TRANSCENDENCE OF THE MINIMUM OF PRIME-REPRESENTING CONSTANTS

KOTA SAITO
GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY

Abstract. Let $(c_k)_{k\in\mathbb{N}}$ be a sequence of integers satisfying that $c_k \geq 2$ for every $k\in\mathbb{N}$ and $\overline{\lim}_{k\to\infty} c_k = \infty$. We investigate the set of $A>1$ such that $\lfloor A^{c_1\cdots c_k}\rfloor$ is a prime number for every $k\in\mathbb{N}$. Let $\mathcal{W}(c_k)$ be the set of all such $A>1$. We show that the minimum of $\mathcal{W}(c_k)$ exists, and is transcendental.

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. Let us denote $\lfloor x\rfloor$ as the integer part of $x\in\mathbb{R}$. Let $\mathcal{P}$ be the set of all prime numbers. A function $f\colon \mathbb{N}\to\mathcal{P}$ is called a *prime-representing function (PRF)*. The construction of PRFs has been studied by many researchers. Mills was the first to propose a PRF of the form $\lfloor A^{C_k}\rfloor$. He showed that there exists $A>1$ such that $\lfloor A^{3^k}\rfloor$ is a PRF [Mil47]. The minimum of such A's is called Mills' constant. Note that we can verify the existence of the minimum in Section 3. It is still open whether Mills' constant is rational or irrational. Recently, The author and Takeda studied the topological properties and algebraic independence of prime-representing constants [ST]. As a corollary, they disclosed the transcendence of some prime-representing constants. We say that $A\in\mathbb{C}$ is transcendental if $f(A)\neq 0$ for all non-zero polynomials f with rational coefficients. The discussions in [ST] are a bit complicated. Thus, the aim of this article is to focus on the simple case of [ST] and give the proof of the following theorem.

Theorem 1.1. *Let $(c_k)_{k\in\mathbb{N}}$ be a sequence of integers satisfying that*

(1) $c_1 = 1$,
(2) $c_k \geq 3$ *for all* $k\in\mathbb{N}$,
(3) $\overline{\lim}_{k\to\infty} c_k = \infty$.

Then the minimum of $\{A>1\colon \lfloor A^{c_1\cdots c_k}\rfloor$ is a PRF$\}$ exists and is transcendental.

Remark that Theorem 1.1 is a special case of Theorem 1.3 in [ST].

Notation 1.2. For all polynomials $f(x) = \sum_{j=0}^d a_j x^j$ with real coefficients, we define $|f|(x) = \sum_{j=0}^d |a_j| x^j$. For all real sequences $(c_k)_{k\in\mathbb{N}}$, we define $C_k = c_1\cdots c_k$ for every $k\in\mathbb{N}$. Define $\theta = 21/40$.

2. Mills' construction and Outline of the proof of Theorem 1.1

For all positive real sequences $(c_k)_{k\in\mathbb{N}}$, we define

$$\mathcal{W}(c_k) = \{A>1\colon \lfloor A^{C_k}\rfloor \text{ is a PRF}\}.$$

We start with the following result given by Baker, Harman, and Pintz:

Theorem 2.1 ([BHP01]). *There exists a constant $d_0 > 0$ such that*

$$\#([x, x + x^\theta] \cap \mathcal{P}) \geq d_0 x^\theta / \log x$$

for sufficiently large $x > 0$.

Lemma 2.2 ([AD04, Theorem 1]). *Let $(c_k)_{k\in\mathbb{N}}$ be a real sequence satisfying that*

(1) $c_1 > 0$,
(2) $c_{k+1} \geq 40/19$ *for all* $k \in \mathbb{N}$.

Then $\mathcal{W}(c_k)$ is non-empty.

