernoulli number B_{k,χ_c}, associated with a shifted primitive character χ_c, if $(X,\widetilde{Y})=(0,\widetilde{1})$. We in partucular write $\mathcal{C}_k(X,\widetilde{Y})=\mathcal{C}_{k,\iota}(X,\widetilde{Y})$ for any $(X,\widetilde{Y})\in\mathbb{C}\times\widetilde{\mathbb{C}^\times}$ $(k=0,1,\dots)$ (cf. [11, Sect.6, (6.2)][15, Sect.2, (2.15)]). It is then seen from (2.6) and (2.7) that

$$\mathcal{C}_{k,\chi_c}(X,\widetilde{Y})=f^{k-1}\sum_{h=0}^{f-1}\chi_c(h)\mathcal{C}_k\Bigl(\frac{X+h}{f},\widetilde{Y}\Bigr)\qquad(k=0,1,\dots). \tag{2.8}$$

The reciprocal relations for $\mathcal{C}_{k,\chi_c}(X,\widetilde{Y})$ are shown in (4.1) and (4.2) below.

We next state our second main result, which gives the asymptotic expansion for (1.2) as $\tau\to0$ through the sector $|\arg\tau|<\pi/2$.

Theorem 2. *Let $\alpha,\beta,\mu,\nu\in\mathbb{R}$ be any parameters, χ_a and ψ_b with any $a,b\in\mathbb{Z}$ arbitrary (shifted) primitive Dirichlet character modulo $f\geq1$ and $g\geq1$ respectively, and write $z=e^{\pi i/2}\tau$ with $|\arg\tau|<\pi/2$ for any $z\in\mathfrak{H}^+$. Set*

$$\begin{aligned}\mathcal{B}_{1,\chi_a}(s,\alpha,\mu)&=i\chi(-1)\sin(\pi s)L_{\chi_{-a}}(s,-\alpha,-\mu)\\&=ie_f\{(\alpha-a)\mu\}G_\chi\frac{(2\pi/f)^s}{2\Gamma(s)}\bigl\{e^{\pi i(1-s)/2}L_{\overline{\chi}}(1-s,-\mu,\alpha-a)\\&\quad+\chi(-1)e^{-\pi i(1-s)/2}L_{\overline{\chi}}(1-s,\mu,-(\alpha-a))\bigr\},\end{aligned} \tag{2.9}$$

$$\begin{aligned}\mathcal{B}_{2,\psi_b}(s,\beta,\nu)&=\psi(-1)e^{\pi is/2}L_{\psi_{-b}}(s,-\beta,-\nu)+e^{-\pi is/2}L_{\psi_b}(s,\beta,\nu)\\&=\psi(-1)e_g\{(\beta-b)\nu\}G_\psi\frac{(2\pi/g)^s}{\Gamma(s)}L_{\overline{\psi}}(1-s,\nu,-(\beta-b)),\end{aligned} \tag{2.10}$$

where the equalities in the second and fourth lines follow from the functional equation for $L_{\chi_c}(r,\gamma,\kappa)$. Then for any integer $J\geq0$, in the region $\sigma>-J$ we have

$$\begin{aligned}F_{\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z)&=\psi_b(-\beta)\mathcal{B}_{1,\chi_a}(s,\alpha,\mu)+\chi_a(-\alpha)\mathcal{B}_{2,\psi_b}(s,\beta,\nu)\tau^{-s}\\&\quad+S_{J,\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z)+R_{J,\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z),\end{aligned} \tag{2.11}$$

where S_{J,χ_a,ψ_b} is the asymptotic series of the form

$$\begin{aligned}S_{J,\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z)&=2\chi(-1)\sin(\pi s)\sum_{j=-1}^{J-1}\frac{i^{j+1}(s)_j}{(j+1)!}L_{\chi_{-a}}(s+j,-\alpha,-\mu)\\&\quad\times\mathcal{C}_{j+1,\psi_{b-\lfloor\beta\rfloor}}(\langle\beta\rangle,\widetilde{e}(\nu))\tau^j,\end{aligned} \tag{2.12}$$

and R_{J,χ_a,ψ_b} is the remainder satisfying the estimate

$$R_{J,\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z)=O(|\tau|^J) \tag{2.13}$$

as $\tau\to 0$ through the sector $|\arg\tau|\le\pi/2-\eta$ with any small $\eta>0$.

Let $\widehat{q}=e(i/\tau)=e(-1/z)=e^{-2\pi/\tau}$, and ${}_1F_1\left({\kappa \atop \lambda};Z\right)$ and $U(\kappa;\lambda;Z)$ denote Kummer's confluent hypergeometric function of the first and second kind, defined respectively by

$${}_1F_1\left({\kappa \atop \lambda};Z\right)=\sum_{k=0}^{\infty}\frac{(\kappa)_k}{(\lambda)_k k!}Z^k$$

for any $(\kappa,\lambda)\in\mathbb{C}\times(\mathbb{C}\setminus\mathbb{Z}_{\le 0})$ and $|Z|<+\infty$, and

$$U(\kappa;\lambda;Z)=\frac{1}{\Gamma(\kappa)\{e(\kappa)-1\}}\int_{\infty}^{(0+)}e^{-Zw}w^{\kappa-1}(1+w)^{\lambda-\kappa-1}dw$$

for any $(\kappa,\lambda)\in\mathbb{C}^2$ and $|\arg Z|<\pi/2$, where the latter can be continued to the whole sector $|\arg Z|<3\pi/2$ by rotating appropriately the path of integration (cf. [7, p.273, 6.11.2(9)]).

