On power series generated by simpler sequences and having strong algebraic independence properties

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1 Introduction and the results

In this paper we consider the algebraic independence of the values of a power series of the form $f(z) = \sum_{k=0}^{\infty} z^{e_k}$, where $\{e_k\}_{k\geq 0}$ is an increasing sequence of nonnegative integers. Let $\alpha_1, \ldots, \alpha_r$ be algebraic numbers with $0 < |\alpha_i| < 1$ $(1 \leq i \leq r)$. We consider the following condition under which the values $f(\alpha_1), \ldots, f(\alpha_r)$ are clearly algebraically dependent.

(∗) There exist a nonempty subset $\{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$ of $\{\alpha_1, \ldots, \alpha_r\}$, roots of unity $\zeta_1, \ldots, \zeta_s$, an algebraic number γ with $\alpha_{i_q} = \zeta_q \gamma$ $(1 \leq q \leq s)$, and algebraic numbers $\xi_1, \ldots, \xi_s$, not all zero, such that

$$\sum_{q=1}^{s} \xi_q \zeta_q^{e_k} = 0 \tag{1}$$

for all sufficiently large k.

In what follows, $\overline{\mathbb{Q}}$ denotes the field of algebraic numbers. Suppose that the condition (∗) is satisfied. Then there exists a nonnegative integer k_0 such that $\sum_{q=1}^{s} \xi_q \zeta_q^{e_k} = 0$ for all $k \geq k_0$. Hence

$$\sum_{q=1}^{s} \xi_q f(\alpha_{i_q}) - \sum_{q=1}^{s} \xi_q \sum_{k=0}^{k_0-1} \alpha_{i_q}^{e_k} = \sum_{k=k_0}^{\infty} \sum_{q=1}^{s} \xi_q \alpha_{i_q}^{e_k} = \sum_{k=k_0}^{\infty} \left(\sum_{q=1}^{s} \xi_q \zeta_q^{e_k} \right) \gamma^{e_k} = 0$$

and so $\sum_{q=1}^{s} \xi_q f(\alpha_{i_q}) \in \overline{\mathbb{Q}}$. Therefore we have the following:

Proposition 1.1. *If the infinite set of the values* $\{f(\alpha) \mid \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ *is algebraically independent, then there exist no distinct roots of unity* $\zeta_1, \ldots, \zeta_s$ *for which* (1) *holds.*

If $\{e_k\}_{k\geq 0}$ increases rapidly, then the converse of Proposition 1.1 also holds. More precisely, the derivatives can be included. In what follows, $\mathbb{Z}_{\geq 0}$ and $f^{(l)}(z)$ denote the set of nonnegative integers and the derivative of an analytic function $f(z)$ of order $l \in \mathbb{Z}_{\geq 0}$, respectively.

Theorem 1.2 (A special case of Nishioka [4])**.** *Let* $f_1(z) = \sum_{k=0}^{\infty} z^{e_k}$, *where* $\{e_k\}_{k\geq 0}$ *is an increasing sequence of nonnegative integers satisfying* $\lim_{k\to\infty} e_{k+1}/e_k = \infty$. *Let* $\alpha_1, \ldots, \alpha_r$ *be algebraic numbers with* $0 < |\alpha_i| < 1$ $(1 \leq i \leq r)$. *Then the following three properties are equivalent*:

- (i) *The infinite set* $\{f_1^{(l)}(\alpha_i) \mid l \in \mathbb{Z}_{\geq 0},\ 1 \leq i \leq r\}$ *is algebraically dependent.*
- (ii) $1, f_1(\alpha_1), \ldots, f_1(\alpha_r)$ *are linearly dependent over* $\overline{\mathbb{Q}}$.
- (iii) *The condition* $(*)$ *holds for* $\alpha_1, \ldots, \alpha_r$.

By the discussion before Proposition 1.1, the main assertion of Theorem 1.2 is that the property (i) implies (iii); the latter property is broken by the following condition as is shown in Proposition 1.4 below.

Definition 1.3. *A sequence of nonnegative integers* $\{e_k\}_{k\geq 0}$ *is said to be distributed infinitely to any of congruence classes if* $\{k \in \mathbb{Z}_{\geq 0} \mid e_k \equiv a \pmod{N}\}$ *is an infinite set for all positive integer* N *and for all* $a \in \{0, 1, \ldots, N-1\}$.

Proposition 1.4. *There exist no distinct roots of unity* $\zeta_1, \ldots, \zeta_s$ *for which* (1) *holds if and only if* $\{e_k\}_{k\geq 0}$ *is distributed infinitely to any of congruence classes.*

Proof. First we prove that, if there exist no distinct roots of unity $\zeta_1, \ldots, \zeta_s$ for which (1) holds, then $\{e_k\}_{k\geq 0}$ is distributed infinitely to any of congruence classes. We show the contrapositive. Suppose that there are a positive integer N and an integer a with $0 \leq a \leq N-1$ such that $\{k \in \mathbb{Z}_{\geq 0} \mid e_k \equiv a \pmod{N}\}$ is a finite set. Then there exist a nonnegative integer k_0 and $\{b_1, \ldots, b_s\} \subset \{0, 1, \ldots, N-1\}$ with $s < N$ such that $\{e_k + N\mathbb{Z} \mid k \geq k_0\} = \{b_1 + N\mathbb{Z}, \ldots, b_s + N\mathbb{Z}\}$. Letting ζ be a primitive N-th root of unity and noting that $s < N$, we can take algebraic numbers $\xi_0, \ldots, \xi_{N-1}$, not all zero, such that

$$\begin{pmatrix} 1 & \zeta^{b_1} & \cdots & \left(\zeta^{N-1}\right)^{b_1} \\ \vdots & \vdots & & \vdots \\ 1 & \zeta^{b_s} & \cdots & \left(\zeta^{N-1}\right)^{b_s} \end{pmatrix} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{N-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then, for any $k \geq k_0$, there exist nonnegative integers $b_{i(k)}$ with $1 \leq i(k) \leq s$ and m_k such that $e_k = b_{i(k)} + m_k N$. Hence

$$\sum_{j=0}^{N-1} \xi_j \left(\zeta^j\right)^{e_k} = \sum_{j=0}^{N-1} \xi_j \left(\zeta^j\right)^{b_{i(k)}} = 0 \quad (k \geq k_0)$$

and so the roots of unity $1, \zeta, \ldots, \zeta^{N-1}$ satisfy (1).

