# COHOMOLOGY OF THE SPACES OF COMMUTING ELEMENTS IN LIE GROUPS OF RANK TWO

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Abstract. Let $G$ be a compact, connected Lie group, and let $\mathrm{Hom}(\mathbb{Z}^m, G)$ denote the space of commuting $m$-tuples in $G$. Baird proved that the cohomology of $\mathrm{Hom}(\mathbb{Z}^m, G)$ is identified with a certain ring of invariants of the Weyl group of $G$. In this paper by using the result of Baird we give the cohomology ring of $\mathrm{Hom}(\mathbb{Z}^2, G)$ for simple, simply connected Lie groups $G$ of rank 2.

## 1. Introduction

Let $G$ be a Lie group and $T$ be a maximal torus of $G$. Let $W(G)$ denote the Weyl group of $G$. The space of commuting elements in $G$, denoted by $\mathrm{Hom}(\mathbb{Z}^m, G)$, is the subspace of the Cartesian product $G^m$ consisting of $(g_1, \dots, g_m) \in G^m$ such that $g_1, \dots, g_m$ are pairwise commutative. Since the space $\mathrm{Hom}(\mathbb{Z}^m, G)$ is identified with the moduli space of based flat $G$-bundles over an $m$-torus, $\mathrm{Hom}(\mathbb{Z}^m, G)$ is studied in not only topology but also geometry and physics. On the other hand the cohomology of $\mathrm{Hom}(\mathbb{Z}^m, G)$ is deeply related with the invariant theory, since Baird proved that the cohomology is identified with a certain ring of invariants of the Weyl group of $G$. Thus the cohomology of $\mathrm{Hom}(\mathbb{Z}^m, G)$ is important for many fields. The general result on the cohomology of $\mathrm{Hom}(\mathbb{Z}^m, G)$ is studied in [2, 4, 10, 11, 9], while there is not much research on specific computation. For example, the space $\mathrm{Hom}(\mathbb{Z}^m, SU(2))$ is deeply studied in [3, 5].

In this paper we give the cohomology ring of $\mathrm{Hom}(\mathbb{Z}^2, G)$ for $G = Sp(2), SU(3), G_2$. Let $\mathbb{F}$ be a field of characteristic zero or prime to the order of $W(G)$, and $\mathbb{F}\langle S\rangle$ denote a free graded commutative algebra generated by a graded set $S$. The main theorem in this paper is the following.

**Theorem 1.1.** *For the simply connected simple Lie group $G$ of rank 2, there is an isomorphism*

$$H^*(\mathrm{Hom}(\mathbb{Z}^2, G); \mathbb{F}) \cong \mathbb{F}\langle a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2\rangle/(a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2)^3 + I,$$

*where $I$ is generated by*

$$b_1b_2, \quad {b_2}^2, \quad a_2^1b_2, \quad a_2^2b_2, \quad a_1^1b_2 + a_2^1b_1, \quad a_1^2b_2 + a_2^2b_1, \quad a_1^1a_2^2 + a_1^2a_2^1,$$

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*and*

$$|a_i^j| = \begin{cases} 2i+1 & (G = SU(3)) \\ 4i-1 & (G = Sp(2)) \\ 8i-5 & (G = G_2), \end{cases}$$

$$|b_i| = \begin{cases} 2i & (G = SU(3)) \\ 4i-2 & (G = Sp(2)) \\ 8i-6 & (G = G_2). \end{cases}$$

To prove this we use the general results of Baird [2]. Let $\mathrm{Hom}(\mathbb{Z}^m, G)_1$ denote the connected component of $\mathrm{Hom}(\mathbb{Z}^m, G)$ containing $(1, \dots, 1)$. We consider the right action of $W(G)$ on $G/T \times T^m$ given by

$$(gT, t_1, t_2, \dots, t_m) \cdot w = (gwT, w^{-1}t_1w, \dots, w^{-1}t_1w)$$

for $w \in W(G)$, $g \in G$, $t_1, \dots, t_m \in T$. Then the map

$$G \times T^m \to \mathrm{Hom}(\mathbb{Z}^m, G)_1, \quad (g, t_1, \dots, t_m) \mapsto (gt_1g^{-1}, \dots, gt_mg^{-1})$$

