YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

Matrices and algebras in the canonical tensor model.

DOCTORAL DISSERTATION

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Abstract

The quantisation of gravity is one of the most challenging problems in theoretical physics, and arguably the biggest missing piece in our fundamental understanding of nature. One of the proposed models attempting to construct a theory for quantum gravity is the canonical tensor model. In this model, a spatial slice of spacetime is described by a completely symmetric real tensor of degree three, and its dynamical structure has algebraic similarities to the Hamiltonian formulation of general relativity. In this dissertation, this model will be thoroughly explained in the newly developed algebraic interpretation of tensor models, where a tensor generates an associative commutative algebra together with a list of eigenvalues of the Laplace-Beltrami operator, giving this model a potential direct spacetime interpretation through a duality between smooth manifolds and their algebra of smooth functions. After this, a matrix model will be introduced and analysed corresponding to a simplification of a wave function of the canonical tensor model. Lastly, some research related to the configuration space of tensor rank decompositions will be explained.

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The content of this thesis may partially be found in [1, 2, 3]:
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Preface

At the time of writing it is almost three years ago, October 1st 2019, that I started my doctorate programme at Kyoto University. Back then I was out of academia for about a year, working as a security consultant at CGI Netherlands. The idea of obtaining a doctorate degree always stuck with me, so when I got the opportunity to do it with the MEXT scholarship I was happy to accept it.

The research area I chose to specialise in is called "quantum gravity". To me this was always the most exciting branch of physics, as it is in some way the most fundamental of all. In quantum gravity one tries to really understand what the nature of space and time itself are. There are many interesting approaches to this, and all use a certain set of underlying principles. In a way, the research is all about choosing and understanding what kind of consistent theories can emerge from a given set of theoretical principles inspired by both quantum mechanics and general relativity. In the absence of current experimental evidence, these principles are the main guidance we have in the search for a theory of quantum gravity. One would hope that a consistent theory could then actually start making predictions, so specific experiments and measurements could then be proposed (though currently we are still quite far from that stage).

My research topic was centered around a specific model for quantum gravity, called the canonical tensor model. I encountered this model during my masters degree which I obtained at the Radboud University in the Netherlands. As a master student, I was interested in doing the research for my master thesis abroad in Japan. I was happy to find a professor at Kyoto University, Naoki Sasakura, who did exactly the kind of research I was interested in: Discrete quantum gravity. I still feel very humbled and fortunate that he agreed to accept me me to do research in Kyoto for one year during my masters. I see this year as a very productive year where I learned a lot, including the publication of my first three co-authored papers [4, 5, 6]. It was when I was writing my masters thesis about this research under the supervision of Renate Loll, that I discovered what I found most exciting about the topic: The model seems really promising and had very nice results so far, but the direct interpretation of it was still not well-established.

That is why, when I got the opportunity a year later to do my doctorate at Kyoto University under the supervision of prof. Sasakura, I already knew exactly what I wanted to do: To find a rigorous mathematical footing of the canonical tensor model. It was certainly not an easy

goal, and I struggled at times for sure. It is quite easy to end up digging yourself too deep into the mathematics, so I was lucky that my professor would sometimes help me to look at the actual physical picture again. About halfway during my research I found myself to be on the right track, with the early ingredients for the definition of the "associative closure" being found. In the end, it turned out that it would be possible to store all of the geometric information of a Riemannian manifold into a tensor P_{abc} and reconstruct it, which indeed makes it possible to treat a tensor model like the canonical tensor model as a model for quantum gravity.

Besides this, I also did research projects together with prof. Sasakura. The first project was analysing a matrix model that may be seen as a simplification of a wave function of the canonical tensor model. The second project was about a mathematical tool that often turns up in the analysis of the canonical tensor model, called the tensor rank decomposition. Both of these projects have been interesting parts of my research, and will be covered in this thesis.

My research done during my doctorate degree is published, or to be published, in [1, 2, 3]. I am really proud of the research that I could do, and I look back to a really great time I had in Japan.

Contents

Chapter 1

Introduction

One of the most impressive achievements in theoretical physics is the development of general relativity, which gives the best understanding of gravity to date. In this theory, gravity is described as the curvature of spacetime, and it has been shown to describe gravitational effects extremely well [7, 8]. Some notable experimental verifications of this theory range from the correct prediction of the precession of mercury at the introduction of the theory in 1915 [9], measurements of the deflection of light due to gravitational lensing around the sun in 1919 [10], all the way up to the recent direct observation of black holes [11] and gravitational waves [12], which have already been predicted over a century ago using Einstein's theory [13, 14].

Even though this theory can be called extremely successful and has yet to be disproven, physicists believe that general relativity is incomplete and has to be replaced by a new theory at high energy levels. This theory is expected to be a quantum theory, and thus it is called quantum gravity. The first reason to believe that such a theory should exist is the believe and experience that the microscopic world is inherently *quantum*. This is supported by the fact that all of the known matter, and the other three fundamental forces, are all combined within the framework of quantum field theory in the standard model of particle physics. Secondly, there is hope that such a theory for quantum gravity might actually be able to give new insights in physical phenomena, and even help solve some puzzles that appear in the study of the universe such as inflation, dark matter, or why the universe has the properties it has.

The last, and maybe most convincing, argument for the need of a theory of quantum gravity is of a more technical nature, namely through the singularity theorems by Penrose and Hawking [15, 16]. These theorems state that, under physically reasonable conditions, the existence of singularities is guaranteed in for instance black holes or at the big bang. At singularities, the geodesics of the theory become incomplete, which by physicists is often interpreted as the theory simply not being a good description of reality in those extreme conditions. It is expected that at these extremely strong gravitational fields, i.e. at very high energies, a more fundamental theory will become relevant, such that general relativity is mainly a low-energy description of gravity.

Usually in physics, theoretical development would be guided by experimental results that give some properties of nature that need explanation. One of the biggest challenges faced in the development of any theory for quantum gravity is that these experimental

results are largely absent. The reason is that performing direct experiments at the energy scale where quantum gravitational effects are expected to play a role is next to impossible. This energy scale is called the *Planck scale*, and it can be constructed by using the fundamental constants c (the speed of light), $\hbar$ (the Planck constant) and G (the gravitational constant)

$$E_P = \sqrt{\frac{c^5\hbar}{G}} \approx 10^{28}\,\text{eV}.$$

This energy scale is considerably higher than the maximal energy of the Large Hadron Collider at $\sim 10^{13}$ eV [17, 18], and even the most energetic cosmic ray ever measured does not come close at $\sim 10^{20}$ eV [19]. Therefore, one is mainly left to physical intuition built from experience with both general relativity and quantum theory when constructing models for quantum gravity.

1.1 Overview of this thesis

This thesis will discuss the construction of an algebraic approach to quantum gravity using tensors, and use this as a mathematical foundation to introduce the canonical tensor model. One of the known wave functions of the canonical tensor model is then analysed, by considering a simplification that effectively turns it into a matrix model.

The current chapter serves as an introduction to the thesis, in order to understand the context of the matter discussed. In the following section, Section 1.2, a short non-exhaustive overview of some approaches to quantum gravity will be given, especially those that have had an influence in the development of the canonical tensor model or the algebraic tensor model approach. Section 1.3 will discuss some reasons as to why an algebraic approach to quantum gravity might be useful. Lastly, Section 1.4 will give a brief summary of the canonical tensor model, in order to understand what the development of the algebraic tensor approach is building towards.

In Chapter 2, the mathematical foundations for the usage of algebras will be explained, as this is crucial in the development of the theory. The geometric structure will be built up throughout the chapter, starting with the duality between topological spaces and (some) commutative associative algebras, then discussing the duality between the algebra of smooth functions and topological manifolds, to finally discussing how a full Riemannian geometry may be reconstructed by adding a Hilbert space structure and spectral data of an operator to the algebra.

Chapter 3 will introduce the main framework used to link tensors of degree three to Riemannian manifolds. In this chapter, it will be explained how one can reconstruct the topological structure from even a finite-dimensional tensor. Furthermore, a way of including the spectral data of an operator, and thus according to Chapter 2 the full Riemannian geometry, is discussed, making these tensor models possible candidates for quantum gravity.

Chapter 4 then gives a full introduction to the canonical tensor model in the context of algebraic tensor models. The classical model in the Hamiltonian framework will be explained, and using canonical quantisation the quantum model is defined. The main results, including some exactly known wave-functions, will be explained.

In Chapter 5, the main results of [1] will be discussed. A matrix model relating to one of the known wave functions of the canonical tensor model is analysed. After this, Chapter 6 will explain the work done in [2], where the configuration space of tensor rank decompositions was introduced and analysed.

1.2 Some approaches to quantum gravity

In this section, some approaches to quantum gravity will be discussed. Since the field is very broad and many approaches exist, the approaches mentioned here are not exhaustive. For some more discussion on quantum gravity, several useful references are [20, 21, 22, 23]. The approaches discussed in this section are the most relevant to the development of quantum gravity, and have been influential in the development of algebraic tensor models and the canonical tensor model.

The approaches may in general be split between perturbative and non-perturbative approaches. The former of these will be discussed first in order to gain some historical perspective, followed by two kinds of non-perturbative approaches: covariant and canonical approaches. The canonical tensor model that this thesis is falls into the canonical non-perturbative category.

1.2.1 Perturbative methods

Perturbative expansions were introduced by Rayleigh for pre-quantum wave theory [24] and later applied by Schrödinger to quantum mechanics [25], but only later made mathematically precise [26]. Perturbation theory has been enormously successful in quantum theory. From using it to explain for instance the Van der Waals interaction in quantum mechanics [27], to being the main tool for calculations in the standard model of particle physics [28] using perturbative quantum field theory [29].

The original attempts for quantising gravity were perturbative [30], mainly due to the enormous success of theories like the standard model for particle physics. As this approach worked for three of the four fundamental forces, it made sense to try to apply this to the fourth one as well. The original idea, now often called the *background field method*, was to expand the gravitational field, i.e. the metric field, around some background metric $\bar{g}_{\mu\nu}(x)$:

$$g_{\mu\nu}(x) \equiv \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x),$$

where $h_{\mu\nu}(x)$ denotes a small perturbation around this background metric so that it can be used for a perturbative expansion. As the Minkowski metric $\eta_{\mu\nu}$ was most-often used in particle physics, this was the most logical choice for a background metric.

From dimensional analysis [29], one may already expect at this point what the result of this exercise will be: The gravitational action is perturbatively non-renormalisable [21]. Dimensional analysis is a qualitative argument which explains whether quantum loop corrections should be expected to diverge or converge if the cutoff energy is taken to be infinite. However, the result of this argument is not necessarily always true. For instance if the theory has a strong symmetry, divergences may be softened or cancelled out completely [31].

In [32], contributions up to one-loop order in perturbative quantum gravity were calculated, and as expected, divergences appeared. However, it was possible to absorb these in field re-definitions up to a total derivative term, which is deemed unphysical and thus not relevant. Therefore, it seemed as if it might be possible for the theory to still be perturbatively renormalisable. This was later found to be incorrect, since calculations showed that quantum gravity coupled to matter diverges at one-loop order, and pure gravity diverges at two-loop order [33].

Modern perturbative approaches most-often try to circumvent these issues by adding additional structure to the theory. As mentioned above, extra structure or symmetry might soften out divergences such that they can be handled properly. In the 1970s it was argued that adding a symmetry called supersymmetry to general relativity might improve the divergent behaviour, and it was found that by adding a spin-$3/2$ field to the gravitational action making the action supersymmetric, the theory would be finite up to two-loop order [34]. However, interest faded in the years later when it was found that the theory had bad UV behaviour after all, even when allowing the extension of the theory to 11 dimensions [35].

Another approach which adds extra structure to the theory is string theory [36]. The idea of string theory is that the fundamental divergence of general relativity is due to the point-particle description of nature in quantum field theory. Spreading out these interactions by using two-dimensional world sheets instead of one-dimensional world-lines circumvents this, and leads to a renormalisable theory that has general relativity as its lower energy limit. One complication of the theory is the so-called string theory landscape, which shows a high level of ambiguity in the theory coming from the enormous amounts of possible compactifications of the theory [37].

1.2.2 The covariant approach

In this subsection, the covariant approach to non-perturbative quantum gravity will be discussed. As explained in Section 1.2.1, general relativity is perturbatively non-renormalisable. The philosophy behind the non-perturbative approaches is that the main issue lies in the application of perturbation theory, even though the theory is not suited for that. If one wants to avoid adding additional structure to the theory, one has to carefully evaluate how to actually define the theory and avoid using perturbation theory in its calculations.

The covariant approach to quantum gravity uses the path integral formulation of quantum field theory, as this allows a non-perturbative starting point for quantum field theories. The method is called covariant because, if the action used is covariant, the theory is described in a *manifestly* covariant way. The path integral formulation of quantum general relativity was already formulated by Misner in 1957 [38], and yields a formal definition of the path integral as

$$\mathcal{Z} := \int \mathcal{D}[g]\ \mathrm{e}^{i\,S_{EH}([g])}, \tag{1.1}$$

where $[g]$ denotes an equivalence class of diffeomorphisms of metrics g, and the Einstein-Hilbert action is used

$$S_{EH}(g) = \frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{|g|}R, \tag{1.2}$$

where $|g|$ denotes the absolute value of the determinant of the metric g, and R is the Ricci scalar.

The main tasks for theories in the path integral approach, and also the main point where they differ from one another, is to attempt to make the expression in Eq. (1.1) well-defined. There are several ways to do this.

One promising example is asymptotic safety [39], originally introduced by Weinberg [40]. With asymptotic safety one assumes the existence of a UV fixed point of the renormalisation group flow. In order to do this, one needs a non-perturbative definition of renormalisation which is given by the functional renormalisation group [41]. In general terms, the asymptotic safety programme boils down to carefully defining renormalisation in a non-perturbative way, and evaluating if there is a UV fixed point making general relativity "asymptotically safe".

Another way of defining the path integral properly is by regularising it using discrete building blocks. This is the general approach of theories like (causal) dynamical triangulation and simplicial tensor models.

In dynamical triangulation [42, 43], the path integral is replaced by a sum over all triangulations of a given topological manifold, weighted with a discrete version of the Einstein-Hilbert action coming from Regge calculus. Dynamical triangulation was really successful in two dimensions, but in three and four dimensions it did not show the emergence of macroscopic spacetimes [44]. The real success of this approach came with the introduction of causal dynamical triangulation [45, 46, 47], where a causal restriction was added to the allowed triangulations such that a Wick rotation between the Euclidean triangulations and their Lorentzian geometries was well-defined. This causal requirement seemed crucial, as now there are some serious signs of the emergence of macroscopic spacetimes in four dimensions [48].

Simplicial tensor models are another discrete approach to gravity. They were originally motived by the success of matrix models to describe two-dimensional quantum gravity similarly to dynamical triangulation [49, 50, 51]. The idea is that d-dimensional simplices are glued together according to the contraction of tensors of degree d, generated by a partition function. The original models were not very satisfactory, since they mainly produced singular spaces. These problems seem similar to the issues that dynamical triangulation was facing before the introduction of a causal requirement, but for tensor models it seems impossible to include such a requirement in a natural way. This motivated the introduction of the canonical tensor model, as it introduces time in a fundamentally different way to tensor models, as discussed below.

1.2.3 Canonical quantum gravity

In this subsection, some elements of canonical quantum gravity will be discussed. First, a brief overview of the canonical formalism of general relativity, called the ADM-formalism, will be given, and some canonical quantisation methods will be briefly discussed.

The ADM-formalism, after Arnowitt, Deser and Misner who originally proposed this form of general relativity [52], is the symplectic geometrical formulation of general relativity. This formulation singles out time in order to describe the evolution of spatial slices through time according to a Hamiltonian. It is important to note that although this

breaks manifest general covariance, general covariance is not broken since the choice of time is taken to be a gauge choice. One might thus already expect that the resulting Hamiltonian will be totally constraint. Here, a short summary will be given, for a more elaborate discussion refer to [53, 54, 55].

The central assumption that makes this description possible is the assumption that the spacetime manifold $(\mathcal{M}, g)$ is globally hyperbolic, meaning that it has a Cauchy surface. A Cauchy surface is a subset of $\mathcal{M}$ such that any nowhere spacelike intersects it exactly once. One might see this as a kind of causality condition, since if one knows the initial conditions on a Cauchy surface there is a unique global solution on the whole manifold. If this is satisfied, it may be shown that the spacetime manifold has the topology [56, 57]

$$\mathcal{M} \cong \mathbb{R} \times \mathcal{S}. \tag{1.3}$$

Here, $\mathcal{S}$ is a three-dimensional manifold of some fixed topology.

The main strategy is now to foliate $\mathcal{M}$ into hypersurfaces $\Sigma_t \cong \mathcal{S}$, parameterised by $t \in \mathbb{R}$, and rewrite the Einstein-Hilbert action in terms of the spatial manifold metric q_{ab}, the shift vector N^a and the lapse function N. Subsequently, one finds the canonical momenta corresponding to these variables and finds the Hamiltonian, and one can show that the final Hamiltonian actually only depends on the spatial metric q_{ab} and its canonical conjugate p^{ab}. The result of this exercise is given by

$$\mathbf{H} := \int_{\mathcal{S}} d^3x [N^a \mathcal{H}_a + N\mathcal{H}],$$

where the *spatial diffeomorphism constraint* $\mathcal{H}_a$ and the *Hamiltonian constraint* $\mathcal{H}$ are given by

$$\begin{aligned} \mathcal{H}_a &:= -2q_{ac}\nabla_b p^{bc}, \\ \mathcal{H} &:= \frac{1}{\det(q)}\left[p_{ab}p^{ab} - \frac{1}{2}p^2\right] + \sqrt{\det(q)}R, \end{aligned}$$

with R the Ricci scalar on $(\mathcal{S}, q)$, ∇_b the covariant derivative, and the shift vector and lapse function act as Lagrange multipliers.

Since the shift vector and lapse function act as Lagrange multipliers, the effective phase space now only consists of the canonically conjugate pair (q_{ab}, p^{ab}). Note that this is an over-countably infinite-dimensional symplectic manifold, since the metric is defined on every point on $\mathcal{S}$. The Poisson structure on the phase space is given by the following Poisson brackets

$$\{q_{ab}(t,x), p^{cd}(t',x')\} = \delta^{cd}_{(ab)}\delta(t,t')\delta^{(3)}(x,x'), \tag{1.4}$$

where $\delta^{cd}_{(ab)} = \frac{1}{2}(\delta^c_a\delta^d_b + \delta^c_b\delta^d_a)$.

The constraints $\mathcal{H}_a$ and $\mathcal{H}$ form the *hypersurface deformation algebra*

$$\begin{aligned} \left\{\mathcal{H}(f), \mathcal{H}(f')\right\} &= \vec{\mathcal{H}}(\vec{K}(f,f')), \\ \left\{\mathcal{H}(f), \vec{\mathcal{H}}(\vec{F}')\right\} &= \mathcal{H}(L_{\vec{F}}f), \\ \left\{\vec{\mathcal{H}}(\vec{F}), \vec{\mathcal{H}}(\vec{F}')\right\} &= \vec{\mathcal{H}}(L_{\vec{F}}\vec{F}'), \end{aligned} \tag{1.5}$$

where $K^a := q^{ab}(f\partial_b f' - f'\partial_b f)$ and the Lie derivative of a vector field is given by $(L_{\vec{F}}\vec{F}')^a = F^b\nabla_b F'^a - F'^b\nabla_b F^a$ and acting on a function gives $L_{\vec{F}}f = F^\mu\partial_\mu f$. Note

here that the right hand side of the first equation depends on q^{ab}, which makes this not a proper Lie-algebra.

The most direct way of canonically quantising general relativity, is by taking the canonical variables $q_{ab}(x)$ and $p^{ab}(x)$ and mapping them to operators $\hat{q}_{ab}(x)$ and $\hat{p}^{ab}(x)$. These operators then are required to satisfy the quantum commutator algebra

$$\left[\hat{q}_{ab}(t,x), \hat{p}^{cd}(t',x')\right] = i\delta^{cd}_{(ab)}\delta(t,t')\delta(x,x'), \tag{1.6}$$

which is a quantum analogue of the Poisson algebra in Eq. (1.4). Formally, one can write these in a representation on a suitable function space over the space of all Riemannian 3-metrics on the spatial manifold $\mathcal{S}$ as

$$\begin{aligned}\hat{q}_{ab}(x)\psi[q_{ab}(x)] &= q_{ab}(x)\psi[q_{ab}(x)],\\ \hat{p}^{cd}(x)\psi[q_{ab}(x)] &= -i\hbar\frac{\delta}{\delta q_{cd}(x)}\psi[q_{ab}(x)].\end{aligned}$$

To find the *physical Hilbert space* of the theory, one has to define the quantum version of the constraints, which we will formally denote here by $\hat{\mathcal{H}}_a$ and $\hat{\mathcal{H}}$, and require the physical states to satisfy the constraint equations

$$\begin{aligned}0 \equiv \hat{\mathcal{H}}\Psi &:= \left(\frac{1}{2\sqrt{q}}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd})\frac{\delta}{\delta q_{ab}}\frac{\delta}{\delta q_{cd}} - \sqrt{q}R\right)\Psi,\\ 0 \equiv \hat{\mathcal{H}}_a\Psi &:= i2\nabla_b q_{ac}\frac{\delta}{\delta q_{bc}}\Psi.\end{aligned} \tag{1.7}$$

The first of these equations is called the Wheeler-deWitt equation [54, 58]. These equations are notoriously difficult to solve, and it is currently not known if the commutator algebra of these equations contains any anomalies, in which case one cannot expect four-dimensional covariance to occur.