Next we prove that, if $\{e_k\}_{k\geq 0}$ is distributed infinitely to any of congruence classes, then there exist no distinct roots of unity $\zeta_1, \ldots, \zeta_s$ for which (1) holds. Assume on the contrary that there exist algebraic numbers $\xi_1, \ldots, \xi_s$, not all zero, distinct roots of unity $\zeta_1, \ldots, \zeta_s$, and a nonnegative integer k_0 such that

$$\sum_{q=1}^{s} \xi_q \zeta_q^{e_k} = 0 \quad (k \geq k_0).$$

Let N be a positive integer satisfying $\zeta_q^N = 1$ $(1 \leq q \leq s)$ with $N \geq s$. For each $a \in \{0, 1, \ldots, N-1\}$, since $\{k \geq k_0 \mid e_k \equiv a \pmod{N}\} \neq \emptyset$ by the assumption, we can take $k(a) = \min\{k \geq k_0 \mid e_k \equiv a \pmod{N}\}$. Then

$$\sum_{q=1}^{s} \xi_q \zeta_q^{a} = \sum_{q=1}^{s} \xi_q \zeta_q^{e_{k(a)}} = 0 \quad (a \in \{0, 1, \ldots, N-1\}).$$

In particular, we have

$$\begin{pmatrix} 1 & \cdots & 1 \\ \zeta_1 & \cdots & \zeta_s \\ \vdots & & \vdots \\ \zeta_1^{s-1} & \cdots & \zeta_s^{s-1} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \vdots \\ \xi_s \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\zeta_1, \ldots, \zeta_s$ are distinct, by the non-vanishing of the Vandermonde determinant, we see that $\xi_q = 0$ $(1 \leq q \leq s)$, a contradiction. $\square$

By Proposition 1.4 we see that, if a sequence $\{e_k\}_{k\geq 0}$ satisfying the assumptions of Theorem 1.2 is distributed infinitely to any of congruence classes, then the property (iii) does not hold for any of the distinct algebraic numbers $\alpha_1, \ldots, \alpha_r$. Since the sequence $\{k!+k\}_{k\geq 0}$ is distributed infinitely to any of congruence classes, we have the following result as the corollary to Theorem 1.2.

Corollary 1.5 (Nishioka [4]). *Let* $f_2(z) = \sum_{k=0}^{\infty} z^{k!+k}$. *Then the infinite set* $\{f_2^{(l)}(\alpha) \mid l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ *is algebraically independent.*

By Hadamard's Gap Theorem (cf. Rudin [6, 16.6 Theorem]), power series $\sum_{k=0}^{\infty} z^{e_k}$ with $\liminf_{k\to\infty} e_{k+1}/e_k > 1$ has the unit circle $|z| = 1$ as its natural boundary. Hence $f_2(z)$ in Corollary 1.5 is one of the concrete examples of power series satisfying the following:

Property 1.6. *The infinite set consisting of all the values of a power series in question and its derivatives of any order, at any nonzero algebraic numbers within its domain of existence, is algebraically independent.*

Here, we introduce another power series which has the unit circle as its natural boundary and satisfies Property 1.6. Let $\lfloor x \rfloor$ denote the integral part of the real number x, namely, the largest integer not exceeding x.

Theorem 1.7 (Tanuma [8]). *Let* $\omega > 0$ *be a quadratic irrational number. Define* $f_3(z) = \sum_{k=1}^{\infty} z^{\lfloor k\omega \rfloor}$. *Then the infinite set* $\{f_3^{(l)}(\alpha) \mid l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ *is algebraically independent.*

The sequence $\{\lfloor k\omega \rfloor\}_{k\geq 0}$ is called the Beatty sequence. By Corollary 1.5 and Theorem 1.7, it is expected that, if $f(z) = \sum_{k=0}^{\infty} z^{e_k}$ has the unit circle as its natural boundary and if $\{e_k\}_{k\geq 0}$ is distributed infinitely to any of congruence classes, then $f(z)$ also has Property 1.6. Before constructing such a power series, we consider power series generated by a geometric progression, which has simpler structure than $\{k!+k\}_{k\geq 0}$ in Corollary 1.5 or the Beatty sequence $\{\lfloor k\omega \rfloor\}_{k\geq 0}$.

Theorem 1.8 (Loxton and van der Poorten [2]). *Let* $d \geq 2$ *be an integer and define* $f_4(z) = \sum_{k=0}^{\infty} z^{d^k}$. *Let* $\alpha_1, \ldots, \alpha_r$ *be algebraic numbers with* $0 < |\alpha_i| < 1$ $(1 \leq i \leq r)$. *Then the following three properties are equivalent*:

(i) *The infinite set* $\{f_4^{(l)}(\alpha_i) \mid l \in \mathbb{Z}_{\geq 0},\ 1 \leq i \leq r\}$ *is algebraically dependent.*

(ii) $1, f_4(\alpha_1), \ldots, f_4(\alpha_r)$ *are linearly dependent over* $\overline{\mathbb{Q}}$.

(iii) *There exist a nonempty subset* $\{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$ *of* $\{\alpha_1, \ldots, \alpha_r\}$, *nonnegative integers* $k_1, \ldots, k_s$, *roots of unity* $\zeta_1, \ldots, \zeta_s$, *an algebraic number* γ *with* $\alpha_{i_q}^{d^{k_q}} = \zeta_q \gamma$ $(1 \le q \le s)$, *and algebraic numbers* $\xi_1, \ldots, \xi_s$, *not all zero, such that*

$$\sum_{q=1}^{s} \xi_q \zeta_q^{d^k} = 0 \quad (k \ge 0).$$

By Hadamard's Gap Theorem, $f_4(z)$ also has the unit circle as its natural boundary. Moreover, the property (iii) of Theorem 1.8 is similar to that of Theorem 1.2. Hence, imitating Corollary 1.5, we expect to construct a sequence which is distributed infinitely to any of congruence classes and which generates a power series satisfying Property 1.6. In this paper we consider the case of the sum of a geometric progression and an arithmetic progression. To begin with, we observe the following:

Proposition 1.9. *Let* $c \ge 1$, $d \ge 2$, *and* m *be integers. Then, if the sequence* $\{cd^k + mk\}_{k \ge 0}$ *is distributed infinitely to any of congruence classes, then* $|m| = 1$.

Proof. We show the contrapositive. Assume that $|m| \ne 1$. Put

$$N := \max\{\gcd(c, m), \gcd(d, m)\}.$$

We distinguish the following two cases:

First we consider the case of $N \ge 2$, which includes the case of $m = 0$ by $d \ge 2$. Since $cd^k + mk \equiv 0 \pmod{N}$ for all $k \ge 1$, the sequence $\{cd^k + mk\}_{k \ge 0}$ is not distributed infinitely to the congruence classes other than 0 modulo N.

Secondly we consider the remaining case of $N = 1$, which implies $|m| \ge 2$. Let p be a prime factor of m and put $N' := p^2$. Then the shortest period modulo N' of the sequence $\{mk\}_{k \ge 0}$ is equal to 1 or p, respectively according as m is divisible by p^2 or not. Since the shortest period modulo N' of the sequence $\{cd^k\}_{k \ge 0}$ is a divisor of $\varphi(p^2) = p(p-1)$, that of the sequence $\{cd^k + mk\}_{k \ge 0}$ is also a divisor of $p(p-1)$, which is smaller than N'. This means that there are some congruence classes modulo N' to which the sequence $\{cd^k + mk\}_{k \ge 0}$ is not distributed infinitely. □

From the discussion above, we have the following result.