Proof. Take a sufficiently large prime number p_1. Then by Theorem 2.1 with $x = p_1^{c_2}$, there exists $p_2 \in \mathcal{P}$ such that $p_1^{c_2} \leq p_2 < p_2 + 1 \leq p_1^{c_2} + p_1^{c_2\theta} + 1$. We see that $c_2\theta \leq c_2 - 1$ since we have $c_2\theta \leq c_2 - 1 \iff c_2(1-\theta) \geq 1 \iff c_2 \geq 1/(1-\theta) = 40/19$. Therefore

$$p_1^{c_2} \leq p_2 < p_2 + 1 \leq p_1^{c_2} + p_1^{c_2 - 1} + 1 < (p_1 + 1)^{c_2}.$$

Assume that there exist prime numbers p_k and p_{k+1} such that

$$p_k^{c_{k+1}} \leq p_{k+1} < p_{k+1} + 1 < (p_k + 1)^{c_{k+1}} \tag{2.1}$$

for some $k \in \mathbb{N}$. Then similarly, we can find a prime number p_{k+2} such that

$$p_{k+1}^{c_{k+2}} \leq p_{k+2} < p_{k+2} + 1 < (p_{k+1} + 1)^{c_{k+2}}.$$

By induction, there exists a sequence $(p_k)_{k\in\mathbb{N}}$ of prime numbers satisfying (2.1) for every $k \in \mathbb{N}$. Therefore for every $k \in \mathbb{N}$ we obtain

$$p_k^{1/C_k} \leq p_{k+1}^{1/C_{k+1}} < (p_{k+1} + 1)^{1/C_{k+1}} < (p_k + 1)^{1/C_k},$$

which implies that

$$p_1^{1/C_1} \leq p_2^{1/C_2} \leq p_3^{1/C_3} \leq \cdots < (p_3 + 1)^{1/C_3} < (p_2 + 1)^{1/C_2} < (p_1 + 1)^{1/C_1}.$$

Hence there exist $A, A' \in \mathbb{R}$ such that $\lim_{k\to\infty} p_k^{1/C_k} = A \leq A' = \lim_{k\to\infty} (p_k + 1)^{1/C_k}$. Since $p_k^{1/C_k} \leq A \leq A' < (p_k + 1)^{1/C_k}$ for every $k \in \mathbb{N}$, we have $p_k \leq A^{C_k} < p_k + 1$. Therefore $\lfloor A^{C_k} \rfloor = p_k$ for every $k \in \mathbb{N}$. □

Let $(c_k)_{k\in\mathbb{N}}$ be a sequence of integers satisfying the conditions in Theorem 1.1. By Lemma 2.2, $\mathcal{W}(c_k)$ is non-empty. Further, we will show that the minimum of $\mathcal{W}(c_k)$ exists in Section 3. Let $A = \min \mathcal{W}(c_k)$. Let $p_k = \lfloor A^{C_k} \rfloor$ and $\alpha_k = p_k^{1/C_k}$ for all $k \in \mathbb{N}$. Fix any non-zero polynomial $P(x)$ with integral coefficients. Let us show that $P(A) \neq 0$. By the triangle inequality, for all $k \in \mathbb{N}$, we have

$$|P(A)| \geq |P(\alpha_k)| - |P(A) - P(\alpha_k)| =: S - T.$$

We firstly evaluate upper bounds for T. Since A is the minimum of $\mathcal{W}(c_k)$, $|A - \alpha_k|$ is very small. Indeed, in Section 3, we will show that there exists $\gamma > 1$ such that

$$|A - \alpha_k| \leq A\gamma^{(\theta-1)C_{k+1}}$$

for every sufficiently large $k \in \mathbb{N}$. By this inequality and the mean value theorem, we can obtain quantitative upper bounds for T in Section 4.