An application of the connection formula

$$\begin{aligned}{}_1F_1\left({\kappa \atop \lambda};Z\right)&=\frac{\Gamma(\lambda)}{\Gamma(\lambda-\kappa)}e^{\varepsilon(Z)\pi i\kappa}U(\kappa;\lambda;Z)+\frac{\Gamma(\lambda)}{\Gamma(\kappa)}e^{\varepsilon(Z)\pi i(\kappa-\lambda)}e^Z\\&\quad\times U(\lambda-\kappa;\lambda;e^{-\varepsilon(Z)\pi i}Z)\end{aligned} \tag{2.14}$$

for $0<|\arg Z|<\pi$ (cf. [7, p.259, 6.7(7)][14, Sect.10, (10.5)]) leads us to extract the exponentially small order terms $\mathcal{S}_{1-s,\chi,\overline{\psi}}(\pm(\alpha-a),\pm\nu;\pm\mu,\mp(\beta-b);\widehat{q}^{1/g})$ (as $\tau\to 0$) from the remainder in (2.11) or (2.13); this eventually yields the following Theorem 3.

Theorem 3. *In the region $\sigma>1-J$ with any $J\ge 1$ and in the sectors $0<|\arg\tau|<\pi/2$, we have the formula*

$$\begin{aligned}&R_{J,\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z)\\&=e_g\{(\beta-b)\nu\}G_\psi\frac{(2\pi/g\tau)^s}{\Gamma(s)}\{\psi(-1)\mathcal{S}_{1-s,\chi,\overline{\psi}}(\alpha-a,\nu;\mu,-(\beta-b);\widehat{q}^{1/g})\\&\quad+\chi(-1)e^{\varepsilon(\tau)\pi is}\mathcal{S}_{1-s,\chi,\overline{\psi}}(-(\alpha-a),-\nu;-\mu,\beta-b;\widehat{q}^{1/g})\}\\&\quad+\chi(-1)(-1)^Je_g\{(\beta-b)\nu\}G_\psi\frac{(s)_J(2\pi/g\tau)^s}{\Gamma(s)\Gamma(1-s)}S^*_{J,\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z),\end{aligned} \tag{2.15}$$

where the expression

$$\begin{aligned}&S^*_{J,\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z)\\&=\psi(-1)\sum_{\substack{\alpha-a<m\\-\nu<n}}\frac{\chi(m)\overline{\psi}(n)e_f\{(-(\alpha-a)+m)(-\mu)\}e_g\{(\nu+n)(-(\beta-b))\}}{(\nu+n)^{1-s}}\\&\quad\times F_{s,J}\{2\pi(-(\alpha-a)+m)(\nu+n)/g\tau\}\\&\quad-e^{\varepsilon(\tau)\pi is}\sum_{\substack{\alpha-a<m\\\nu<n}}\frac{\chi(m)\overline{\psi}(n)e_f\{(-(\alpha-a)+m)(-\mu)\}e_g\{(-\nu+n)(\beta-b)\}}{(-\nu+n)^{1-s}}\\&\quad\times F_{s,J}\{2\pi e^{\varepsilon(\tau)\pi i}(-(\alpha-a)+m)(-\nu+n)/g\tau\}\end{aligned} \tag{2.16}$$

holds with

$$F_{s,J}(Z)=U(s+J;s+J;Z). \tag{2.17}$$

Furthermore, for any integers J and K with $J\geq 1$ and $K\geq 0$, in the region $\sigma>1-J-K$, we have the formula

$$\begin{aligned} &S^*_{J,\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z) \\ &\quad=\frac{\psi(-\varepsilon(\tau))e_g\{-(\beta-b)\nu\}\overline{G_\psi}}{(2\pi/e^{\varepsilon(\tau)\pi i/2}g)^{s-1}}\sum_{k=0}^{K-1}\frac{(\varepsilon(\tau)i)^{J+k+1}(s+J)_k}{(J+k+1)!} \\ &\qquad\times L_{\overline{\chi}}(s+J+k,-(\alpha-a),-\mu)\mathcal{C}_{J+k+1,\psi_{-\varepsilon(\tau)b-\lfloor\varepsilon(\tau)\beta\rfloor}}(\langle\varepsilon(\tau)\beta\rangle,\widetilde{e}(\varepsilon(\tau)\nu)) \\ &\qquad\times(e^{\varepsilon(\tau)\pi i/2}\tau)^{s+J+k}+R^*_{J,K,\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z)\end{aligned} \tag{2.18}$$

in the sectors $0<|\arg\tau|<\pi/2$, where R^*_{J,K,χ_a,ψ_b} is the remainder satisfying the estimate

$$R^*_{J,K,\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z)=O(|\tau|^{\sigma+J+K}) \tag{2.19}$$

as $\tau\to 0$ through $\eta\leq|\arg\tau|\leq\pi/2-\eta$ with any small $\eta>0$. Here the constant implied in the O-symbol depends at most on s, a, b, μ, ν, J, K and η.

Remark. The formulae for R_{J,χ_a,ψ_b} and S^*_{J,χ_a,ψ_b} above in fact reveal that the instances of 'exponentially improved asymptotics' and 'Stokes' phenomena' respectively, which are normally observed in the theory of differential equations in the complex domain, also occur in the present situation of generalized holomorphic Eisenstein series.

3. Character analogues of Ramanujan's formula and of classical Eisenstein series

The combination of Theorems 1 and 2 with Theorem 3 in fact yields several character analogues of Ramanujan's formula for $\zeta(2k+1)$ as well as (quasi) modular relations for the classical Eisenstein series of integer weights.