A prominent canonical theory of gravity that uses a different set of variables is called loop quantum gravity [54]. In this theory, the so-called Ashtekar variables are quantised instead of the metric. These variables are related to the metric, but rewritten in such a way that the three-dimensional metric degrees of freedom are represented as an $SU(2)$ gauge field.

1.3 The case for algebras

When developing a theory for quantum gravity, one has to look at the principles behind the theories of both general relativity and quantum mechanics, and decide which of these principles one deems important to keep in the final theory for quantum gravity. For all the differences in the approaches in Section 1.2, they do have one thing in common: They all attempt to quantise the metric tensor of (pseudo-)Riemannian manifolds, or other variables from differential geometry. Arguably one of the big difficulties in the quantisation of gravity is that this configuration space of all possible geometries represented in this way is extremely large and complicated.

An intriguing development, and a big enabler of the field of algebraic geometry, is a theorem due to Gelfand [59, 60], stating that commutative C^* algebras are dual to topological Hausdorff spaces. The C^* algebra is then isomorphic to the continuous

functions over a topological space X, $C(X)$. Later, Gelfand and Naimark generalised this result to more general cases [61], which is often seen as the starting point for non-commutative geometry [62].

For the commutative case, it turns out that even more structure of the geometry can directly be linked to the algebra. If one considers an algebra $\mathcal{A}$ which is isomorphic to the algebra of smooth functions over a manifold $\mathcal{M}$, $C^\infty(\mathcal{M})$, then one can reconstruct the full smooth manifold structure purely from the abstract algebra $\mathcal{A}$ [63]. This means that if one wants to describe a smooth manifold, one can choose to represent this as the smooth manifold $\mathcal{M}$, or equivalently one can represent the same information as the abstract algebra $\mathcal{A}$.

Generally, one can equip the algebra with a Banach space structure, and in many cases, for instance for compact Riemannian manifolds, one can create a Banach space with a Schauder basis. This basis is countable, and thus suddenly one can describe the degrees of freedom of a topological manifold with a countably infinite-dimensional object. What is more, as will be argued in Section 2.3, in the case of a compact Riemannian manifold the algebra will get a natural Hilbert space structure, and together with the spectrum of an operator - a countably infinite-dimensional list of numbers - one can reconstruct the full geometry of the Riemannian manifold.

If one denotes the basis of the Hilbert space mentioned above by $\{f_a\}$, the algebra of functions may then be described as

$$f_a \cdot f_b = P_{ab}{}^c f_c.$$

The objects $P_{ab}{}^c$ are known as the structure constants of the algebra, but they may also collectively be seen as a tensor. If one has a measure on the space, as is the case for Riemannian manifolds, the tensor may be written with a lower index P_{abc}. Together with a way of encoding the spectrum of an operator in this tensor which is explained in Section 3.3, one thus has a countably infinite-dimensional configuration space of symmetric tensors P_{abc} which may encode the full geometry of a Riemannian manifold. This configuration space is relatively easy to handle, and as will be shown below, it is even possible to make sense of finite-dimensional tensors as a representation of Riemannian manifolds.

The relatively simple configuration space, and the fact that an algebraic representation of a manifold is manifestly diffeomorphism invariant, makes this approach an approach worth investigating. After all, a priori there is no way of knowing which representation of reality is a more appropriate starting point for a quantum theory of gravity. Besides fundamentally changing the description of a manifold, which might be beneficial in order to quantise the theory, there is a more philosophical argument to make for the usage of this representation [63]. Whenever one makes a measurement on space, one actually uses functions for this. For instance, separating two points on some manifold $x_1, x_2 \in \mathcal{M}$ may be done by finding any function that has a different value at these points $f(x_1) \neq f(x_2)$. If such a function exists, the points x_1 and x_2 are not the same. If one takes a ruler and measures the distance between two points, one is basically measuring the value of a function. Functions are the foundation of measurements, and it makes sense from the point of physics to take these as the fundamental variables for the theory.

1.4 A sneak peek into the canonical tensor model

This section serves as a short explanation of the ideas, principles and interpretation of the canonical tensor model, in order to understand what kinds of models may be thought of as algebraic tensor models. It is not meant as a full introduction into the model, which will be properly defined and examined in Chapter 4.

The canonical tensor model was introduced in 2011 in an attempt to build a tensor model for quantum gravity with a local notion of time [64]. This was mainly motivated by the success of causal dynamical triangulation over dynamical triangulation, suggesting that the key element that makes a quantum theory of gravity feasible is a notion of causality, or time. In simplicial tensor models, it is hard to incorporate a notion of causality on a fundamental tensor level. Therefore, the idea of the model was to place the model in the Hamiltonian formalism, in order to explicitly introduce time generated by some Hamiltonian.

The fundamental variables of the model are a canonically conjugate pair of real N-dimensional symmetric tensors of degree three, Q_{abc} and P_{abc}. Though they have finite dimension N, it is understood that in the full quantum theory - or at least in the classical limit - one has to take some $N \to \infty$ limit. The phase space of the model is thus isomorphic to $\mathbb{R}^{2\mathcal{N}}$, where $\mathcal{N} = \frac{1}{6}N(N+1)(N+2)$ is the amount of degrees of freedom of a real N-dimensional symmetric tensor.

The dynamical structure of the theory is such that it is algebraically similar to the ADM-formalism of general relativity. The Hamiltonian of the theory is fully constraint, and may be written as

$$\mathbf{H} := n_a \mathcal{H}_a + n_{ab} \mathcal{J}_{ab}.$$

Here, n_a and n_{ab} are Lagrange multipliers and $\mathcal{J}_{ab}$ is an analogue to the spatial diffeomorphism constraint, and comes from the assumed $SO(N)$ rotation symmetry of the theory. $\mathcal{H}_a$ is called the Hamiltonian constraint, and it is determined by the requirement that $\mathcal{H}_a$ and $\mathcal{J}_{ab}$ span an algebra similar to the ADM constraint algebra.

The quantum mechanical version of the model is obtained by canonical quantisation. The fundamental variables have operator-versions $\hat{Q}_{abc}$ and $\hat{P}_{abc}$, which satisfy canonical commutation relations. Interestingly, the constraint algebra of the now operators $\hat{\mathcal{H}}_a$ and $\hat{\mathcal{J}}_{ab}$ is consistent with its classical analogue. The physical states Ψ of the theory are determined as one would do in canonical quantum gravity, by requiring

$$\hat{\mathcal{J}}_{ab}\Psi = \hat{\mathcal{H}}_a\Psi = 0.$$

This equation has several known exact solutions, some with interesting properties like the emergence of symmetries.

The model has been shown to be unique under certain physical assumptions [65], and has been shown to have some connections to general relativity in both the $N = 1$ case and a formal $N \to \infty$ limit [66, 67, 68]. Despite this encouraging evidence of a connection to gravity, and analysis using data analytic tools showing some signs of geometry [6], a complete connection to a spacetime picture has been absent until recently when the algebraic tensor model picture was developed [3].[1]

[1] Note that it has not been completely established whether or not this is the best spacetime interpretation for the model.

In the algebraic tensor model picture, one of the tensors Q_{abc} or P_{abc} is supposed to play the role of the tensor defining the algebra of functions of a Riemannian manifold. The tensor, say P_{abc}, would then represent a spatial slice of spacetime as a Riemannian manifold, and the time evolution using the Hamiltonian would determine the full pseudo-Riemannian manifold.

Chapter 2

Algebras and Riemannian manifolds

In this chapter, the description of Riemannian manifolds in an algebraic language is discussed in order to set the stage for an algebraic interpretation of tensor models. Section 2.1 a short introduction to this part of algebraic geometry will be given. Section 2.2 then explains the duality between algebras and topological spaces to some detail, including the reconstruction of smooth manifolds. Section 2.3 then explains how the full geometry of a Riemannian manifold can be reconstructed by introducing an inner product structure with some operator.

2.1 The duality between topological spaces and algebras

In this section, the original Gelfand duality and generalisations thereof will be explained to provide the context in order to understand the place of this work within algebraic geometry. This will be the starting point for the discussion in the following sections, where the focus is to limit ourselves to a specific set of algebras which correspond to real smooth manifolds and what extra structure is needed to create a link to Riemannian manifolds.

The Gelfand duality is a duality between commutative C^*-algebras and topological spaces. An algebra is a vectorspace $\mathcal{A}$ equipped with a bilinear product:

$$\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}.$$

In general, this product is not assumed to be commutative nor associative. A $*$-algebra is a complex algebra with an involution operation $*$, while a C^*-algebra additionally has a Banach space structure with the condition that $||AA^*|| = ||A||^2$.

For a locally compact Hausdorff space X, the (complex) continuous functions that vanish at infinity on that space, denoted by $C_0(X)$, are closed under the following norm

$$||f|| = \sup\{|f(z)| \,|\, z \in X\},$$

and thus form a Banach space. Moreover, it may be verified that it is an associative commutative C^*-algebra under the pointwise product and pointwise complex conjugation for the involution.

The Gelfand duality, cited here without proof, now claims the existence of a, rather surprising, duality between C^* algebras and locally compact Hausdorff spaces X [59, 60]:

Theorem 2.1. *Let $\mathcal{A}$ be a commutative C^*-algebra. There exists a unique (up to homeomorphisms) locally compact Hausdorff space X, for which $\mathcal{A}$ is isomorphic to $C_0(X)$. If $\mathcal{A}$ is unital, X is compact (and thus $\mathcal{A} \cong C(X)$).*

A natural question to ask is whether one can actually reconstruct the space X from the algebra. The answer to this is positive. Consider the set of nonzero homomorphisms of the algebra, which are a subset of the linear dual space $\mathcal{A}^*$, denoted by $|\mathcal{A}|$

$$|\mathcal{A}| := \{p : \mathcal{A} \to \mathbb{C} \,|\, \forall\, f, g \in \mathcal{A},\, p(f)p(g) = p(f \cdot g)\}. \tag{2.1}$$

This space may then be equipped with the weak-$*$ topology.[1] This space is actually the topological space one looks for. This can be seen by defining the *Gelfand transform* $\hat{f}$ of an element $f \in \mathcal{A}$ as a function on $|\mathcal{A}|$ such that

$$\hat{f}(p) := p(f),$$

for all $p \in |\mathcal{A}|$. The weak-$*$ topology on $|\mathcal{A}|$ exactly corresponds to the weakest topology such that the functions on $|\mathcal{A}|$ defined in this way are continuous. In the case of real algebras below, the same topology will be introduced.

It is good to appreciate at this point that this is a true duality between C^* algebras and locally compact Hausdorff spaces. In the case of real algebras, as will be discussed later, this will not be the case anymore and one has to require an algebra to correspond to an algebra of functions.

Theorem 2.1 was the starting point for several generalisations of these concepts. One notable example is non-commutative geometry, where the complex numbers $\mathbb{C}$ in Eq. (2.1) are replaced by operators algebras. This opens the door to relaxing the commutative requirement of the algebra, $f \cdot g = g \cdot f$, as operators that are represented on a Hilbert space may well be defined without this property. Generalising the C^* structure may be done by only requiring the algebra to be a vectorspace, but not needing any Banach space structure. In algebraic geometry, this is done using the language of schemes, which correspond to algebras generated by a finite amount of elements [70].

In this work the main focus is on real commutative algebras. In this chapter they will always be associative as well, though Chapter 3 will also consider a certain class of non-associative algebras. These non-associative algebras are generated by tensors, and the question will be how to properly define an associative counterpart to it.

2.2 Associative commutative algebras and topological manifolds

This section aims to describe topological manifolds in an algebraic way. For this, the content of [63] will be used in combination with some additional insights. In order to

[1] See for instance [69] for an introduction to these concepts.

make the concepts in this section more clear, some examples will be used throughout. Note that some of the examples will be somewhat abstract, since they mainly serve to explain the mathematical concepts. Towards the end of the section, when smooth algebras will be discussed, more physically relevant examples will be discussed, notably the algebra of functions on the circle, which will be the prime example for the next chapter as well.

2.2.1 Algebras and topological spaces

In this part, the duality between commutative associative real algebras and topological spaces will be reviewed. For this, first one needs a proper definition of an $\mathbb{R}$-algebra.

Definition 2.1. An $\mathbb{R}$-*algebra* $\mathcal{F}$ is a linear space $(\mathcal{F}, +)$ over $\mathbb{R}$ equipped with a bilinear product

$$\cdot : \mathcal{F} \times \mathcal{F} \to \mathcal{F}. \tag{2.2}$$

In this chapter, Chapter 2, the $\mathbb{R}$-algebra is always assumed to be commutative and associative, though the latter restriction will be relaxed in Chapter 3. Two examples of commutative associative $\mathbb{R}$-algebras are the following.

Example 2.1. An elementary example is the real numbers $\mathbb{R}$ with the usual product. This one-dimensional $\mathbb{R}$-algebra is generated by the set $\{1\}$, and is isomorphic to $\mathbb{R}$ as a field.

By "generated by" one means to take linear combinations and products of the elements in the generated set to arrive at the full algebra.

Example 2.2. A little more sophisticated example is the algebra of real polynomials generated by $\{1, x\}$, denoted by $\mathbb{R}[x]$. This algebra is infinite-dimensional and has a Hamel basis given by $\{1, x, x^2, ...\}$.

An $\mathbb{R}$-algebra homomorphism is defined as follows:

Definition 2.2. An $\mathbb{R}$-*algebra homomorphism* is a homomorphism $\alpha : \mathcal{F}_1 \to \mathcal{F}_2$ that preserves the operations $+$ and $\cdot$, preserves multiplication with $\lambda \in \mathbb{R}$ and is unital, i.e. sends the unit of $\mathcal{F}_1$ to the unit of $\mathcal{F}_2$. In this dissertation, $\mathcal{F}_2$ will usually be taken to be the real numbers $\mathbb{R}$.

Example 2.3. An example of an $\mathbb{R}$-algebra homomorphism for the algebra in example 2.2 may be constructed as

$$\begin{aligned} \alpha_0 : \mathbb{R}[x] &\to \mathbb{R}, \\ f &\to f(x = 0), \end{aligned}$$

where by $f(x = 0)$ means to set x to 0. Take for example $f = \sqrt{2} + x + 3x^2$, then $\alpha_0(f) = \sqrt{2}$.

Replacing x by a real number is called the *evaluation map*, so here the polynomial f is evaluated at 0. Note that any other real number could also have been chosen and would be a valid $\mathbb{R}$-algebra homomorphism.

In order to view the algebra $\mathcal{F}$ as an algebra of functions over some topological space $\mathcal{M}$, one defines the algebraic dual space $|\mathcal{F}|$ as

Definition 2.3. The *algebraic dual space* of an algebra $\mathcal{F}$, denoted by $|\mathcal{F}|$, is defined as the set of all $\mathbb{R}$-algebra homomorphisms of $\mathcal{F}$ into $\mathbb{R}$:

$$|\mathcal{F}| := \{p : \mathcal{F} \to \mathbb{R} \,|\, \forall f, g \in \mathcal{F},\, p(f \cdot g) = p(f) \cdot p(g)\}.$$

Remark 2.1. The algebraic dual space is a subset of the linear dual space $\mathcal{F}^*$

$$|\mathcal{F}| \subset \mathcal{F}^*.$$

Elements of the algebraic dual space, $p \in |\mathcal{F}|$, will be referred to as "points".

Example 2.4. The dual space of $\mathcal{F} = \mathbb{R}$ from Example 2.1 is a single point $|\mathcal{F}| = \{p\}$. This can be seen as follows. For any $\mathbb{R}$-homomorphism, the unit of $\mathcal{F}$ has to be sent to the unit in $\mathbb{R}$. However, since the $+$ and $\cdot$ operations and multiplication by $\lambda \in \mathbb{R}$ have to be preserved, the only $\mathbb{R}$-algebra homomorphism possible is

$$\begin{aligned} p : \mathcal{F} &\to \mathbb{R}, \\ x &\to p(x) = x. \end{aligned}$$

Example 2.5. The dual space of $\mathcal{F} = \mathbb{R}[x]$ is the real numbers $|\mathcal{F}| = \mathbb{R}$. This may be seen as follows. As already mentioned in Example 2.3, the evaluation of a polynomial at a point in $\mathbb{R}$ is always an $\mathbb{R}$-algebra homomorphism. The only thing to prove now is that every $\mathbb{R}$-algebra homomorphism α is determined by a single real number λ, which holds true as follows

$$\begin{aligned} \alpha(f) &= \alpha\left(\sum_k a_k x^k\right), \\ &= \sum_k a_k \alpha(x^k), \\ &= \sum_k a_k \alpha(x)^k \equiv \sum_k a_k \lambda^k. \end{aligned}$$

Here, $\lambda \equiv \alpha(x)$ indeed has to be a real number, and it determines the action of the homomorphism uniquely.

In order to view $\mathcal{F}$ as an algebra of functions over the dual space $|\mathcal{F}|$, one needs to define the Gelfand transform of an algebra element

Definition 2.4. The *Gelfand transform* $\tilde{f} : |\mathcal{F}| \to \mathbb{R}$ of an element $f \in \mathcal{F}$ is a function on $|\mathcal{F}|$ defined as

$$p \in |\mathcal{F}|,\ \tilde{f}(p) = p(f).$$

Definition 2.5. The algebra of functions on $|\mathcal{F}|$ canonically induced by $\mathcal{F}$ are all the Gelfand transforms of $\mathcal{F}$,

$$\tilde{\mathcal{F}} := \left\{\tilde{f} : |\mathcal{F}| \to \mathbb{R}\ ,\ \tilde{f}(p) = p(f) \mid f \in \mathcal{F}\right\},$$

where we have the natural map from the algebra $\mathcal{F}$ to $\tilde{\mathcal{F}}$

$$\begin{aligned} \tau : \mathcal{F} &\to \tilde{\mathcal{F}}, \\ f &\to (p \to p(f))\,. \end{aligned}$$

Remark 2.2. τ is a homomorphism, and thus, $\tilde{\mathcal{F}}$ may indeed be interpreted as an algebra of functions over $|\mathcal{F}|$. Furthermore it is surjective.

As the goal is to create a connection between algebras and topological spaces, one wants to interpret the algebra $\mathcal{F}$ as an algebra of functions over a topological space $|\mathcal{F}|$. It is worth noting that the algebraic structures in this section are slightly different from the case of the Gelfand duality as discussed in Section 2.1. For one, the algebras discussed are real, and there is no Banach space structure assumed. In general, a real algebra $\mathcal{F}$ does not directly correspond to an algebra of functions over the space $|\mathcal{F}|$, justifying the following definition

Definition 2.6. A $\mathbb{R}$-algebra $\mathcal{F} \ni f$ is *geometric* if it is isomorphic to the space of Gelfand transforms of $\mathcal{F}$,

$$\mathcal{F} \cong \tilde{\mathcal{F}}.$$

In other words: $\mathcal{F}$ is geometric if it can be seen as an algebra of functions over some space $M = |\mathcal{F}|$.

Remark 2.3. Note that if $\mathcal{F}$ is geometric, the algebra $\tilde{\mathcal{F}}$ induced by $\mathcal{F}$ is pointwise as

$$(\tilde{f} \cdot \tilde{g})(p) = p(f \cdot g) = p(f)p(g) = \tilde{f}(p)\tilde{g}(p).$$

Example 2.6. The algebra $\mathcal{F} = \mathbb{R}$, discussed in Examples 2.1 and 2.4, is geometric. The (clearly bijective) map τ is given by

$$\begin{aligned} \tau : \mathcal{F} &\to \tilde{\mathcal{F}}, \\ x &\to \tilde{x}(p) = p(x) = x. \end{aligned}$$

Example 2.7. The algebra $\mathcal{F} = \mathbb{R}[x]$, discussed in Examples 2.2 and 2.5 is also geometric, as every element $\tilde{f} \in \tilde{\mathcal{F}}$ can be written as

$$\tilde{f}(p_\lambda) = p_\lambda(f) = \sum_k a_k \lambda^k,$$

where $p_\lambda \in |\mathcal{F}|$ exists for any $\lambda \in \mathbb{R}$. Therefore this exactly corresponds to a polynomial given by the coefficients a_k, and thus

$$\tau(f)(p) = \tilde{f}(p)$$

is bijective.

Example 2.8. Consider the field $\mathbb{R}^2$ endowed with the product

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, x_1 y_2 + x_2 y_1).$$

Note that the unit of this algebra is $(1, 0)$. This means that for $\forall p \in |\mathcal{F}|$, $p((x, 0)) = x$, since p should be a homomorphism. This means that for general points $(x, y) \in \mathbb{R}^2$:

$$p((x, y)) = p((x, 0) + (0, y)) = x + p((0, y)).$$

This means that for the product of two elements for homomorphisms:

$$\begin{aligned} p((x_1,y_1)\cdot(x_2,y_2)) &= p((x_1,y_1))p((x_2,y_2)),\\ &= x_1x_2 + x_1p((0,y_2)) + x_2p((0,y_1)) + p((0,y_1))p((0,y_2)),\\ &= x_1x_2 + p((0,x_1y_2+x_2y_1)) + p((0,y_1))p((0,y_2)). \end{aligned}$$

However, from the multiplcation rule of the algebra follows:

$$\begin{aligned} p((x_1,y_1)\cdot(x_2,y_2)) &= p((x_1x_2, x_1y_2+x_2y_1)),\\ &= x_1x_2 + p((0,x_1y_2+x_2y_1)). \end{aligned}$$

So the only way for which p can be a homomorphism is if $p((x,y)) = p((x,0)) + p((0,y)) = x$: The dual space consists of only a single point. It is now clear that the map $\tau : \mathcal{F} \to \tilde{\mathcal{F}}$ cannot be injective, as

$$\tilde{f}(p) = \tau(f)(p) = p(f) = p((x,y)) = p((x,0)).$$

So this algebra is not geometric.

