Differential Galois Theory and Quantum Physics

Juan José Morales-Ruiz
Depto. de Matemática Aplicada, Universidad Politécnica de Madrid, Madrid, Spain*

Abstract

We survey my 2020 paper [18], on the relevance of the differential Galois theory of linear differential equations for the exact semiclassical computations in path integrals in quantum mechanics. The main tool will be a necessary condition for complete integrability of classical Hamiltonian systems obtained by Ramis and myself, formulated in the framework of differential Galois theory. A corollary of this result is that, for finite dimensional integrable Hamiltonian systems, the semiclassical approach is computable in closed form. This explains in a very precise way the success of quantum semiclassical computations for integrable Hamiltonian systems. Moreover, I will point out several of the many open problems motivated from the above simple result: problems from quantum mechanics to quantum field theory.

Key words: *Differential Galois theory, Action Integral, Complete Integrability, Path Integrals, Semiclassical Approximation*

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Introduction

In 1942 Feynman in his Phd Thesis discovered a new formulation to quantum mechanics, as an alternative to the more classical formulations: the Schrödinger's operator and the Heisenberg's matrix ones ([10]). Today Feynman's approach is one of the most successful ways to study quantum systems, either finite dimensional (quantum mechanics) or not (quantum fields), including bosons, fermions, strings or even fields with gauge symmetries, today necessary in any reasonable quantum field theory.

Feynman motivation was to find a formulation of the quantum mechanics closet to the classical mechanics. In fact, Feynman recognizes in 1964 the difficulties to understand quantum mechanics:

"I think I can safely say that nobody really understands quantum mechanics" ([11]).

On the other hand, DeWitt pointed out:

*juan.morales-ruiz@upm.es

"The quantum theory is basically a theory of small disturbances" ([7]).

Thus, it is possible to consider the quantum mechanics as essentially (quantum) fluctuations around the (classical) solutions given by the classical mechanics. The same idea works for the connection between classical field theory and quantum field theory. Then one can approach to the quantum physics by considering classical solutions and small perturbation of this classical solutions with respect to initial conditions and parameters. Here we concentrate our attention to the variations with respect to the initial conditions: the small variations with respect to the initial conditions are given by a very well-known object in dynamical systems, the *variational equation*, and in variational calculus it is called the Jacobi equation (and its solutions Jacobi fields). In quantum physics the variational equation is also called the "small disturbance equation" ([7, 8]), in agreement with the above, but it is an equation that comes from the classical (non-quantum) physics.

Related with the above is the semiclassical (or WKB) approximation in quantum systems. Since Feynman, the semiclassical expansion is one of the most effective methods to compute propagators in quantum systems. The idea is as follows, starting from a *classical path* γ, $\mathbf{x} = \mathbf{x}(t)$ (ie, solution of the Lagrange equation), to consider paths given by small quantum fluctuations around it, $\mathbf{x} + \boldsymbol{\xi}$, and then expand the amplitudes around the classical solution in powers of $\hbar$, being the first term of this expansion the semiclassical approximation. The fluctuations in this approximation are then expressed by means of the determinant of the differential operator of the variational equation (a functional determinant) with suitable boundary conditions. By the so-called Gelfand-Yaglom method, this infinite determinant can be obtained as the determinant of a block of the fundamental matrix of the variational equation with standard initial conditions: we move from an apparently hard spectral problem to an apparently more treatable initial value problem. A similar, but more complex situation, is given in the case of fields, where the classical fields are defined by infinite dimensional problems. For functional determinants see [9].

An ideal goal in quantum physics is to solve the equations of quantum mechanics and of quantum field theory. No body known a general approach to study this goal. Instead of the above, our goal will be *to solve the equations of the semiclassical approximation of quantum physics.*

In agreement with Dewitt remark, the semiclassical approximation is precisely very related with quantum fluctuations around classical solutions, and hence with the variational equations.

Two problems are hidden in the above goal:

1) Is it possible to solve the equations by closed analytical formulas?

2) If yes, then how to obtain the solutions.

The tool for both problems will be the *differential Galois theory.*

When it is possible to solve the equations by closed analytical formulas we say that *the problem is integrable.*

As we will see, there are three different, but very related, integrability notions along the paper:

i) The integrability in the sense of the differential Galois theory.

ii) The integrability of the classical system.

iii) The integrability of the semiclassical quantum approximation.