for $g \in G$, $t_1, \dots, t_m \in T$ defines a map

$$\phi\colon G/T \times_{W(G)} T^m \to \mathrm{Hom}(\mathbb{Z}^m, G)_1.$$

In [2] Baird proved that the map $\phi$ is an isomorphism in cohomology with $\mathbb{F}$ coefficients. Moreover by Theorem 4.1 in [2] (proved by Kac and Smilga [8]) the space $\mathrm{Hom}(\mathbb{Z}^2, G)$ is connected for a 1-connected Lie group $G$. Thus for a 1-connected Lie group $G$ there is a ring isomorphism

$$H^*(\mathrm{Hom}(\mathbb{Z}^2, G); \mathbb{F}) \cong (H^*(G/T; \mathbb{F}) \otimes H^*(T; \mathbb{F})^{\otimes 2})^{W(G)}. \tag{1}$$

Using the result of Baird, we can see another meaning of the main theorem. We apply the result of Baird for $m = 1$, we obtain the isomorphism

$$(H^*(G/T; \mathbb{F}) \otimes H^*(T; \mathbb{F}))^{W(G)} \cong H^*(G).$$

This isomorphism has long been known(cf. [6]), and can be proved by using the theorem of Solomon(cf. [12, Theorem 9.3.2]) and the theorem of Shepard-Todd (cf. [12, Theorem 7.4.3]) in representation theory. We know that $H^*(G; \mathbb{F})$ is isomorphic to the exterior algebra generated by $\mathrm{rank}(G)$ elements. Therefore the ungraded ring structure of $(H^*(G/T; \mathbb{F}) \otimes H^*(T; \mathbb{F}))^{W(G)}$ only depends on the rank of $G$. We have not known the generalization of the correspondence of ring structure, but the main theorem in this paper supports the existence of the generalization.

We prove the main theorem for each case that $G = SU(3), Sp(2), G_2$.

## 2. Cohomology of $\mathrm{Hom}(\mathbb{Z}^2, SU(3))$

In this section we compute the ring structure of $H^*(\mathrm{Hom}(\mathbb{Z}^2, SU(3)))$. The result of this ring structure is recorded in [9]. By (1), there is an isomorphism

$$H^*(\mathrm{Hom}(\mathbb{Z}^2, U(3))_1; \mathbb{F}) \cong \left(\mathbb{F}[x_1, x_2, x_3]_{W(U(3))} \otimes \Lambda(y_1^1, y_2^1, y_3^1) \otimes \Lambda(y_1^2, y_2^2, y_3^2)\right)^{W(U(3))},$$

where $|x_i| = 2$ and $|y_i^j| = 1$ for any $i, j$, $\mathbb{Z}[x_1, x_2, x_3]_{W(U(3))}$ is the ring of coinvariants of $W(U(3))$, and the action of $W(U(3))$ on $\{x_1, x_2, x_3\}$ and $\{y_1^i, y_2^i, y_3^i\}$ is the permutation action. By the isomorphism in Kishimoto and Takeda [9, p.12 (4)], there is an isomorphism

$$H^*(\mathrm{Hom}(\mathbb{Z}^2, SU(3))) \cong H^*(\mathrm{Hom}(\mathbb{Z}^2, U(3)))/(y_1^1 + y_2^1 + y_3^1, y_1^2 + y_2^2 + y_3^2).$$

By Theorem 1.1 in Ramras and Stafa [10], we obtain the Poincaré series of $\mathrm{Hom}(\mathbb{Z}^2, SU(3))$.