Remark 2.4. An important observation of Example 2.8 which will be important for the following is that the algebra has an ideal, given by $\mathcal{I} = \text{Ker } p = \{(0,y) \mid y \in \mathbb{R}\}$. This ideal is the root of the problem, since every element of this ideal is mapped to 0 by elements in the dual space. In a slightly less rigorous phrasing: The dependence on elements of this ideal disappears in the algebra of functions $\tilde{\mathcal{F}}$, as under τ for $f = (x,y)$; $\tilde{f}(p) = p(f) = p((x,y)) = p((x,0))$.

This remark inspires the following definition and theorem.

Definition 2.7. The *central ideal* $\mathcal{I}(\mathcal{F})$ of an $\mathbb{R}$-algebra $\mathcal{F}$ is

$$\mathcal{I}(\mathcal{F}) := \bigcap_{x\in|\mathcal{F}|} \text{Ker } x.$$

Theorem 2.2. *An $\mathbb{R}$-algebra $\mathcal{F}$ is geometric iff its central ideal $\mathcal{I}(\mathcal{F})$ is trivial.*

Proof. Note that $\mathcal{F}$ is geometric iff Ker $\tau = 0$. So assume

$$\begin{aligned} f \in \text{Ker } \tau &\Leftrightarrow \tau(f) = \tilde{f} = 0,\\ &\Leftrightarrow \tilde{f}(x) = x(f) = 0 \;\forall x \in |\mathcal{F}|,\\ &\Leftrightarrow f \in \bigcap_{x\in|\mathcal{F}|} \text{Ker } x = \mathcal{I}(\mathcal{F}) \end{aligned}$$

□

Remark 2.5. With this theorem, the examples in Examples 2.6 and 2.7 become much easier. Example 2.6 is trivial, and for Example 2.7 the only polynomial that evaluates to zero everywhere is the zero-polynomial. The theorem also makes sense of Example 2.8 combined with remark 2.4. The central ideal of the algebra is non-trivial, so the algebra is not geometric.

As explained before, not every $\mathbb{R}$-algebra is geometric. However, with the following proposition one can turn every arbitrary $\mathbb{R}$-algebra into a geometric $\mathbb{R}$-algebra.

Proposition 2.1. *For an arbitrary $\mathbb{R}$-algebra, the quotient $\mathbb{R}$-algebra*

$$\mathcal{F}/\mathcal{I}(\mathcal{F})$$

is geometric and has the same dual space:

$$|\mathcal{F}| = |\mathcal{F}/\mathcal{I}(\mathcal{F})|.$$

Proof. First it will be shown that the dual spaces are isomorphic. For this one needs the quotient map

$$\begin{aligned} \pi : \mathcal{F} &\to \mathcal{F}/\mathcal{I}(\mathcal{F}), \\ f &\to [f]. \end{aligned}$$

With this one can define the map

$$\begin{aligned} \varphi : |\mathcal{F}/\mathcal{I}(\mathcal{F})| &\to |\mathcal{F}|, \\ b \quad &\to b \circ \pi. \end{aligned}$$

This is well-defined since taking an element $f \in \mathcal{F}$ and projecting it to the quotient space, and then using a homomorphism of the dual space of the quotient space is a homomorphism of $\mathcal{F}$ as well.

In order to show that φ is an isomorphism, consider injectivity and surjectivity seperately.

- **Injective:** Take $b_1, b_2 \in |\mathcal{F}/\mathcal{I}(\mathcal{F})|$, with $b_1 \neq b_2$. Then, since π is surjective,

  $$\varphi(b_1) = b_1 \circ \pi \neq b_2 \circ \pi = \varphi(b_2).$$

  So indeed φ is injective.

- **Surjective:** To show surjectivity, take an element $a \in |\mathcal{F}|$. Note that $\operatorname{Ker} a \supseteq \mathcal{I}(\mathcal{F})$. This means that all elements that are in an equivalence class $[f]$ of the quotient space $\mathcal{F}/\mathcal{I}(\mathcal{F})$ are mapped to the same value under a. So

  $$\bar{a}([f]) = a(f),$$

  is a well-defined homomorphism of $\mathcal{F}/\mathcal{I}(\mathcal{F})$, i.e. $\bar{a} \in |\mathcal{F}/\mathcal{I}(\mathcal{F})|$. But that means that a, by the definition of $\bar{a}$, is nothing more than

  $$a = \bar{a} \circ \pi = \varphi(\bar{a}),$$

  so indeed φ is surjective.

Thus we can conclude that φ is an isomorphism.

Now to show that the central ideal of this quotient algebra is trivial, take an element $[f] \in \mathcal{I}\left(\mathcal{F}/\mathcal{I}(\mathcal{F})\right)$. Then:

$$\begin{aligned} [f] \in \mathcal{I}\left(\mathcal{F}/\mathcal{I}(\mathcal{F})\right) &\Leftrightarrow \forall b \in |\mathcal{F}/\mathcal{I}(\mathcal{F})|,\ b([f]) = 0, \\ &\Leftrightarrow \forall a \in |\mathcal{F}|,\ a(f) = 0, \\ &\Leftrightarrow f \in \mathcal{I}(\mathcal{F}), \\ &\Leftrightarrow [f] = 0. \end{aligned}$$

□

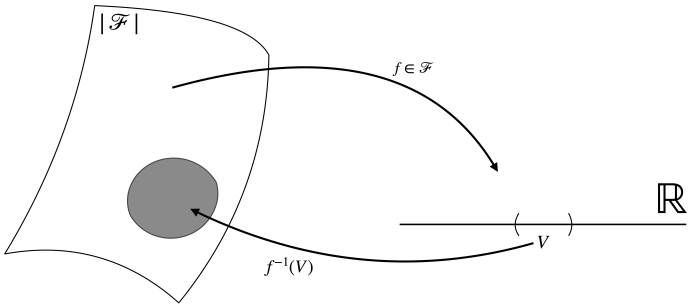

Figure 2.1: An illustration of the construction of a topology on $|\mathcal{F}|$ as described in Remark 2.6. Figure taken from [3].

From now on, the algebra $\mathcal{F}$ is generally assumed to be geometric, as one can always construct a geometric algebra from an arbitrary $\mathbb{R}$-algebra. Furthermore, the algebras $\mathcal{F}$ and $\tilde{\mathcal{F}}$ will be used interchangeably as they are isomorphic in that case.

Remark 2.6. Given a geometric $\mathbb{R}$-algebra $\mathcal{F}$, one can define a topology on $|\mathcal{F}|$. This topology is given by the basis of the form $f^{-1}(V)$, where $V \subset \mathbb{R}$ is open and $f \in \mathcal{F}$. Note that $f^{-1}(V)$ denotes the preimage of V under f, and is a meaningful expression since f can be identified with a function over $|\mathcal{F}|$ by the Gelfand transform.

This topology in $|\mathcal{F}|$ is the *weakest* topology for which all functions in $\mathcal{F}$ are continuous.

Proposition 2.2. *If $\mathcal{F}$ is a geometric $\mathbb{R}$-algebra, the dual space $|\mathcal{F}|$ with the topology introduced in Remark 2.6 is a* Hausdorff space.

Proof. $|\mathcal{F}|$ with this topology is a Hausdorff space if for every two distinct points $x, y \in |\mathcal{F}|$, there exist neighborhoods $U \ni x$ and $V \ni y$ that are disjoint, i.e. $U \cap V = \emptyset$.

Suppose $x, y \in |\mathcal{F}|$ with $x \neq y$. This means that there is an $f \in \mathcal{F}$ such that $f(x) \neq f(y)$. Without loss of generality one can assume that $f(x) < f(y)$. Two neighborhoods in $|\mathcal{F}|$ may then be constructed by, for instance,

$$U = f^{-1}\left(\left(-\infty, \frac{f(x)+f(y)}{2}\right)\right), \quad V = f^{-1}\left(\left(\frac{f(x)+f(y)}{2}, \infty\right)\right),$$

which are indeed neighborhoods in the topology by the definition in Remark 2.6, and indeed $U \cap V = \emptyset$. □

Example 2.9. Continuing with Example 2.7, where $\mathcal{F} = \mathbb{R}[x]$ and $|\mathcal{F}| = \mathbb{R}$, one can now construct a topology on this dual space. Take for instance the polynomial $f = x$. Now any open set $V \subseteq \mathbb{R}$ corresponds by $U = f^{-1}(V)$. This already constitutes the usual basis for $\mathbb{R}$, and this actually already defines the whole topology on $|\mathcal{F}|$.

Example 2.10. A little more advanced is the example of the algebra of polynomials of n variables, $\mathcal{F} = \mathbb{R}[x_1, \ldots, x_n]$. It can be checked with similar arguments as before that $|\mathcal{F}| = \mathbb{R}^n$. The topology defined in Remark 2.6 coincides with the usual topology of $\mathbb{R}^n$.

It is good to reflect for a moment what has been achieved. From an *abstract* algebra $\mathcal{F}$, one can now unambiguously define an associated Hausdorff space to it, where the

algebra now represents an algebra of functions on this Hausdorff space. The question now is whether one can go even further; can one get any more structure from the algebra? The answer to this is yes, in the following will be discussed what requirements the algebra has to satisfy in order to be able to fully reconstruct a smooth manifold.

2.2.2 Smooth manifolds and algebras

Here, the algebraic definition of smooth manifolds will be given. For an algebra to correspond to an algebra of functions over a smooth manifold, two more notions will be needed: The notions of *completeness* and *smoothness*.

Definition 2.8. Suppose $\mathcal{F}$ is a geometric $\mathbb{R}$-algebra and $A \subset |\mathcal{F}|$ a subset of its dual space. The *restriction* of $\mathcal{F}$ to A, denoted by $\mathcal{F}|_A$, are all functions $f : A \to \mathbb{R}$, such that for any $a \in A$ there exists an open neighborhood $U \subset A$ and a function $f' \in \mathcal{F}$, such that the restriction of f to U coincides with the restriction of f' to U.

The *restriction homomorphism* for a subset $A \subset |\mathcal{F}|$ is

$$\rho_A : \mathcal{F} \to \mathcal{F}|_A,$$
$$f \to f|_A,$$

where $f|_A$ is the usual restriction of a function f to region A.

In order to see why this is useful, consider the following example.

Example 2.11. Suppose $F = C^\infty(\mathbb{R})$ and $A = \mathbb{R}_+ \subset \mathbb{R}$. With the definition in Definition 2.8,

$$C^\infty(\mathbb{R})|_{\mathbb{R}_+} = C^\infty(\mathbb{R}_+).$$

Take for instance the function $f(x) = 1/x \in C^\infty(\mathbb{R}_+)$, it may be constructed in the following way. Choose any $a > 0$, one can choose a function α which vanishes for $x \leq a/3$ and equals 1 whenever $x > a/2$. Then the function $\alpha/x \in C^\infty(\mathbb{R})$ and coincides with $1/x$ in a neighbourhood of a.

Note that there is no function $g(x) \in C^\infty(\mathbb{R})$ such that $g(x)|_{\mathbb{R}_+} = f(x)$, which makes Definition 2.8 useful.

This leads to the definition of a complete algebra.

Definition 2.9. A geometric $\mathbb{R}$-algebra $\mathcal{F}$ is called *complete* if the restriction homomorphism $\rho : \mathcal{F} \to \mathcal{F}|_{|\mathcal{F}|}$ is an isomorphism. In other words, every function on $|\mathcal{F}|$ that locally overlaps with elements of $\mathcal{F}$, is also included in $\mathcal{F}$.

Remark 2.7. If an algebra is not closed, one can take the restriction $\mathcal{F}|_{|\mathcal{F}|}$ to naturally construct a closed algebra.

Example 2.12. In this example, the algebra $\mathcal{F} = C^\infty(U)$ of smooth functions over a subset $U \subset \mathbb{R}^n$ will be discussed. While the aforementioned structures will all be discussed, some proofs will be omitted.

The first thing one needs to establish, is what is the dual space of $\mathcal{F}$? It might not be a big surprise, but still something that one formally has to prove [63], that the dual space is homeomorphic to U

$$|\mathcal{F}| = U.$$

In particular, $|C^\infty(\mathbb{R}^n)| = \mathbb{R}^n$. It now is also clear that the algebra is geometric, as only the zero-function will evaluate to zero everywhere.

From the nature of smooth functions, it is also clear that the restriction of the algebra to U will result in the same algebra

$$\mathcal{F}|_U = \mathcal{F},$$

since every function that overlaps with a smooth function around any point, is necessarily smooth, and thus the algebra is complete as well.

The last definition that makes the link to smooth manifolds possible is the following.

Definition 2.10. A complete geometric $\mathbb{R}$-algebra $\mathcal{F}$ is called smooth if there is an, at most countable, open covering $\{U_k\}$ of $|\mathcal{F}|$ such that all restricted algebras $\mathcal{F}|_{U_k} \cong C^\infty(\mathbb{R}^n)$ for some fixed n.

It should come to no surprise that a smooth algebra $\mathcal{F}$ corresponds to the smooth functions over a manifold $\mathcal{M} \cong |\mathcal{F}|$. Actually proving that such an algebra indeed corresponds to a manifold with a specific atlas is far from trivial, and for this proof I would like to refer to chapter 7 of [63]. Here, the main theorem will be quoted:

Theorem 2.3. *Suppose $\mathcal{F}$ is any smooth $\mathbb{R}$-algebra. Then there exists a smooth atlas A on the dual space $\mathcal{M} = |\mathcal{F}|$, such that the map*

$$\mathcal{F} \to C^\infty(\mathcal{M}),\ f \to (p \to p(f)),$$

is an isomorphism.

In the following, the example of the circle will be discussed for all of the structures explained above. This example will be the main example discussed and developed further in the following section and chapter as well.

Example 2.13. The algebra of the circle. The circle is a smooth manifold, denoted by S^1, and it will serve as the main example throughout the following chapters. Therefore, it is useful to algebraically define the circle, and show that this algebra indeed corresponds to a geometric, complete and smooth algebra. In the next section, a Hilbert space structure will be introduced, making this algebra much simpler to describe explicitly.

The easiest way to define this algebra is as a sub-algebra of the algebra of smooth functions over $\mathbb{R}$, $C^\infty(\mathbb{R})$ (see Example 2.12), namely by only considering periodic functions of period 2π:

$$\mathcal{F} := \{f \in C^\infty(\mathbb{R}) \,|\, x \in \mathbb{R},\ f(x + 2\pi) = f(x)\}.$$

Proving that the dual space equals the topology of the circle S^1 is done as follows. First note that every evaluation map of an element $x \in \mathbb{R}$, $p_x(f) = f(x)$, by the pointwise definition of the algebra constitutes an $\mathbb{R}$-homomorphism. What is thus left to prove, is that every $p \in |\mathcal{F}|$ corresponds to some evaluation map for a point $x \in \mathbb{R}$.

Consider an element from the dual space $p \in |\mathcal{F}|$, and assume that it does not correspond to some point $x \in \mathbb{R}$. In that case, there must exist a function $f_x \in \mathcal{F}$ such

that $f_x(x) \neq p(f_x)$. One can now find an open covering of the interval $[0, 2\pi]$ by sets of the form

$$U_x = \{y \in \mathbb{R} |\, f_x(y) \neq p(f_x)\}.$$

Since $[0, 2\pi]$ is compact, by definition there is a finite subcover, here denoted by $U_{x_1}, \ldots, U_{x_n}$. One can now construct a function which is nowhere zero on $[0, 2\pi]$, for instance

$$g = \sum_{i=1}^{n} (f_{x_i} - p(f_{x_i}))^2.$$

Therefore, the function $1/g$ is also a smooth periodic function, and thus $1/g \in \mathcal{F}$. From the definition of the algebraic dual space, p is a unital $\mathbb{R}$-algebra homomorphism, and therefore

$$1 = p(g \cdot (1/g)) = p(g)p(1/g),$$

however, from the way that g is defined,

$$p(g) = \sum_{i=1}^{n} (p(f_{x_i}) - p(f_{x_i}))^2 = 0,$$

which is in contradiction with the statement above. Therefore, every point $p \in |\mathcal{F}|$ needs to be a point in $p \in \mathbb{R}$. Furthermore, since the points that are $2\pi m$ apart (for m an integer) evaluate exactly the same, the dual space is given by $|\mathcal{F}| = \mathbb{R}/\mathbb{Z} = S^1$.

The fact that the algebra $\mathcal{F}$ is complete is a easier to establish. One has to proof that every function on $S^1 = |\mathcal{F}|$, $f : |\mathcal{F}| \to \mathbb{R}$, that coincides in a neighbourhood of every point $a \in |\mathcal{F}|$ with some smooth function $f_a \in \mathcal{F}$, is also an element of $\mathcal{F}$. However, since $C^\infty(\mathbb{R}) \supset C^\infty(S^1)$ is complete, surely there is a smooth function $g \in C^\infty(\mathbb{R})$ that for every $a \in |\mathcal{F}|$ has $g(a) = f(a)$. If one then takes the restriction of this function $g|_{|\mathcal{F}|}$, one has found the element $g|_{|\mathcal{F}|} \in C^\infty(S^1) = \mathcal{F}$ that overlaps with f on $S^1 = |\mathcal{F}|$.

In order to show that this algebra is smooth, one has to find an open covering $\{U_i\}$ such that all $C^\infty(U_i) \cong C^\infty(\mathbb{R})$. If one picks two functions $f_1, f_2 \in \mathcal{F}$

$$f_1(x) = \sin^2 \frac{x}{2},\ f_2(x) = \cos^2 \frac{x}{2},$$

where $x \in \mathbb{R}/\mathbb{Z} = \mathbb{R}/x \sim x + 2\pi$, one can find a covering of $|\mathcal{F}|$ as

$$U_i = \{x \in |\mathcal{F}| \cong \mathbb{R}/\mathbb{Z} |\, f_i(x) \neq 0\},\ i \in \{1, 2\}.$$

These U_i are homeomorphic to $(0, 2\pi)$, and thus we see

$$\mathcal{F}|_{U_i} \cong C^\infty\left((0, 2\pi)\right) \cong C^\infty(\mathbb{R}).$$

Since we explicitly found a covering as in Definition 2.10, $\mathcal{F}$ is a smooth algebra.

A final note about this example is that this algebra may also be constructed in different ways. Another way is to consider the set of points of $S^1 := \{(x, y) \in \mathbb{R}^2 |\, x^2 + y^2 = 1\}$, and construct the above algebra as $\mathcal{F} = C^\infty(\mathbb{R}^2)|_{S^1}$ using the restriction of an algebra in Definition 2.8.

2.3 Spectral representation of Riemannian manifolds

In Section 2.2 the duality between geometric complete smooth algebras and topological manifolds was discussed. It was found that the algebra, a real vectorspace with a product operator, actually contains the full information of the topological manifold. In this section, additional structure will be added to the algebra, namely a Hilbert space and a self-adjoint operator, and it will be shown that this actually contains the full data of a Riemannian manifold. In order to prepare for the discussion in later chapters, extra emphasis will be placed on the case where the Hilbert space is seperable.

A Riemannian manifold $(\mathcal{M}, g)$ is a smooth topological manifold $\mathcal{M}$, together with a positive-definite metric tensor field g. The metric defines an inner product on every point $p \in \mathcal{M}$

$$g : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}.$$

Using this metric, one can define a measure, infinitesimally $\mathrm{d}\mu(x) = \mathrm{d}^d x\sqrt{|g|}$. Here, $|g|$ denotes the determinant of the metric g_{ab}.