All this notions are used to clarify what exactly means to solve one of the problems in "closed form".

Our proposal is based onto two key ideas:

Key Idea 1: *A necessary condition for integrability of Hamiltonian systems is the integrability of the variational equation along a particular solution of the system.*
Key Idea 2:*The integrability of the variational equation implies the integrability of the semiclassical quantum approximation defined by the particular solution.*

Hence:
The integrability of the Hamiltonian system implies the integrability of the quantum semiclassical approximation defined by the particular solution.

As will becomes clear later, the quantum semiclassical approximation is not only very close to the classical mechanics, but *it only depends on classical mechanics constructions: the classical functional action and the variational equations.*

1 Differential Galois Theory

The differential Galois theory of linear ordinary differential equations is also called the Picard-Vessiot theory, because it was discovered by Picard at the end of the XIX century and with relevant contributions by Vessiot, a Picard's student, some years later. It was formalized by Kolchin in the middle of the XX century. Two standard monographs about it are [6, 21], and for an analytic introduction I recommend [23]. As complementary references, see also Martinet and Ramis's nice presentation [16] and the second chapter of the book [17].

We will assume that we are in the complex analytical category: the coordinates are over a complex analytical manifold, etc. In particular, time is analytically prolonged to the complex plane, which is in complete agreement with today's quantum physics.

A differential field K is a field with a derivative (or derivation) $\partial = '$, ie, an additive mapping satisfying the Leibniz rule. From now on we will assume that $K = \mathcal{M}(\Gamma)$, the meromorphic functions over a connected Riemann surface Γ. If t is a local coordinate over Γ, we consider $\frac{d}{dt}$ as derivation. Particular cases are $K = \mathbf{C}(t) = \mathcal{M}(\mathbf{P^1})$, the field of rational functions (the field of meromorphic functions over the Riemann sphere $\mathbf{P^1}$) or that of meromorphic functions on a genus one Riemann surface (a field of elliptic functions).

We can define differential subfields and differential extensions in a direct way by requiring that inclusions commute with the derivation. Analogously, a differential automorphism in K is an automorphism commuting with the derivative.

Let

$$\mathbf{y}' = A\mathbf{y}, \quad A = A(t) \in Mat(m, K) \tag{1.1}$$

be a system of linear differential equations. We now proceed to associate with (1.1) the so-called Picard-Vessiot extension of K. The Picard-Vessiot extension L of (1.1) is an extension of K, such that if $\phi_1, ..., \phi_m$ is a "fundamental" system of solutions of the equation (1.1) (ie, linearly independent over $\mathbf{C}$), then $L = K(\phi_{ij})$ (rational functions in K in the coefficients of the "fundamental" matrix $\Phi = (\phi_1 \cdots \phi_m)$). This is the extension of K generated by K together with the m^2 elements ϕ_{ij} of the fundamental matrix. We observe that L is a differential field (by (1.1)).

As in classical Galois theory of algebraic equations, we define the Galois group of (1.1), $G := Gal(L/K)$, as the group of all the (differential) automorphisms of L leaving the elements of K fixed. In a concrete way, that means the group that leaves invariant the rational relations of the matrix elements ϕ_{ij} of the fundamental matrix with coefficients in K. This group must be linear, because it leaves invariant (1.1) over the fundamental matrix, writing as

$$A = \Phi' \Phi^{-1}.$$

Then one of the main results of the theory is that the Galois group of (1.1) is faithfully represented as an *algebraic* linear group over $\mathbf{C}$, the representation being given by the action $\sigma \in G$,

$$\sigma(\Phi) = \Phi B_\sigma, \tag{1.2}$$

$B_\sigma \in GL(m, \mathbf{C})$. We recall that a linear algebraic group is a linear group that is an algebraic variety and the structures of group and algebraic varieties are compatible, ie, the group multiplication and the inversion transform are morphisms of algebraic varieties.

We remark that if the equation (1.1) has some additional structure, the Galois group preserves it. For example, if it is symplectic, ie, $A = JS(t)$ with S symmetric, then the Galois group is contained in the symplectic group (for a proof see [17]).