**Lemma 2.1.** *The Poincaré series of* $\mathrm{Hom}(\mathbb{Z}^2, SU(3))$ *is given by*

$$P(\mathrm{Hom}(\mathbb{Z}^2, SU(3)); t) = 1 + t^2 + 2t^3 + 2t^4 + 4t^5 + t^6 + 2t^7 + 3t^8.$$

In this section we define $a_i^j = x_1^i y_1^j + x_2^i y_2^j + x_3^i y_3^j$ and $b_i = x_1^{i-1} y_1^1 y_1^2 + x_2^{i-1} y_2^1 y_2^2 + x_3^{i-1} y_3^1 y_3^2$. A generating set is called minimal if the set doesn't properly contain any generating set.

**Theorem 2.2.** $H^*(\mathrm{Hom}(\mathbb{Z}^2, SU(3)))$ *is minimally generated by* $\{a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2\}$.

*Proof.* This follows from the result of Kishimoto and Takeda [9, Corollary 6.18]. □

**Theorem 2.3.** *There is an isomorphism*

$$H^*(\mathrm{Hom}(\mathbb{Z}^2, SU(3)); \mathbb{F}) \cong \mathbb{F}\langle a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2\rangle/(a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2)^3 + I,$$

*where $I$ is generated by*

$$b_1 b_2, \quad {b_2}^2, \quad a_2^1 b_2, \quad a_2^2 b_2, \quad a_1^1 b_2 + a_2^1 b_1, \quad a_1^2 b_2 + a_2^2 b_1, \quad a_1^1 a_2^2 + a_1^2 a_2^1.$$

*Proof.* By Theorem 2.2 the natural map

$$\mathbb{F}\langle a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2\rangle \to H^*(\mathrm{Hom}(\mathbb{Z}^2, SU(3)); \mathbb{F}).$$

is a surjection. By Lemma 2.1 we can obtain the generators of $(a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2)^3 + I$ are 0 in $H^*(\mathrm{Hom}(\mathbb{Z}^2, SU(3)); \mathbb{F})$ except for

$$b_1 b_2,\ a_1^1 b_2 + a_2^1 b_1,\ a_1^2 b_2 + a_2^2 b_1,\ a_1^1 a_2^2 + a_1^2 a_2^1,\ a_1^1 a_2^1 b_1,\ b_1^3.$$

There is a following equation

$$\begin{aligned} b_1 b_2 &= (y_1^1 y_1^2 + y_2^1 y_2^2 + y_3^1 y_3^2)(x_1 y_1^1 y_1^2 + x_2 y_2^1 y_2^2 + x_3 y_3^1 y_3^2) \\ &= (x_1 + x_2) y_1^1 y_1^2 y_2^1 y_2^2 + (x_2 + x_3) y_2^1 y_2^2 y_3^1 y_3^2 + (x_3 + x_1) y_3^1 y_3^2 y_1^1 y_1^2 \\ &= 2(x_1 + x_2 + x_3) y_1^1 y_1^2 y_2^1 y_2^2 \\ &= 0, \end{aligned}$$

and we obtain $b_1 b_2 = 0$. By the similar calculation we can show the other elements are 0 in $H^*(\mathrm{Hom}(\mathbb{Z}^2, SU(3)); \mathbb{F})$.

Therefore the surjection induces the following map

$$\mathbb{F}\langle a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2\rangle/(a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2)^3 + I \to H^*(\mathrm{Hom}(\mathbb{Z}^2, SU(3)); \mathbb{F}).$$

The Poincaré series of $\mathbb{F}\langle a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2\rangle/(a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2)^3 + I$ is equal to $1+t^2+2t^3+2t^4+4t^5+t^6+2t^7+3t^8$, and it corresponds with the Poincaré series of $\mathrm{Hom}(\mathbb{Z}^2, SU(3))$ in Lemma 2.1. Therefore the map is an isomorphism, and the proof is complete. □

## 3. Cohomology of $\mathrm{Hom}(\mathbb{Z}^2, Sp(2))$

In this section we compute the ring structure of $H^*(\mathrm{Hom}(\mathbb{Z}^2, Sp(2)))$. The Weyl group $W(Sp(2))$ is isomorphic to $\langle a, b \mid a^2 = b^2 = (ab)^4 = 1\rangle$. Let $\{x_1, x_2\}$ be the generator of $H^*(Sp(2); \mathbb{F})$ and $\{y_1^i, y_2^i\}$ be the generator of the cohomology of $i$-th torus. Then the $W(Sp(2))$-action on $H^2(Sp(2))$ and $H^*(T; \mathbb{F})$ is the signed permutation i.e.