As $\mathcal{M}$ is a smooth manifold, one can consider the algebra of smooth functions $\mathcal{F} = C^\infty(\mathcal{M})$. This is a real infinite-dimensional vectorspace with a product operation, but in the case of a compact Riemannian manifold one can define an inner product on this vectorspace as

$$\langle f \,|\, g \rangle := \int_{\mathcal{M}} \mathrm{d}^d x \sqrt{|g|}\, f(x) g(x), \tag{2.3}$$

where $f, g \in C^\infty(\mathcal{M})$. It is important to note that $C^\infty(\mathcal{M})$ is not closed with respect to this inner product, however if one takes the closure of $C^\infty(\mathcal{M})$ with respect to this inner product, one arrives at a Hilbert space called the square integrable functions[2]

$$L^2(\mathcal{M}) := \overline{C^\infty(\mathcal{M})}.$$

For compact Riemannian manifolds, the Hilbert space $L^2(\mathcal{M})$ is actually seperable, meaning that there is a countable basis $\{f_a\}$ which is also called a Schauder basis. Any function $f \in C^\infty(\mathcal{M}) \subset L^2(\mathcal{M})$ may thus be written by

$$f = \sum_{a \geq 1} \alpha^a f_a.$$

The Hilbert space structure makes the algebra easier to describe, because if one knows the product rules for all the basis functions $f_a \cdot f_b$, one can reconstruct the whole algebra. Take for instance $f = \sum_a \alpha^a f_a \in C^\infty(\mathcal{M}),\ g = \sum_b \beta^b f_b \in C^\infty(\mathcal{M})$, then

$$f \cdot g = \left(\sum_{a \geq 1} \alpha^a f_a \right) \left(\sum_{b \geq 1} \beta^b f_b \right) = \sum_{a \geq 1} \sum_{b \geq 1} \alpha^a \beta^b f_a \cdot f_b.$$

On a Riemannian manifold, one can define a natural self-adjoint operator: The Laplace-Beltrami operator. This operator may be defined as

$$\Delta := -\mathrm{div} \circ \nabla,$$

[2]The discussion here is mainly geared towards compact Riemannian manifolds, as these are the main objects of study in this work. For non-compact cases, one can still construct a Hilbert space by considering the smooth functions with compact support, $C_c^\infty(\mathcal{M})$, though this will not be a seperable Hilbert space.

where ∇ is the gradient of a function, which is a map from the smooth functions to the tangent bundle

$$\nabla : C^\infty(\mathcal{M}) \to T\mathcal{M},$$

which may be defined by first taking the exterior derivative of a function, and then using the metric to map this element to the tangent bundle. In local coordinates this is given by (acting on a function $f \in C^\infty(\mathcal{M})$)

$$\nabla f = g^{ab}\partial_a f \partial_b.$$

The operator div is the divergence operator, which maps tangent vectorfields to the smooth functions again

$$\text{div} : T\mathcal{M} \to C^\infty(\mathcal{M}).$$

The divergence is usually defined as the formal dual of the gradient with respect to the inner product $\langle .|\, . \rangle$. Requiring this, one finds

$$\text{div}\, X = \frac{1}{\sqrt{|g|}}\partial_a(X^a\sqrt{|g|}), \tag{2.4}$$

where $X \in T\mathcal{M}$. In local coordinates, acting on a function $f \in C^\infty(\mathcal{M})$, the resulting Laplace-Beltrami operator is then given by

$$\Delta f = -\frac{1}{\sqrt{|g|}}\partial_a(g^{ab}\sqrt{|g|}\partial_b f). \tag{2.5}$$

A very useful theorem for compact Riemannian manifold is the following [71], which will be stated here without proof.

Theorem 2.4. ***The Hodge theorem for functions.*** *Let $(\mathcal{M}, g)$ be a compact Riemannian manifold. Then, there exists an orthonormal basis of $L^2(\mathcal{M})$ consisting of eigenfunctions of the Laplace-Beltrami operator. All eigenvalues are negative (except for zero which has multiplicity one), and have finite multiplicity.*

Therefore, the Laplace-Beltrami operator is very useful in order to find a useful basis of the Hilbert space $L^2(\mathcal{M})$.

From (2.5) it is very clear that the metric is crucial for the construction of this operator, and that the operator is in this sense determined by the metric. However, this works also the other way around: The action of the Laplace-Beltrami operator on smooth functions $f \in C^\infty(\mathcal{M})$ fully determines the metric. This may be readily seen by expanding the expression (2.5)

$$\Delta f \approx g^{ab}\partial_a\partial_b f + \text{lower order terms}. \tag{2.6}$$

Therefore, if one takes a function that is locally $f \sim x^a x^b$, one can reconstruct g^{ab}. Combining this with the knowledge that from the algebra of functions $\mathcal{F} = C^\infty(\mathcal{M})$ one can reconstruct the smooth manifold structure, this means that for a Riemannian manifold $(\mathcal{M}, g)$, the triple $(\mathcal{F}, \mathcal{H}, \Delta)$ actually contains the exact same information, where $\mathcal{F} \cong C^\infty(\mathcal{M})$ is a real commutative associative algebra, $\mathcal{H} = L^2(\mathcal{M})$ is a Hilbert space that is the closure of $\mathcal{F}$ with respect to some inner product, and $\Delta : \mathcal{H} \to \mathcal{H}$ a densely defined operator on $\mathcal{H}$.

This statement is similar to the spectral triple approach from non-commutative geometry [62], where the so-called canonical triple contains a C^*-algebra, a Hilbert space and a Dirac-operator [72, 73]. The difference to this approach is the type of algebraic structures used.

Example 2.14. Continuation of the circle. In Example 2.13, the smooth algebra of the circle has been introduced as a sub-algebra of the smooth functions on $\mathbb{R}$. Here, the example will be built out more, by not considering only the algebra of smooth functions on S^1, $C^\infty(S^1)$, but adding a Hilbert space structure and the Laplace-Beltrami operator.

The circle is a one-dimensional manifold (see Example 2.13), and one can take a local coordinate system where $\theta \in [0, 2\pi]$. The Laplace-Beltrami operator on a unit circle, in terms of these coordinates, is defined as

$$\Delta := \frac{\mathrm{d}^2}{\mathrm{d}\theta^2},$$

and as one might expect from the discussion above, this actually fixes the metric to be

$$g_{\theta\theta} = 1.$$

The eigenfunctions, $\{f_a(\theta)\}$, of the Laplace-Beltrami operator on the circle are well-known, and are given by

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\sin(\theta), \frac{1}{\sqrt{\pi}}\cos(\theta), \frac{1}{\sqrt{\pi}}\sin(2\theta), \frac{1}{\sqrt{\pi}}\cos(2\theta), \ldots \right\}, \tag{2.7}$$

where they have been normalised with respect to the inner product of Eq. (2.3),

$$\delta_{ab} = \langle f_a | \, f_b \rangle = \int_0^{2\pi} \mathrm{d}\theta f_a(\theta) f_b(\theta).$$

The eigenvalues of these eigenfunctions have multiplicity 2 (except for the constant function), and are given by n^2 for $n \in \mathbb{N}$. In order of the eigenfunctions above they are

$$\{0, 1, 1, 4, 4, 9, 9, \ldots\}.$$

One might wonder if it is possible to use Eq. (2.6) to reconstruct the metric above. This is indeed possible, if one takes a point $\theta_0 \in [0, 2\pi]$, one finds the inverse metric at that point by choosing a function that locally equals $f_{\theta_0}(\theta) \approx (\theta - \theta_0)^2$. For $f_{\theta_0}(\theta)$ one may take the following function

$$\begin{aligned} f_{\theta_0}(\theta) &= 2(1 - \cos(\theta - \theta_0)), \\ &= 2(1 - \sin(\theta_0)\sin(\theta) - \cos(\theta_0)\cos(\theta))). \end{aligned}$$

Now, consider the action of the Laplacian at point θ_0 to find the (inverse) metric at that point

$$\left[\frac{1}{2}\Delta f_{\theta_0}(\theta)\right]_{\theta=\theta_0} = (\sin(\theta_0)^2 + \cos(\theta_0)^2) = 1 = g^{\theta\theta}(\theta_0),$$

so the (inverse) metric has been correctly reconstructed.

In the following chapter, this fact will be used to build a framework that is fundamentally using these three elements. A tensor, acting on a Hilbert space, will define the algebra of functions and contain the spectral information of the Laplace-Beltrami operator. It is thus a single (countable) object that includes all information of a compact Riemannian manifold.

Chapter 3

Algebraic tensor models

In this chapter, the focus will be on the construction of algebraic tensor models. The idea is to use a tensor P_{abc} to generate an algebra, and then use the algebraic insights developed in Chapter 2 to relate this tensor to a geometric space. While generating a finite-dimensional commutative algebra is relatively straightforward for finite-dimensional tensors, assuring associativity and finding infinite-dimensional algebras is more difficult. In Section 3.1, the main topological aspects of this will be discussed. The associative closure of a tensor will be defined, which gives the opportunity to connect a finite-dimensional tensor to an infinite-dimensional associative commutative algebra of functions. In Section 3.2 it will be found that demanding the tensor to be totally symmetric may be interpreted as introducing a Hilbert space structure on which the product operator acts as a self-adjoint operator, which is then shown to introduce a measure on the topological space. Section 3.3 then shows how one can encode the full geometric information of a Riemannian manifold into a tensor, and discusses possibilities to extract this information from the tensor.

The formalism explained here has been introduced in [3]. Though the general explanation remains the same, in this chapter the emphasis will be to explain why all the mathematical structures involved are used and building the theory up layer by layer. For instance, while a Banach space structure of the algebra for compact spaces comes for free, and this is extensively used in Section 3.1 to make a tensor-representation of the algebra possible, taking the tensor to be fully symmetric will be shown to introduce a Hilbert space structure and, by that, a measure on the topological space in Section 3.2. If the algebra generated by the tensor does not have an exact unit, this is then interpreted in Section 3.3 as including geometric information into the tensor, yielding the potential reconstruction of a full Riemannian manifold.

3.1 Tensors, algebras and topology

In this section, a tensor describing a product operation of an abstract algebra will be introduced. This (possibly infinite-dimensional) tensor naturally exists for any algebra $(\mathcal{A}, \cdot)$, and since it completely fixes the algebra it is an interesting object to consider. First, it will be shown how to, in quite general cases, define this tensor for an algebra. After that, the other way around will be explored. Since not every tensor immediately

corresponds to an associative algebra, it is necessary to define the notion of associative closure in order for the algebra to correspond to a pointwise product.

For the real continuous functions over a compact space,[1] one can consider the linear dual space $C(X)^*$ of all bounded linear functionals $\mu : C(X) \to \mathbb{R}$. The Singer-Riesz representation theorem for Banach spaces now states the following [74, 75, 76, 77]:

Theorem 3.1. ***The Singer-Riesz representation theorem.*** *For every bounded linear functional on $C(X)$, there exists a unique (signed) Radon measure μ on X such that the functional can be represented as*

$$\mu(f) = \int_X f(x)\mathrm{d}\mu(x).$$

This theorem will be useful later on.

3.1.1 Tensors corresponding to functional algebras

Consider a compact topological space X, and the real continuous functions on this space $C(X)$. Since here we consider compact spaces, the real continuous functions have a (sensible) norm; namely for $f \in C(X)$

$$||f|| = \sup\{|f(x)| \,|\, x \in X\}. \tag{3.1}$$

The continuous functions form an algebra by a pointwise definition (hereafter called pointwise algebras). This product operation

$$\cdot : C(X) \times C(X) \to C(X),$$

is defined for every $x \in X$ as

$$(f \cdot g)(x) = f(x)g(x), \tag{3.2}$$

utilising the usual product on the real numbers $\mathbb{R}$. Because this algebra is defined pointwise, and the product on the real numbers has these properties, the algebra is commutative and associative. Furthermore, since the unit function is a continuous function, it is also a unital algebra. An important fact about these algebras is that they are seperable. From now on, abstract Banach $\mathbb{R}$-algebras $\mathcal{F}$ will be discussed which are unital, associative, commutative and seperable.[2] In Chapter 2 is already explained how to reconstruct the topological space X from only the abstract algebra.

In order to arrive at a tensor-representation of this algebra, it is useful to introduce the canonical product map.

$$\begin{aligned} P : \mathcal{F} \times \mathcal{F} &\to \mathcal{F}, \\ P(f, g) &\to f \cdot g. \end{aligned} \tag{3.3}$$

From the definition of the product, by construction, this is a symmetric bilinear map. While this map is trivial in the sense that it acts exactly as the operator $\cdot$, it will be more straightforward to link this to a tensor.

[1] Or continuous functions with compact support on locally compact spaces.

[2] In the finite dimensional case, assuming a Banach space structure is not strictly necessary, since the main element used here is the existence of a countable (Schauder) basis. But for the finite-dimensional case the Hamel basis is already finite. However, in the construction of the associative closure, we will formally assume this finite basis to be a subset of a Schauder basis.

Let us denote a Schauder basis of the algebra $\mathcal{F}$ by $\{f_a\}$. It is known that there is a dual linearly independent set of elements in the linear dual space $\{\alpha^a\} \subset \mathcal{F}^*$, such that they form a bi-orthogonal system [78], i.e.

$$\alpha^a(f_b) = \delta^a_b.$$

Using this, one can decompose any given function $f \in \mathcal{F}$ in terms of the basis f_a, and explicitly find its decomposition by using these elements $\{\alpha^a\}$

$$f = \sum_{a\geq 1} \alpha^a(f) f_a.$$

Now, since the product of two elements of the algebra is again an element of the algebra, one can decompose this resulting element in terms of this basis too

$$\mathcal{P}(f,g) = f \cdot g = \sum_{c\geq 1} \alpha^c(f \cdot g) f_c.$$

Using the bilinearity of the product (or the linearity of α^c), one can fully decompose this as

$$\mathcal{P}(f,g) = \sum_{a\geq 1}\sum_{b\geq 1}\sum_{c\geq 1} \alpha^a(f)\alpha^b(g)\alpha^c(f_a \cdot f_b) f_c. \tag{3.4}$$

This shows that the product operator $\mathcal{P}$ is fully determined by the set of scalars

$$P_{ab}{}^c := \alpha^c(f_a \cdot f_b) \in \mathbb{R},$$

as one can reconstruct any product $f \cdot g$ using Eq. (3.4). Thus, given this (countably infinite) set of numbers and assuming the existence of a Schauder-basis of the underlying vectorspace $\mathcal{F}$, one can thus reconstruct every product of the algebra $(\mathcal{F}, \cdot)$, and thus using the methods developed earlier reconstruct the topological space. These numbers may be recognised to be a tensor, acting as

$$P : \mathcal{F} \times \mathcal{F} \times \mathcal{F}' \to \mathbb{R},$$

where $\mathcal{F}' = \overline{\text{span}\{\alpha^a\}} \subseteq \mathcal{F}^*$, where $\mathcal{F}' = \mathcal{F}^*$ if $\mathcal{F}$ is reflexive.[3] The fact that this is a tensor, i.e. a multilinear map, is due to the bi-linearity of $\mathcal{P}$ and the linearity of the α's.

Note that the scalars $P_{ab}{}^c$ may also be interpreted as the structure constants of the algebra, since

$$f_a \cdot f_b = \sum_{c\geq 1} P_{ab}{}^c f_c. \tag{3.5}$$

Some interesting properties of the maps above will now be considered due to the structure of the $\mathbb{R}$-algebra, besides the multi-linearity.[4] Firstly, since the product is commutative the map $\mathcal{P}$ is symmetric,

$$\mathcal{P}(f,g) = (f \cdot g) = (g \cdot f) = \mathcal{P}(g,f),$$

which translates to the tensor P as a symmetry in the first two indices

$$P_{ab}{}^c = P_{ba}{}^c \tag{3.6}$$

[3] Later, when Riemannian manifolds are considered, a Hilbert space structure will be introduced which makes this reflexive.

[4] Though the multi-linearity ultimately is also due to the definition of the algebra.

Secondly, the $\mathbb{R}$-algebra product is associative. This translates to the property that

$$\mathcal{P}(\mathcal{P}(f,h),g) = ((f \cdot h) \cdot g) = (f \cdot (g \cdot h)) = \mathcal{P}(f,\mathcal{P}(g,h)).$$

For the tensor P, one may define a new object Γ describing the non-associativity of the tensor P, which for associative algebras equals zero

$$\Gamma_{abc}{}^{d} := \alpha^{d}((f_a \cdot f_b) \cdot f_c - f_a \cdot (f_b \cdot f_c)) = P_{ab}{}^{e} P_{ec}{}^{d} - P_{bc}{}^{e} P_{ae}{}^{d} = 0. \tag{3.7}$$

Lastly, the $\mathbb{R}$-algebra has a unit, this means that there exists an element $1 = \sum_{a\geq 1} \gamma^a f_a$ such that, for all $f \in \mathcal{F}$,

$$1 \cdot f = f,$$

in particular for every element f_a of the basis. For the tensor P this translates to

$$\begin{aligned} 1 \cdot f_b = \sum_{a\geq 1} \gamma^a (f_a \cdot f_b) = \sum_{a\geq 1}\sum_{c\geq 1} \gamma^a P_{ab}{}^{c} f_c \overset{!}{=} f_b, \\ \Rightarrow \gamma^a P_{ab}{}^{c} = \delta^c_b \end{aligned} \tag{3.8}$$

From now on, the opposite approach will be taken, where one assumes a tensor $P_{ab}{}^{c}$ and the generated algebra is given by the vectorspace $\mathcal{F} = \mathrm{span}\{f_a\}$ and the product operation defined by the tensor $P_{ab}{}^{c}$

$$f_a \cdot f_b = P_{ab}{}^{c} f_c.$$

Note that one does not need to assume the vectorspace to have a norm or to be closed, as with the algebra one can already define the dual space $|\mathcal{F}|$ with a topology [63]. Note that as long as the dual space is compact, or the algebra represents functions with compact support, one can reconstruct the Banach space structure by taking the norm of Eq. (3.1). Similarly to the names of algebras, a tensor $P_{ab}{}^{c}$ is called commutative if Eq. (3.6) holds, associative if Eq. (3.7) holds and unital if there is an element γ^a such that Eq. (3.8) holds.

Note that the algebraic dual space (see Definition 2.3) of a tensor $P_{ab}{}^{c}$ is given by elements $p \in \mathcal{F}^*$ which satisfy

$$p_a p_b = \sum_{c\geq 1} P_{ab}{}^{c} p_c, \tag{3.9}$$

where the notation $p_a \equiv p(f_a)$ is often used throughout this thesis. Note that the full homomorphism p in this notation is given by $p = \sum_{a\geq 1} p_a \alpha^a$.

Example 3.1. The tensor corresponding to a circle. In Examples 2.13 and 2.14, the algebra of smooth functions on the circle has been examined already. Here, a tensor will be constructed that corresponds to this algebra. Taking the basis of the algebra defined in Eq. (2.7), one can evaluate the pointwise product. For instance, a well known geometric identity gives

$$f_2 \cdot f_2 = \frac{1}{\pi}\sin(x)\cdot\sin(x) = \frac{1}{2\pi}\left(1-\cos(2x)\right) = \frac{1}{\sqrt{2\pi}} f_1 - \frac{1}{2\sqrt{\pi}} f_5.$$

This product determines the following elements of P

$$P_{22}{}^{1} = \frac{1}{\sqrt{2\pi}},\ P_{22}{}^{5} = -\frac{1}{2\sqrt{\pi}},\ P_{22}{}^{a} = 0 \text{ (for } a \neq 1,5).$$

By considering the general product rules

$$\begin{aligned}\sin(n\,x)\cdot\sin(m\,x) &= \frac{1}{2}\left(\cos\left((m-n)x\right)-\cos\left((m+n)x\right)\right),\\ \cos(n\,x)\cdot\sin(m\,x) &= \frac{1}{2}\left(\sin\left((m-n)x\right)+\sin\left((m+n)x\right)\right),\\ \cos(n\,x)\cdot\cos(m\,x) &= \frac{1}{2}\left(\cos\left((m-n)x\right)+\cos\left((m+n)x\right)\right),\end{aligned} \tag{3.10}$$

all elements of ${P_{ab}}^c$ may be found.

The idea is now to use this tensor ${P_{ab}}^c$ as the fundamental variable of the theory. While one might be tempted to be satisfied with this, after all there is a clear formalism relating tensors to topological spaces, there is some work to be done to make this useful for a quantum theory of gravity:

- Even if Eq. (3.7) holds for certain tensors, quantum perturbations might affect the tensor such that it is not associative anymore. As pointwise-defined algebras are necessarily associative, this needs to be treated carefully. In the following sections, the algebra will be linked to a bigger associative algebra, called the *associative closure* of the tensor ${P_{ab}}^c$.

- The prominent theory that this formulation is built for uses a totally symmetric tensor P_{abc}, as opposed to the tensor ${P_{ab}}^c$ which is only assumed to be symmetric in the first two indices. In Section 3.2 this is solved by assuming a Hilbert space structure, which connects commutative algebras to completely symmetric tensors P_{abc}, and has the interpretation of generating a measure on the dual space.

- Similarly to the first point, not every algebra has a well-defined unit as in Eq. (3.8). In Section 3.3 it will be shown that the deviation of the unit may be used to add the full geometric information of a Riemannian manifold to the tensor.

Note that similarly to the first point, one might also be worried about commutativity, but restricting the configuration space of tensors to symmetric tensors solves this issue.

3.1.2 Partial algebras and associative extensions

In this section, the notions of a *partial algebra* and *associative extension* will be introduced. In order to understand the motivation for the introduction of a partial algebra, the example of the circle will first be explained more.

Example 3.2. Finite-dimensional circle algebra. The circle, already discussed in Examples 2.13, 2.14 and 3.1, is a one-dimensional smooth manifold and thus has an algebra of smooth functions that is infinite-dimensional. In the following, this algebra will be restricted so that there is only a five-dimensional tensor left, to in the end recover the original algebra $C^\infty(S^1)$.

In order to at least contain the information of the products of the generating set of the algebra, given by $\{f_1, f_2, f_3\}$, one can see from Eq. (3.10) that one at least needs to take into account the tensor ${P_{ab}}^c$ with labels reaching up to $N = 5$.

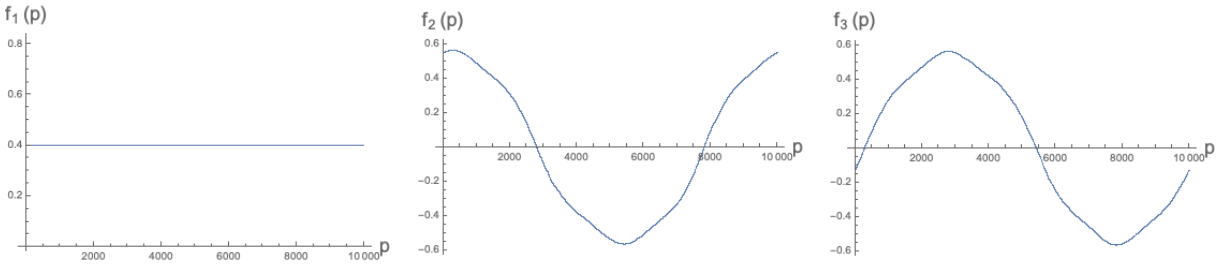

Figure 3.1: $10,000$ homomorphisms of the first three functions f_1, f_2, f_3 of the circle, plotted as the list $p^i(f_a)$ using the arranging of the points as described in Example 3.2. The homomorphisms where calculated using Mathematica by minimising $(p_a p_b - \sum_{c=1}^{5} P_{ab}{}^c p_c)^2$ and only keeping the ones that evaluated to zero (excluding the trivial zero-map). Note that the first point is arbitrary, thus there is an arbitrary shift. Images taken from [3].