Let us next evaluate lower bounds for S. Since $[\mathbb{Q}(\alpha_k) : \mathbb{Q}] = C_k$, if we take k such that $C_k > \deg P$, then $|P(\alpha_k)| > 0$. Moreover, we will be able to obtain quantitative lower bounds for S. Indeed, in Section 4, we will have

$$S = |P(\alpha_k)| \geq (4 + 4|P|(A))^{-C_k}$$

Therefore by combining bounds for S and T, and using $\overline{\lim}_{k\to\infty} c_k = \infty$, there exists a suitable large $k \in \mathbb{N}$ such that $S > T$. Therefore A is transcendental.

3. Lemmas

In this section, we present some lemmas to evaluate S and T.

Lemma 3.1 ([ST, Lemma 4.1])**.** *Let $(c_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers. Suppose that $\mathcal{W}(c_k)$ is non-empty. Let $(A_r)_{r\in\mathbb{N}}$ be a sequence of $\mathcal{W}(c_k)$. If $A_1 \geq A_2 \geq \cdots$, then $\lim\limits_{r\to\infty} A_r$ exists in $\mathcal{W}(c_k)$. In particular,* $\min \mathcal{W}(c_k)$ *exists.*

Proof. Since A_j is monotonically decreasing and $A_j > 1$ for all $j \in \mathbb{N}$, the limit $\lim_{j\to\infty} A_j$ exists. Let A be this limit. Fix any $k \in \mathbb{N}$. Let $\epsilon = (1 - \{A^{C_k}\})/2 > 0$, where $\{x\}$ is the fractional part of $x \in \mathbb{R}$. We have $0 \leq \{A^{C_k}\} + \epsilon = (1 + \{A^{C_k}\})/2 < 1$. Therefore, $\lfloor A^{C_k} \rfloor = \lfloor A^{C_k} + \epsilon \rfloor$. There exists a large $j \in \mathbb{N}$ such that $0 \leq A_j^{C_k} - A^{C_k} \leq \epsilon$, which implies that $\lfloor A^{C_k} \rfloor \leq \lfloor A_j^{C_k} \rfloor \leq \lfloor A^{C_k} + \epsilon \rfloor = \lfloor A^{C_k} \rfloor$. Therefore,

$$\lfloor A^{C_k} \rfloor = \lfloor A_j^{C_k} \rfloor \in \mathcal{P}.$$

Let $\xi = \inf \mathcal{W}(c_k)$. Such ξ exists since $\mathcal{W}(c_k)$ is lower bounded and non-empty. Then for every $N \in \mathbb{N}$ there exists $A_N \in \mathcal{W}(c_k)$ such that $\xi \leq A_N \leq \xi + 1/N$. Therefore $\xi = \lim_{N\to\infty} A_N \in \mathcal{W}(c_k)$. Hence $\xi = \min \mathcal{W}(c_k)$. □

Lemma 3.2 ([ST, Lemma 4.5])**.** *Let $(c_k)_{k\in\mathbb{N}}$ be a sequence of positive integers. Suppose that $\mathcal{W}(c_k)$ is non-empty. Let $A \in \mathcal{W}(c_k)$ and $p_k = \lfloor A^{C_k} \rfloor$ for every $k \in \mathbb{N}$. Then for every $k \in \mathbb{N}$*

$$p_k^{c_{k+1}} \leq p_{k+1} < (p_k + 1)^{c_{k+1}} - 1. \tag{3.1}$$

Proof. Fix any $k \in \mathbb{N}$. Then we have $p_k \leq A^{C_k} < p_k + 1$ by $A \in \mathcal{W}(c_k)$. This implies that $p_k^{c_{k+1}} \leq A^{C_{k+1}} < (p_k + 1)^{c_{k+1}}$. By $c_{k+1} \in \mathbb{N}$, we have $p_k^{c_{k+1}} \leq \lfloor A^{C_{k+1}} \rfloor \leq (p_k + 1)^{c_{k+1}} - 1$. Note that $p_{k+1} = \lfloor A^{C_{k+1}} \rfloor$. If $p_{k+1} = (p_k + 1)^{c_{k+1}} - 1$, then $p_k + 1 = 2$ since $c_{k+1} \geq 3$ is an integer. This is a contradiction. Therefore, (3.1) holds. □