$$x_1^a = x_2,\ x_2^a = x_1,\ x_1^b = -x_1,\ x_2^b = x_2,$$
$$(y_1^i)^a = y_2^i,\ (y_2^i)^a = y_1^i,\ (y_1^i)^b = -y_1^i,\ (y_2^i)^b = y_2.$$

By (1) there is an isomorphism

$$H^*(\mathrm{Hom}(\mathbb{Z}^2, Sp(2))) \cong \left(\mathbb{Z}[x_1, x_2]_{W(Sp(2))} \otimes \Lambda(y_1^1, y_2^1) \otimes \Lambda(y_1^2, y_2^2)\right)^{W(Sp(2))},$$

where $|x_i| = 2$, $|y_i| = 1$, $\mathbb{Z}[x_1, x_2]_{W(Sp(2))}$ is the ring of coinvariant of $W(Sp(2))$. In this section we define $a_i^j = x_1^{2i-1}y_1^j + x_2^{2i-1}y_2^j$ and $b_i = x_1^{2i-2}y_1^1y_1^2 + x_2^{2i-2}y_2^1y_2^2$.

**Lemma 3.1.** *$H^*(\mathrm{Hom}(\mathbb{Z}^2, Sp(2)))$ is minimally generated by $\{a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2\}$.*

*Proof.* This follows from the result of Kishimoto and Takeda [9, Theorem 6.28]. □

On the other hand, by Theorem 1.1 in Ramras and Stafa [10] we can determine the Poincaré series of $\mathrm{Hom}(\mathbb{Z}^2, Sp(2))$.

**Lemma 3.2.** *The Poincaré series of $\mathrm{Hom}(\mathbb{Z}^2, Sp(2))$ is given by*

$$P(\mathrm{Hom}(\mathbb{Z}^2, Sp(2)); t) = 1 + t^2 + 2t^3 + t^4 + 2t^5 + 2t^6 + 2t^7 + 2t^9 + 3t^{10}.$$

**Theorem 3.3.** *There is an isomorphism*

$$H^*(\mathrm{Hom}(\mathbb{Z}^n, Sp(2)); \mathbb{F}) \cong \mathbb{F}\langle a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2\rangle/(a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2)^3 + I,$$

*where $I$ is generated by*

$$b_1b_2, \quad {b_2}^2, \quad a_2^1b_2, \quad a_2^2b_2, \quad a_1^1b_2 + a_2^1b_1, \quad a_1^2b_2 + a_2^2b_1, \quad a_1^1a_2^2 + a_1^2a_2^1.$$

*Proof.* By Theorem 3.1 the natural map

$$\mathbb{F}\langle a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2\rangle \to H^*(\mathrm{Hom}(\mathbb{Z}^2, Sp(2)); \mathbb{F}).$$

is a surjection. By Lemma 3.2 we can obtain the generators of $(a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2)^3 + I$ are 0 in $H^*(\mathrm{Hom}(\mathbb{Z}^2, Sp(2)); \mathbb{F})$ except for

$$a_1^1b_2 + a_2^1b_1,\ a_1^2b_2 + a_2^2b_1,\ a_1^1a_2^2 + a_1^2a_2^1,\ b_1^3.$$

There is a following equation

$$\begin{aligned} a_2^1 b_1 &= (x_1 y_1^1 + x_2 y_2^1)(x_1^2 y_1^1 y_1^2 + x_2^2 y_2^1 y_2^2) \\ &= x_1 x_2^2 y_1^1 y_2^1 y_2^2 + x_1^2 x_2 y_2^1 y_1^1 y_1^2 \\ &= -x_1^3 y_1^1 y_2^1 y_2^2 - x_2^3 y_2^1 y_1^1 y_1^2 \\ &= -(x_1^3 y_1^1 + x_2^3 y_2^1)(y_1^1 y_1^2 + y_2^1 y_2^2) \\ &= -a_1^1 b_2, \end{aligned}$$

and we obtain $a_1^1 b_2 + a_2^1 b_1 = 0$. By the similar calculation we can show the other elements are 0 in $H^*(\mathrm{Hom}(\mathbb{Z}^2, Sp(2)); \mathbb{F})$.