Main Theorem 1.10. *Let $c \geq 1$, $d \geq 2$, and m be integers. Define*

$$f(z) = \sum_{k=0}^{\infty} z^{cd^k + mk}.$$

Then, the infinite set $\{f^{(l)}(\alpha) \mid l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ is algebraically independent if and only if $|m| = 1$.

Proof. If the infinite set $\{f^{(l)}(\alpha) \mid l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ is algebraically independent, then so is the infinite subset $\{f(\alpha) \mid \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$. By Proposition 1.1, there exist no distinct roots of unity $\zeta_1, \ldots, \zeta_s$ for which (1) holds. Then by Proposition 1.4, the sequence $\{cd^k + mk\}_{k \geq 0}$ is distributed infinitely to any of congruence classes. Hence by Proposition 1.9, we have $|m| = 1$.

The converse is immediate from Theorem 1.11 below. □

Theorem 1.11. *Let $c \geq 1$ and $d \geq 2$ be integers. Define*

$$g(z) = \sum_{k=0}^{\infty} z^{cd^k + k} \qquad \text{and} \qquad g^{-}(z) = \sum_{k=0}^{\infty} z^{cd^k - k}.$$

Then, each of the infinite sets $\{g^{(l)}(\alpha) \mid l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ and $\{g^{-(l)}(\alpha) \mid l \in \mathbb{Z}_{\geq 0},\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ is algebraically independent.

By Propositions 1.1, 1.4 and Theorems 1.7, 1.11, we have the following:

Corollary 1.12. *Each of the sequences $\{\lfloor k\omega \rfloor\}_{k \geq 0}$, where $\omega > 0$ is a quadratic irrational number, and $\{cd^k + k\}_{k \geq 0}$, $\{cd^k - k\}_{k \geq 0}$, where $c \geq 1$, $d \geq 2$ are integers, is distributed infinitely to any of congruence classes.*

2 Lemmas

In this section we prepare several lemmas for proving Theorem 1.11. The following lemma is a well-known fact on linear recurrences.

Lemma 2.1 (cf. Shorey and Tijdeman [7, Theorem C.1])**.** *Let $p_1(Y), \ldots, p_s(Y)$ be nonzero polynomials with algebraic coefficients and $\theta_1, \ldots, \theta_s$ nonzero algebraic numbers. Let*

$$r_k = \sum_{i=1}^{s} p_i(k)\theta_i^k \quad (k \geq 0). \tag{2}$$

Put $d_i = \deg p_i$ $(1 \le i \le s)$, $m = \sum_{i=1}^{s}(d_i+1)$, *and define algebraic numbers* $b_1, \ldots, b_m$ *by*

$$\prod_{i=1}^{s}(X-\theta_i)^{d_i+1} =: X^m - b_1X^{m-1} - \cdots - b_m.$$

Then

$$r_{k+m} = b_1 r_{k+m-1} + \cdots + b_m r_k \quad (k \ge 0)$$

holds.

The following lemma plays a crucial role in making the descent method work in the proof of Theorem 1.11.

Lemma 2.2 (A special case of Lemma 2.3 of Ide, Tanaka, and Toyama [1]). *Let* $d \ge 2$ *be an integer. Then, for any integer* $N \ge 2$, *there exist a positive integer* N_1 *and a nonnegative integer* u_1 *such that* $N_1, u_1 < N$ *and* $d^{k+N_1} \equiv d^k \pmod{N}$ *for any* $k \ge u_1$.

The following lemma is deduced from Lemma 2.1.

Lemma 2.3 (A special case of Lemma 2.2 of Ide, Tanaka, and Toyama [1]). *Let* d, N, N_1, *and* u_1 *be integers as in Lemma 2.2. Define*

$$R_k = r_k \zeta^{d^k} \quad (k \ge 0),$$

where $\{r_k\}_{k\ge 0}$ *is a linear recurrence of algebraic numbers of the form* (2) *and* ζ *is an* N*-th root of unity. Then* $\{R_{N_1k+u_1+\sigma}\}_{k\ge0}$ $(0 \le \sigma \le N_1 - 1)$ *are linear recurrences satisfying the same recurrence relation*

$$R_{N_1(k+m)+u_1+\sigma} = b_1 R_{N_1(k+m-1)+u_1+\sigma} + \cdots + b_m R_{N_1k+u_1+\sigma} \quad (k \ge 0),$$

where $b_1, \ldots, b_m$ *are algebraic numbers defined by*

$$\prod_{i=1}^{s}(X-\theta_i^{N_1})^{d_i+1} =: X^m - b_1X^{m-1} - \cdots - b_m$$

and d_i $(1 \le i \le s)$, m *are as in Lemma 2.1.*

Let $z_1, \ldots, z_s$ be variables and $d \ge 2$ an integer. Denote $\boldsymbol{z} = (z_1, \ldots, z_s)$,

$$\boldsymbol{d}(k) = (\underbrace{d^k, \ldots, d^k}_{s}), \quad \text{and} \quad \boldsymbol{z}^{\boldsymbol{d}(k)} = (z_1^{d^k}, \ldots, z_s^{d^k}). \tag{3}$$

Lemma 2.4 (A special case of Lemma 2.4 of Ide, Tanaka, and Toyama [1]). *Let N be a positive integer and let $\{R_k^{(\sigma)}\}_{k\ge 0}$ $(0 \le \sigma \le N-1)$ be linear recurrences of algebraic numbers satisfying the same recurrence relation*

$$R_{k+m}^{(\sigma)} = b_1 R_{k+m-1}^{(\sigma)} + \cdots + b_m R_k^{(\sigma)} \quad (k \ge 0).$$

Define

$$f(\boldsymbol{z}) = \sum_{k=0}^{\infty} \sum_{\sigma=0}^{N-1} R_k^{(\sigma)} \prod_{j=1}^{s} \left(z_j^{d^{Nk+u+\sigma}} \right)^{\ell_j},$$

where u is a nonnegative integer and $\ell_1, \dots, \ell_s$ are nonnegative integers not all zero. Then $f(\boldsymbol{z})\left(= f(\boldsymbol{z}^{\boldsymbol{d}(0)})\right), f(\boldsymbol{z}^{\boldsymbol{d}(N)}), \dots, f(\boldsymbol{z}^{\boldsymbol{d}((m-1)N)})$ satisfy the functional equation

$$\begin{pmatrix} f(\boldsymbol{z}) \\ f(\boldsymbol{z}^{\boldsymbol{d}(N)}) \\ \vdots \\ f(\boldsymbol{z}^{\boldsymbol{d}((m-1)N)}) \end{pmatrix} = \begin{pmatrix} b_1 & \cdots & \cdots & b_m \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ \text{\huge 0} & & 1 & 0 \end{pmatrix} \begin{pmatrix} f(\boldsymbol{z}^{\boldsymbol{d}(N)}) \\ f(\boldsymbol{z}^{\boldsymbol{d}(2N)}) \\ \vdots \\ f(\boldsymbol{z}^{\boldsymbol{d}(mN)}) \end{pmatrix} + \begin{pmatrix} b(\boldsymbol{z}) \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $b(\boldsymbol{z})$ is a polynomial in variables $z_1, \dots, z_s$ with algebraic coefficients.