Now we will define integrability in the Picard-Vessiot theory. We call an extension of differential fields $K \subset L$ a *Liouville (or Liouvillian) extension* over K if there exists a chain of differential extensions $K_1 := K \subset K_2 \subset \cdots \subset K_r := L$, where each extension is given by the adjunction of one element a, $K_i \subset K_{i+1} = K_i(a)$, such that a satisfies one of the following conditions:

(i) $a' \in K_i$,

(ii) $a' = ba$, $b \in K_i$,

(iii) a is algebraic over K_i.

A Liouvillian function over K is a function that belongs to a Liouville extension L of K.

It can be proven that *the Picard-Vessiot extension of a linear differential equation is a Liouville extension if, and only if, the identity component G^0 of its Galois group is a solvable group.* In particular, if G^0 is abelian, then the Picard-Vessiot extension is a Liouville one.

Then we define *a linear differential equation as integrable if the associated Picard-Vessiot extension is Liouvillian.* This is a very precise integrability statement.

We only consider here gauge transformations with coefficients that remain in the differential field of coefficients K. Then a gauge transform of (1.1) is a linear change of the dependent variables $P(t) \in GL(n, K)$,

$$\mathbf{y} = P(t)\mathbf{z}.$$

Furthermore, if the linear equation has more structure, it is natural to consider gauge transformations that preserve this structure. For example, if the equation is symplectic, symplectic gauge transformation P over K are the natural gauge transforms to be considered. The transformed equation becomes

$$\mathbf{z}' = P[A](t)\mathbf{z},\ P[A] = P^{-1}AP - P^{-1}P'. \tag{1.3}$$

Then *the Galois group is invariant by the gauge transformation*, ie, as the Picard-Vessiot extensions are the same, the Galois groups of (1.1) and (1.3) are also the same. As a particular example, we can interpret geometrically d'Alembert classical reduction of order (see for instance [14], p. 121), when a particular solution is known: take the particular solution as one of the columns of P.

2 The variational equations

Let $H = H(\mathbf{x}, \mathbf{y}, t)$ be a (classical) real analytic Hamiltonian function with n degrees of freedom, defining the Hamiltonian system

$$\dot{x}_i = \partial H/\partial y_i, \dot{y}_i = -\partial H/\partial x_i,\ i = 1, ..., n, \tag{2.1}$$

then we can write the variational equation of (2.1) along an integral curve $\mathbf{x} = \mathbf{x}(t)$, $\mathbf{y} = \mathbf{y}(t)$,

$$\begin{pmatrix} \dot{\boldsymbol{\xi}} \\ \dot{\boldsymbol{\eta}} \end{pmatrix} = JH''(\mathbf{x}(t), \mathbf{y}(t)) \begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix}, \tag{2.2}$$

where $H''(\mathbf{x}(t), \mathbf{y}(t))$ is the Hessian matrix of H evaluated at the integral curve and

$$J = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix},$$

is the symplectic matrix, with $\mathbf{1}_n$ the identity matrix of dimension n.

Now let $\Phi(t, t_0)$ be the fundamental matrix of the variational equation of the Hamiltonian system along the integral curve, with initial condition $\Phi(t_0, t_0) = \mathbf{1}_{2n}$. Then, it will be relevant later the decomposition

$$\Phi(t,t_0)=\begin{pmatrix} H(t,t_0) & J(t,t_0) \\ L(t,t_0) & P(t,t_0) \end{pmatrix} \tag{2.3}$$

of the above matrix in four squared boxes of dimension n.

In order to apply the differential Galois theory, we need to extend analytically our real analytic Hamiltonian system to complex coordinates, including time. Then the integral curves (solutions on phase space) are given geometrically by Riemann surfaces Γ parameterized by complex time.

Let now (2.1) be a *complex analytical* Hamiltonian system defined over a complex symplectic manifold M. For simplicity of notation, we denote again by H the Hamiltonian function, and by x_i, y_i, t, the local complex symplectic coordinates and complex time.