Therefore the surjection induces the following map

$$\mathbb{F}\langle a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2\rangle/(a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2)^3 + I \to H^*(\mathrm{Hom}(\mathbb{Z}^2, Sp(2)); \mathbb{F}).$$

The Poincaré series of $\mathbb{F}\langle a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2\rangle/(a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2)^3 + I$ is equal to $1 + t^2 + 2t^3 + t^4 + 2t^5 + 2t^6 + 2t^7 + 2t^9 + 3t^{10}$, and it corresponds with the Poincaré series of $\mathrm{Hom}(\mathbb{Z}^2, Sp(2))$ in Lemma 3.2. Therefore the map is an isomorphism, and the proof is complete. □

## 4. Cohomology of $\mathrm{Hom}(\mathbb{Z}^2, G_2)$

In this section we compute the ring structure of $H^*(\mathrm{Hom}(\mathbb{Z}^2, G_2))$. The Weyl group $W(G_2)$ is isomorphic with the dihedral group $D_6 = \langle a, b \mid a^6 = b^2 = abab = 1\rangle$. Let $V = \mathrm{span}_{\mathbb{F}}\{z_1, z_2, z_3\}$ be the three dimension vector space on $\mathbb{F}$ spanned by $z_1, z_2, z_3$ and the action of $D_6$ on $V$ be the homomorphism $\phi\colon D_6 \to GL_{\mathbb{F}}(V)$ such that $\phi_a(s_1 z_1 + s_2 z_2 + s_3 z_3) = -s_1 z_3 - s_2 z_1 - s_3 z_2$ and $\phi_b(s_1 z_1 + s_2 z_2 + s_3 z_3) = s_1 z_1 + s_2 z_3 + s_3 z_2$ for $s_1, s_2, s_3 \in \mathbb{F}$. Since $\phi_a(z_1 + z_2 + z_3) = -(z_1 + z_2 + z_3)$, $\phi_b(z_1 + z_2 + z_3) = z_1 + z_2 + z_3$, $\mathrm{span}_{\mathbb{F}}\{z_1 + z_2 + z_3\}$ is an invariant subspace for the action. Therefore there is a $D_6$-action on $V/\mathrm{span}_{\mathbb{F}}\{z_1 + z_2 + z_3\}$ induced by $\phi$, and let $\bar{\phi}$ denote this action. By the definition of $\bar{\phi}$ and the definition of the canonical representation of $W(G_2)$ (see [1, Section 7]), we obtain the next lemma.

**Lemma 4.1.** *There are $W(G_2)$-equivariant isomorphisms from $V/\mathrm{span}_{\mathbb{F}}\{z_1 + z_2 + z_3 = 0\}$ to $H^1(T; \mathbb{F})$ and $H^2(G_2/T; \mathbb{F})$.*

Next we compute the ring of coinvariants of $W(G_2)$.

**Lemma 4.2.** *There is an isomorphism*

$$\mathbb{F}[x_1, x_2, x_3]/(e_1, e_2, e_3^2) \to H^*(G_2/T; \mathbb{F}),$$

*where $|x_i| = 2$ and $e_i$ is the $i$-th elementary symmetric polynomial in $x_1, x_2, x_3$.*

*Proof.* The cohomology $H^*(G_2/T; \mathbb{F})$ is isomorphic with the ring of coinvariant of $W(G_2)$. By the definition of $\phi$, the polynomial $e_2, e_3^2$ is in the invariant ring $\mathbb{F}[x_1, x_2, x_3]^{D_6}$. By Lemma 4.1 there is a surjection

$$\alpha\colon \mathbb{F}[x_1, x_2, x_3]/(e_1, e_2, e_3^2) \to H^*(G_2/T; \mathbb{F}).$$