Theorem 4. *Let k be any integer, α,β,μ and ν any real parameter, χ and ψ the primitive Dirichlet characters modulo $f\geq 1$ and $g\geq 1$ respectively, a and b be any integers, write $q=e^{-2\pi\tau}$ and $\widehat{q}=e^{-2\pi/\tau}$, and suppose further that $f,g\geq 2$ or $\alpha,\beta\notin\mathbb{Z}$ if $k=1$. Then we have*

$$\begin{aligned} &e_f\{(\alpha-a)\mu\}f^{k-1}G_\chi\Big\{\psi_b(-\beta)L_{\overline{\chi}}(k,-\mu,\alpha-a)+\mathcal{S}_{k,\psi_b,\overline{\chi}}\big(\beta,-\mu;\nu,\alpha-a;q^{1/f}\big) \\ &\qquad+(-1)^{k-1}\chi(-1)\psi(-1)\mathcal{S}_{k,\psi_{-b},\overline{\chi}}\big(-\beta,\mu;-\nu,-(\alpha-a);q^{1/f}\big)\Big\} \\ &\qquad-(-2\pi)^k\sum_{j=0}^{k+1}\frac{(-i)^j\mathcal{C}_{j,\chi_{a-\lfloor\alpha\rfloor}}(\langle\alpha\rangle,\widetilde{e}(\mu))\mathcal{C}_{k+1-j,\psi_{b-\lfloor\beta\rfloor}}(\langle\beta\rangle,\widetilde{e}(\nu))}{j!(k+1-j)!}\tau^{k-j} \\ &\quad=\psi(-1)e_g\{(\beta-b)\nu\}(-ig\tau)^{k-1}G_\psi \\ &\qquad\times\Big\{\chi_a(-\alpha)L_{\overline{\psi}}\big(k,\nu,-(\beta-b)\big)+\mathcal{S}_{k,\chi,\overline{\psi}}(\alpha-a,\nu;\mu,-(\beta-b);\widehat{q}^{1/g}) \\ &\qquad+(-1)^{k-1}\chi(-1)\psi(-1)\mathcal{S}_{k,\chi,\overline{\psi}}\big(-(\alpha-a),-\nu;-\mu,\beta-b;\widehat{q}^{1/g}\big)\Big\},\end{aligned} \tag{3.1}$$

which is transformed through the replacement $(\tau, q, \chi, \psi) \mapsto (1/\tau, \widehat{q}, \overline{\chi}, \overline{\psi})$ *into*

$$
\begin{aligned}
(3.2)\qquad & e_g\{(\beta-b)\nu\}g^{k-1}\overline{G_\psi}\Big\{\overline{\chi}_a(-\alpha)L_\psi(k,\nu,-(\beta-b)) \\
&+\mathcal{S}_{k,\overline{\chi},\psi}\big(\alpha-a,\nu,\mu;-(\beta-b);q^{1/g}\big) \\
&+(-1)^{k-1}\chi(-1)\psi(-1)\mathcal{S}_{k,\overline{\chi},\psi}\big(-(\alpha-a),-\nu;-\mu,\beta-b;q^{1/g}\big)\Big\} \\
&-(-2\pi)^k\sum_{j=0}^{k+1}\frac{i^j\mathcal{C}_{j,\overline{\psi}_{b-\lfloor\beta\rfloor}}(\langle\beta\rangle,\widetilde{e}(\nu))\mathcal{C}_{k+1-j,\overline{\chi}_{a-\lfloor\alpha\rfloor}}(\langle\alpha\rangle,\widetilde{e}(\mu))}{j!(k+1-j)!}\tau^{k-j} \\
&=\chi(-1)e_f\{(\alpha-a)\mu\}(if\tau)^{k-1}\overline{G_\chi}\Big\{\overline{\psi}_b(-\beta)L_\chi(k,-\mu,\alpha-a) \\
&+\mathcal{S}_{k,\overline{\psi}_b,\chi}\big(\beta,-\mu;\nu,\alpha-a;\widehat{q}^{1/f}\big) \\
&+(-1)^{k-1}\chi(-1)\psi(-1)\mathcal{S}_{k,\overline{\psi}_{-b},\chi}\big(-\beta,\mu;-\nu,-(\alpha-a);\widehat{q}^{1/f}\big)\Big\}.
\end{aligned}
$$

Remark. It is worth noting here that a hidden (but crucial) rôle is played by the connection formula (2.14) in producing various Ramanujan type formulae as in Theorem 4 and Corollary 4.1 below for specific values of the Riemann zeta-function.

Let $L_{\chi_c,\kappa}(r)$ for any $\kappa \in \mathbb{R}$ and $c \in \mathbb{Z}$ denote the exponential Dirichlet L-function attached to a (shifted) primitive Dirichlet character χ_c modulo $f(\geq 1)$, defined by

$$L_{\chi_c,\kappa}(r) = L_{\chi_c}(r,0,\kappa) = \sum_{k=1}^{\infty}\frac{\chi_c(k)e_f(k\kappa)}{k^r} \qquad (\rho = \operatorname{Re} r > 1),$$

and B_{k,χ_c} $(k = 0, 1, \ldots)$ the k-th Bernoulli number attached to χ_c, defined by

$$B_{k,\chi_c} = \mathcal{C}_{k,\chi_c}(0,\widetilde{1}) = f^{k-1}\sum_{h=0}^{f-1}\chi_c(h)B_k\Big(\frac{h}{f}\Big). \qquad (k = 0, 1, \ldots).$$

The case $(\alpha, \beta, \mu, \nu) = (0, 0, 0, 0)$ of Theorem 4 implies the following result.