The integrability of the Hamiltonian systems is the Liouville complete integrability. Thus, *we say that the Hamiltonian field* $X_H = (\partial H/\partial y_i, -\partial H/\partial x_i)$ $i = 1, ..., n$, *or the corresponding Hamiltonian system, is integrable* if there are n functions $f_1 = H$, f_2,..., f_n, such that

(1) they are functionally independent ie, the 1-forms df_i $i = 1, 2, ..., n$, are linearly independent over a dense open set $U \subset M$, $\bar{U} = M$;

(2) they form an involutive set, $\{f_i, f_j\} = 0$, i, $j = 1, 2, ..., n$.

We recall that in canonical coordinates the Poisson bracket has the classical expression

$$\{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i}.$$

Then by (2) above the functions f_i, $i = 1, ..., n$ are first integrals of the Hamiltonian field X_H. It is very important to be precise regarding the degree of regularity of these first integrals. Along this paper we assume that the first integrals are meromorphic. Unless otherwise stated, *this is the only type of integrability of Hamiltonian systems that we consider here.* Sometimes, to recall this fact people talk about meromorphic integrability.

Some integrable Hamiltonian systems are:

- 1-degree of freedom ones(harmonic oscillator, simple pendulum, etc.),
- Kepler problem in celestial mechanics and the hydrogen atom,
- spherical pendulum,
- several rigid bodies with a fixed point (Euler, Lagrange and Kovalevskaya),
- geodesics over revolution surfaces,
- Schwarzschild black-hole, etc.

But we remark that the majority of the Hamiltonian systems are not integrable.

Now we are in situation of stay in a precise way the **Key Idea 1** of the introduction. It is given by a joint result with Ramis around twenty-two years ago. It is a necessary condition for integrability of complex analytical Hamiltonian systems given by the Galois group of the variational equation around any particular integral curve Γ.

Theorem 2.1 ([19], see also [17]). *Assume a complex analytic Hamiltonian system is meromorphically completely integrable in a neighborhood of the integral curve Γ. Then the identity component of the Galois group of the variational equations (2.2) is an abelian group.*

In particular, as an abelian group is solvable, the variational equations are integrable in the sense of the Picard-Vessiot theory and the Picard-Vessiot extension is a Liouville extension.

Along the last twenty-two years this theorem has been applied to a considerable amount of Hamiltonian systems, as a non-integrability criterium. However, here we are mainly interested to apply it in a direct way to the integrable classical mechanical systems.

Example We illustrate the above ideas with an application to an elementary example, but with some relevance in path integrals. It is well-known that the variational equation of a 1-degree of freedom Hamiltonian system is solved in closed form, but we would like to look at this from the point of view of the Picard-Vessiot theory. We remark that what follows is a very particular simple case of the method of reduction to the normal variational equations (see [19], and also [17], pp. 75-77). For simplicity we assume that the Hamiltonian is natural

$$H = \frac{1}{2m} y^2 + V(x).$$

Then the variational equation around the solution $x = x(t)$, $y = y(t)$ is given by

$$\frac{d}{dt}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ -V'' & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{2.4}$$

denoting $V'' = V''(x(t))$. A particular solution of (2.4) is

$$\begin{pmatrix} \frac{y}{m} \\ -V' \end{pmatrix},$$

being $y = y(t)$ and $V' = V'(x(t))$. It seems clear that the coordinates of this solution belong to K, the meromorphic functions over the Riemann surface defined by the particular solution, because we assume V analytical in some domain, etc. Hence we consider the simple symplectic gauge transformation taking as the last column the above particular solution

$$P = \begin{pmatrix} 0 & \frac{y}{m} \\ -\frac{m}{y} & -V' \end{pmatrix} \in SL(2, K)$$

(this is not the only possible choice for P, but it is a simple symplectic one).

Then the matrix of the transformed system is triangular

$$P[A] = P^{-1}AP - P^{-1}\dot{P} = \begin{pmatrix} 0 & 0 \\ -\frac{m}{y^2} & 0 \end{pmatrix}.$$