Since $e_1, e_2, e_3^2$ is a regular sequence, the Poincaré series of $\mathbb{F}[x_1, x_2, x_3]/(e_1, e_2, e_3^2)$ is given by

$$P(\mathbb{F}[x_1, x_2, x_3]/(e_1, e_2, e_3^2); t) = \left(\frac{1}{1-t^2}\right)^3 (1-t^2)(1-t^4)(1-t^{12})$$
$$= (1+t^2)(1+t^2+t^4+t^6+t^8+t^{10}),$$

and of $H^*(G_2/T)$ is

$$P(G_2/T; t) = \left(\frac{1}{1-t^2}\right)^2 (1-t^4)(1-t^{12})$$
$$= (1+t^2)(1+t^2+t^4+t^6+t^8+t^{10}).$$

Since these Poincaré series are finite type, the map $\alpha$ is isomorphism.

□

**Lemma 4.3.** *The set* $\{x_1^i x_2^j \mid 0 \le i \le 5, 0 \le j \le 1\}$ *is a basis of* $\mathbb{F}[x_1, x_2, x_3]/(e_1, e_2, e_3^2)$.

*Proof.* In $\mathbb{F}[x_1, x_2, x_3]/(e_1, e_2, e_3^2)$, there are equations

$$x_1 + x_2 + x_3 = e_1 = 0$$
$$x_1^2 + x_1x_2 + x_2^2 = -e_2 + x_1e_1 + x_2e_1 = 0.$$

Therefore $x_3$ and $x_2^2$ can be replaced to $-x_1 - x_2$ and $-x_1^2 - x_1x_2$ respectively. Since $x_2^3 = -x_1^2x_2 - x_1x_2^2 = x_1^3$, there is a equation

$$x_1^6 = x_1^3x_2^3 = x_1^2x_2^2(x_1+x_2)^2 - x_1^2x_2^2(x_1^2 + x_1x_2 + x_2^2)$$
$$= x_1^2x_2^2x_3^2 = e_3^2 = 0.$$

By considering the Poincaré series of $\mathbb{F}[x_1, x_2, x_3]/(e_1, e_2, e_3^2)$ which is given by

$$P(\mathbb{F}[x_1, x_2, x_3]/(e_1, e_2, e_3^2); t) = 1 + 2t^2 + 2t^4 + 2t^6 + 2t^8 + 2t^{10} + t^{12},$$

We obtain this lemma.

□

By (1) and Lemma 4.2, there is an isomorphism

$$H^*(\mathrm{Hom}(\mathbb{Z}^2, G_2)) \cong \left(\mathbb{F}[x_1, x_2, x_3]/(e_1, e_2, e_3^2) \otimes \bigotimes_{i=1}^{2} \Lambda(y_1^i, y_2^i, y_3^i)/(y_1^i + y_2^i + y_3^i)\right)^{D_6}.$$

On the other hand, by Theorem 1.1 in Ramras and Stafa [10] we can compute the Poincaré series of $\mathrm{Hom}(\mathbb{Z}^2, G_2)$.

**Lemma 4.4.** *The Poincaré series of* $\mathrm{Hom}(\mathbb{Z}^2, G_2)$ *is given by*

$$P(\mathrm{Hom}(\mathbb{Z}^2, G_2)_1; t) = 1 + t^2 + 2t^3 + t^4 + 2t^5 + t^6 + t^{10} + 2t^{11} + 2t^{13} + 3t^{14}.$$

In this section we define $a_i^j = \sum_{l=1}^3 x_l^{4i-3} y_l^j$ and $b_i = \sum_{l=1}^3 x_l^{4i-4} y_l^1 y_l^2$.

