Bratteli–Vershik models for zero-dimensional systems

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Abstract. Herman, Putnam and Skau (R. H. Herman, I. F. Putnam and C. F. Skau (1992)) showed some correspondence between the pointed topological conjugacy classes of essentially minimal compact zero-dimensional systems (X, φ, y) and the equivalence classes of essentially simple ordered Bratteli diagrams. In fact, using these, they made deep investigations into C^*-algebraic theories. Later Medynets (K. Medynets (2006)) showed that every Cantor aperiodic system is homeomorphic to the Vershik map acting on the space of infinite paths of an ordered Bratteli diagram. He produced an equivalent class of ordered Bratteli diagrams from a topologically conjugacy classes of a triple (X, φ, B), in which B is a particular closed set that is called a basic set. In this manuscript, we explain some basic concepts that can extend some topology of their works to the case in which there may be a lot of periodic orbits. In doing this, the work by Downarowicz and Karpel (T. Downarowicz and O. Karpel(2019)) plays an important role.

1. Introduction

In [HPS92, Theorem 4.7], R. H. Herman, I. F. Putnam and C. F. Skau established the bijective correspondence between equivalence classes of essentially simple ordered Bratteli diagrams and pointed topological conjugacy classes of essentially minimal systems. Later in [M06], K. Medynets introduced the notion of basic sets. He showed in [M06, Theorem 2.4] that if a Cantor system (X, f) is aperiodic then it has a basic set B, and the triple (X, f, B) is topologically conjugate to a Bratteli–Vershik system $(V, E, \geqslant)$ such that B corresponds to the set of minimal paths. Following the work [DK19], by T. Downarowicz and O. Karpel, I in [S20] could extend the bijective correspondence to almost all zero-dimensional systems. In this manuscript, I would like to explain how this is done. Owing to [DK19, Theorem 3.1], it was natural to exclude the zero-dimensional systems that have the set of periodic orbit with non-empty interior. In this manuscript, (X, f) is a *zero-dimensional system* if X is compact metrizable zero-dimensional space, and $f : X \to X$ is a homeomorphism. For the definition of the ordered Bratteli diagrams, see for example [HPS92, M06, BY17].

2. Key definitions and main results

An ordered Bratteli diagram is represented as $(V, E, \geqslant)$, where V is the set of the vertices, E is the set of the edges, and $\geqslant$ is the partial order on the set E. The vertex set V is partitioned as $V = \bigcup_{i=0}^{\infty} V_i$, where V_i's are disjoint finite set of vertices. It is assumed that V_0 consists of a unique vertex v_0. For each edge $e \in E_n$, there exists the source map $s : E_n \to V_{n-1}$ and the range map $r : E_n \to V_n$, i.e., each edge $e \in E_n$ is from the vertex $s(e)$ to the vertex $r(e)$. In this manuscript, the set of infinite paths from the vertices of V_n is denoted as $E_{n,\infty}$. Thus, we

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mainly consider $E_{0,\infty}$. The paths of $E_{0,\infty}$ are naturally partially ordered by the lexicographic order that arise from $\geqslant$ on the set E. It is well known that there exists the non-empty set $E_{0,\infty,\max}$ of all maximal paths and the non-empty set $E_{0,\infty,\min}$ of all minimal paths. For each $p \in E_{0,\infty} \backslash E_{0,\infty,\max}$, there exists the least $p' > p$. The *Vershik map* ψ on $E_{0,\infty} \backslash E_{0,\infty,\max}$, is defined as $\phi(p) = p'$. If it is possible to define $\psi : E_{0,\infty,\max} \to E_{0,\infty,\min}$ such that there exists a topological conjugacy between (X, f, B) and $(E_{0,\infty}, \psi, E_{0,\infty,\min})$, then $(V, E, \geqslant, \psi)$ is called a *Bratteli–Vershik model for* (X, f, B). In [DK19], Downarowicz and Karpel defined that an ordered Bratteli diagram $(V, E, \geqslant)$ is *decisive* if the Vershik map $\psi : E_{0,\infty} \backslash E_{0,\infty,\max} \to E_{0,\infty}$ prolongs in a unique way to a homeomorphism $\psi : E_{0,\infty} \to E_{0,\infty}$. In this manuscript, for a decisive ordered Bratteli diagram $(V, E, \geqslant)$, the unique Bratteli–Vershik model $(V, E, \geqslant, \psi)$ is also said to be *decisive*. They also defined that a zero-dimensional system (X, f) is *Bratteli–Vershikizable* if it is conjugate to $(E_{0,\infty}, \psi)$ for a decisive ordered Bratteli diagram $(V, E, \geqslant)$. They have shown that a zero-dimensional system (X, f) is Bratteli–Vershikizable if and only if either the set of aperiodic points is dense or its closure misses one periodic orbit (see [DK19, Theorem 3.1]).