The general solution of the transformed system is

$$\begin{pmatrix} c_1 \\ -c_1 m \int \frac{dt}{y^2} + c_2 \end{pmatrix},$$

with c_1, c_2 integration constants. As the fundamental matrix of this system is

$$\begin{pmatrix} 1 & 0 \\ -m \int \frac{dt}{y^2} & 1 \end{pmatrix},$$

the Picard-Vessiot extension is given by

$$K \subset K(\int \frac{dt}{y^2}) = L. \tag{2.5}$$

The Galois group is represented as an algebraic subgroup of the additive group

$$B_\sigma = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix},$$

with $\alpha \in \mathbb{C}$, coming from its action on the integral, ie,

$$\sigma(\int \frac{dt}{y^2}) = \int \frac{dt}{y^2} + \alpha$$

(see formula (1.2)). In fact, only two cases are possible, either the integral $\int \frac{dt}{y^2}$ belongs to K or either it does not. In the first case the group reduce to the identity, as follows from the definition of the Galois group, and in the second case the Galois group is the complete additive group. In any case the Galois group is abelian and coincides with its identity component.

Coming back to the initial system (2.4), its general solution is

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -y \int \frac{dt}{y^2} & \frac{y}{m} \\ -\frac{m}{y} + mV' \int \frac{dt}{y^2} & -V' \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{2.6}$$

We observe that the fundamental matrix in (2.6) is symplectic with coefficients in the Picard-Vessiot extension (2.5). In (2.6) we correct some typos of paper [18].

3 Semiclassical quantum mechanics

The (integral kernel) propagator, $K(\mathbf{x}_1, t_1|\mathbf{x}_0, t_0)$ in the coordinate representation, between the points $(\mathbf{x}_0, t_0)$ and $(\mathbf{x}_1, t_1)$ gives the solution of the Cauchy problem for the time-dependent Schrödinger equation, ie,

$$\psi(\mathbf{x}_1, t_1) = \int_{\mathbf{R}^n} K(\mathbf{x}_1, t_1|\mathbf{x}_0, t_0)\, \psi(\mathbf{x}_0, t_0)\, d\mathbf{x}_0 \tag{3.1}$$

is the solution of

$$i\hbar \frac{\partial}{\partial t}\psi(t) = \hat{H}\psi(t), \quad \psi_{t_0}(\mathbf{x}) = \psi(\mathbf{x}, t_0), \tag{3.2}$$

being $\hat{H}$ the Hamiltonian operator corresponding to the classical Hamiltonian H.

We recall that, for a 1-degree of freedom system with

$$H = \frac{1}{2m}y^2 + V(x)$$

then, as the momentum corresponds to the operator $\hat{y} = -i\hbar\frac{\partial}{\partial x}$, and the position to the operator $\hat{x} = xId$ (denoted again by x),

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x).$$

Then the Schrödinger equation becomes

$$i\hbar\frac{\partial}{\partial t}\psi = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x))\psi,$$

and separating variables

$$(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x))\varphi = E\varphi.$$

As this is a linear ODE, it is possible to apply to it the differential Galois theory (see my work in collaboration with Acosta-Humanez and Weil, [1]). But for more degrees of freedom is not possible to apply *in a direct way* the differential Galois theory to the Schrödinger equation. However, it will be possible to apply it to the semiclassical approach.

Furthermore, $K(\mathbf{x}_1, t_1|\mathbf{x}_0, t_0)$ represents the probability amplitude (or simply amplitude) for a quantum system to go from the point $(\mathbf{x}_0, t_0)$ to the point $(\mathbf{x}_1, t_1)$ in the configuration space. The quantum-mechanical problem is reduced to compute this propagator. For the path integral approach to quantum mechanics see [12].

We recall that the propagator have an asymptotic expansion in the Planck constant. Thus, the *semiclassical expansion* of the propagator around the *classical path* γ from $(\mathbf{x}_0, t_0)$ to $(\mathbf{x}, t)$ is (see [8])

$$K(\mathbf{x}_1, t_1|\mathbf{x}_0, t_0) = K_{WKB}(\mathbf{x}_1, t_1|\mathbf{x}_0, t_0)(1 + O(\hbar)), \tag{3.3}$$

where $K_{WKB}(\mathbf{x}_1, t_1|\mathbf{x}_0, t_0)$ is the semiclassical approximation of the propagator, *this is the function we are interested to compute*. In fact, we start from a particular *configuration classical path* γ joining the points $(\mathbf{x}_1, t_1)$ and $(\mathbf{x}_0, t_0)$ in the configuration space. Then the propagator "operates" over the wave function along the path γ. This path is a projection of the integral curve Γ, $(\mathbf{x}(t), \mathbf{y}(t))$ onto the configuration space. We assume that this projection is well-defined, ie, that only one path γ corresponds to Γ, because we do not consider focal (or conjugate) points. Another assumption is that we assume that the integral curve is a fixed one and it is explicitly done in closed form. For simplicity we use the same notation Γ either for the real or for the complex integral curve.