**Theorem 4.5.** *There is an isomorphism*

$$H^*(\mathrm{Hom}(\mathbb{Z}^n, G_2); \mathbb{F}) \cong \mathbb{F}\langle a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2\rangle/(a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2)^3 + I,$$

*where $I$ is generated by*

$$b_1b_2, \quad {b_2}^2, \quad a_2^1b_2, \quad a_2^2b_2, \quad a_1^1b_2 + a_2^1b_1, \quad a_1^2b_2 + a_2^2b_1, \quad a_1^1a_2^2 + a_1^2a_2^1.$$

*Proof.* First we prove that $H^*(\mathrm{Hom}(\mathbb{Z}^n, G_2); \mathbb{F})$ is generated by $a_1^1, a_2^1, a_1^2, a_2^2, b_1, b_2$. By the calculation similar to the proof in Lemma 4.3, there is a equation

$$\begin{aligned}
a_1^1b_2 &= (\sum_{l=1}^{3} x_l y_l^1)(\sum_{l=1}^{3} x_l^4 y_l^1 y_l^2) \\
&= ((x_1 - x_3)y_1^1 + (x_2 - x_3)y_2^1)((x_1^4 - x_3^4)y_1^1y_1^2 + (x_2^4 - x_3^4)y_2^1y_2^2) \\
&= (2x_2 + x_1)(x_1^4 - (x_1 + x_2)^4)y_1^1y_2^1y_2^2 + (2x_1 + x_2)(x_2^4 - (x_1 + x_2)^4)y_2^1y_1^1y_1^2 \\
&= 3x_1^4x_2y_1^1y_2^1y_2^2 + 3x_1x_2^4y_2^1y_1^1y_1^2
\end{aligned}$$

By Lemma 4.3 we obtain $a_1^1b_2 \neq 0$. By the similar calculation we can show that $a_1^2 \neq 0, a_1^1a_1^2 \neq 0, a_1^ib_1 \neq 0, a_1^ib_2 \neq 0, a_1^ia_2^j \neq 0$ for $i, j = 1, 2$. By Lemma 4.4 and considering the degree with respect to the exterior algebra, the first statement is proved.

We can prove the rest part of this theorem similar to Theorem 2.3.

∎


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## References


[1] A.V. Borovik, and A. Borovik, *Mirrors and reflections. The geometry of finite reflection groups.* Universitext. Springer, New York, 2010.

[2] T.J. Baird, *Cohomology of the space of commuting n-tuples in a compact Lie group*, Algebr. Geom. Topol. **7** (2007), 737-754.

[3] T. Baird, L.C. Jeffrey, and P. Selick, *The space of commuting n-tuples in SU(2)*, Illinois J. Math. **55** (2011), 805-813.

[4] F.R. Cohen and M. Stafa, *On spaces of commuting elements in Lie groups*, Math. Proc. Camb. Philos. Soc. **161** (2016), 381-407.

[5] M.C. Crabb, *Spaces of commuting elements in SU(2)*, Proc. Edinburgh Math. Soc. **54** (2011) 67-75.

[6] W. Greub, S. Halperin, R. Vanstone, *Connections, curvature, and cohomology. Vol. II: Lie groups, principal bundles, and characteristic classes.* Pure and Applied Mathematics, Vol. 47-II. Academic Press, New York-London, 1973.

[7] J.E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, **29** Camb. Univ. Press, Cambridge, 1990.

[8] V. Kac and A. Smilga, *Vacuum structure in supersymmetric Yang-Mills theories with any gauge group*, The many faces of the superworld (2000), 185-234, World Sci. Publishing, River Edge, NJ, 2000.

[9] D. Kishimoto and M. Takeda, *Spaces of commuting elements in the classical groups*, Adv. Math. **386** (2021),

[10] D.A. Ramras and M. Stafa, *Hilbert-Poincaré series for spaces of commuting elements in Lie groups*, Math. Z. **292** (2019), 591-610.

[11] D.A. Ramras and M. Stafa, *Homological stability for spaces of commuting elements in Lie groups*, Int. Math. Res. Not. IMRN 2021, **5**, 3927-4002.

[12] L. Smith, *Polynomial invariants of finite groups*, Research Notes in Mathematics, 6. A K Peters, Ltd., Wellesley, MA, 1995.

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