I denote that

$$\mathcal{M} := \{ M \subseteq X \mid (M, f|_M) \text{ is a minimal set.} \},$$

and, I say that a closed set $A \subseteq X$ is a *quasi-section* if $A \cap M \neq \varnothing$ for all $M \in \mathcal{M}$. If B is a quasi-section of a zero-dimensional system (X, f), then I say that (X, f, B) is a *triple of zero-dimensional systems with quasi-sections*. I define as follows:

Definition 2.1. For a zero-dimensional system (X, f) and a quasi-section $B \subseteq X$, the triple (X, f, B) is said to be *continuously decisive* if $\mathrm{int} B = \varnothing$.

With these definitions, the following theorem holds:

Theorem 2.2. *Let (X, f) be a zero-dimensional system and B be a quasi-section. Then, there exists a Bratteli–Vershik model $(V, E, \geqslant, \psi)$ such that $B = E_{0,\infty,\min}$ with respect to a topological conjugacy. Furthermore, if (X, f, B) is continuously decisive, then the map $\psi : E_{0,\infty,\max} \to E_{0,\infty,\min}$ is uniquely determined for ψ to be continuous.*

Therefore, the existence of continuously decisive quasi-section is a matter of question. Following [DK19, Theorem 3.1], I define as follows:

Definition 2.3. A zero-dimensional system (X, f) is *densely aperiodic* if the set $\{ x \in X \mid f^i(x) \neq x \text{ for all } \mathbb{Z} \ni i \neq 0 \}$ is dense in X.

Definition 2.4. An ordered Bratteli diagram $(V, E, \geqslant)$ is *continuously decisive* if <u>it is decisive</u> and $E_{0,\infty} = \overline{E_{0,\infty} \backslash E_{0,\infty,\max}}$.

Then, we get a one-to-one correspondence as follows:

Theorem 2.5. *There exists a one-to-one correspondence between the equivalence classes of continuously decisive ordered Bratteli diagrams and the topological conjugacy classes of continuously decisive triples of zero-dimensional systems with quasi-sections.*

Generally, even if B is a continuously decisive quasi-section, there may exists a point $x \in X$ such that x enters B twice and more. Therefore, we are led to the following definition:

Definition 2.6. Let (X, f) be a zero-dimensional system. A closed set $A \in X$ is called a *basic set* if (1) A is a quasi-section and (2) for every $x \in X$, the orbit of x enters A at most once, i.e., $|O(x) \cap A| \leqslant 1$, in which $O(x) := \{ f^i(x) \mid i \in \mathbb{Z} \}$.

If B is a basic set of (X, f), then the corresponding ordered Bratteli diagrams must have some special quality. So, we define as follows:

Definition 2.7. Let (V, E) be a Bratteli diagram and $n \geqslant 0$. We say that an infinite path $(e_{n+1}, e_{n+2}, \dots) \in E_{n,\infty}$ is *constant* if $|r^{-1}(r(e_i))| = 1$ for all $i > n$. A Bratteli–Vershik model $(V, E, \geqslant, \psi)$ has the *closing property* if every constant path $(e_{n+1}, e_{n+2}, \dots) \in E_{n,\infty}$ with $n \geqslant 0$ is a periodic orbit.

Using the basic sets and the ordered Bratteli diagrams with closing property, we can get a little more fine results as follows:

Theorem 2.8. *Let (X, f) be a Bratteli–Vershikizable zero-dimensional system. Then, there exists a decisive Bratteli–Vershik model $(V, E, \geqslant, \psi)$ of (X, f) with closing property.*

Theorem 2.9. *There is a one-to-one correspondence between the equivalence classes of continuously decisive ordered Bratteli diagrams with closing property and the topological conjugacy classes of continuously decisive triples of zero-dimensional systems with basic sets.*

Very roughly, the existence of basic sets is remained. Let (X, f) be a zero-dimensional system. Of course, X can be embedded in the Cantor set in $[0, 1]$. Therefore, X is linearly ordered with the linear order $\geqslant$ in $[0, 1]$. For every $x \in X$, let $\inf(x) := \inf(O(x))$. It can be proved that the set $\{ \inf(x) \mid x \in X \}$ is closed. The next result is gotten directly:

Theorem 2.10. *The set $\{ \inf(x) \mid x \in X \}$ is a basic set.*

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