The following lemma is used for constructing the Mahler functions in the proof of Theorem 1.11.

Lemma 2.5 (Loxton and van der Poorten [2]). *Let $\alpha_1, \dots, \alpha_r$ be algebraic numbers with $0 < |\alpha_i| < 1$ $(1 \le i \le r)$. Then there exist multiplicatively independent algebraic numbers $\beta_1, \dots, \beta_s$ with $0 < |\beta_j| < 1$ $(1 \le j \le s)$ such that*

$$\alpha_i = \zeta_i \prod_{j=1}^{s} \beta_j^{\ell_{ij}} \quad (1 \le i \le r), \tag{4}$$

where ζ_i $(1 \le i \le r)$ are roots of unity and ℓ_{ij} $(1 \le i \le r, 1 \le j \le s)$ are nonnegative integers.

Remark 2.6. In Lemma 2.5, at least one of $\ell_{i1}, \dots, \ell_{is}$ is positive for any i.

Lemma 2.7 (Ide, Tanaka, and Toyama [1]). *Let $\{b_k^{(i)}\}_{k\ge 0}$ $(1 \le i \le p)$ be sequences of complex numbers which are eventually periodic with period N. Let $\delta_1, \dots, \delta_p$ be complex numbers with $|\delta_i| = 1$ $(1 \le i \le p)$ and δ_i/δ_j $(1 \le i < j \le p)$ are not N-th roots of unity. If*

$$\sum_{i=1}^{p} b_k^{(i)} \delta_i^k \to 0 \quad (k \to \infty),$$

then $b_k^{(i)} = 0$ $(1 \le i \le p)$ *for all sufficiently large* k.

3 Proof of Theorem 1.11

We denote by $F[z_1, \dots, z_s]$ and by $F[[z_1, \dots, z_s]]$ the ring of polynomials and that of formal power series in variables $z_1, \dots, z_s$ with coefficients in a field F, respectively.

Proof of Theorem 1.11. First we prove the algebraic independency of $\{g^{(l)}(\alpha) \mid l \in \mathbb{Z}_{\ge 0},\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ and secondly we verify that the algebraic independency of $\{g^{-(l)}(\alpha) \mid l \in \mathbb{Z}_{\ge 0},\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ can be proved in the similar way. We assume on the contrary that there exist distinct algebraic numbers $\alpha_1, \dots, \alpha_r$ with $0 < |\alpha_i| < 1$ $(1 \le i \le r)$ and a nonnegative integer L such that $\{g^{(l)}(\alpha_i) \mid 0 \le l \le L,\ 1 \le i \le r\}$ is algebraically dependent. For each l $(0 \le l \le L)$, let

$$g_l(z) = \left(z\frac{d}{dz}\right)^l g(z) = \sum_{k=0}^{\infty} (cd^k + k)^l z^{cd^k + k}.$$

Then we see that $\{g_l(\alpha_i) \mid 0 \le l \le L,\ 1 \le i \le r\}$ is algebraically dependent. Let $\zeta_i, \beta_j, \ell_{ij}$ $(1 \le i \le r,\ 1 \le j \le s)$ be as in Lemma 2.5. Then the algebraic numbers $\alpha_1, \dots, \alpha_r$ are expressed as (4). Let $z_1, \dots, z_s$ be variables and $\boldsymbol{z} = (z_1, \dots, z_s)$. For each l, i $(0 \le l \le L,\ 1 \le i \le r)$, define

$$g_{li}(\boldsymbol{z}) = \sum_{k=0}^{\infty} (cd^k + k)^l \alpha_i^k \zeta_i^{cd^k} \left(\prod_{j=1}^{s} z_j^{\ell_{ij}}\right)^{cd^k}.$$

Then by (4) we have $g_{li}(\boldsymbol{\beta}) = g_l(\alpha_i)$ $(0 \le l \le L,\ 1 \le i \le r)$, where $\boldsymbol{\beta} = (\beta_1, \dots, \beta_s)$. Thus $\{g_{li}(\boldsymbol{\beta}) \mid 0 \le l \le L,\ 1 \le i \le r\}$ is algebraically dependent. Take a positive integer N_0 such that $\zeta_i^{N_0} = 1$ for any i $(1 \le i \le r)$. Then by Lemma 2.2 there exist a positive integer N_1 and a nonnegative integer u_1 such that

$$cd^{k+N_1} \equiv cd^k \pmod{N_0} \tag{5}$$

for any $k \geq u_1$. Then, using Lemma 2.3 with (5) and applying Lemma 2.4 to

$$\sum_{k=0}^{\infty}\sum_{\sigma=0}^{N-1}(cd^{N_1k+u_1+\sigma}+N_1k+u_1+\sigma)^l\alpha_i^{N_1k+u_1+\sigma}\zeta_i^{cd^{N_1k+u_1+\sigma}}\left(\prod_{j=1}^{s}z_j^{\ell_{ij}}\right)^{cd^{N_1k+u_1+\sigma}}$$
$$=g_{li}(\boldsymbol{z})-\sum_{k=0}^{u_1-1}(cd^k+k)^l\alpha_i^k\zeta_i^{cd^k}\left(\prod_{j=1}^{s}z_j^{\ell_{ij}}\right)^{cd^k}\qquad(0\leq l\leq L,\ 1\leq i\leq r),$$

we see that $g_{li}(\boldsymbol{z}^{\boldsymbol{d}(pN_1)})$ $(0 \leq l \leq L,\ 1 \leq i \leq r,\ 0 \leq p \leq m(l)-1)$ satisfy the Mahler type functional equation of the form

$$\begin{pmatrix} g_{li}(\boldsymbol{z}) \\ g_{li}(\boldsymbol{z}^{\boldsymbol{d}(N)}) \\ \vdots \\ g_{li}(\boldsymbol{z}^{\boldsymbol{d}((m(l)-1)N)}) \end{pmatrix} - \begin{pmatrix} b_1 & \cdots & \cdots & b_{m(l)} \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{pmatrix} \begin{pmatrix} g_{li}(\boldsymbol{z}^{\boldsymbol{d}(N)}) \\ g_{li}(\boldsymbol{z}^{\boldsymbol{d}(2N)}) \\ \vdots \\ g_{li}(\boldsymbol{z}^{\boldsymbol{d}(m(l)N)}) \end{pmatrix}$$
$$\in \left(\overline{\mathbb{Q}}[\boldsymbol{z}]\right)^{m(l)}, \tag{6}$$

where $m(l)$ $(0 \leq l \leq L)$ are some positive integers. Moreover, by the vanishing theorem of Masser [3], all the conditions required for Mahler functions are satisfied. Then by Theorem 2 of Nishioka [5], $\{g_{li}(\boldsymbol{z}^{\boldsymbol{d}(0)}) = g_{li}(\boldsymbol{z}) \mid 0 \leq l \leq L,\ 1 \leq i \leq r\}$ is linearly dependent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}[\boldsymbol{z}]$. Thus there exist algebraic numbers c_{li} $(0 \leq l \leq L,\ 1 \leq i \leq r)$, not all zero, such that

$$\sum_{l=0}^{L}\sum_{i=1}^{r}c_{li}g_{li}(\boldsymbol{z})=\sum_{l=0}^{L}\sum_{i=1}^{r}c_{li}\sum_{k=0}^{\infty}(cd^k+k)^l\alpha_i^k\zeta_i^{cd^k}\left(\prod_{j=1}^{s}z_j^{\ell_{ij}}\right)^{cd^k}\in\overline{\mathbb{Q}}[\boldsymbol{z}]. \tag{7}$$