Lemma 3.3 ([ST, Lemmas 5.1 and 5.2])**.** *Let $(c_k)_{k\in\mathbb{N}}$ be a sequence of positive integers satisfying that*

1. *$c_1 \geq 1$,*
2. *$c_{k+1} \geq 3$ for every $k \in \mathbb{N}$.*

Let $A = \min \mathcal{W}(c_k)$, and let $p_k = \lfloor A^{C_k} \rfloor$ for every $k \in \mathbb{N}$. Then there exists $k_0 > 0$ such that for every $k \geq k_0$ we have

$$p_k^{c_{k+1}} \leq p_{k+1} \leq p_k^{c_{k+1}} + p_k^{\theta c_{k+1}}. \tag{3.2}$$

Further, there exists $\gamma > 1$ such that for all $k \geq k_0$ we have

$$|A - p_k^{1/C_k}| \leq A\gamma^{(\theta-1)C_{k+1}}. \tag{3.3}$$

Proof. By Lemma 3.2, we have $p_k^{c_{k+1}} \leq p_{k+1}$ for every $k \in \mathbb{N}$. Assume that for infinitely many $k \in \mathbb{N}$ such that $p_k^{c_{k+1}} + p_k^{\theta c_{k+1}} < p_{k+1}$. Then take a sufficiently large $\ell \in \mathbb{N}$ such that $p_\ell^{c_{\ell+1}} + p_\ell^{\theta c_{\ell+1}} < p_{\ell+1}$. By Theorem 2.1, there exists $q_{\ell+1} \in \mathcal{P}$ such that

$$p_\ell^{c_{\ell+1}} \leq q_{\ell+1} \leq p_\ell^{c_{\ell+1}} + p_\ell^{\theta c_{\ell+1}}. \tag{3.4}$$

Similarly with the proof of Lemma 2.2, we can construct a sequence of prime numbers $(q_k)_{k=\ell+1}^{\infty}$ such that

$$q_k^{c_{k+1}} \le q_{k+1} < q_{k+1} + 1 < (q_k + 1)^{c_{k+1}} \tag{3.5}$$

for every $k = \ell+1, \ell+2, \ldots$. Let $q_k = p_k$ for every $k = 1, 2, \ldots, \ell$. Then by Lemma 3.2 and (3.4), for every $k \in \mathbb{N}$, (3.5) holds. Therefore, $B = \lim_{k\to\infty} q_k^{1/C_k}$ exists and $\lfloor B^{C_k} \rfloor = q_k$ for every $k \in \mathbb{N}$. Hence $B \in \mathcal{W}(c_k)$. Further, $\lfloor B^{C_{\ell+1}} \rfloor = q_{\ell+1} \le p_\ell^{c_{\ell+1}} + p_\ell^{\theta c_{\ell+1}} < p_{\ell+1} = \lfloor A^{C_{\ell+1}} \rfloor$. This implies that $B < A$ which is a contradiction to $A = \min \mathcal{W}(c_k)$. Thus there exists $k_0 > 0$ such that for every $k \ge k_0$, (3.2) is true.

Let us next show (3.3). Fix any $k \ge k_0$. Since $\lfloor A^{C_k} \rfloor = p_k$, we have $p_k^{1/C_k} \le A$. By (3.2) and the definition of A, we obtain

$$\begin{aligned}|A - p_k^{1/C_k}| &= A - p_k^{1/C_k} < (p_{k+1} + 1)^{1/C_{k+1}} - p_k^{1/C_k} \le (p_k^{c_{k+1}} + p_k^{\theta c_{k+1}} + 1)^{1/C_{k+1}} - p_k^{1/C_k} \\ &\le (p_k^{c_{k+1}} + 2p_k^{\theta c_{k+1}})^{1/C_{k+1}} - p_k^{1/C_k} = p_k^{1/C_k} \left((1 + 2p_k^{(\theta-1)c_{k+1}})^{1/C_{k+1}} - 1 \right).\end{aligned}$$