Corollary 4.1. *Let k be any integer, χ and ψ any primitive Dirichlet characters modulo $f \geq 1$ and $g \geq 1$ respectively, a and b any integer, and suppose further that $f, g \geq 2$ or $\alpha, \beta \notin \mathbb{Z}$ if $k = 1$. The we have*

$$
\begin{aligned}
(3.3)\qquad & f^{k-1}G_\chi\Big\{\psi(b)L_{\overline{\chi},-a}(k)+\mathcal{S}_{k,\psi_b,\overline{\chi}}\big(0,0;0,-a;q^{1/f}\big)+(-1)^{k-1}\chi(-1)\psi(-1) \\
&\times\mathcal{S}_{k,\psi_{-b},\overline{\chi}}\big(0,0;0,a;q^{1/f}\big)\Big\}-(-2\pi)^k\sum_{j=0}^{k+1}\frac{(-i)^jB_{j,\chi_a}B_{k+1-j,\psi_b}}{j!(k+1-j)!}\tau^{k-j} \\
&=\psi(-1)(-i\tau g)^{k-1}G_\psi\Big\{\chi(a)L_{\overline{\psi},b}(k)+\mathcal{S}_{k,\chi,\overline{\psi}}\big(-a,0;0,0;\widehat{q}^{1/g}\big) \\
&+(-1)^{k-1}\chi(-1)\psi(-1)\mathcal{S}_{k,\chi,\overline{\psi}}\big(a,0;0,0;\widehat{q}^{1/g}\big)\Big\},
\end{aligned}
$$

which is transformed through the replacement $(\tau, q, \chi, \psi) \mapsto (1/\tau, \widehat{q}, \overline{\chi}, \overline{\psi})$ *into*

$$g^{k-1}\overline{G_\psi}\Big\{\overline{\chi}(a)L_{\psi,b}(k) + \mathcal{S}_{k,\overline{\chi},\psi}\big(-a,0;0,0;q^{1/g}\big) + (-1)^{k-1}\chi(-1)\psi(-1) \times \mathcal{S}_{k,\overline{\chi},\psi}\big(a,0;0,0;q^{1/g}\big)\Big\} - (-2\pi)^k \sum_{j=0}^{k+1} \frac{i^j B_{j,\overline{\psi}_b} B_{k+1-j,\overline{\chi}_a}}{j!(k+1-j)!}\tau^{k-j}$$
$$= \chi(-1)(i\tau f)^{k-1}\overline{G_\chi}\Big\{\overline{\psi}(b)L_{\chi,-a}(k) + \mathcal{S}_{k,\overline{\psi}_b,\chi}\big(0,0;0,-a;\widehat{q}^{1/f}\big) + (-1)^{k-1}\chi(-1)\psi(-1)\mathcal{S}_{k,\overline{\psi}_{-b},\chi}\big(0,0;0,a;\widehat{q}^{1/f}\big)\Big\}. \tag{3.4}$$

Remark. The values of exponential Dirichlet L-functions appear in (3.3) or (3.4) only whien $(a, f) = 1$ or $(b, g) = 1$, othewise the formula merely gives the relation between the double q-series and the polynomials (in τ) with the coeficients of the Bernoulli numbers attached to Dirichlet characters; the original Dirichlet L-function cases are thus excluded.

We next state the (quasi) modular relations of generalized Eisenstein series. Let β and ν are any real parameters, χ any primitive Dirichlet character modulo $f \geq 1$, and define

$$a_{k,\chi} = \begin{cases} 2\zeta'(0) & \text{if } (\chi, f, k) = (\iota, 1, 0), \\ L(1-k,\chi) & \text{otherwise,} \end{cases}$$

$$b_k(\nu,\beta) = \begin{cases} 2\mathcal{C}_1(0,\widetilde{e}(\nu)) + 1 + \zeta'_\nu(0) + \zeta'_{-\nu}(0) & \text{if } k = 0 \text{ and } \beta \in \mathbb{Z}, \\ \phi(1-k,\nu,-\beta) & \text{otherwise,} \end{cases}$$

where $\zeta_\kappa(r) = \phi(r, 0, \kappa) = \psi(r, \gamma, \kappa)$ for $\gamma \in \mathbb{Z}$ denotes the exponential zeta-function. We then introduce the generalizations, $E_{j,k,\chi}(\beta,\nu;z)$ $(j = 1, 2)$ for any $k \in \mathbb{Z}$ satisfying $(-1)^k = \chi(-1)$, of the classical Eisenstein series $E_k(z)$, defined by

$$E_{1,k,\chi}(\beta,\nu;z) = 1 + \frac{2}{a_{k,\chi}}\mathcal{T}_{1,k,\chi}(\beta,\nu;q), \tag{3.5}$$

$$E_{2,k,\chi}(\beta,\nu;z) = \delta_{f1}\frac{b_k(\nu,\beta)}{a_{k,\iota}} + \frac{2f^k}{a_{k,\overline{\chi}}G_\chi}\mathcal{T}_{2,k,\chi}(\beta,\nu;q), \tag{3.6}$$

where $\mathcal{T}_{j,k,\chi}(\beta,\nu;q)$ $(j = 1, 2)$ are the generalized Lambert series of the form

$$\mathcal{T}_{1,k,\chi}\big(\beta,\nu;q\big) = \frac{1}{2}\Big\{\mathcal{S}_{1-k,\iota,\chi}\big(\beta,0;\nu,0;q^{1/f}\big) + \mathcal{S}_{1-k,\iota,\chi}(-\beta,0;-\nu,0;q^{1/f})\Big\}$$
$$= \frac{f^{k-1}}{2}\left[\sum_{h=1}^{f}\chi(h)\sum_{n=0}^{\infty}\frac{e(\langle\beta\rangle'\nu)(h/f+n)^{k-1}q^{\langle\beta\rangle'(h/f+n)}}{1-e(\nu)q^{h/f+n}} + \sum_{h=1}^{f}\chi(h)\sum_{n=0}^{\infty}\frac{e\{\langle-\beta\rangle'(-\nu)\}(h/f+n)^{k-1}q^{\langle-\beta\rangle'(h/f+n)}}{1-e(-\nu)q^{h/f+n}}\right], \tag{3.7}$$