We may assume that c_{li} $(0 \leq l \leq L)$ are not all zero for any i $(1 \leq i \leq r)$. Hence there exists a sufficiently large integer R such that the roots of the polynomials $\sum_{l=0}^{L} c_{li}X^l$ $(1 \leq i \leq r)$ are inside the circle $|X| = R$. Then by (7) we have

$$\sum_{l=0}^{L}\sum_{i=1}^{r}c_{li}\sum_{k=R}^{\infty}(cd^k+k)^l\alpha_i^k\zeta_i^{cd^k}\left(\prod_{j=1}^{s}z_j^{\ell_{ij}}\right)^{cd^k}=:G(\boldsymbol{z})\in\overline{\mathbb{Q}}[\boldsymbol{z}]. \tag{8}$$

Take an integer k_1 so large that

$$cd^{k_1}>\max\left\{\deg_{\boldsymbol{z}}G(\boldsymbol{z}),\max_{1\leq i\leq r,1\leq j\leq s}\{\ell_{ij}d^R\}\right\}. \tag{9}$$

Since ℓ_{1j} $(1 \le j \le s)$ are not all zero by Remark 2.6, we have

$$\deg_{\boldsymbol{z}} \left(\prod_{j=1}^{s} z_j^{\ell_{1j}} \right)^{cd^{k+k_1}} = cd^{k+k_1} \sum_{j=1}^{s} \ell_{1j} \ge cd^{k_1} > \deg_{\boldsymbol{z}} G(\boldsymbol{z})$$

for all $k \ge 0$. Let S be the subset of $\{1, \ldots, r\}$ consisting of the indices i such that

$$\left(\prod_{j=1}^{s} z_j^{\ell_{ij}} \right)^{cd^{k_i}} = \left(\prod_{j=1}^{s} z_j^{\ell_{1j}} \right)^{cd^{k_1}}$$

for some $k_i \ge R$. Then for all $i \in S$ and $k \ge 0$,

$$\left(\prod_{j=1}^{s} z_j^{\ell_{ij}} \right)^{cd^{k+k_i}} = \left(\prod_{j=1}^{s} z_j^{\ell_{1j}} \right)^{cd^{k+k_1}}, \tag{10}$$

and for all $i \in \{1, \ldots, r\} \setminus S$, $k \ge 0$, and $k' \ge R$,

$$\left(\prod_{j=1}^{s} z_j^{\ell_{ij}} \right)^{cd^{k'}} \ne \left(\prod_{j=1}^{s} z_j^{\ell_{1j}} \right)^{cd^{k+k_1}}.$$

Therefore, comparing the coefficients of $(\prod_{j=1}^{s} z_j^{\ell_{1j}})^{cd^{k+k_1}}$ in (8), we have

$$\sum_{i \in S} \sum_{l=0}^{L} c_{li} (cd^{k+k_i} + k + k_i)^l \alpha_i^{k+k_i} \zeta_i^{cd^{k+k_i}} = 0 \tag{11}$$

for all $k \ge 0$. Expanding (11), we have

$$\begin{aligned} &\sum_{i \in S} \sum_{l=0}^{L} c_{li} \zeta_i^{cd^{k+k_i}} \alpha_i^{k+k_i} \sum_{u+v+w=l} \binom{l}{u\ v\ w} (cd^{k+k_i})^u k^v k_i^w \\ &= \sum_{i \in S} \sum_{u=0}^{L} \sum_{v=0}^{L-u} \sum_{l=u+v}^{L} \binom{l}{u\ v\ l-u-v} c_{li} c^u \zeta_i^{cd^{k+k_i}} (\alpha_i d^u)^{k_i} k_i^{l-u-v} (\alpha_i d^u)^k k^v \\ &= 0 \end{aligned} \tag{12}$$

for all $k \ge 0$. Let $\theta_T > \theta_{T-1} > \cdots > \theta_1$ be the distinct absolute values of $\alpha_i d^u$ $(i \in S,\ 0 \le u \le L)$ and let

$$X_t = \{(i, u) \mid i \in S,\ 0 \le u \le L,\ |\alpha_i d^u| = \theta_t\}$$

for each t $(1 \le t \le T)$. Then we have $S \times \{0, \dots, L\} = X_1 \sqcup \cdots \sqcup X_T$. By (12), we obtain

$$\begin{aligned}
&\sum_{t=1}^{T} \sum_{(i,u)\in X_t} \sum_{v=0}^{L-u} \sum_{l=u+v}^{L} \binom{l}{u\ v\ l-u-v} c_{li} c^u \zeta_i^{cd^{k+k_i}} (\alpha_i d^u)^{k_i} k_i^{l-u-v} (\alpha_i d^u)^k k^v \\
&= \sum_{t=1}^{T} \sum_{v=0}^{L} a_k^{(t,v)} \theta_t^k k^v \\
&= 0 \qquad (13)
\end{aligned}$$

for all $k \ge 0$, where

$$a_k^{(t,v)} = \sum_{\substack{(i,u)\in X_t,\\ u+v\le L}} \sum_{l=u+v}^{L} \binom{l}{u\ v\ l-u-v} c_{li} c^u \zeta_i^{cd^{k+k_i}} (\alpha_i d^u)^{k_i} k_i^{l-u-v} \left(\frac{\alpha_i d^u}{\theta_t}\right)^k. \qquad (14)$$