It is good to note that once one restricts the algebra to finite dimensions, the algebra will become non-associative. In this case, one example of a non-associative product is

$$f_2 \cdot (f_3 \cdot f_4) = f_2 \cdot \sum_{c=1}^{5} P_{34}{}^c f_c = \frac{1}{2\sqrt{\pi}} f_2 \cdot f_2 = \frac{1}{2\sqrt{2}\pi} f_1 - \frac{1}{4\pi} f_5,$$
$$(f_2 \cdot f_3) \cdot f_4 = \sum_{c=1}^{5} P_{23}{}^c f_c \cdot f_4 = \frac{1}{2\sqrt{\pi}} f_4 \cdot f_4 = \frac{1}{2\sqrt{2}\pi} f_1.$$

This non-associativity is due to finite cutoff $N = 5$ that was introduced, as for $C^\infty(S^1)$

$$\sin(\theta)(\cos(\theta)\sin(2\theta)) = \frac{1}{2}\sin(\theta)(\sin(\theta) + \sin(3\theta)) = \frac{1}{4}(1 - \cos(4\theta)),$$
$$(\sin(\theta)\cos(\theta))\sin(2\theta) = \frac{1}{2}\sin(2\theta)\sin(2\theta) = \frac{1}{4}(1 - \cos(4\theta)).$$

In the finite-dimensional case the $\sin(3\theta)$ contribution "drops out", because of which the product is not associative anymore since $f_5 \sim \cos(2\theta)$ should not be present.

If one now wants to find "homomorphisms" that are only homomorphisms for a subset $\mathcal{S} \cong \{f_1, f_2, f_3\}$ of the algebra, one could define them as

$$p \in \mathcal{F}^* \cong \mathbb{R}^5,\ a, b \in \{1, 2, 3\} :\ p_a p_b = \sum_{c=1}^{5} P_{ab}{}^c p_c. \tag{3.11}$$

Note that these are not homomorphisms of the algebra $\mathcal{F}$ generated by P, but only of a part of the algebra, namely $\mathcal{S}$, and they are elements of $\mathcal{F}^*$, not $\mathcal{S}^*$. Let us denote this "dual space" as $|\mathcal{S}|$, then it is still possible to introduce a topology in the same way as one would usually do for the algebraic dual space by demanding all elements of $\mathcal{S} = \{f_1, f_2, f_3\}$ to be continuous functions.

Solutions to Eq. (3.11) may be found by using a computer. In [3], Mathematica was used to calculate $10,000$ points in this way. These points are labeled by $p^i,\ i \in \{1, \ldots, 10,000\}$. In order to visualise the topology, consider the elementary definition of continuity,

$$\lim_{\epsilon \to 0^+} f_a(p) - f_a(p + \epsilon) \to 0.$$

For a discrete set of points, the most straightforward version of the continuous definition above is by arranging the points such that the absolute value of

$$f_a(p^i) - f_a(p^{i+1}) = p_a^i - p_a^{i+1}$$

is as small as possible. For this, one starts with a point, say p^1. Then, take p^2 to be the point that minimises $\sum_{a=1}^{5} |p_a^1 - p_a^2|^2$. This is then done for every point p^i, such that

$$\sum_{a=1}^{5} |p_a^i - p_a^{i+1}|^2 \tag{3.12}$$

will be minimised. The result of this is shown in Fig. 3.1, where the resemblance, up to deformations, with the functions $\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\sin(x), \frac{1}{\sqrt{\pi}}\cos(x)\}$ is clear.

This example justifies the following definitions.

Definition 3.1. Consider a vectorspace $\mathcal{F}$. A sub-vectorspace $\mathcal{S} \subset \mathcal{F}$ with a bilinear product operator on $\mathcal{F}$, $\mathcal{P} : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, is called a *partial algebra*.

The partial algebra is unital if $\exists\, 1 \in \mathcal{S}, \forall f \in \mathcal{S} : \mathcal{P}(1, f) = f$. Similarly, the partial algebra is commutative if $\forall f, g \in \mathcal{S}, \mathcal{P}(f, g) = \mathcal{P}(g, f)$ and associative if $\forall f, g, h \in \mathcal{S}, \mathcal{P}(f, \mathcal{P}(g, h)) = \mathcal{P}(\mathcal{P}(f, g), h)$.

Note that for the partial algebra, for the commutativity and associativity conditions, the result of $\mathcal{P}(f, g)$ (and the others) can be lie outside of $\mathcal{S}$, in $\mathcal{F}$. This $\mathcal{P}$ is here taken to be generated by a tensor $P_{ab}{}^c$, as described in Section 3.1.1.

Definition 3.2. The *algebraic dual-space* of a *partial* algebra $\mathcal{S} \subset \mathcal{F}$ with operator $\mathcal{P}$, denoted $|\mathcal{S}|$, are all $p \in \mathcal{F}^*$ that are homomorphisms of the partial algebra in the sense that

$$|\mathcal{S}| := \{p \in \mathcal{F}^* | \, \forall f, g \in \mathcal{S},\ p\left(\mathcal{P}(f, g)\right) = p(f)p(g)\}\,. \tag{3.13}$$

The elements of this dual space are called *partial homomorphisms*.

There is an important difference between this definition, and the original definition of the algebraic dual space of an algebra in Definition 2.3, namely that the resulting product $\mathcal{P}(f, g)$ of two elements $f, g \in \mathcal{S}$ is only required to be an element of the bigger algebra $\mathcal{F}$, and thus may lie outside of $\mathcal{S}$. Furthermore, the elements $p \in |\mathcal{S}|$ are elements of $\mathcal{F}^*$, but not $\mathcal{S}^*$, in general. Note that these definitions properly define the notions used in Example 3.2.

In the case of Example 3.2, the partial algebra $\mathcal{S} \cong \mathrm{span}\{f_1, f_2, f_3\}$ "reached" every element of the full algebra $\mathcal{F} \cong \mathrm{span}\{f_1, \ldots, f_5\}$. The following definition serves to make this notion precise.

Definition 3.3. The *range* of a unital partial algebra is defined as the sub-vectorspace $\mathcal{K}^{(\mathcal{S})}$ of $\mathcal{F}$, where $\mathcal{S} \subseteq \mathcal{K}^{(\mathcal{S})} \subseteq \mathcal{F}$, which is reached by evaluating products of elements of $\mathcal{S}$.

$$\mathcal{K}^{(\mathcal{S})} := \{\mathcal{P}(f, g) | \, f, g \in \mathcal{S}\}.$$

A partial algebra is called *covering* if $\mathcal{K}^{(\mathcal{S})} = \mathcal{F}$.

Note that the partial homomorphisms of $\mathcal{S}$ are well-defined on $\mathcal{K}^*$ as projections $p|_{\mathcal{K}^*}$, since there they will be restricted by the action of the product. According to the definition above, the partial algebra of Example 3.2 is a covering partial algebra. The benefit of a covering partial algebra is that the partial homomorphisms are unambiguously defined on the whole dual space $\mathcal{F}^*$.

In order to be more general than only single partial algebras, the following notion is useful.

Definition 3.4. A *system* of partial algebras is a set of partial algebras $\{\mathcal{S}_i | \, i \in \mathcal{I}\}$ with ranges $\{\mathcal{K}_i | \, i \in \mathcal{I}\}$, where $\mathcal{I}$ is some index set, such that for every pair $(\mathcal{S}_i, \mathcal{S}_j)$ their dual spaces have a nontrivial intersection, i.e. $|S_i| \cap |S_j| \neq \emptyset$. The range of the system is defined as the union of the ranges of the partial algebras

$$\mathcal{K}^{(\{\mathcal{S}_i\})} := \bigcup_{\{i \in \mathcal{I}\}} \mathcal{K}^{(\mathcal{S}_i)}.$$

The system $\{\mathcal{S}_i\}$ is called *maximal* if there is no system, say $\{\mathcal{T}_j\}$, with a larger range, i.e. $\mathcal{K}^{(\{\mathcal{S}_i\})} \subset \mathcal{K}^{(\{\mathcal{T}_j\})}$. This system is said to be *covering* $\mathcal{F}$ if its range spans the whole algebra, i.e. $\mathcal{K}^{(\{\mathcal{S}_i\})} = \mathcal{F}$. The *algebraic dual-space* of a system of partial algebras $\{\mathcal{S}_i\}$ is the intersection of all of the dual spaces, i.e.

$$|\{\mathcal{S}_i\}| := \bigcap_i |\mathcal{S}_i|.$$

Now, all ingredients needed are present to define the *associative extension*, which will be the basis of the associative closure later on.

Definition 3.5. An *associative extension* of a tensor ${P_{ab}}^c$ acting on a vectorspace $\mathcal{F}$ with basis $\{f_a\}$ and dual elements $\{\alpha^a\}$, is an algebra $(\mathcal{A}, \cdot)$, consisting of a seperable Banach space $\mathcal{A}$, which is an extension of $\mathcal{F} \subset \mathcal{A}$, and a product operation $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfying:

1. The algebra is unital, associative and commutative.
2. The product operation $\cdot$ reduces to ${P_{ab}}^c$ on $\mathcal{F}$ in the sense that

$$\forall f_a, f_b \in \{f_a\}, \, \alpha^c \in \{\alpha^c\} \, : \, {P_{ab}}^c = \alpha^c(f_a \cdot f_b), \tag{3.14}$$

3. Every element of the algebraic dual space $p \in |\mathcal{A}|$ projected to $\mathcal{F}^*$, i.e. $p|_{\mathcal{F}^*}$, is contained in the dual space of some maximal system of partial algebras $|\{\mathcal{S}_i\}| \subset \mathcal{F}^*$. Furthermore this projection is injective.

The definition above has several requirements that have either mathematical or physical reasons. Firstly, the fact that the algebra should be unital, associative and commutative is simply because the associative extension $\mathcal{A}$ is supposed to describe a pointwise algebra which naturally has these properties. Secondly, the main information about the algebra one has is given by the tensor ${P_{ab}}^c$, so the second requirement ensures this information is present. The last condition has two main reasons for being included in this way. Physically, it is important that the functions that are measured, basically the partial algebra $\mathcal{S}$,[5] do not change their form once the extension of the algebra is considered. Therefore, the elements of the dual space of this associative extension have to be proper extensions of the partial homomorphisms. The second reason relates to the injectivity requirement, as one does not want to consider algebras that are not connected to the algebra generated by ${P_{ab}}^c$.

[5] Or system of partial algebras

Example 3.3. Partial algebras and associative extension of the circle. In Example 3.1, an example of a covering unital commutative associative partial algebra $\mathcal{S}$ of the five-dimensional circle algebra was already given. There are other partial algebras possible as well.

Consider for instance the partial algebra $\mathcal{S}_1 = \mathrm{span}\{f_1, f_2\}$. This partial algebra has range

$$\mathcal{K}_1 = \mathrm{span}\{f_1, f_2, f_5\},$$

and is thus not a covering partial algebra. Another example is a one-parameter family of partial algebras $\mathcal{S}_\alpha = \mathrm{span}\{f_1, \sin(\alpha) f_2 + \cos(\alpha) f_3\}$, for $\alpha \in [0, 2\pi)$, with ranges

$$\mathcal{K}_\alpha = \mathrm{span}\{f_1, \sin(\alpha) f_2 + \cos(\alpha) f_3, 2\sin(\alpha)\cos(\alpha) f_4 + (\cos(\alpha)^2 - \sin(\alpha)^2) f_5\}.$$

Note that this reduces to $\mathcal{S}_1$ if $\alpha = \pi/2$. Using this one-parameter family, one can construct a system of partial algebras $\{\mathcal{S}_\alpha\}$ which is actually also a covering set of partial algebras. Note that $\bigcup_\alpha \mathcal{S}_\alpha = \mathcal{S}$, and in particular $\mathcal{K}^{\{S_\alpha\}} = \mathcal{K}$ and $|\{\mathcal{S}_\alpha\}| = |\mathcal{S}|$.

Let us prove that the smooth functions over the circle are an associative extension of the five-dimensional algebra $\mathcal{F}$.

Proposition 3.1. *Consider the algebra $\mathcal{F} \cong \mathrm{span}\{f_1, \ldots, f_5\}$ with the tensor ${P_{ab}}^c$ acting on it as in Example 3.1, $\mathcal{A} \cong C^\infty(S^1)$ is an associative extension of this algebra.*

Proof. Since the algebra $\mathcal{F}$ is simply the span of the first few elements of the algebra, $\mathcal{A}$ is naturally an extension, and it is known already to be a Banach space. Furthermore the algebra is unital, associative and commutative, and the tensor ${P_{ab}}^c$ is defined exactly in the same way as required by Definition 3.5. Therefore the only thing that needs to be proven is that the third condition of Definition 3.5 is satisfied.

To do this, one needs to show that every $p \in |C^\infty(S^1)|$, projected to $\mathcal{F}^*$, is an element of the dual space of a maximal partial algebra. For this, one can take the partial algebra $\mathcal{S} \cong \mathrm{span}\{f_1, f_2, f_3\}$. However, this can easily be seen, since this is equivalent to checking whether

$$f_a(p) f_b(p) = \sum_{c=1}^{5} {P_{ab}}^c f_c(p),$$

for $a, b \leq 3$. This is exactly the case by construction, since all information for the products of $\{f_1, f_2, f_3\}$ is contained in ${P_{ab}}^c$. □

3.1.3 The associative closure

The associative extension defined in Section 3.1.2 links an associative algebra to a tensor ${P_{ab}}^c$ which does not directly generate an associative algebra. In Example 3.2 the example of a five-dimensional tensor representing the low-energy limit of the algebra of the circle.

The problem with the associative extension is that, while the smooth algebra of functions indeed is an associative extension, there are many other extensions. For instance, there are many associative extensions of the five-dimensional algebra mentioned above which have finite dimension, though the associative algebra that is looked for, $C^\infty(S^1)$, is infinite-dimensional. This final algebra will be called the "associative closure".

When looking for a candidate of such an algebra, it is useful to look for the dual space of this candidate, since one would like this dual space to be as large as possible in order to come closest to the notion of a continuous space. The strategy to define the associative closure, is to "merge" all the dual spaces of associative extensions together. The following notion will be useful for this.

Definition 3.6. Let $p \in |\mathcal{A}|$ for any associative extension $(\mathcal{A}, \cdot)$ of ${P_{ab}}^c$. The projection of p to the linear dual space of $\mathcal{F}$,

$$p^* \equiv p|_{\mathcal{F}^*},$$

is called a *potential homomorphism* of ${P_{ab}}^c$.

The *space of potential homomorphisms* of $\mathcal{F}$, denoted $|\mathcal{F}|^{(P)}$, is the collection of all homomorphisms of all associative extensions projected to $\mathcal{F}^*$.

The key point of this definition is to allow one to not only consider the homomorphisms of $\mathcal{F}$ under P, but consider all homomorphisms of some bigger algebra that could generate P. The space of potential homomorphisms may then be thought of as the actual space of points. The associative closure may then finally be defined.

Definition 3.7. An associative closure of a tensor ${P_{ab}}^c$ acting on $\mathcal{F}$ is an associative extension $(\mathcal{A}, \cdot)$ of ${P_{ab}}^c$ such that the restriction of its algebraic dual space to the linear dual space of $\mathcal{F}$ is exactly the space of potential homomorphisms of $\mathcal{F}$, i.e. $|\mathcal{A}||_{\mathcal{F}^*} = |\mathcal{F}|^{(P)}$.

One can readily see that this lemma holds.

Lemma 3.1. *The algebraic dual space of an algebraic closure is isomorphic to the space of potential homomorphisms.*

Proof. Since the projection map is surjective for the associative closure, and injectivity is already assumed in the definition of the associative extension, the projection map is an isomorphism. □

With this definition, the associative closure essentially is the associative extension with the largest possible dual space. Note that while the associative closure is not necessarily unique, this is physically not really a problem. The underlying assumption is that the functions one can actually measure are the functions included in $\mathcal{F}$, and the physical space is the space of potential homomorphisms. This space will be exactly the same for every associative closure. The associative closure is in that sense mainly a mathematical tool to be sure that there exists a topology on the space of potential homomorphisms.

In the following will be proved that the algebra $C^\infty(S^1)$ is an associative closure of the algebra of Example 3.2.

Example 3.4. The associative closure, $C^\infty(S^1)$, for the circle. In this example it will be shown that the smooth functions $C^\infty(S^1)$ are an associative closure for the five-dimensional tensor from Example 3.2.

For this, the following lemma will be useful.

Lemma 3.2. *The space of potential homomorphisms is included in the dual space of the partial algebra* $\mathcal{S} = \text{span}\{f_1, f_2, f_3\}$. *I.e.*

$$|\mathcal{F}|^{(P)} \subseteq |\mathcal{S}|.$$

Proof. Note that the cases that $\mathcal{S}' \supset \mathcal{S}$, the requirement for elements of $|\mathcal{S}'|$ will be at least as restrictive as for $|\mathcal{S}|$, and thus in those cases the statement is trivial since then $|\mathcal{S}'| \subset |\mathcal{S}|$. Furthermore, a two-dimensional partial algebra can never cover a five-dimensional vectorspace. Therefore, the most important type of partial algebra to consider are three-dimensional ones. There are two possibilities, either the partial algebra overlaps with $\mathcal{S}$, or it does not.

Let us assume the latter. In that case the partial algebra has the form

$$\mathcal{W} = \text{span}\{f_1, f_4, f_5\},$$

but note that this gives

$$f_4 \cdot f_4 \sim f_5 \cdot f_5 \sim f_1,$$
$$f_4 \cdot f_5 = 0,$$

so this partial algebra is not a covering algebra. F

The most general form of three-dimensional partial algebras that overlap with $\mathcal{S}$ is

$$\mathcal{V} = \text{span}\{f_1, g = \alpha f_2 + \beta f_3, h = \gamma f_4 + \delta f_5\},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. However, one wants partial algebras that are associative. One could construct the Γ-tensor of Eq. (3.7) and see for which triplets of vectors this would evaluate to zero. However, it is instructive to simply calculate the product here and see which products are associative. If one does this exercise to compare $(g \cdot g) \cdot h$ with $g \cdot (g \cdot h)$, one gets

$$(g \cdot g) \cdot h = \frac{\delta(\beta^2 - \alpha^2) + 2\alpha\beta\gamma}{2\sqrt{2}\pi} f_1 + \frac{(\alpha^2 + \beta^2)\gamma}{2\pi} f_4 + \frac{(\alpha^2 + \beta^2)\delta}{2\pi} f_5,$$

whereas

$$g \cdot (g \cdot h) = \frac{\delta(\beta^2 - \alpha^2) + 2\alpha\beta\gamma}{2\sqrt{2}\pi} f_1 + \frac{(\alpha^2 + \beta^2)\gamma}{4\pi} f_4 + \frac{(\alpha^2 + \beta^2)\delta}{4\pi} f_5.$$

Comparing the two equations above, it is clear that this can only be associative if either $\alpha = \beta = 0$ or $\gamma = \delta = 0$, but then the case reduces to a two-dimensional non-covering algebra.

So there is no other associative covering partial algebras.

Therefore, every potential homomorphism has to be included in $|\mathcal{S}|$, since they cannot come from other partial algebras. □

Proposition 3.2. $C^\infty(S^1)$ *is an associative closure of the five-dimensional tensor* ${P_{ab}}^c$ *of Example 3.2.*

Proof. As has already been proven in Proposition 3.1, $C^\infty(S^1)$ is an associative extension of ${P_{ab}}^c$. What one needs to show is that every potential homomorphism of ${P_{ab}}^c$ is an element of $|C^\infty(S^1)||_{\mathcal{F}^*}$.

Using Lemma 3.2, it is clear that every potential homomorphism must lie in $|\mathcal{S}|$. This means that, if $|C^\infty(S^1)||_{\mathcal{F}^*} = |\mathcal{S}|$, then necessarily every potential homomorphism lies in $|C^\infty(S^1)||_{\mathcal{F}^*}$ and thus $C^\infty(S^1)$ is an associative closure.

To confirm this, take an element $p \in |\mathcal{S}|$. From Definition 3.2, this is any $p \in \mathbb{R}^5$ such that $\forall a, b \leq 3$

$$p_a p_b = \sum_{c=1}^{5} P_{ab}{}^c p_c.$$

For p_1, p_2, p_3 this already fixes the components as evaluation maps of $f_1 \sim 1, f_2 \sim \sin(x), f_3 \sim \cos(x)$. This is because the evaluation maps of these functions are precisely all the homomorphisms $p : C^\infty(S^1) \to \mathbb{R}$, and since all the information of their products is contained within the five-dimensional tensor P_{abc}, the solutions to the above equation are nothing more but the restriction of these homomorphisms to $\mathbb{R}^3$, so $|C^\infty(S^1)||_{\mathbb{R}^3} = |\mathcal{S}||_{\mathbb{R}^3}$. One now needs to check the last two components, p_4 and p_5, and show that these correspond to restrictions of the full $C^\infty(S^1)$ homomorphisms.