By the mean value theorem, there exists $\eta \in (0, 2p_k^{(\theta-1)c_{k+1}})$ such that

$$(1 + 2p_k^{(\theta-1)c_{k+1}})^{1/C_{k+1}} - 1 = \frac{2}{C_{k+1}} p_k^{(\theta-1)c_{k+1}} (1+\eta)^{1/C_{k+1}-1} \le p_k^{(\theta-1)c_{k+1}}.$$

By $p_1^{1/C_1} \le p_k^{1/C_k}$, we obtain $p_k^{(\theta-1)c_{k+1}} \le (p_1^{1/C_1})^{(\theta-1)C_{k+1}}$. Let $\gamma = p_1^{1/C_1}$. Therefore we conclude that $|A - p_k^{1/C_k}| \le A\gamma^{(\theta-1)C_{k+1}}$. □

Let $f(x)$ be a polynomial with rational coefficients, and suppose that

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 = a_d \prod_{j=1}^{d} (x - \alpha_j)$$

for some $a_d, a_{d-1}, \ldots, a_0 \in \mathbb{Q}$ and $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$. Then we define

$$L(f) = \sum_{j=0}^{d} |a_j|, \quad M(f) = |a_d| \prod_{j=1}^{d} \max(1, |\alpha_j|).$$

Remark that $L(f)$ is called *the length of* f and $M(f)$ is called *the Mahler measure of* f.

Lemma 3.4. *In the above notations, we have* $L(f) \le 2^d M(f)$.

Proof. By expanding $f(x) = a_d \prod_{j=1}^{d} (x - \alpha_j)$,

$$f(x) = a_d \sum_{N=0}^{d} (-1)^N \left(\sum_{1 \le j_1 < \cdots < j_N \le d} \alpha_{j_1} \cdots \alpha_{j_N} \right) x^{d-N}.$$

Therefore, we have

$$L(f) \le |a_d| \sum_{N=0}^{d} \left| \sum_{1 \le j_1 < \cdots < j_N \le d} \alpha_{j_1} \cdots \alpha_{j_N} \right| \le M(f) \sum_{N=0}^{d} \binom{d}{N} = 2^d M(f). \tag{3.6}$$

□

We refer the reader to the book written by Everest and Ward [EW99, Chapter 1] for more details on the Mahler measure.

4. Proof of Theorem 1.1

Let $(c_k)_{k\in\mathbb{N}}$ be a sequence of integers satisfying the conditions in Theorem 1.1. By Lemma 2.2, $\mathcal{W}(c_k)$ is non-empty. Further, the minimum of $\mathcal{W}(c_k)$ exists by Lemma 3.1. Let $A = \min\mathcal{W}(c_k)$. Let $p_k = \lfloor A^{C_k}\rfloor$ and $\alpha_k = p_k^{1/C_k}$ for all $k \in \mathbb{N}$. Fix any non-zero polynomial $P(x)$ with integral coefficients. Let us show that $P(A) \neq 0$. Fix any large $k \in \mathbb{N}$. By the triangle inequality, for all $k \in \mathbb{N}$, we have

$$|P(A)| \geq |P(\alpha_k)| - |P(A) - P(\alpha_k)| := S - T.$$

Evaluating T. By the mean value theorem and Lemma 3.3, there exist $\eta \in (\alpha_k, A)$, $k_0 \in \mathbb{N}$, and $\gamma > 1$ such that if $k \geq k_0$, then we have

$$T = |A-\alpha_k||P'(\eta)| \leq |A-\alpha_k||P'|(\eta) \leq F\gamma^{(\theta-1)C_{k+1}} \tag{4.1}$$

where $F = |P'|(A)$.