Note that each sequence $\{a_k^{(t,v)}\}_{k\ge 0}$ is bounded since $|\alpha_i d^u/\theta_t| = 1$ and $|\zeta_i| = 1$. We give a lexicographical order to $(t,v) \in \{1,\dots,T\} \times \{0,\dots,L\}$, namely, $(T,L) > (T,L-1) > \cdots > (T,0) > (T-1,L) > \cdots > (1,1) > (1,0)$. We prove by induction on (t,v) from (T,L) down to $(1,0)$ that

$$a_k^{(t,v)} = 0 \qquad (15)$$

for all sufficiently large k. First, we see that $a_k^{(T,L)} = 0$. Indeed, if $(i,u) \in X_T$, then $u = L$, since $d \ge 2$. Thus, if $v = L$, then we have $u+v = 2L > L$, and so $a_k^{(T,L)}$ is an empty sum. Hence $a_k^{(T,L)} = 0$. Assume that for all (t,v) with $(t,v) > (t_0, v_0)$,

$$a_k^{(t,v)} = 0$$

for all sufficiently large k. Then by (13), we have

$$\sum_{(t,v)\le(t_0,v_0)} a_k^{(t,v)} \theta_t^k k^v = 0$$

and hence

$$\begin{aligned}
a_k^{(t_0,v_0)} \theta_{t_0}^k k^{v_0} &= -\sum_{(t,v)<(t_0,v_0)} a_k^{(t,v)} \theta_t^k k^v \\
&= -\sum_{0\le v<v_0} a_k^{(t_0,v)} \theta_{t_0}^k k^v - \sum_{t=1}^{t_0-1} \sum_{v=0}^{L} a_k^{(t,v)} \theta_t^k k^v \qquad (16)
\end{aligned}$$

for all sufficiently large k. Note that if $v_0 = 0$, the first sum of the rightmost side of (16) is an empty sum. Dividing both sides by $\theta_{t_0}^k k^{v_0}$, we have

$$a_k^{(t_0,v_0)} = -\sum_{0\le v<v_0} a_k^{(t_0,v)} k^{v-v_0} - \sum_{t=1}^{t_0-1}\sum_{v=0}^{L} a_k^{(t,v)}\left(\frac{\theta_t}{\theta_{t_0}}\right)^k k^{v-v_0}.$$

Since $v - v_0 < 0$, the first sum of the right-hand side tends to zero as k tends to infinity. The second sum of the right-hand side also tends to zero as k tends to infinity, since $\theta_t < \theta_{t_0}$ if $1 \le t \le t_0 - 1$. Therefore, we have

$$a_k^{(t_0,v_0)} \to 0 \quad (k \to \infty). \tag{17}$$

Here we write $(i,u) \sim (i',u')$ if $(\alpha_i d^u)^{N_1} = (\alpha_{i'} d^{u'})^{N_1}$. Then $\sim$ is an equivalence relation on $Y := \{(i,u) \in X_{t_0} \mid u + v_0 \le L\}$. Let $Y = Y_1 \sqcup \cdots \sqcup Y_p$ be the partition of Y with respect to $\sim$. For each q $(1 \le q \le p)$, we fix a representative (i_q, u_q) of Y_q and let

$$\delta_q := \frac{\alpha_{i_q} d^{u_q}}{\theta_{t_0}}.$$

Then $|\delta_q| = 1$ $(1 \le q \le p)$, and $\delta_q/\delta_{q'}$ $(1 \le q < q' \le p)$ are not N_1-th roots of unity. In addition, for each $(i,u) \in Y_q$, letting

$$\frac{\alpha_i d^u}{\theta_{t_0}} = \xi_{iu}\delta_q$$

with ξ_{iu} an N_1-th root of unity, we have by (14) the expression

$$\begin{aligned} a_k^{(t_0,v_0)} &= \sum_{(i,u)\in Y}\sum_{l=u+v_0}^{L} \binom{l}{u\ v_0\ l-u-v_0} c_{li} c^u \zeta_i^{cd^{k+k_i}} (\alpha_i d^u)^{k_i} k_i^{l-u-v_0} \left(\frac{\alpha_i d^u}{\theta_{t_0}}\right)^k \\ &= \sum_{q=1}^{p} b_k^{(q)} \delta_q^k, \end{aligned}$$

where

$$b_k^{(q)} = \sum_{(i,u)\in Y_q}\sum_{l=u+v_0}^{L} \binom{l}{u\ v_0\ l-u-v_0} c_{li} c^u \zeta_i^{cd^{k+k_i}} (\alpha_i d^u)^{k_i} k_i^{l-u-v_0} \xi_{iu}^k.$$

Since $\{\zeta_i^{d^{k+k_i}}\}_{k\ge 0}$ $(1 \le i \le r)$ are eventually periodic with period N_1 by (5), so are $\{b_k^{(q)}\}_{k\ge 0}$ $(1 \le q \le p)$. By (17) and Lemma 2.7, we have

$$b_k^{(q)} = 0 \quad (1 \le q \le p)$$

for all sufficiently large k. Thus

$$a_k^{(t_0,v_0)} = 0$$

for all sufficiently large k, and (15) is proved. In particular, noting that $(i,u) \in X_1$ if and only if $|\alpha_i| = \min_{i' \in S} |\alpha_{i'}| = \theta_1$ and $u = 0$, we have

$$a_k^{(1,0)} = \sum_{\substack{i \in S,\\ |\alpha_i| = \theta_1}} \sum_{l=0}^{L} c_{li} \zeta_i^{cd^{k+k_i}} \alpha_i^{k_i} k_i^l \left(\frac{\alpha_i}{\theta_1}\right)^k = 0 \tag{18}$$

for all sufficiently large k. Renumbering the indices i $(1 \le i \le r)$, we may assume that $\{i \in S \mid |\alpha_i| = \theta_1\} = \{1, \ldots, r_0\}$. Put $\lambda_i := \sum_{l=0}^{L} c_{li} k_i^l$ $(1 \le i \le r_0)$. Then $\lambda_i \ne 0$ $(1 \le i \le r_0)$ since $k_i \ge R$. Multiplying both sides of (18) by θ_1^k, we have

$$\sum_{i=1}^{r_0} \lambda_i \alpha_i^{k+k_i} \zeta_i^{cd^{k+k_i}} = 0$$

for all $k \ge k_0$, where k_0 is taken to be sufficiently large. Then, using (10), we have

$$\begin{aligned}\sum_{i=1}^{r_0} \lambda_i g_{0i}(\boldsymbol{z}) &= \sum_{i=1}^{r_0} \lambda_i \sum_{k=0}^{\infty} \alpha_i^k \zeta_i^{cd^k} \left(\prod_{j=1}^{s} z_j^{\ell_{ij}}\right)^{cd^k} \\ &= \sum_{i=1}^{r_0} \lambda_i \sum_{k=0}^{k_0+k_i-1} \alpha_i^k \zeta_i^{cd^k} \left(\prod_{j=1}^{s} z_j^{\ell_{ij}}\right)^{cd^k} \\ &\in \overline{\mathbb{Q}}[\boldsymbol{z}],\end{aligned} \tag{19}$$

namely $\{g_{0i}(\boldsymbol{z}) \mid 1 \le i \le r_0\}$ is linearly dependent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}[\boldsymbol{z}]$. Noting that

$$g_{0i}(\boldsymbol{z}) = \sum_{k=0}^{\infty} \alpha_i^k \zeta_i^{cd^k} \left(\prod_{j=1}^{s} z_j^{\ell_{ij}}\right)^{cd^k}, \tag{20}$$

we see by (3) and (5) that

$$g_{0i}(\boldsymbol{z}) - \alpha_i^{N_1} g_{0i}(\boldsymbol{z}^{\boldsymbol{d}(N_1)}) \in \overline{\mathbb{Q}}[\boldsymbol{z}] \tag{21}$$

for each i $(1 \le i \le r_0)$. Multiplying (21) by $\lambda_i \alpha_i^{-N_1}$, we get

$$\lambda_i \alpha_i^{-N_1} g_{0i}(\boldsymbol{z}) - \lambda_i g_{0i}(\boldsymbol{z}^{\boldsymbol{d}(N_1)}) \in \overline{\mathbb{Q}}[\boldsymbol{z}]. \tag{22}$$