$$\mathcal{T}_{2,k,\chi}(\beta,\nu;q) = \frac{e(\beta\nu)}{2}\Big\{\mathcal{S}_{1-k,\chi,\iota}\big(0,\nu;0,-\beta;q\big) + \mathcal{S}_{1-k,\chi,\iota}\big(0,-\nu;0,\beta;q\big)\Big\}$$
$$= \frac{e(\beta\nu)}{2}\left[\sum_{h=1}^{f}\chi(h)\sum_{n=0}^{\infty}\frac{e\{(\langle\nu\rangle'+n)(-\beta)\}(\langle\nu\rangle'+n)^{k-1}q^{h(\langle\nu\rangle'+n)}}{1-q^{f(\langle\nu\rangle'+n)}} + \sum_{h=1}^{f}\chi(h)\sum_{n=0}^{\infty}\frac{e\{(\langle-\nu\rangle'+n)\beta\}(\langle-\nu\rangle'+n)^{k-1}q^{h(\langle-\nu\rangle'+n)}}{1-q^{f(\langle-\nu\rangle'+n)}}\right]. \tag{3.8}$$

Then the combination of Theorems 1 and 2 with Theorem 3 yields the following (quasi) modular relation.

Corollary 4.2. *For any real parameters β and ν, for any primitive Dirichlet character χ modulo $f \geq 1$, any integer k satisfying $(-1)^k = \chi(-1)$, and for any $z \in \mathfrak{H}^+$ we have*

$$E_{1,k,\overline{\chi}}\Big(\beta, \nu; -\frac{1}{z}\Big) = z^k E_{2,k,\chi}(\beta, \nu; z) + \frac{(-2\pi i)^{1-k}}{a_{k,\chi} G_\chi}\Bigg[\delta_{f1}\delta_{k0}\Big\{\delta(\beta)\Big(\frac{\log z}{2\pi i} - \frac{1}{4}\Big) - \frac{1}{4}\Big\} + \sum_{j=0}^{2-k} \frac{(-1)^j B_{j,\chi} \mathcal{C}_{2-k-j}(\langle\beta\rangle, \widetilde{e}(\nu))}{j!(2-k-j)!} z^{j+k-1}\Bigg], \tag{3.9}$$

which reduces in particular when $(\beta, \nu) = (0, 0)$ to

$$E_{1,0,\overline{\chi}}\Big(0, 0; -\frac{1}{z}\Big) = E_{2,0,\chi}(0, 0; z) - \frac{2\pi i}{a_{0,\overline{\chi}} G_\chi}\Big\{\delta_{f1}\Big(\frac{\log z}{2\pi i} - \frac{1}{2} + \frac{1}{12z}\Big) + B_{2,\chi} z\Big\} \tag{3.10}$$

for $k = 0$, while for $k = 1$ to

$$E_{1,1,\chi}\Big(0, 0; -\frac{1}{z}\Big) = z E_{2,1,\chi}(0, 0; z) + \frac{B_{1,\chi} z}{B_{1,\overline{\chi}} G_\chi}, \tag{3.11}$$

and for $k = 2$ to

$$E_{1,2,\overline{\chi}}\Big(0, 0; -\frac{1}{z}\Big) = z^2 E_{2,2,\chi}(0, 0; z) + \delta_{f1}\frac{6z}{\pi i}. \tag{3.12}$$

4. Basic properties of Dirichlet-Hurwitz-Lerch L-functions

We present in this section several basic (but new) properties Dirichlet-Hurwitz-Lerch L-functions, together with several associated results, which play underlying rôles in establishing our main formulae. Let δ_{hk} denote hereafter Kronecker's symbol. Then the reciprocal relations for $\mathcal{C}_{k,\chi_c}(X, \widetilde{Y})$ are given as follows.

Lemma 1. *For any $(X, \widetilde{Y}) \in \mathbb{C} \times \widetilde{\mathbb{C}^\times}$, and any $c, k \in \mathbb{Z}$ with $k \geq 0$ we have*

$$\mathcal{C}_{k,\chi_{1-c}}(1 - X, \widetilde{1}/\widetilde{Y}) = (-1)^k \chi(-1) \mathcal{C}_k(X, \widetilde{Y}), \tag{4.1}$$

$$\begin{aligned} \mathcal{C}_{k,\chi_{-c}}(0, \widetilde{1}/\widetilde{Y}) &= (-1)^k \chi(-1)\big\{\mathcal{C}_{k,\chi_c}(X, \widetilde{Y}) + \delta_{k1}\chi_c(0)\big\} \\ &= (-1)^k \chi(-1) \mathcal{C}_{k,\chi_c}(X, \widetilde{Y}) - \delta_{k1}\chi_{-c}(0). \end{aligned} \tag{4.2}$$

where $\widetilde{1}/\widetilde{Y} \in \widetilde{\mathbb{C}^\times}$ is the point defined by $|\widetilde{1}/\widetilde{Y}| = 1/|\widetilde{Y}|$ and $\arg(\widetilde{1}/\widetilde{Y}) = -\arg \widetilde{Y}$.