However, since the $C^\infty(S^1)$ algebra is defined pointwise, this is necessarily the case. For instance, if one takes the case of f_5. The product defined by $P_{22}{}^c$ gives

$$f_5 = \sqrt{2} f_1 - 2\sqrt{\pi} f_2 \cdot f_2,$$

and taking the same homomorphism $p \in \mathbb{R}^5$

$$f_5(p) = p_5 = \sqrt{2} p_1 - 2\sqrt{\pi} p_2 p_2 = \sqrt{2} f_1(p) - 2\sqrt{\pi} f_2(p) f_2(p),$$

one gets a pointwise definition of the function $f_5 \sim \cos(2x)$. A similar statement holds for $f_4 \sim \sin(2x)$. Since every homomorphism of $C^\infty(S^1)$ necessarily respects this, we see that actually

$$|C^\infty(S^1)||_{\mathcal{F}^*} = |\mathcal{S}|.$$

□

3.2 A measure from a Hilbert space

In this section, the structure introduced in Section 3.1 will be extended. In Section 3.1, a Banach space structure was always assumed. In the case of a (locally) compact Hausdorff space, this is a fair assumption to make. For a (canonical) theory of gravity however, one would like to ideally describe Riemannian manifolds, or at least spaces that look like it. Furthermore, this section will put the notation of the tensor $P_{ab}{}^c$ more in line with the canonical tensor model assumption that the main tensor of interest is a totally-symmetric tensor P_{abc}. After this, the construction of an associative closure will be discussed using a space of potential homomorphisms, after which it will be explained how these potential homomorphisms may be generated.

3.2.1 Hilbert space, totally symmetric tensors and a measure

In Section 2.1 was already explained that the space of compactly supported smooth functions on a Riemannian manifold has a natural inner product, as defined in Eq. (2.3). Therefore, assuming an inner product structure on the algebra $\mathcal{A}$ seems like a logical

first step towards describing a Riemannian manifold in terms of a tensor. Here, it will be shown that in this case the natural measure on a Riemannian manifold,

$$\mu(f) := \int_{|\mathcal{A}|} \mathrm{d}^d x \sqrt{|g|} f(x),$$

may be reconstructed. In the following, the vectorspace $\mathcal{A}$ with basis $\{f_a\}_{a\geq 1}$ will be assumed with an inner product

$$\langle f_a | \, f_b \rangle = \delta_{ab}.$$

First, let us note that for a Hilbert space $\mathcal{A}$, the Riesz representation theorem of Theorem 3.1 has an even stronger notion [69], namely that for every $\alpha \in \mathcal{A}^*$, there is a $g \in \mathcal{A}$ such that for every $f \in \mathcal{A}$

$$\alpha(f) = \langle g | \, f \rangle \, .$$

Specifically, for a real measured space of functions (like the functions on a Riemannian manifold), this means that there is a single measure μ such that

$$\alpha(f) = \langle g | \, f \rangle = \int_{|\mathcal{A}|} \mathrm{d}\mu(p) f(p) g(p),$$

where the measure $\mathrm{d}\mu(p) = \mathrm{d}^d x \sqrt{|g(x)|}$ for Riemannian manifolds, with x the local coordinate representation of p. Let us define the product operator $P_g : \mathcal{A} \to \mathcal{A}$ of a function $g \in \mathcal{A}$ as follows

$$P_g(f) = f \cdot g.$$

This operator is self-adjoint under the inner product, as may be seen as follows

$$\langle h | \, P_g(f) \rangle = \langle h | \, f \cdot g \rangle = \int_{|\mathcal{A}|} \mathrm{d}\mu(p) h(p) f(p) g(p) = \langle h \cdot g | \, f \rangle = \langle P_g(h) | \, f \rangle \, .$$

Note that the definition of ${P_{ab}}^c$ may now be rewritten to

$${P_{ab}}^c := \alpha^c(f_a \cdot f_b) = \int_{|\mathcal{A}|} \mathrm{d}\mu(p) f_a(p) f_b(p) f_c(p). \tag{3.15}$$

From this expression it is clear that the tensor ${P_{ab}}^c$ is totally symmetric. More abstractly, this comes from the fact that the product operator $P_g(f)$ above is self-adjoint, so for an abstract algebra one would like (or impose) this to be the case. If one now writes

$$g_{cd} := \langle f_d | \, f_c \rangle \, , \tag{3.16}$$

and

$$P_{abc} := g_{cd} {P_{ab}}^d,$$

one arrives at the totally-symmetric tensor of degree three that is used in the canonical tensor model. This implies that the restriction of the tensor P_{abc} to have this form in the model, together with the total symmetric requirement, implies that the tensor actually represents a product on an inner-product space. This seemingly simple, and almost trivial, requirement actually makes the interpretation of a measured space possible.

From here on, when it is clear from the context that the Hilbert space is considered, the Einstein summation convention will be assumed for lower indices as well unless specified otherwise. This means that for instance,

$$P_{abc} f_c \equiv \sum_{c \leq 1} P_{abc} f_c.$$

It is useful to separately define the associative extension and closure of a tensor P_{abc} with three lower indices, just to be clear that the Hilbert space structure is assumed for tensors with all-lower indices, while only a Banach space is assumed for tensors of the form $P_{ab}{}^{c}$.

Definition 3.8. An *associative extension* of a tensor $P_{abc} := \delta_{cd} P_{ab}{}^{d}$, is an associative extension as in Definition 3.5 which is a Hilbert space. Here,

$$P_{abc} = \langle f_c |\, f_a \cdot f_b \rangle \,.$$

Definition 3.9. A *potential homomorphism* of a tensor P_{abc} acting on $\mathcal{F}$ is a homomorphism of an associative extension $(\mathcal{A}, \cdot)$ of P_{abc} projected to $\mathcal{F}^*$. The *space of potential homomorphisms* of $\mathcal{F}$, denoted $|\mathcal{F}|^{(P)}$, is the collection of all homomorphisms of all associative extensions projected to $\mathcal{F}^*$.

Definition 3.10. An associative closure of a tensor P_{abc} acting on $\mathcal{F}$ is an associative extension $(\mathcal{A}, \cdot)$ of P_{abc} such that the restriction of its algebraic dual space to the linear dual space of $\mathcal{F}$ is exactly the space of potential homomorphisms of $\mathcal{F}$, i.e. $|\mathcal{A}||_{\mathcal{F}^*} = |\mathcal{F}|^{(P)}$.

In order to fully define the relationship to a measure, the following definition is useful.

Definition 3.11. A tensor P_{abc} acting on $\mathcal{F}$ is said to be *measure-generated* if there is an associative closure $\mathcal{A}$ equipped with a measure μ on $|\mathcal{A}|$ such that

$$\forall f, g \in \mathcal{A} : \langle f |\, g \rangle = \int_{|\mathcal{A}|} \mathrm{d}\mu(p) f(p) g(p). \tag{3.17}$$

With the following proposition can be seen that this is equivalent to the existence of an associative closure.

Proposition 3.3. *A unital tensor P_{abc} is measure-generated iff there is an associative closure of P_{abc}.*

Proof. If there is an associative closure of P_{abc}, then P_{abc} may be written as

$$P_{abc} = \langle f_c |\, f_a \cdot f_b \rangle = \int_{|\mathcal{A}|} \mathrm{d}\mu(p) f_a(p) f_b(p) f_c(p).$$

In particular, if one uses the bi-linearity of the inner product one can show that, for $f = \alpha^a f_a$, $g = \beta^b f_b$ and $1 = \gamma^c f_c$

$$\langle f |\, g \rangle = \alpha^a \beta^b \gamma^c \langle f_c |\, f_a \cdot f_b \rangle = \int_{|\mathcal{A}|} \mathrm{d}\mu(p) f(p) g(p).$$

□

It is currently not known whether every tensor has an associative closure, but this is the current conjecture. The following statements utilise the isomorphism $|\mathcal{A}| \cong |\mathcal{F}|^{(P)}$.

Proposition 3.4. *For any $f, g \in \mathcal{F}$*

$$\langle f |\, g \rangle = \int_{|\mathcal{A}|} \mathrm{d}\mu(p) f(p) g(p) = \int_{|\mathcal{F}|^{(P)}} \mathrm{d}\mu(p) f(p) g(p),$$

Proof. $|\mathcal{F}|^{(P)}$ is the projection of homomorphisms $p \in |\mathcal{A}|$ to $\mathcal{F}^*$ by the definition of the associative closure, so for $f \in \mathcal{F}, p(f) = p^*(f)$, where p^* denotes the projection of p to $\mathcal{F}^*$. Since we are only considering functions in $\mathcal{F}$, and the spaces are isomorphic, we only have to consider $|\mathcal{F}|^{(P)}$. □

Corollary 3.1.1. *In particular, for any two basis elements f_a, f_b*

$$\delta_{ab} = \int_{|\mathcal{F}|^{(P)}} \mathrm{d}\mu(p) f_a(p) f_b(p). \tag{3.18}$$

Proposition 3.5. *A measure-generated unital tensor is given by*

$$P_{abc} = \int_{|\mathcal{F}|^{(P)}} \mathrm{d}\mu(p) f_a(p) f_b(p) f_c(p). \tag{3.19}$$

Proof. Take $\mathcal{A}$ to be an associative closure of P_{abc}. For an element of its algebraic dual space $p \in |\mathcal{A}|$ we have

$$p(f_a)p(f_b) = p(f_a \cdot f_b).$$

Furthermore, for any product of the basis function $f_a, f_b \in \mathcal{B}_{\mathcal{F}}$

$$f_a \cdot f_b = P_{abc} f_c + g,$$

where $P_{abc} f_c \in \mathcal{F}$ and $g \in \mathcal{F}^{\perp}$. This comes from the definition of an associative extension, such that $P_{abc} = \langle f_c |\, f_a \cdot f_b \rangle$, so $\langle f |\, g \rangle = 0$ for all $f \in \mathcal{F}$. In particular, for any $p \in |\mathcal{A}|$

$$p(f_a)p(f_b) = p(f_a \cdot f_b) = p(P_{abc} f_c) + p(g).$$

Since $p(f) = p^*(f)$ for $f \in \mathcal{F}$, where p^* denotes the projection of p to $\mathcal{F}^*$, we find

$$\begin{aligned}\int_{|\mathcal{F}|^{(P)}} \mathrm{d}\mu(p) p(f_a)p(f_b)p(f_c) &= \int_{|\mathcal{A}|} \mathrm{d}\mu(p) p(f_a)p(f_b)p(f_c), \\ &= \int_{|\mathcal{A}|} \mathrm{d}\mu(p) p(P_{abd} f_d)p(f_c) + \int_{|\mathcal{A}|} \mathrm{d}\mu(p) p(g)p(f_c).\end{aligned}$$

On the right-hand-side we have the inner product as defined in (3.17), so we get

$$\int_{|\mathcal{F}|^{(P)}} \mathrm{d}\mu(p) p(f_a)p(f_b)p(f_c) = P_{abd} \langle f_d |\, f_c \rangle + \langle g |\, f_c \rangle = P_{abc},$$

where the last inner product is zero since $g \notin \mathcal{F}$. □

Given the above mathematical notions, it is interesting to explore how one could construct the measure in practice. Note that simple functions play a central role in measure theory, and they will be useful here as well. First, assume that the basis functions may be written as simple functions. These simple functions are linear combinations of indicator functions $\mathbf{1}_{A_i}$, where $A_i \subset |\mathcal{F}|^{(P)}$ are measurable sets. Denoting R as the amount of disjoint regions A_i, the function is then given by

$$f_a = \sum_{i=1}^{R} p_a^i \mathbf{1}_{A_i}, \tag{3.20}$$

With the measure on $|\mathcal{F}|^{(P)}$ above one can now perform the integration

$$\int_{|\mathcal{F}|^{(P)}} \mathrm{d}\mu(p) f_a = \sum_{i=1}^{R} p_a^i \mu(A_i) \equiv \sum_{i=1}^{R} p_a^i \beta_i,$$

with $\beta_i \equiv \mu(A_i) > 0$. One assumption of the above is that all of the basis functions are decomposed in terms of the same indicator functions. This means that one can evaluate a product of two functions by simply writing

$$f_a \cdot f_b = \sum_{i=1}^{R} p_a^i p_b^i \mathbf{1}_{A_i}.$$

If one then integrates this quantity as shown above, one gets a representation of the inner product

$$\langle f_a |\, f_b \rangle = \int_{|\mathcal{F}|^{(P)}} \mathrm{d}\mu(p) f_a(p) f_b(p) = \sum_{i=1}^{R} \mu(A_i) p_a^i p_b^i \equiv \sum_{i=1}^{R} \beta_i p_a^i p_b^i. \tag{3.21}$$

Moreover, by using (3.19) one finds

$$P_{abc} = \sum_{i=1}^{R} \beta_i p_a^i p_b^i p_c^i. \tag{3.22}$$

This expression is a tensor rank decomposition of the tensor P_{abc}, and shows why the tensor rank decomposition has been so useful when applied to the canonical tensor model in the past. Below it will be conjectured that taking a positive minimal tensor rank decomposition, notions which will be defined below, corresponds to potential homomorphisms $p^i \in |\mathcal{F}|^{(P)}$ with a measure-value for the region they represent $\beta_i \in \mathbb{R}_+$.

Example 3.5. Constructing the measure of the circle. For the circle, it was found in Example 3.4 that the space of potential homomorphisms is actually the dual space of the partial algebra $\mathcal{S} \cong \mathrm{span}\{f_1, f_2, f_3\}$, $|\mathcal{S}|$. In Example 3.2 was already explained how the points of $|\mathcal{S}|$ may be found. In this example, these points will be used, and a consistent measure will be found for this space.

First note that, since the circle is one-dimensional, the measurable sets A_i of Eq. (3.20) are generated by line segments $[a_i, b_i)$, where $b > a$. Therefore, suppose one has a solution to Eqs. (3.21) and (3.22), then the β_i actually are describing the length of each line-segment $\beta_i = b_i - a_i$. If one assumes this approximation of the functions f_a as simple functions, the function may be written as

$$f_a(x) = \begin{cases} p_a^1 \,,\ 0 \leq x < \beta_1, \\ p_a^2 \,,\ \beta_1 \leq x < \beta_1 + \beta_2, \\ \dots \\ p_a^R \,,\ \sum_{i=1}^{R-1} \beta_i \leq x < \sum_{i=1}^{R} \beta_i. \end{cases} \tag{3.23}$$

These β_i may be found by finding solutions to Eqs. (3.21) and (3.22), for instance by using Mathematica. In [3] this exercise was performed using 1.000 randomly selected points of the 10.000 points from Example 3.2.The result of this procedure may be found in Fig. 3.2. Note that the circumference of the circle exactly matches 2π, and the functions look exactly like the functions used to construct the tensor. It should be noted that this is merely an approximation of the functions, but in the limit they will represent the original functions $\{1, \sin(x), \cos(x)\}$.

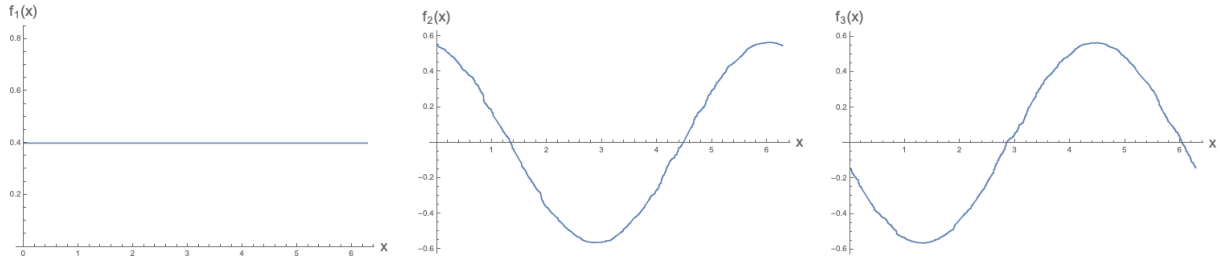

Figure 3.2: A plot of the first three basis functions of the circle, as given by Eq. (3.23), taken from [3]. As mentioned in the text, 1.000 points are used. Note that the deformations of Fig. 2.1 disappear due to the inclusion of the information of the measure.

In order to reconstruct the full measure, one needs to take all possible finite subsets of $|\mathcal{S}| \cong |\mathcal{F}|^{(P)}$, and take all possible solutions to Eq. (3.21). The collection of all these different solutions will then correspond to the measure over different regions. This means that one has a well-defined Lebesgue integral for any function.

Note that in practice, it usually suffices to take only one set of points and find a solution like in Fig. 3.2, similarly to the fact that for many purposes taking a finite Riemann sum gives a good approximation of many integrals. This then makes it possible to do numerical calculations, as for instance will be shown below in Example 3.7.

3.2.2 Generating potential homomorphisms

So-far, the mathematical background has been given to link a tensor P_{abc} to an associative closure. In this section, it will be shown how to generate a space of potential homomorphisms using a technique used in data analysis which has also been extensively used in the canonical tensor model [6, 1], the tensor rank decomposition. Once one has this space of potential homomorphisms, Section 3.2.3 will explain how one can construct the associative closure.

The straightforward way of generating potential homomorphisms is the one used sofar in Examples 3.1 to 3.4, namely find all (covering) partial algebras $\mathcal{S}_k$, construct their dual spaces $|\mathcal{S}_k|$, and and check which $p \in |\mathcal{F}|^{(P)} \subseteq \bigcup_k |\mathcal{S}_k|$ have a finite measure contribution in Eqs. (3.21) and (3.22).

The tensor rank decomposition is a data analysis tool which has many applications in various fields, such as machine learning [79], signal processing [80], neuroscience [81], mathematics [82] and many more [83]. Notably, it was used in the context of the canonical tensor model several times [6, 2, 84, 85]. In [6] it was shown that the tensor rank decomposition, combined with other techniques such as persistent homology [86, 87, 88, 89], may be used to recover topological and geometrical information from a tensor P_{abc} that is constructed from eigenfunctions of the Laplace-Beltrami operator as explained above. From Eq. (3.22) it can be seen why the tensor rank decomposition might be so useful, since one recovers a tensor rank decomposition if one uses a suitable simple function to approximate the basis functions $p_a = f_a(p)$.

In [3], a peculiar connection between the tensor rank decomposition and potential homomorphisms was discovered. It was conjectured that a certain class of tensor rank decompositions, namely positive minimal tensor rank decompositions, generates potential homomorphisms. Consider a rank-R tensor rank decomposition of a unital

tensor P_{abc} acting on a vectorspace $\mathcal{F}$,

$$P_{abc} = \sum_{i=1}^{R} \phi_a^i \phi_b^i \phi_c^i, \tag{3.24}$$

where $\phi^i \in \mathcal{F}^*$. Note that often the definition of a tensor rank decomposition includes the requirement that R is the lowest possible integer for which such a decomposition may be found. Here, however, this is not required and instead defined seperately (as in [2]).

Definition 3.12. A tensor rank decomposition

$$P_{abc} = \sum_{i=1}^{R_c} \phi_a^i \phi_b^i \phi_c^i,$$

is called *minimal* if there is no integer $R < R_c$ for which such a decomposition is possible.

Since the tensor P_{abc} is unital, there is a unit element $1 = \gamma_a f_a$ such that

$$\delta_{bc} = \gamma_a P_{abc} = \sum_{i=1}^{R} (\gamma_a \phi_a^i) \phi_b^i \phi_c^i \equiv \sum_{i=1}^{R} \lambda_i \phi_b^i \phi_c^i. \tag{3.25}$$

This leads to the following definition

Definition 3.13. A tensor rank decomposition

$$P_{abc} = \sum_{i=1}^{R_c} \phi_a^i \phi_b^i \phi_c^i,$$

for a unital tensor P_{abc} with unit $1 = \gamma_a f_a$ is called positive if for all i

$$\lambda_i \equiv (\gamma_a \phi_a^i) > 0.$$

If one now writes $\beta_i \equiv (\lambda_i)^{1/3}$ and $p_a^i \equiv (\beta_i)^{-1/3} \phi_a^i$, one finds that Eqs. (3.24) and (3.25) may be written as

$$\begin{aligned} P_{abc} &= \sum_{i=1}^{R} \beta_i p_a^i p_b^i p_c^i, \\ \delta_{ab} &= \sum_{i=1}^{R} \beta_i p_a^i p_b^i, \end{aligned} \tag{3.26}$$

which are exactly of the same form as Eqs. (3.21) and (3.22). This similarity suggests that there might be a relationship between positive tensor rank decompositions and the space of potential homomorphisms with a measure on it.

From the above, there are some clear connections between positive tensor rank decompositions and the space of potential homomorphisms. For one, both the building blocks of the tensor rank decomposition, p^i, and elements of the space of potential homomorphisms r, are elements of the linear dual space $\mathcal{F}^*$. Furthermore, there is a natural association of the β_i in the tensor rank decomposition with a measure, where $\beta_i \equiv \mu(A_i)$ for a region $A_i \subset |\mathcal{F}|^{(P)}$.