Evaluating S. Let us show that there exists $\beta = \beta(A,P) > 1$ such that

$$S = |P(\alpha_k)| \geq \beta^{-C_k}. \tag{4.2}$$

We may assume that $|P(\alpha_k)| \leq 1$. Let $K = \mathbb{Q}(\alpha_k)$. Then $g(x) = x^{C_k} - p_k$ is the minimal polynomial of α_k over $\mathbb{Q}$. Therefore $[K,\mathbb{Q}] = \deg g = C_k$. Let G be the set of all field homomorphisms from K to $\mathbb{C}$. We define

$$\varphi(x) = \prod_{\sigma\in G}\left(x - \sigma(P(\alpha_k))\right).$$

Then $\varphi(0) \in \mathbb{Z}$ since $\varphi(0)$ is the norm of $P(\alpha_k)$ on $K/\mathbb{Q}$, and $P(\alpha_k)$ is an algebraic integer. Moreover, $\varphi(0) \neq 0$, if not, then there exists $\sigma \in G$ such that $\sigma(P(\alpha_k)) = 0$. By the injection of σ, $P(\alpha_k) = 0$. This contradicts $[\mathbb{Q}(\alpha_k):\mathbb{Q}] = C_k$ if k is enough large to satisfy $C_k > \deg P$. Therefore, by retaking a larger k, we see that $\varphi(0) \in \mathbb{Z}\setminus\{0\}$. Hence by the mean value theorem, there exists $\theta \in (-|P(\alpha_k)|, |P(\alpha_k)|) \subseteq (-1,1)$ such that

$$1 \leq |\varphi(0)| = |\varphi(0) - \varphi(P(\alpha_k))| = |P(\alpha_k)||\varphi'(\theta)| \leq |P(\alpha_k)|(\deg\varphi)L(\varphi).$$

Lemma 3.4 yields that

$$L(\varphi) \leq 2^{\deg\varphi}M(\varphi) = 2^{\deg\varphi}\prod_{\sigma\in G}\max(1, |\sigma(P(\alpha_k))|) \leq 2^{\deg\varphi}\prod_{\sigma\in G}(1 + |P|(|\sigma(\alpha_k)|)).$$

By $|\sigma(\alpha_k)| = \alpha_k$, we have

$$\prod_{\sigma\in G}(1 + |P|(|\sigma(\alpha_k)|)) = (1 + |P|(\alpha_k))^{\deg\varphi}.$$

Therefore, by $\deg\varphi = C_k$, we obtain

$$1 \leq |P(\alpha_k)|C_k 2^{C_k}(1 + |P|(\alpha_k))^{C_k} \leq |P(\alpha_k)|\left(4 + 4|P|(A)\right)^{C_k}.$$

Setting $\beta = \beta(A,P) = 4 + 4|P|(A)$, we conclude (4.2).

Completing the proof of Theorem 1.1. By combining (4.1) and (4.2), we have

$$|P(A)| \geq S - T \geq \beta^{-C_k} - F\gamma^{(\theta-1)C_{k+1}}.$$

Further, we observe that

$$\begin{aligned}\beta^{-C_k} - F\gamma^{(\theta-1)C_{k+1}} > 0 &\Leftrightarrow (-C_k)\log\beta > \log F + (\theta-1)C_{k+1}\log\gamma \\ &\Leftrightarrow -\log\beta > (\log F)/C_k + (\theta-1)c_{k+1}\log\gamma.\end{aligned}$$

Hence there exists a large $k \in \mathbb{N}$ such that $\beta^{-C_k} - F\gamma^{(\theta-1)C_{k+1}} > 0$ since $\overline{\lim_{k\to\infty}}\, c_{k+1} = \infty$. Therefore, $|P(A)| > 0$ which means that A is transcendental.

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Kota Saito, Graduate School of Mathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, 464-8602, Japan

Email address: m17013b@math.nagoya-u.ac.jp