Then, replacing $\boldsymbol{z}$ with $\boldsymbol{z}^{\boldsymbol{d}(N_1)}$ in (19) and substituting (22) into it, we have

$$\sum_{i=1}^{r_0} \lambda_i \alpha_i^{-N_1} g_{0i}(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}].$$

We write $i \sim i'$ if $\alpha_i^{N_1} = \alpha_{i'}^{N_1}$. Then $\sim$ is an equivalence relation on $Z := \{1, \ldots, r_0\}$. Let $Z = Z_1 \sqcup \cdots \sqcup Z_p$ denote the partition of Z with respect to $\sim$ such that $|Z_1| \leq \cdots \leq |Z_p|$. Renumbering the indices i $(1 \leq i \leq r_0)$, we may assume that $Z_1 = \{1, \ldots, r_1\}$. Letting i_p be the representative of Z_p and subtracting

$$\alpha_{i_p}^{N_1} \sum_{i=1}^{r_0} \lambda_i \alpha_i^{-N_1} g_{0i}(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}]$$

from (19), we have

$$\sum_{\substack{1 \leq i \leq r_0, \\ \alpha_i^{N_1} \neq \alpha_{i_p}^{N_1}}} \lambda_i \left(1 - \frac{\alpha_{i_p}^{N_1}}{\alpha_i^{N_1}}\right) g_{0i}(\boldsymbol{z}) = \sum_{i \in Z \setminus Z_p} \lambda_i \left(1 - \frac{\alpha_{i_p}^{N_1}}{\alpha_i^{N_1}}\right) g_{0i}(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}]$$

and thus $\{g_{0i}(\boldsymbol{z}) \mid i \in Z \setminus Z_p\}$ is linearly dependent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}[\boldsymbol{z}]$. Continuing this process, we see that $\{g_{0i}(\boldsymbol{z}) \mid i \in Z_1\} = \{g_{0i}(\boldsymbol{z}) \mid 1 \leq i \leq r_1\}$ is linearly dependent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}[\boldsymbol{z}]$. Here, either of the following two cases holds:

(i) There exists $1 \leq i' \leq r_0$ such that $\alpha_i^{N_1} \neq \alpha_{i'}^{N_1}$ $(i \in \{1, \ldots, r_0\} \setminus \{i'\})$.

(ii) For each $1 \leq i \leq r_0$, there exists $i' \neq i$ such that $\alpha_i^{N_1} = \alpha_{i'}^{N_1}$.

If (i) holds, then $r_1 = 1$ and so $g_{01}(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}]$, which contradicts (20). Hence (ii) holds, namely $\alpha_1^{N_1} = \cdots = \alpha_{r_1}^{N_1}$ and the functions $g_{0i}(\boldsymbol{z})$ $(1 \leq i \leq r_1)$ are linearly dependent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}[\boldsymbol{z}]$, which imply respectively that $N_1 \geq r_1 \geq 2$ since $\alpha_1, \ldots, \alpha_{r_1}$ are distinct and that the values $g_{0i}(\boldsymbol{\beta})$ $(1 \leq i \leq r_1)$ are algebraically dependent. Since $\alpha_1^{N_1} = \cdots = \alpha_{r_1}^{N_1}$, we can write $\alpha_i = \xi_i \alpha_1$ $(1 \leq i \leq r_1)$, where ξ_i $(1 \leq i \leq r_1)$ are N_1-th roots of unity with $\xi_1 = 1$. Let z_0 be a variable. For each i $(1 \leq i \leq r_1)$, we define

$$h_i(z_0) = \sum_{k=0}^{\infty} \alpha_i^k \xi_i^{cd^k} z_0^{cd^k}. \tag{23}$$

Since $h_i(\alpha_1) = \sum_{k=0}^{\infty} \alpha_i^{cd^k+k} = g_{0i}(\boldsymbol{\beta})$ $(1 \le i \le r_1)$, the values $h_i(\alpha_1)$ $(1 \le i \le r_1)$ are algebraically dependent. By $N_1 \ge 2$, from Lemma 2.2 there exist a positive integer N_2 and a nonnegative integer u_2 such that $N_2, u_2 < N_1$ and

$$cd^{k+N_2} \equiv cd^k \pmod{N_1} \tag{24}$$

for any $k \ge u_2$. Then by (23) and (24) we see that

$$h_i(z_0) - \alpha_i^{N_2} h_i(z_0^{d^{N_2}}) \in \overline{\mathbb{Q}}[z_0]$$

for each i $(1 \le i \le r_1)$. Hence $h_i(z_0)$ $(1 \le i \le r_1)$ are Mahler functions of one variable and so, only by Theorem 2 of Nishioka [5], they are linearly dependent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}[z_0]$. We may assume that all the coefficients of the linear dependence relation of $h_i(z_0)$ $(1 \le i \le r_1)$ are nonzero. Then, similarly to the case (i) above, if there exists $1 \le i' \le r_1$ such that $\alpha_i^{N_2} \ne \alpha_{i'}^{N_2}$ $(i \in \{1, \ldots, r_1\} \setminus \{i'\})$, then $h_{i'}(z_0) \in \overline{\mathbb{Q}}[z_0]$, which contradicts (23). Otherwise, renumbering the indices i $(1 \le i \le r_1)$, we see that there exists some r_2 $(2 \le r_2 \le N_2)$ such that $\alpha_1^{N_2} = \cdots = \alpha_{r_2}^{N_2}$ and $h_i(z_0)$ $(1 \le i \le r_2)$ are linearly dependent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}[z_0]$. Iterating this process finite times, we reach $N_1 > N_2 > N_3 > \cdots > N_M = 1$ by Lemma 2.2. Since $\alpha_1, \ldots, \alpha_{r_{M-1}}$ are distinct, similarly to the case (i) above, we lead to a contradiction.