Now it is well-known that the semiclassical approximation K_{WKB} is given by

$$K_{WKB}(\mathbf{x_1}, t_1|\mathbf{x}_0, t_0) = \frac{1}{(2\pi i\hbar)^{n/2}} \frac{1}{\sqrt{\det J(t_1, t_0)}} \exp(\frac{i}{\hbar} S(\gamma)), \tag{3.4}$$

where

$$S(\gamma) := \int_{t_0}^{t_1} (\sum_{1}^{n} y_i dx_i - H dt) \tag{3.5}$$

is the action on the classical path (extremal of the Hamilton functional action: it depends on the initial and final points of the path), and $J(t_1, t_0)$ is the $n \times n$ block of the fundamental matrix of the variational equation defined in section 2. We recall that the determinant $\det J$ is very related with the Morette-Van Hove determinant, $\det M$,

$$\det J = -\frac{1}{\det M}.$$

We call to the formula (3.4) the *Pauli-Morette-Van Hove formula* and it is obtained by means of some differential calculus starting from the quantum mechanical Feynmann principle of path integrals, see [18] for some idea of the proof and [8] for a detailed study.

In [18] we call Van Vleck-Morette determinant to $\det M$, but it seems that the names of Morette and Van Hove reflected better the origins of this determinant in connection with the semiclassical approach of the Feymann propagator (see [5]).

Now we can implement easily the **Key Idea 2** of the introduction, ie, that the integrability of the variational equations implies integrability of the semi-classical approximation: the determinant $\det J$ belong to the Picard-Vessiot extension L of the variational equation, also the action (3.5) is of course a Liouvillian function over the field of coefficients K of the variational equation (it is given by quadratures), being as well the exponential in (3.4) a Liouvillian function over K. Hence, K_{WKB} becomes a Liouvillian function over K, considered as a function of t_1 (in the autonomous case we can take $t_0 = 0$). Finally, we obtain our main result:

Theorem 3.1. *([18]) Assume that the Hamiltonian system is meromorphically completely integrable in a neighborhood of the complex integral curve Γ. Then the semiclassical approximation of the propagator $K_{WKB}(t_1)$ around γ is a Liouvillian function over the field K of meromorphic functions over Γ.*

The above theorem says that the integrability of the semiclassical approximation is a necessary condition for the integrability of the Hamiltonian system. We recall that it is not a sufficient condition. In fact, there are Hamiltonian systems that are not integrable but the variational equation becomes integrable, and hence the semiclassical approximation will be integrable as well.

Example. The 1-degree of freedom Hamiltonian

$$H = \frac{1}{2m}y^2 + V(x). \tag{3.6}$$

is completely integrable.

Then in section 2 the space of solutions of the variational equation around any integral curve was computed in closed form. To obtain the semiclassical approximation from the formula (3.4), we only need to compute the classical action $S(\gamma)$ integrating the Lagrangian over the classical path, and to obtain the function $J(t,0)$ (as the system is autonomous we take $t_0 = 0$). Now $J(t,0)$ is the $\xi(t)$ "position" solution of the variational equation with initial conditions $\xi(0) = 0$, $\eta(0) = 1$. Looking at the formula (2.6) of the general solution, it is not difficult to obtain

$$J(t_1, 0) = \frac{1}{m}y(0)y(t_1)\int_0^{t_1}\frac{dt}{y^2(t)}. \tag{3.7}$$

Hence, the semiclassical approximation is given as a Liouvillian function over the field of meromorphic functions on the Riemann surface defined by the classical solution:

$$K_{WKB}(x_1, t_1|x_0, 0) = \sqrt{\frac{m}{2\pi i\hbar}}\frac{1}{\sqrt{y(0)y(t_1)\int_0^{t_1}\frac{dt}{y^2(t)}}}\exp(\frac{i}{\hbar}S(\gamma)). \tag{3.8}$$