Proof. First (4.1) is obtained by comparing the coefficients of Z^k on both sides of the equality

$$F_{\chi_{1-c}}(1 - X, \widetilde{1}/\widetilde{Y}; Z) = \chi(-1) F_{\chi_c}(X, \widetilde{Y}; -Z), \tag{4.3}$$

which is derived as follows: The left side of (4.3) equals

$$\begin{aligned} \sum_{h=0}^{f-1} \chi_{1-c}(h) \frac{Z(\widetilde{1}/\widetilde{Y})^{(1-X+h)/f} e^{(1-X+h)Z}}{(\widetilde{1}/\widetilde{Y})^1 e^{fZ} - 1} &= \sum_{h=0}^{f-1} \chi(-h-c) \frac{Z\widetilde{Y}^{(X+h-f)/f} e^{(f-X-h)Z}}{\widetilde{Y}^{-1} e^{fZ} - 1} \\ &= \sum_{h=0}^{f-1} \chi(-1)\chi_c(h) \frac{Z\widetilde{Y}^{(X+h)/f} e^{-(X+h)Z}}{1 - \widetilde{Y}^1 e^{-fZ}}, \end{aligned}$$

where the summation index is changed as $h \mapsto f-1-h$ in the second member; this readily concludes (4.1).

Next (4.2) is obtained by comparing the coefficients of Z^k on both sides of the equality

$$F_{\chi_{-c}}(0, \widetilde{1}/\widetilde{Y}; Z) = \chi(-1)\bigl\{F_{\chi_c}(0, \widetilde{Y}; -Z) + \chi_c(0)Z\bigr\}, \tag{4.4}$$

which is derived as follows. The left side of (4.4) equals

$$\begin{aligned}
&\sum_{h=0}^{f-1} \chi_{-c}(h) \frac{Z(\widetilde{1}/\widetilde{Y})^{h/f} e^{hZ}}{(\widetilde{1}/\widetilde{Y})^1 e^{fZ} - 1} \\
&\quad = \sum_{h=1}^{f} \chi(-h-c) \frac{Z\widetilde{Y}^{h/f-1} e^{(f-h)Z}}{\widetilde{Y}^{-1} e^{fZ} - 1} = \sum_{h=1}^{f} \chi(-1)\chi_c(h) \frac{(-Z)\widetilde{Y}^{h/f} e^{-hZ}}{\widetilde{Y}^1 e^{-fZ} - 1} \\
&\quad = \chi(-1)\biggl\{\sum_{h=1}^{f-1} \chi_c(h) \frac{(-Z)\widetilde{Y}^{h/f} e^{-hZ}}{\widetilde{Y}^1 e^{-fZ} - 1} + \chi_c(f)\Bigl(\frac{-Z}{\widetilde{Y}^1 e^{-fZ} - 1} - Z\Bigr)\biggr\} \\
&\quad = \chi(-1)\biggl\{\sum_{h=0}^{f-1} \chi_c(h) \frac{(-Z)\widetilde{Y}^{h/f} e^{-hZ}}{\widetilde{Y}^1 e^{-fZ} - 1} + \chi_c(0)(-Z)\biggr\},
\end{aligned}$$

where the summation index is changed as $h \mapsto f-h$ in the second member; this readily concludes (4.0). □

Lemma 2. *For any $\gamma, \kappa \in \mathbb{R}$, any (shifted) primitive Dirichlet character χ_c modulo $f \geq 1$, and any $c, k \in \mathbb{Z}$ with $k \geq 0$ we have*

$$\begin{aligned}
\mathcal{C}_{k,\chi_{-c-\lfloor -\gamma \rfloor}}(\langle -\gamma \rangle, \widetilde{e}(-\kappa)) &= (-1)^k \chi(-1)\bigl\{\mathcal{C}_{k,\chi_{c-\lfloor \gamma \rfloor}}(\langle \gamma \rangle, \widetilde{e}(\kappa)) + \delta_{k1}\chi_c(-\gamma)\bigr\} \\
&= (-1)^k \chi(-1)\mathcal{C}_{k,\chi_{c-\lfloor \gamma \rfloor}}(\langle \gamma \rangle, \widetilde{e}(\kappa)) - \delta_{k1}\chi_{-c}(\gamma).
\end{aligned} \tag{4.5}$$

Proof. We first treat the case $\gamma \notin \mathbb{Z}$. Then $-\gamma = (-1-\lfloor \gamma \rfloor) + (1 - \langle \gamma \rangle)$ shows that $\lfloor -\gamma \rfloor = -1 - \lfloor \gamma \rfloor$ and $\langle -\gamma \rangle = 1 - \langle \gamma \rangle$, and hence

$$\mathcal{C}_{k,\chi_{-c-\lfloor -\gamma \rfloor}}(\langle -\gamma \rangle, \widetilde{e}(-\kappa)) = \mathcal{C}_{k,\chi_{1-c+\lfloor \gamma \rfloor}}(1 - \langle \gamma \rangle, \widetilde{1}/\widetilde{e}(\kappa)),$$

which concludes (4.5) in this case, by (4.1).

Next for $\gamma \in \mathbb{Z}$, we see $\gamma = \lfloor \gamma \rfloor$, $-\gamma = \lfloor -\gamma \rfloor$ and $\langle -\gamma \rangle = 0$, giving

$$\begin{aligned}
\mathcal{C}_{k,\chi_{-c-\lfloor -\gamma \rfloor}}(0, \widetilde{e}(-\kappa)) &= \mathcal{C}_{k,\chi_{-c+\lfloor \gamma \rfloor}}(0, \widetilde{1}/\widetilde{e}(\kappa)) \\
&= (-1)^k \chi(-1)\bigl\{\mathcal{C}_{k,\chi_{c-\lfloor \gamma \rfloor}}(0, \widetilde{e}(\kappa)) + \delta_{k1}\chi_{c-\lfloor \gamma \rfloor}(0)\bigr\},
\end{aligned}$$

which concludes (4.5) in this case, since $\chi_{c-\lfloor \gamma \rfloor}(0) = \chi_c(-\gamma)$ for $\gamma \in \mathbb{Z}$. □