The question is whether or not these elements are actually potential homomorphisms. To classify this, another definition will come in handy

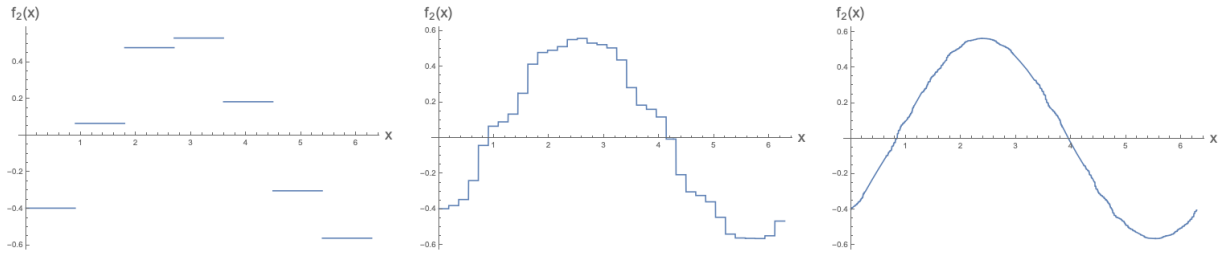

Figure 3.3: A plot of the points of the circle generated by the tensor rank decomposition of the second basis function $p_2^i = f_2(p^i)$, using the simple function representation described in Eq. (3.23). From left to right, $M = 1, 5$ and 150 tensor rank decompositions were used. Figure taken from [3].

Definition 3.14. A positive tensor rank decomposition

$$P_{abc} = \sum_{i=1}^{R_c} \beta_i p_a^i p_b^i p_c^i,$$

is called *pointwise*, if all elements are potential homomorphisms, i.e. every $p_a^i \in |\mathcal{F}|^{(P)}$.

The underlying idea of the formalism described in this section is that if one considers all pointwise tensor rank decompositions, one finds the whole space of potential homomorphisms $|\mathcal{F}|^{(P)}$. The main question is thus under what conditions does a tensor rank decomposition correspond to a pointwise decomposition. The example of the circle will lay the groundwork for a conjecture about this very question below.

Example 3.6. Tensor rank decompositions of the circle. In this example, the tensor rank decomposition of the tensor corresponding to the circle, as already discussed in Examples 3.1 to 3.5.

This $N = 5$ dimensional tensor P_{abc} has a (positive) rank of $R = 7$. Therefore it is useful to first look at the elements p_a^i which make up these (minimal) tensor rank decompositions. This may be done by finding solutions to

$$0 = \left(\sum_{i=1}^{R} \phi_a^i \phi_b^i \phi_c^i - P_{abc} \right)^2,$$

for instance with a computer. After this, one can use rewrite this tensor rank decomposition in the form of Eq. (3.26), in order to get to a set of "points" $\{p_a^i\}$ and their associated "measure" $\{\beta_i\}$. In [3], this was done using Mathematica by finding $M = 150$ positive minimal tensor rank decompositions. Here, M denotes the amount of tensor rank decompositions generated.

For any element $p^i \in \mathcal{F}^*$, one can check if it is a potential homomorphism by verifying whether they are elements of the partial algebra $\mathcal{S}$ defined in Example 3.3. This verification for a point p_a^i is straightforwardly done by checking if

$$s_a^1 s_b^2 p_a^i p_b^i = s_a^1 s_b^2 P_{abc} p_c^i, \tag{3.27}$$

for every $s^1, s^2 \in \mathcal{S}$. Rather surprisingly, if one performs this check for all the points generated by the minimal positive tensor rank decomposition above, $\{p_a^i\}$, every point

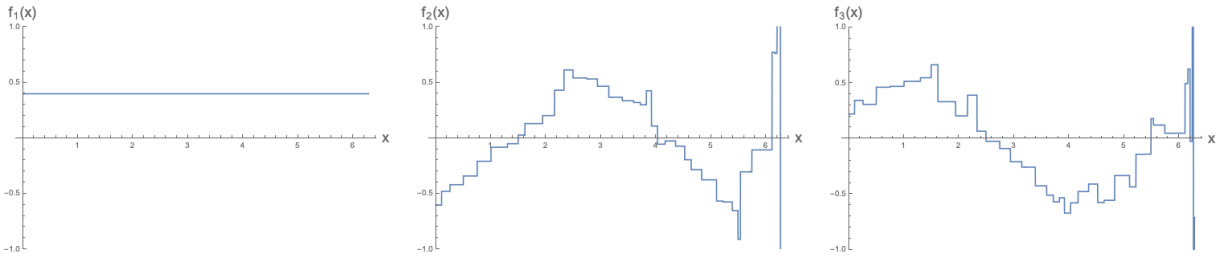

Figure 3.4: A plot of the "points" generated by a non-minimal tensor rank decomposition, showing how they deviate from the results in figure 3.3. In this image, 10 tensor rank decompositions of rank 8 are combined into the plot, using the same procedure as before. Some points still look similar to figure 3.3, but there are clear discrepancies from what is expected. The figure was taken from [3].

actually corresponds to a potential homomorphism. Furthermore, for every tensor rank decomposition, the sum of the measure elements exactly evaluates to

$$\sum_{i=1}^{R} \beta_i = 2\pi,$$

which is exactly what one expects for a measure corresponding to a circle of circumference 2π.

To visually represent all of these points together as simple functions, it is useful to use the notation $(\beta_i^{(I)}, {p_a^{(I)}}^i)$ denoting the i-th point and measure in the I-th tensor rank decomposition. If one now rewrites the tensor in terms of the M tensor rank decompositions one finds

$$\begin{aligned} P_{abc} &= \frac{1}{M}\left(\sum_{i=1}^{R} \beta_i^{(1)} {p_a^{(1)}}^i {p_b^{(1)}}^i {p_c^{(1)}}^i + \ldots + \sum_{i=1}^{R} \beta_i^{(M)} {p_a^{(M)}}^i {p_b^{(M)}}^i {p_c^{(M)}}^i\right), \\ &= \frac{1}{M}\sum_{I=1}^{M}\sum_{i=1}^{R} \beta_i^{(I)} {p_a^{(I)}}^i {p_b^{(I)}}^i {p_c^{(I)}}^i. \end{aligned}$$

This implies that, using M tensor rank decompositions, the correct measure values to take are actually

$$\frac{1}{M}\beta_i^{(I)}. \tag{3.28}$$

This is what was used in the demonstration in Fig. 3.3, where this approach was demonstrated for three cases: Using only $M = 1$ tensor rank decomposition, using $M = 5$ tensor rank decompositions, and using $M = 150$ tensor rank decompositions.

In this example, thus-far only the *minimal* tensor rank decomposition was discussed. This was for a reason, namely for a non-minimal tensor rank decomposition the elements p_a^i are not necessarily potential homomorphisms anymore. This might not be clear from the discussion above, and a deeper understanding of this might be useful in the future. In the case of this example, one might try to find positive tensor rank decompositions of rank $R = 8$. When finding solutions to this, there always seem to be elements where Eq. (3.27) is not satisfied quite substantially (of the order $|s_a^1 s_b^2 p_a^i p_b^i - s_a^1 s_b^2 P_{abc} p_c^i| \sim O(1)$).

This fact is seen more clearly in a visual representation. If one generates $M = 10$ tensor rank decompositions of rank $R = 8$ and uses the same intepretation as points as for $R = 7$, the result is given in Fig. 3.4. Clearly this points towards the requirement

that the tensor rank decomposition is supposed to be *minimal*. Note that this does not mean that there are no pointwise tensor rank decompositions of higher rank, but only for minimal tensor rank decompositions does every tensor rank decomposition seem to be pointwise.

From Example 3.6, one is led to the following conjecture.

Conjecture 3.1. *For a unital tensor P_{abc} that admits a covering partial algebra, any minimal positive tensor rank decomposition is a pointwise decomposition.*

If this conjecture is true, it makes generating potential homomorphisms relatively simple, as many techniques for finding tensor rank decompositions already exist [83]. Currently it is not known if this conjecture hold more generally or not. It is an interesting potential future research direction.

An interesting aspect to consider is the case when a tensor P_{abc} has some symmetry. In [84] it was already pointed out that, if a tensor P_{abc} has a symmetry such that for a Lie-group transformation G_{ab}

$$G_{aa'}G_{bb'}G_{cc'}P_{a'b'c'} = P_{abc},$$

there is a continuous degeneracy of the tensor rank decomposition. This is because for every p^i_a belonging to a tensor rank decomposition, the element $G_{aa'}p^i_{a'}$ belongs to another tensor rank decomposition

$$G_{aa'}G_{bb'}G_{cc'}P_{a'b'c'} = \sum_{i=1}^{R} \beta_i (G_{aa'}p^i_{a'})(G_{bb'}p^i_{b'})(G_{cc'}p^i_{c'}) = P_{abc}. \tag{3.29}$$

Therefore, if such a symmetry exists, it implies together with Conjecture 3.1 that the tensor P_{abc} corresponds to a continuous space.

One last thing to discuss about the construction of a measured space from tensor rank decompositions is the choice of the individual measure elements as in Eq. (3.28). It should be noted that this is nothing more than a choice. Since the tensor rank decompositions are generated randomly, it is not clear beforehand if this would be the most natural choice. The only real restriction for the final measure elements is that the sum of the elements remains unchanged $\sum_{i=1}^{M*R} \beta_i = \sum_{i=1}^{R} \beta_i^{(I)} \, \forall_{I \leq M}$. One could write Eq. (3.28) as the rule

$$\beta_{(I-1)*R+j} = \frac{1}{M}\beta_j^{(I)}. \tag{3.30}$$

There is a certain ambiguity in doing so. The most general form of this would be given by

$$\beta_{(I-1)*R+j} = A_I \beta_j^{(I)},$$

where $A_I > 0$ and $\sum_{I=1}^{M} A_I = 1$. It is expected that such a "transformation" corresponds to performing a diffeomorphism on the manifold, since it can be interpreted as a deformation.

3.2.3 Constructing the associative closure

In this section, it will be shown how an associative closure of a tensor P_{abc} acting on vectorspace $\mathcal{F}$ may be reconstructed if one has the space of potential homomorphisms $|\mathcal{F}|^{(P)}$. This space may be found by using the tensor rank decomposition, as explained in Section 3.2.2.

Consider two basis functions f_a, f_b. A potential homomorphism $p \in |\mathcal{F}|^{(P)}$ will, in the associative closure, correspond to the evaluation of the functions f_a and f_b at a point p. The strategy will be to already treat this point as an evaluation map, and the part of the pointwise product that is not described by the tensor P_{abc} yet should then correspond to a new function. Performing this decomposition, one gets

$$f_a(p)f_b(p) = P_{abc}f_c(p) + g(p) = \langle f_c |\, f_a \cdot f_b \rangle\, f_c(p) + g(p), \tag{3.31}$$

where $g(p)$ corresponds to the difference between the pointwise product to the product induced by P_{abc}. There are two options, either $g(p) = 0\,\forall_{p\in|\mathcal{F}|^{(P)}}$ or $g(p)$ is non-trivial. In the first case, the pointwise product is already exactly described by P_{abc}, and f_a and f_b span a partial algebra. One has to pick new f_a and f_b until one finds a non-trivial result. If such f_a, f_b do not exist, this means P_{abc} already describes an associative algebra and one is done.

In the other case, where $g(p)$ is non-trivial, the idea is to see $g(p)$ as a new function that is not in the algebra yet. By construction, g is a map

$$g : |\mathcal{F}|^{(P)} \to \mathbb{R}.$$

In a sense, the algebra is now "closed" with respect to this function if it is added as a new basis element. Note that, again by construction,

$$\int_{|\mathcal{F}|^{(P)}} \mathrm{d}\mu(p) f_a(p) g(p) = 0.$$

This shows that the inner product may be extended to g trivially as well, keeping the basis orthogonal. In order to make the basis orthonormal, introduce the final new basis element

$$f_{N+1}(p) = \frac{g(p)}{\|g\|},$$

where $\|g\| = \sqrt{\langle g |\, g \rangle}$ is the norm induced by the inner product. f_{N+1} together with $\{f_a\}$ forms an orthonormal basis with the same inner product

$$\delta_{ab} = \langle f_a |\, f_b \rangle = \int_{|\mathcal{F}|^{(P)}} \mathrm{d}\mu(p) f_a(p) f_b(p), \tag{3.32}$$

where a now runs from 1 to $N+1$. Let us now denote this vectorspace by $F^{(N+1)}$, and introduce a new tensor $P^{(N+1)}_{abc} = \langle f_c |\, f_a \cdot f_b \rangle$ for all $a, b, c \leq N+1$ with the inner product of Eq. (3.32).

Next, one takes new $f_a, f_b \in \mathcal{F}^{(N+1)}$ and restarts this process, starting with Eq. (3.31). In this case, $\mathcal{F}^{(N+1)}$ is treated as the starting vectorspace using the tensor $P^{(N+1)}_{abc}$. The associative closure is now found as the inductive limit of this process, producing a countably infinite-dimensional algebra.

One has to show that this indeed produces an associative closure.

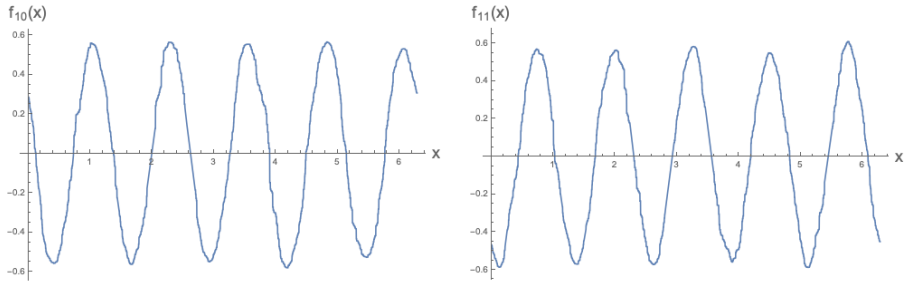

Figure 3.5: A plot of the 10th and 11th basis functions of the exact circle using the procedure described in the text, with the same 1.000 points as before in figure 3.2.

Proposition 3.6. *The construction above produces an associative closure.*

Proof. The proof will commence in two parts. First it will be shown that this construction leads to an associative extension, and after that it will be shown that this extension is actually an associative closure.

First note that for every element in $p^* \in |\mathcal{F}|^{(P)}$, an element was constructed in the algebraic dual space of $\mathcal{A}$, $p \in |\mathcal{A}|, p(f_a) = f_a(p^*)$, since basically a new algebra was constructed where the original potential homomorphisms become proper homomorphisms. Referring to Definition 3.5, it is clear that the first two points are satisfied by construction. Furthermore, since every element of the dual space $p \in |\mathcal{A}|$ projected to $\mathcal{F}^*$ must be an element of some associative extension by the definition of potential homomorphisms, it necessarily lies in $|\mathcal{S}|$ for a maximal system of partial algebras $\mathcal{S}_i \subset \mathcal{F}$. Since every element $p \in |\mathcal{A}|$ is constructed from a $p^* \in |\mathcal{F}|^{(P)}$, this is injective.

Similarily it follows that it is an associative closure, since for every $p \in |\mathcal{A}|$ it must hold that $p^* \equiv p|_{\mathcal{F}^*} \in |\mathcal{F}|^{(P)}$ by definition of the space of potential homomorphisms, and $p \in |\mathcal{A}|$ was directly constructed from every $p^* \in |\mathcal{F}|^{(P)}$. Therefore, this is an isomorphism and thus $|\mathcal{A}||_{\mathcal{F}^*} = |\mathcal{F}|^{(P)}$. □

To end this section, the example of the circle will once more be used.

Example 3.7. Constructing the associative closure of the circle. In this example, the exact circle from Examples 3.1 to 3.6 is used again.

As has already been shown, there is an infinite-dimensional associative closure $(\mathcal{A}, \cdot)$ of the five-dimensional tensor P_{abc} given my the smooth functions

$$\mathcal{A} \cong C^\infty(S^1).$$

It is useful, however, to show this using this more general procedure. In [3], this procedure was used to construct an extension of the algebra up to $\mathcal{A} \cong \mathbb{R}^{11}$, which gives the opportunity to display the funcitons $f_{10}(x) \sim \sin(5x)$ and $f_{11}(x) \sim \cos(5x)$. To this end, a number of 1000 points was used, defining the pointwise inner product as Eq. (3.21). The result of this is shown in Fig. 3.5.

Note that technically, the construction using only a finite number of points will result in an associative extension. Therefore, if one wants to generate high frequency functions reliably, one needs to use an approximation with more and more points.

3.3 Generating a unit and reconstructing geometry

Thus far, in this chapter, a topological and measure-theoretic understanding of algebras generated by a tensor P_{abc} has been shown. It might be noted that the general strategy followed here is similar to that used in the spectral triple approach [72], though the mathematical objects used differ slightly, namely

- An associative commutative algebra $\mathcal{A}$ generates the topological space $|\mathcal{A}|$.
- A Hilbert space $\mathcal{H}$ on which the elements of the algebra are represented yields a measure.
- An operator D acting on the Hilbert space $\mathcal{H}$ which contains the full information of the Riemannian metric q_{ab}.

This section will now explain how the last step may be added in this formalism. It will be shown that it is possible to do this without adding any additional objects to the theory, meaning that a symmetric tensor of degree three, P_{abc}, will be all that is needed to describe the full geometry of a Riemannian manifold.

The approach taken in this section is in two steps. First a method will be given to include the information of an operator into the tensor P_{abc}, and then it will be explained how this information may be reconstructed from the tensor P_{abc} afterwards without losing the important topological and measure-theoretic results obtained thusfar. As already alluded to in Chapter 2, the operator that plays a central role in Riemannian geomtry is the Laplace-Beltrami operator [71]. Therefore, this is the main operator that will be used.

Before resuming, it is useful to introduce some notation for this section. The "naked" tensor that corresponds to a measured topological space as explained in the sections above will be denoted by $\bar{P}_{abc}$, where the tensor corresponding to the full geometric information will be denoted by P_{abc}. The tensor $\bar{P}_{abc}$ will also be called the topological tensor, and the tensor P_{abc} the geometric tensor.

3.3.1 Spectral information in algebraic tensor models

In this section, it will be explained how the full information of the Laplace-Beltrami operator may be included in a tensor P_{abc} by altering the topological tensor $\bar{P}_{abc}$. In Section 3.3.2 it will then be shown that one can actually recover the original tensor $\bar{P}_{abc}$ The reason for wanting the information of the Laplace-Beltrami operator is the close connection it has with Riemannian geomtry.

Recall from Section 2.3 that the Laplace-Beltrami operator Δ, for a compact Riemannian manifold $(\mathcal{M}, q)$, has a countable number of eigenfunctions $\{f_a\}$ satisfying

$$\Delta f_a := \lambda_a f_a. \tag{3.33}$$

Moreover, these eigenfunctions form a Schauder basis for $L^2(\mathcal{M})$. This is useful in this setup, since $L^2(\mathcal{M})$ is also the Hilbert space that the tensor $\bar{P}_{abc}$ is defined on. Therefore, in principle, if one knows the tensor $\bar{P}_{abc}$ and the eigenvalues of the Laplace-Beltrami operator $\{\lambda_a\}$, then one can reconstruct the full geometry of $(\mathcal{M}, q)$.

However, in practice it will be difficult to use the Laplace-Beltrami operator directly, as it is not compact. The idea is to use a positive compact self-adjoint operator $\mathcal{O}$. From the spectral theorem [69], it is known that compact self-adjoint operators on a seperable Hilbert space have some nice properties. Firstly, their eigenvectors $\{f_a\}$ form a countable (Schauder) basis,[6] and the eigenvalues, μ_a, have the property that $\mu_a \to 0$ for $a \to \infty$.

Note that one has several choices of an operator which directly correspond to the Laplace-Beltrami operator one can choose. In Section 3.3.2, it will be clear that a positive compact self-adjoint operator makes sense, but besides that the choice of operator will be theory dependent. One example of such an operator would be the heat operator

$$\mathrm{e}^{-t\Delta} : L^2(\mathcal{M}) \to L^2(\mathcal{M}),$$

for $t > 0$. One can prove that this is a compact operator on $L^2(\mathcal{M})$ [71]. Here, this operator will be assumed, though other choices can be considered in exactly the same way. The eigenvalues of this operator μ_a may be written in terms of the eigenvalues of the Laplace-Beltrami operator

$$\mu_a = \mathrm{e}^{-t\lambda_a} .$$

This means that knowing the eigenvalues μ_a of $\mathrm{e}^{-t\Delta}$ is enough to identify the eigenvalues λ_a of Δ. This is the key property that is important for a general operator $\mathcal{O}$, the eigenvalues of the Laplace-Beltrami operator have to be determined by the eigenvalues of the operator $\mathcal{O}$, and not add additional degeneracies to the spectrum.

Assuming the tensor $\bar{P}_{abc}$ to already be defined in the same basis $\{f_a\}$ of eigenfunctions of the operator $\mathcal{O}$, one can define a new tensor P_{abc} as

$$P_{abc} := \langle \mathcal{O}f_c | \, (\mathcal{O}f_a) \cdot (\mathcal{O}f_b) \rangle \, . \tag{3.34}$$

Because the eigenfunctions are already assumed to be aligned, this may also be written in terms of the eigenvalues and the topolotical tensor $\bar{P}_{abc}$

$$\begin{aligned} P_{abc} &:= \langle \mathcal{O}f_c | \, (\mathcal{O}f_a) \cdot (\mathcal{O}f_b) \rangle \, , \\ &= \mu_a \mu_b \mu_c \, \langle f_c | \, f_a \cdot f_b \rangle \, , \\ &= \mu_a \mu_b \mu_c \bar{P}_{abc}, \end{aligned} \tag{3.35}$$

where the original topological definition of the tensor was used. Note that this tensor P_{abc} is still real and completely symmetric. In the specific case of the heat operator above, this reduces to

$$P_{abc} = \mathrm{e}^{-t(\lambda_a+\lambda_b+\lambda_c)} \, \bar{P}_{abc}. \tag{3.36}$$

Note that this is exactly an expression that was used in [6], where a "damping factor" $\mathrm{e}^{-\lambda_a/\Lambda}$ was introduced in the definition of the eigenfunctions. This damping factor was motivated by the fact that the results for reconstructing the topology and geometry became more stable. The current discussion gives this factor a whole new interpretation, since this factor basically introduces the full geometric information into the tensor. From the discussion above, it is clear that the topological tensor $\bar{P}_{abc}$ simply does not contain any information on the geometry of the Riemannian manifold yet except for the measure,

[6] It might be that the kernel of $\mathcal{O}$ is non-trivial. It will not be the case here, but in that case one has to add a basis for the kernel of $\mathcal{O}$ in order to arrive at a countable basis.

so this kind of "damping factor" is a way of ensuring that the information is included in the tensor. This also explains the success that was had reconstructing part of the geometry and the topology of the manifolds considered using a discrete version of the heat equation.