A concrete example is given by the harmonic oscillator, where

$$y(t) = \frac{m\omega}{\sin\omega t_1}(x_1 - x_0\cos\omega t_1)\cos\omega t - m\omega x_0\sin\omega t,$$

and

$$y(0)y(t_1) = \frac{m^2\omega^2}{\sin^2\omega t_1}(x_1 - x_0\cos\omega t_1)(x_1\cos\omega t_1 - x_0),$$

$$\int_0^{t_1}\frac{dt}{y^2(t)} = \frac{\sin^3\omega t_1}{m^2\omega^3(x_1^2\cos\omega t_1 - x_1x_0 - x_1x_0\cos^2\omega t_1 + x_0^2\cos\omega t_1)}. \tag{3.9}$$

Thus,

$$y(0)y(t_1)\int_0^{t_1}\frac{dt}{y^2(t)} = \frac{\sin\omega t_1}{\omega},$$

obtaining the well-known expression of the Feynman propagator for the harmonic oscillator ([12]):

$$K_{WKB}(x_1, t_1|x_0, 0) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega t_1}} \exp\left(\frac{im\omega}{2\hbar \sin \omega t_1}[({x_0}^2 + {x_1}^2)\cos \omega t_1 - 2x_0 x_1]\right).$$

We remark that as the Hamiltonian is quadratic, then the semiclassical approximation is exact: $K = K_{WKB}$. Furthermore, for the harmonic oscillator with constant coefficients variational equation, formula (3.7) is not the best way to obtain the function J, because we know directly the general solution of the variational equation with no need of computing any quadrature.

4 Open problems: a Program

Our proposal open new lines of research that in our opinion deserve to be developed. Some of them were already pointed out in [18].

1) As far as we know the only previous applications of the differential Galois theory to the computation of the Feynmann propagators are obtained by Acosta-Humánez and Suazo for some one-dimensional time-dependent harmonic oscillators ([2, 3]). We are convinced that these works could be naturally included in our approach here.

2) To apply our results to systems with more than one degree of freedom. In a forthcoming paper with P. B. Acosta-Humánez, C. Pantazi and J. T. Lázaro we will give explicit computations on some concrete two-degrees of freedom families.

3) Is it possible to interpret some of the dynamical aspects of semiclassical quantum spectral properties of the Hamiltonian systems in the framework of the differential Galois theory? For instance, to include the focal (conjugate) points in our Galoisian approach.

4) My result with Ramis was extended in 2007 to higher order variational equations ([20]). Then it is natural to consider the application of this extension to higher order in the semiclassical expansion (3.3). It seems that J.-P. Ramis have some idea of how to approach this problem ([22]).

5) There exist some kind of "dictionary" between the quantum mechanics path integrals and the Wiener random processes: by an analytic prolongation to imaginary complex time (a Wick rotation), the Schrödinger equation is transformed in a "diffusion like equation". Motivated by the Einstein work on the Brownian motion, Wiener considered path integrals to study such diffusion equations almost 20 years before Feynman. Hence , taking into account this "dictionary" between Feynman path integrals and the Wiener ones, the approach of this paper could also be applied to Wiener classical path integrals for Wiener processes (see for instance, [4, 15]).

6) In a similar way there is a "dictionary" between the statistical mechanics and the quantum mechanics, because the density matrix also satisfy a "diffusion like equation" (see [13, 4]). Thus, it will be also possible to apply our Galoisian methods to the statistical mechanics.

7) Is it possible to extend our results to quantum field theory? To do that it is convenient to extend my theorem with Ramis to integrable classical fields. We remark that we include the gauge fields in the classical fields, where some integrable families of systems are well-known (the so-called self and anti-self dual Yang Mills fields), and also some integrable string theories in the framework of the AdS/CFT correspondence. My idea is that if the particular solution of a classical integrable field is of the so-called "algebro-geometric" type, then this extension will be possible, in some way. With J.-P. Ramis and M. A. Zurro we are working on this difficult problem.

We would like to observe that the above list of problems is possibly not exhaustive. We believe that *in any field where the path integral methods are relevant and where closed analytical semiclassical computations are possible, the differential Galois theory will plays some role in these computations.*

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