Lemma 3. *For any integers c and n, and any (shifted) primitive Dirichlet character χ_c modulo $f \geq 1$ we have*

$$\sum_{h=0}^{f-1} \chi_c(h) e_f(nh) = e_f(-nc) G_\chi \overline{\chi}(n). \tag{4.6}$$

Proof. Let $d \in \mathbb{Z}$ satisfy $c \equiv d \pmod f$ with $0 \le d < f$. Then the left side of (4.6) equals

$$\begin{aligned}\sum_{h=0}^{f-1} \chi_d(h) e_f(nh) &= \biggl(\sum_{h=d}^{f-1} + \sum_{h=f}^{f+d-1}\biggr) \chi(h) e_f\{n(h-d)\} \\ &= e_f(-nd) \biggl[\sum_{h=d}^{f-1} \chi(h) e_f(nh) + \sum_{k=0}^{d-1} \chi(k+f) e_f\{n(k+f)\}\biggr] \\ &= e_f(-nc) \sum_{k=0}^{f-1} \chi(k) e_f(nk),\end{aligned}$$

which, together with the standard evaluation of Gauß' sum, concludes (4.6). □

The following functional equation holds for $K_{\chi_c}(r,\gamma,\kappa)$ or $L_{\chi_c}(r,\gamma,\kappa)$ (cf. [16, Theorem 4]).

Theorem 5. *For any real γ and κ, any integer c, and any (shifted) primitive Dirichlet character χ_c modulo $f \ge 1$, we have the functional equation, in the whole r-plane,*

$$\begin{aligned}L_{\chi_c}(r,\gamma,\kappa) &= \frac{G_\chi \Gamma(1-r)}{f^r (2\pi)^{1-r}} \bigl\{\chi(-1) e^{\pi i(1-r)/2} K_{\overline{\chi}}(1-r,\kappa,-(\gamma-c)) \\ &\qquad + e^{-\pi i(1-r)/2} K_{\overline{\chi}}(1-r,-\kappa,\gamma-c)\bigr\} \\ &= e_f\{\kappa(\gamma-c)\} \frac{G_\chi \Gamma(1-r)}{f^r (2\pi)^{1-r}} \bigl\{\chi(-1) e^{\pi i(1-r)/2} L_{\overline{\chi}}(1-r,\kappa,-(\gamma-c)) \\ &\qquad + e^{-\pi i(1-r)/2} L_{\overline{\chi}}(1-r,-\kappa,\gamma-c)\bigr\},\end{aligned} \tag{4.7}$$

or equivalently,

$$\begin{aligned}K_{\chi_c}(r,\gamma,\kappa) &= e_f(-\kappa c) \frac{G_\chi \Gamma(1-r)}{f^r (2\pi)^{1-r}} \bigl\{\chi(-1) e^{\pi i(1-r)/2} L_{\overline{\chi}}(1-r,\kappa,-(\gamma-c)) \\ &\qquad + e^{-\pi i(1-r)/2} L_{\overline{\chi}}(1-r,-\kappa,\gamma-c)\bigr\} \\ &= e_f(-\kappa\gamma) \frac{G_\chi \Gamma(1-r)}{f^r (2\pi)^{1-r}} \bigl\{\chi(-1) e^{\pi i(1-r)/2} K_{\overline{\chi}}(1-r,\kappa,-(\gamma-c)) \\ &\qquad + e^{-\pi i(1-r)/2} K_{\overline{\chi}}(1-r,-\kappa,\gamma-c)\bigr\}.\end{aligned} \tag{4.8}$$

Proof. Suppose first that $\operatorname{Re} r > 1$. Rewriting the summation index in (2.2) as $k \mapsto h + fk$ ($h = 0, 1, \ldots, f-1$; $k = 0, 1, \ldots$), we have

$$\begin{aligned}L_{\chi_c}(r,\gamma,\kappa) &= f^{-r} \sum_{h=0}^{f-1} \chi_c(h) \psi\Bigl(r, \frac{\gamma+h}{f}, \kappa\Bigr) \\ &= f^{-r} \sum_{h=0}^{f-1} \chi_c(h) \frac{\Gamma(1-r)}{(2\pi)^{1-r}} \biggl\{e^{\pi i(1-r)/2} \phi\Bigl(1-r, \kappa, -\frac{\gamma+h}{f}\Bigr) \\ &\qquad + e^{-\pi i(1-r)/2} \phi\Bigl(1-r, -\kappa, \frac{\gamma+h}{f}\Bigr)\biggr\},\end{aligned} \tag{4.9}$$

by the functional equation for $\phi(r,\gamma,\kappa)$ or $\psi(r,\gamma,\kappa)$ (cf. [15, Proposition 1]).

Suppose next that $\operatorname{Re} r < 0$. Then the h-sums on the rightmost side of (4.9) equal, by changing the order of the h-sum and the inner k-sums (for the defining series of

$\phi(1-r,\pm\kappa,\mp(\gamma+h)/f))$,

$$\sum_{\mp\kappa<k}\frac{1}{(\pm\kappa+k)^{1-r}}\sum_{h=0}^{f-1}\chi_c(h)e_f\{k(\mp(\gamma+h))\}$$
$$=\sum_{\mp\kappa<k}\frac{1}{(\pm\kappa+k)^{1-r}}e_f\{k(\mp(\gamma-c))\}G_\chi\overline{\chi}(\mp k)=\chi(\mp1)K_{\overline{\chi}}\bigl(1-r,\pm\kappa,\mp(\gamma-c)\bigr)$$

respectively, where the second equality holds by (4.6) and the third by (2.1); this readily concludes (4.7). □