Now we verify that a similar argument can be applied to $g^-(z) = \sum_{k=0}^{\infty} z^{d^k-k}$ in proving the algebraic independency of $\{g^{-(l)}(\alpha) \mid l \in \mathbb{Z}_{\ge 0},\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$. Assume that $\{g^{-(l)}(\alpha_i) \mid 0 \le l \le L,\ 1 \le i \le r\}$ is algebraically dependent. For each l, i $(0 \le l \le L,\ 1 \le i \le r)$, define

$$g_{li}^-(\boldsymbol{z}) = \sum_{k=0}^{\infty} (cd^k - k)^l \alpha_i^{-k} \zeta_i^{cd^k} \left(\prod_{j=1}^{s} z_j^{\ell_{ij}} \right)^{cd^k}.$$

Then $\{g_{li}^-(\boldsymbol{\beta}) \mid 0 \le l \le L,\ 1 \le i \le r\}$ is algebraically dependent. By (5), Lemma 2.3, and Lemma 2.4, we see that $g_{li}^-(\boldsymbol{z}^{\boldsymbol{d}(pN_1)})$ $(0 \le l \le L,\ 1 \le i \le r,\ 0 \le p \le m(l) - 1)$ satisfy the functional equation of the form (6) with g_{li}^- in place of g_{li}. Then by the vanishing theorem of Masser [3] and by Theorem 2 of Nishioka [5], there exist algebraic numbers c_{li} $(0 \le l \le L,\ 1 \le i \le r)$, not all zero, such that

$$\sum_{l=0}^{L} \sum_{i=1}^{r} c_{li} g_{li}^-(\boldsymbol{z}) = \sum_{l=0}^{L} \sum_{i=1}^{r} c_{li} \sum_{k=0}^{\infty} (cd^k - k)^l \alpha_i^{-k} \zeta_i^{cd^k} \left(\prod_{j=1}^{s} z_j^{\ell_{ij}} \right)^{cd^k} \in \overline{\mathbb{Q}}[\boldsymbol{z}].$$

We may assume that c_{li} $(0 \le l \le L)$ are not all zero for any i $(1 \le i \le r)$. Let R be a sufficiently large integer such that the roots of the polynomials $\sum_{l=0}^{L} c_{li} X^l$ $(1 \le i \le r)$ are inside the circle $|X| = R$. Then we have

$$\sum_{l=0}^{L} \sum_{i=1}^{r} c_{li} \sum_{k=R}^{\infty} (cd^k - k)^l \alpha_i^{-k} \zeta_i^{cd^k} \left(\prod_{j=1}^{s} z_j^{\ell_{ij}} \right)^{cd^k} =: G^-(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}].$$

Take an integer k_1 so large that

$$cd^{k_1} > \max\Big\{ \deg_{\boldsymbol{z}} G^-(\boldsymbol{z}), \max_{1 \le i \le r, 1 \le j \le s} \{\ell_{ij} d^R\} \Big\}.$$

Similarly to the case of $g(z) = \sum_{k=0}^{\infty} z^{cd^k + k}$, we define the subset S of $\{1, \ldots, r\}$ and the nonnegative integers k_i $(i \in S)$. Then, similarly to (11), we see that

$$\sum_{i \in S} \sum_{l=0}^{L} c_{li} (cd^{k+k_i} - k - k_i)^l \alpha_i^{-k-k_i} \zeta_i^{cd^{k+k_i}} = 0$$

for all $k \ge 0$. Expanding this equation, we have

$$\sum_{i \in S} \sum_{u=0}^{L} \sum_{v=0}^{L-u} \sum_{l=u+v}^{L} \binom{l}{u\ v\ l-u-v} \times c_{li} c^u \zeta_i^{cd^{k+k_i}} (\alpha_i^{-1} d^u)^{k_i} (-k_i)^{l-u-v} (\alpha_i^{-1} d^u)^k (-k)^v = 0$$

for all $k \ge 0$. Let $\theta_T > \theta_{T-1} > \cdots > \theta_1$ be the distinct absolute values of $\alpha_i^{-1} d^u$ $(i \in S,\ 0 \le u \le L)$ and let

$$X_t = \{(i,u) \mid i \in S,\ 0 \le u \le L,\ |\alpha_i^{-1} d^u| = \theta_t\}$$

for each t $(1 \le t \le T)$. Define

$$a_k^{-(t,v)} = \sum_{\substack{(i,u) \in X_t, \\ u+v \le L}} \sum_{l=u+v}^{L} \binom{l}{u\ v\ l-u-v} c_{li} c^u \zeta_i^{cd^{k+k_i}} (\alpha_i^{-1} d^u)^{k_i} (-k_i)^{l-u-v} \left(\frac{\alpha_i^{-1} d^u}{\theta_t} \right)^k.$$

Then, similarly to the case of $g(z)$, we can prove that

$$a_k^{-(1,0)} = \sum_{\substack{i \in S, \\ |\alpha_i^{-1}| = \theta_1}} \sum_{l=0}^{L} c_{li} \zeta_i^{cd^{k+k_i}} \alpha_i^{-k_i} (-k_i)^l \left(\frac{\alpha_i^{-1}}{\theta_1} \right)^k = 0$$

for all sufficiently large k. Renumbering the indices i $(1 \leq i \leq r)$, we may assume that $\{i \in S \mid |\alpha_i^{-1}| = \theta_1\} = \{1, \ldots, r_0\}$. Then we see that

$$\sum_{i=1}^{r_0} \lambda_i g_{0i}^{-}(\boldsymbol{z}) = \sum_{i=1}^{r_0} \lambda_i \sum_{k=0}^{k_0+k_i-1} \alpha_i^{-k} \zeta_i^{cd^k} \left(\prod_{j=1}^{s} z_j^{\ell_{ij}} \right)^{cd^k} \in \overline{\mathbb{Q}}[\boldsymbol{z}],$$

where $\lambda_i = \sum_{l=0}^{L} c_{li}(-k_i)^l \neq 0$ $(1 \leq i \leq r_0)$. This implies that $\{g_{0i}^{-}(\boldsymbol{z}) \mid 1 \leq i \leq r_0\}$ is linearly dependent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}[\boldsymbol{z}]$, and the proof is completed in a similar way to (19) and thereafter. $\square$

Acknowledgements

This work was supported by JSPS KAKENHI Grant Number JP20K03519.

References

[1] H. Ide, T. Tanaka, and K. Toyama, *Algebraic independence of the values of power series and their derivatives generated by linear recurrences*, to appear in Tokyo J. Math.

[2] J. H. Loxton and A. J. van der Poorten, *Algebraic independence properties of the Fredholm series*, J. Austral. Math. Soc. Ser. A **26** (1978), 31–45.

[3] D. W. Masser, *A vanishing theorem for power series*, Invent. math. **67** (1982), 275–296.

[4] K. Nishioka, *Conditions for algebraic independence of certain power series of algebraic numbers*, Compositio Math. **62** (1987), 53–61.

[5] K. Nishioka, *Algebraic independence of Mahler functions and their values*, Tohoku Math. J. **48** (1996), 51–70.

[6] W. Rudin, *Real and complex analysis*, Third edition, McGraw-Hill, New York, 1987.

[7] T. N. Shorey and R.Tijdeman, *Exponential Diophantine Equations*, Cambridge University Press, 1986.

[8] Y. Tanuma, *Algebraic independence of the values of certain series and their derivatives involving the Hecke-Mahler series*, J. Number Theory **210** (2020), 231–248.