Theorem 6. *For any $\gamma,\kappa\in\mathbb{R}$, any (shifted) primitive Dirichlet character χ_c modulo $f\geq1$, and any $c,j\in\mathbb{Z}$ with $j\geq0$ we have*

$$\mathrm{Res}_{r=1}L_{\chi_c}(r,\gamma,\kappa)=\mathcal{C}_{0,\chi_{c-\lfloor\gamma\rfloor}}(\langle\gamma\rangle,\widetilde{e}(\kappa))=\frac{e_f\{(\gamma-c)\kappa\}}{f}G_\chi\overline{\chi}(\kappa),\tag{4.10}$$

$$L_{\chi_c}(-j,\gamma,\kappa)=-\frac{1}{j+1}\mathcal{C}_{j+1,\chi_{c-\lfloor\gamma\rfloor}}(\langle\gamma\rangle,\widetilde{e}(\kappa))-\delta_{j0}\chi_c(-\gamma)\tag{4.11}$$

Proof. It follows from (2.2), (4.9) and

$$L_{\chi_c}(r,\gamma,\kappa)=L_{\chi_{c-\lfloor\gamma\rfloor}}(r,\langle\gamma\rangle,\kappa)\tag{4.12}$$

that the left side of (4.10) equals, by [15, Lemma 4, (4.4)] and (2.8),

$$f^{-1}\sum_{h=0}^{f-1}\chi_{c-\lfloor\gamma\rfloor}(h)\mathrm{Res}_{r=1}\psi\Bigl(r,\frac{\langle\gamma\rangle+h}{f},\kappa\Bigr)=f^{-1}\sum_{h=0}^{f-1}\chi_{c-\lfloor\gamma\rfloor}(h)\mathcal{C}_0\Bigl(\frac{\langle\gamma\rangle+h}{f},\widetilde{e}(\kappa)\Bigr)$$
$$=\mathcal{C}_{0,\chi_{c-\lfloor\gamma\rfloor}}(\langle\gamma\rangle,\widetilde{e}(\kappa)),$$

which further equals, by (4.6),

$$f^{-1}\sum_{h=0}^{f-1}\chi_{c-\lfloor\gamma\rfloor}(h)e_f\{(\langle\gamma\rangle+h)\kappa\}\delta(\kappa)=f^{-1}e_f(\langle\gamma\rangle\kappa)e_f\{-(c-\lfloor\gamma\rfloor)\kappa\}G_\chi\overline{\chi}(\kappa)\delta(\kappa),$$

and this readily concludes (4.10).

Next from (4.9), the left side of (4.11) equals, by [15, Lemma 4, (4.5)],

$$f^j\sum_{h=0}^{f-1}\chi_{c-\lfloor\gamma\rfloor}(h)\psi\Bigl(-j,\frac{\langle\gamma\rangle+h}{f},\kappa\Bigr)$$
$$=f^j\sum_{h=0}^{f-1}\chi_{c-\lfloor\gamma\rfloor}(h)\Bigl\{-\frac{1}{j+1}\mathcal{C}_{j+1}\Bigl(\frac{\langle\gamma\rangle+h}{f},\widetilde{e}(\kappa)\Bigr)-\delta_{j0}\delta\Bigl(\frac{\langle\gamma\rangle+h}{f}\Bigr)\Bigr\},$$

which concludes (4.11) from (2.9), since $\delta\{(\langle\gamma\rangle+h)/f\}=\delta_{h0}\delta(\gamma)$ and $\delta(\gamma)\chi_{c-\lfloor\gamma\rfloor}(0)=\chi_c(-\gamma)$ for any $c,\gamma\in\mathbb{Z}$. □

5. Outline of the proofs

Let (u) for $u\in\mathbb{R}$ denote the vertical straight path from $u-i\infty$ to $u+i\infty$. The key ingredient of the proofs is the formula

$$F_{\chi_a,\psi_b}(s;\alpha,\beta;\mu,\nu;z)=\psi_b(-\beta)\mathcal{A}_{\chi_a}(s,\alpha,\mu)+\Sigma_1(s;z)+\Sigma_+(s;z),\tag{5.1}$$

obtained by applying a modification of (so called) Atkinson's Dissection device (cf. [1]), where $\Sigma_{\pm}(s;z)$ are the Mellin-Barnes type integrals of the form

$$\Sigma_{\pm}(s;z)=\chi(\pm 1)\psi(\pm 1)G_{\chi}\frac{(2\pi e^{\mp\pi i/2}/f)^s}{\Gamma(s)}\frac{1}{2\pi i}\int_{(u')}\Gamma(-w) \times K_{\overline{\chi}}(1-s-w,\mp\mu,\pm(\alpha-a))L_{\psi_{\pm b}}(-w,\pm\beta,\pm\nu)(2\pi\tau/f)^w dw \tag{5.2}$$

with a constant u' satisfying $u'<\min(-\sigma,-1)$; these converge absolutely over the whole s-plane, and hence the right side of (5.1) provides there the holomorphic continuation of the left side.

Theorem 1 can be shown by substituting the defining series of $K_{\overline{\chi}}(1-s-w,\mp\mu,\pm(\alpha-a))$ and $L_{\psi_{\pm b}}(-w,\pm\beta,\pm\nu)$ into the integrands on the right side of (5.2), while Theorem 2 by moving the path in (5.2) to the right from (u') to (u_J) with $J-1<u_J<J$ for $J\in\mathbb{Z}_{\geq 0}$, and then collecting the residues of the relevant poles at $w=j$ $(j=-1,0,\dots,J-1)$.

Theorem 3, on the other hand, can be derived by substituting the functional equations for $K_{\overline{\chi}}(1-s-w,\mp\mu,\pm(\alpha-a))$ and $L_{\psi_{\pm b}}(-w,\pm\beta,\pm\nu)$ into the integrands of $\Sigma_{\pm}(s;z)$ after their paths are moved to (u_J), and then apply the connection formula (2.14) to the resulting expressions.

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