A last note before ending this section is on the new tensor P_{abc}. It is important to realise that one cannot use the tensor P_{abc} directly anymore in order to reconstruct the topology and measure by the methods developed in Sections 3.1 and 3.2, one has to use the topological tensor $\bar{P}_{abc}$. Even though all information is included in the tensor P_{abc}, in practice one needs to have the topological tensor $\bar{P}_{abc}$ and a list of eigenvalues λ_a of the Laplace-Beltrami operator in order to reconstruct the Riemannian manifold.

One thing that will be important is the fact that P_{abc} will not have a well-defined unit anymore. Say, for instance, the unit of $\bar{P}_{abc}$ is given by $1 = \gamma f_1$, meaning that

$$\gamma \bar{P}_{1bc} = \delta_{bc}.$$

However, for P_{abc}, this will clearly not be a unit anymore

$$\gamma P_{1bc} = \gamma \mu_1 \mu_b \mu_c \delta_{bc}.$$

In Section 3.3.2 it will be shown how to recover a topological tensor $\bar{P}_{abc}$ and a list of eigenvalues $\{\mu_a\}$ of a compact self-adjoint operator from a large class of tensors P_{abc}.

3.3.2 Distilling the unit and the spectrum of a positive compact self-adjoint operator.

In Section 3.3.1 a method was shown to include geometric information into a symmetric tensor. This section serves to explain how to extract this information again from the tensor. The result of this procedure applied to a tensor P_{abc} will be a topological tensor $\bar{P}_{abc}$ and a list of positive eigenvalues $\{\mu_a\}$, which then can be used to reconstruct the Riemannian manifold. This makes the inclusion of the geometric information as explained in Section 3.3.1 unambiguous, as one has a way to recover the ingredients used, except that one still has to specify the operator one uses to interpret the eigenvalues $\{\mu_a\}$, which will be left as a model dependent property. For instance, the canonical tensor model will be assumed to have some specific operator $\mathcal{O}^{\text{CTM}}$ which will link the dynamics of the model to general relativity.

The key to extracting the spectrum of a positive compact self-adjoint operator lies in the absence of a unit in this kind of tensor P_{abc}, as has already be hinted towards in the end of Section 3.3.1. In Sections 3.1 and 3.2 it was always assumed that the tensor $\bar{P}_{abc}$ has a unit, meaning that $\exists\, \mathbf{1} = \gamma_a f_a \in \mathcal{F}$ such that

$$\gamma_a \bar{P}_{abc} = \delta_{bc}. \tag{3.37}$$

Generally for a symmetric tensor P_{abc}, and in particular for the construction in Section 3.3.1, there will be no such unit element. In order to generate a new tensor $\bar{P}_{abc}$ which *is* unital, it is instructive to go back to the original definition of the unit. Namely a unit $1 \in \mathcal{F}$ has the property that $\forall f = \alpha_a f_a \in \mathcal{F}$

$$\mathbf{1} \cdot f = \gamma_a \alpha_b P_{abc} f_c = \alpha_c f_c = f,$$

which holds iff Eq. (3.37) holds. If one wants to find a candidate for a unit, it is useful to generalise this notion somewhat. This is done by the requirement that the unit is its own unit

$$\mathbf{1} \cdot \mathbf{1} = \gamma_a \gamma_b P_{abc} f_c = \gamma_c f_c = \mathbf{1},$$

which implies

$$\gamma_a \gamma_b P_{abc} = \gamma_c. \tag{3.38}$$

This is exactly the Eigen-problem of a tensor P_{abc}, which always has at least one real solution for real symmetric tensors [90]. This solution might not be unique, but it will be explained later what the canonical choice will be.

Motivated by Eq. (3.37), one can construct a matrix using the solution γ_a,

$$M_{bc} := \gamma_a P_{abc}.$$

If γ_a would correspond to a true unit, this matrix would simply reduce to δ_{bc} as Eq. (3.37). Currently, the interest lies in the case where it does not correspond to the unit. In that case, one can diagonalise the matrix to find a matrix of the form (without Einstein-summation)

$$M_{ab} = w_a \delta_{ab}. \tag{3.39}$$

From now on, P_{abc} is already taken to be in terms of the basis that diagonalises the matrix M_{ab}. Furthermore, w_a is assumed to be positive. In this case, one can define a new tensor (without using Einstein-summation)

$$\bar{P}_{abc} := \frac{1}{\sqrt{w_a w_b w_c}} P_{abc}, \tag{3.40}$$

which is the candidate for the topological tensor. This tensor indeed is unital, with unit

$$\mathbf{1} \equiv \sum_a \bar{\gamma}_a f_a := \sum_a \sqrt{w_a} \gamma_a f_a,$$

as may be seen by (without using Einstein-summation)

$$\sum_a \gamma_a \sqrt{w_a} \bar{P}_{abc} = \sum_a \gamma_a P_{abc} \frac{1}{\sqrt{w_b w_c}} = \frac{w_b \delta_{bc}}{\sqrt{w_b w_c}} = \delta_{bc}.$$

Since the assumption was that the w_a are all positive, the tensor in Eq. (3.40) is real. Equation (3.40) may be rewritten as

$$P_{abc} := \sqrt{w_a w_b w_c} \bar{P}_{abc}.$$

By comparing this equation to Eq. (3.35), it is easily seen that the w_a may be interpreted in terms of the eigenvalues $\{\mu_a\}$ of an operator $\mathcal{O}$ by identifying

$$w_a := \mu_a^2, \tag{3.41}$$

and indeed, if one uses Eq. (3.35) to define a new tensor P_{abc} from the topological tensor $\bar{P}_{abc}$, the procedure described here exactly reproduces the topological tensor $\bar{P}_{abc}$ and spectrum $\{\mu_a\}$.

Since the object that one starts with it P_{abc}, it is interesting to see under which conditions the matrix M_{ab} is positive definite. Consider the problem where one tries to fiend the extremal values of the functional

$$f(\gamma) := \gamma_a \gamma_b \gamma_c P_{abc}, \tag{3.42}$$

under condition $|\gamma|^2 = \gamma_a \gamma_a = 1$. For this, one can introduce a Lagrange multiplier k,

$$g(\gamma) := \gamma_a \gamma_b \gamma_c P_{abc} + k(1 - |\gamma|^2).$$

If one now takes the derivative with respect to γ_a

$$\partial_{\gamma_c} g(\gamma) = 3\gamma_a \gamma_b P_{abc} - 2k\gamma_c = 0,$$

one should solve this equation to find the extremal values. Defining $\gamma'_a = \frac{2k}{3}\gamma_a$, this exactly reproduces the Eigen-problem Eq. (3.38),

$$\gamma'_a \gamma'_b P_{abc} = \gamma'_c.$$

Therefore, by definition of M_{ab}, the extremal value γ_a is an eigenvector of M_{ab}

$$\gamma_a M_{ab} = \gamma_a \gamma'_c P_{abc} = \frac{2k}{3}\gamma_a \gamma_c P_{abc} = \left(\frac{2k}{3}\right)^2 \gamma_b.$$

It can be seen that the eigenvalues are positive for real solutions. This also implies

$$\gamma_a \gamma_b \gamma_c P_{abc} = \frac{2k}{3}.$$

This sets the first eigenvalue to $\left(\frac{2k}{3}\right)^2$. In order to find the other eigenvalues, consider a second order perturbation of (3.42) denoted by $\epsilon\delta_a$, with δ_a a unit-size vector. It follows from the restriction $|\gamma|^2 = 1$ that $\gamma_a \delta_a = 0$. The first-order contribution is zero since for extremal values. The second-order contribution is given by

$$3\gamma_a \delta_b \delta_c P_{abc} = \frac{9}{2k} M_{bc} \delta_b \delta_c,$$

this means that the requirement that M_{ab} is positive-definite coincides with the functional (3.42) either having a local minimum with a positive value ($k > 0$), or a local maximum with a negative value ($k < 0$).

If there are still several candidates left, one can consider to look for "almost unit" functions, thus minimising

$$\sum_{a=1}^{N} (w_a - 1)^2.$$

This would then produce a candidate unit which is as close to the real unit as possible.

Chapter 4

The canonical tensor model and its wave functions

The canonical tensor model was introduced in [64], and is a tensor model set in the Hamiltonian framework. In this chapter, the model will be extensively reviewed. Both the kinematics and dynamics of the classical model will be introduced, and the quantum version of the model will be explained.

4.1 The motivation for a new kind of tensor model

In this thesis, the canonical tensor model is introduced as a prime example of a model that could be interpreted as an algebraic tensor model. In this section the main motivation behind this model will be briefly reviewed, both from the algebraic tensor model perspective and a more historical perspective.

The predecessors of the original tensor models are matrix models. Matrix models appeared in the study of planar maps [91, 92, 93], and were very useful tools in the study of two-dimensional dynamical triangulation since they made explicit calculations possible [94]. In these matrix models, perturbative expansions yield ribbon graphs which are extensions of the Feynmann graphs appearing in field theories. The dual of these ribbon graphs then yields a triangulation of a two-dimensional surface. One of the critical factors to make this breakthrough possible was an observation by 't Hooft that matrix models of this kind using very large $N \times N$ matrices have an $1/N$-expansion, characterising the topology of the resulting graphs [95].

In [49, 50, 51], matrix models were then extended to tensor models in an attempt to reproduce the success of two-dimensional dynamical triangulation in higher dimensions. The logic was that since matrices have two indices they reproduce two-dimensional spaces, whereas tensors of degree d could reproduce d-dimensional spaces. However, a proper $1/N$ expansion was not available for these kind of models, and the emergent spaces are highly singular. In order to improve this, group field theories were introduced shortly after, but they still were difficult to analyse [96, 97, 98]. An apparent breakthrough came with the introduction of coloured tensor models [99, 100], which are a class of tensor models that allow a certain type of $1/N$ expansion [101]. Even though this $1/N$-expansion is not topological as in the matrix model case, this was the first time that tensor

models could systematically be analysed. However, it was found that these models still do not generate macroscopic spacetimes [102], and the main hope is to connect these models to gravity via holography due to the dominance of so-called melonic graphs [103, 104].

The results of tensor models are similar to that of the dynamical triangulation approach to quantum gravity already mentioned in Section 1.2. In dynamical triangulation, the inclusion of a causal requirement to the set of allowed triangulations seemed to solve many of the issues [46]. This is the main insight that led to the introduction of the canonical tensor model [64]. A key point in the causal requirement of causal dynamical triangulation is the assumption that the spacetime manifold has the form of

$$\mathcal{M} \cong \mathbb{R} \times \mathcal{S},$$

i.e. that spacetime is globally hyperbolic as in Eq. (1.3). However, the topology in tensor models is supposed to be an emergent phenomenon, and fixing such a topological requirement would be very unnatural. Therefore, it was proposed to build a tensor model which was set in the Hamiltonian framework built in by the ADM-formalism of general relativity, because this would implicitly implement the causal requirement already and use a Hamiltonian to generate time translations. The Hamiltonian of the model is then chosen by certain first principles in order to assure that its constraint algebra resembles the ADM constraint algebra, as explained in Section 4.2.

One caveat of this approach is that the interpretation of tensors relating to simplices that triangulate a space is lost. Therefore, the strategy has mainly been to investigate this abstract tensor model, and leave the direct interpretation of the tensors in terms of spatial manifolds for later, backed up by some strong evidence suggesting a connection to general relativity [66, 67, 68]. A potential interpretation in the form of algebraic tensor models was then introduced in [3], motivated by data-analytic results in [6].

The interpretation of this model in the algebraic tensor model description of Chapter 3, introduced in [3], uses some ideas that were introduced already before [105, 106, 107, 108, 109, 110, 111]. The absence of a notion of associative closure however, made this approach hard to give some real physical interpretation. After all, any finite-dimensional associative algebra would necessarily reduce to a finite set of disconnected points, and non-associative algebras could not correspond to functions over a manifold. This realisation in rather natural from the algebraic tensor model point of view; if the algebra is finite-dimensional and associative, the algebraic dual space will consist of a finite amount of points with the discrete topology on them. The non-associativity of a finite-dimensional algebra makes it possible to link a finite-dimensional tensor to an infinite-dimensional algebra.

4.2 The classical canonical tensor model

In this section, the classical canonical tensor model will be defined as a starting point to define the full quantum mechanical model. As explained before, it is expected that the tensor P_{abc} represents a spatial slice of the full spacetime manifold through the interpretation of Chapter 3. Therefore, a natural starting point is the Hamiltonian formalism, where all possible tensors P_{abc} constitute the configuration space, and a Hamiltonian defines the time evolution.

4.2.1 The phase space

The canonical tensor model is defined using finite-dimensional real symmetric tensors. The tensors are multilinear maps from a finite-dimensional vectorspace $\mathcal{F}$ to the real numbers

$$Q : \mathcal{F} \times \mathcal{F} \times \mathcal{F} \to \mathbb{R}.$$

The symmetry requirement means that, for any $f, g, h \in \mathcal{F}$,

$$Q(f,g,h) = Q(f,h,g) = Q(g,f,h) = Q(g,h,f) = Q(h,f,g) = Q(h,g,f).$$

As is well known, given a basis $\{f_a\}$ of this vectorspace, a tensor is fully characterised by a real multidimensional array

$$Q_{abc} := Q(f_a, f_b, f_c).$$

It is therefore easy to see that the configuration-space, the space of all possible N-dimensional totally symmetric tensors of degree three, is given by

$$\mathcal{C} = \mathbb{R}^{\mathcal{N}},$$

where

$$\mathcal{N} \equiv \frac{1}{6}N(N+1)(N+2) \tag{4.1}$$

is the amount of independent entries of a symmetric tensor of degree three. This fact may be used to readily find the cotangent space, and thus the canonical phase space of the model with symplectic structure. First, note that the phase space Γ is isomorphic to the cotangent bundle

$$\Gamma := T^*\mathcal{C} \cong \mathbb{R}^{2\mathcal{N}}.$$

If one takes $Q \in \mathcal{C}$, then choose an element $P \in T^*_Q\mathcal{C}$. Since $T^*_Q\mathcal{C} \cong \mathbb{R}^{\mathcal{N}}$, this may again be represented by a tensor P_{abc}. The local coordinates on Γ are now denoted by (Q_{abc}, P_{abc}). In these coordinates, a natural one-form is given by

$$w := P_{abc}\mathrm{d}Q_{abc} \tag{4.2}$$

which gives rise to the symplectic two-form

$$\mathrm{d}w = \mathrm{d}Q_{abc} \wedge \mathrm{d}P_{abc}. \tag{4.3}$$

The Poisson bracket is given by

$$\{Q_{abc}, P_{def}\} := \sum_{\sigma} \delta_{a\sigma_d}\delta_{b\sigma_e}\delta_{c\sigma_f}\ , \tag{4.4}$$

where σ is a permutation of $\{d, e, f\}$ and δ the Kronecker delta.

As mentioned in Section 3.2, the reason why one can consider totally symmetric tensors Q_{abc} is because of an inner product structure present on the vector space $\mathcal{F}$. The choice of inner product given a basis $\{f_a\}$ is taken to be

$$\langle f_a | \, f_b \rangle := \delta_{ab}. \tag{4.5}$$

Note that this means that if one takes a tensor Q_{abc} one implicitly implies the above inner product structure, meaning that with only a tensor one automatically constructs a (finite-dimensional) Hilbert space structure. Having such an inner product, it is thus relevant to investigate the orthogonal transformations of the vectorspace.

By definition, an orthogonal transformation $T : \mathcal{F} \to \mathcal{F}$ preserves the inner product $\langle f | f \rangle = \langle Tf | Tf \rangle$. Doing a basis transformation $f_a \to T_{aa'} f_{a'}$ induces a kinematical symmetry in the model:

$$\begin{aligned} Q_{abc} &\to T_{aa'} T_{bb'} T_{cc'} Q_{a'b'c'}, \\ P_{abc} &\to T_{aa'} T_{bb'} T_{cc'} P_{a'b'c'}. \end{aligned} \tag{4.6}$$

Here, T are matrices of the defining representation of the $O(N)$ orthogonal group.[1] The generators, $\mathcal{J}_{ab}$, of the $SO(N)$ subgroup span the skew-symmetric matrices ($\mathcal{J}_{ab} = -\mathcal{J}_{ab}$). One can show that the generators of the $SO(N)$ subgroup are given by

$$\mathcal{J}_{ab} = -\frac{1}{4}(Q_{acd}P_{bcd} - Q_{bcd}P_{acd}). \tag{4.7}$$

To summarise the kinematical setup, the phase space of the canonical tensor model is given by all symmetric tensors of degree three Q_{abc}. The phase space is then given by $T^*\mathcal{C} \cong \mathbb{R}^{2\mathcal{N}}$, with the Poisson bracket in Eq. (4.4). The inner product assumed on the vector space $\mathcal{F}$ induces a symmetry generator $\mathcal{J}_{ab}$, which will be used below when building a Hamiltonian for the theory.

4.2.2 A Hamiltonian from first principles

The dynamics of a physical model in the Hamiltonian formalism are given by the Hamiltonian $\mathbf{H}$, which fundamentally is a map

$$\mathbf{H} : \Gamma \to \mathbb{R},$$

generating time translations of observables. Observables are given by phase-space functions

$$f : \Gamma \to \mathbb{R},$$

and their time translation is then given by (in case there is no explicit time dependence of f)

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, \mathbf{H}\}.$$

There are clearly many possible functionals one could choose from as a Hamiltonian. However, one can find a good candidate model by making some physical assumptions, mainly coming from the ADM-formalism of general relativity.

The first important observation is that in the canonical formalism of general relativity, the Hamiltonian is totally constraint. This reflects the free choice of coordinates, in order to fully restore the four-dimensional diffeomorphism invariance. This is an desirable aspect of the theory as well, therefore the most general Hamiltonian could be written as a linear combination of first class constraints

$$H := n^I_a \mathcal{H}^I_a + n^I_{ab} \mathcal{H}^I_{ab} + n^I_{abc} \mathcal{H}^I_{abc} + \dots. \tag{4.8}$$

[1] As an aside; there is an extension of the canonical tensor model where the symmetry group is replaced by the orthogonal symplectic group, which might be useful to describe fermionic degrees of freedom [112].

In order to stay as close as possible to the ADM-formalism as possible, it makes sense to focus on just two first-class constraints: A Hamiltonian constraint and a kinematical constraint (analogous to the spatial diffeomorphism constraint of the ADM-formalism). For the kinematical constraint, the most natural choice is the $SO(N)$ symmetry generator, as this represents the fundamental symmetry of the theory one wishes to describe. While in manifold language, coordinate independence is the fundamental underlying symmetry, a theory of an algebra of functions should naturally be basis independent.

Since the $SO(N)$ constraint already takes a care of the potential rearranging of functions, it makes sense to assume a one-index structure $\mathcal{H}_a$ for the remaining "Hamiltonian constraint". Physically, this corresponds to a sort of local time requirement. All in all, the general form of the Hamiltonian is now

$$\mathbf{H} := n_a \mathcal{H}_a + n_{ab} \mathcal{J}_{ab}, \tag{4.9}$$

where n_a and n_{ab} are a vector and an anti-symmetric two-tensor which act like Lagrange multipliers, similar to the lapse function and shift vector in the ADM-formalism.

What is left is to determine the Hamiltonian constraint $\mathcal{H}_a$. For this, one needs to make some more assumptions in order to fully fix it. The first major assumption is that all terms need to be connected via contractions. This means that a term like $Q_{abc}Q_{bcd}Q_{dee}$ is allowed, but $Q_{abb}Q_{cde}Q_{cde}$ is not. Furthermore, for simplicity, the Hamiltonian constraint is a polynomial in terms of Q and P, up to third order.

An important observation of the ADM-formalism is that the Hamiltonian is given by a linear combination of *first class* constraints. Therefore, a physically reasonable requirement on the Hamiltonian is that the constraint algebra closes, such that the Hamiltonian is indeed first-class. If one furthermore assumes that the Hamiltonian is invariant under a time-reversal symmetry (such that $P \to -P$), one arrives at the unique Hamiltonian constraint of the canonical tensor model [65]:

$$\mathcal{H}_a := \frac{1}{2}(P_{abc}P_{bde}Q_{cde} - \lambda Q_{abb}). \tag{4.10}$$

This Hamiltonian was built from first principles to resemble closely the ADM formalism of general relativity. From the point of view of algebraic tensor models it might actually be possible to derive this Hamiltonian directly from general relativity, if one takes the interpretation of Section 3.3 into account. It is not known presently if this is possible, but the Poisson-algebraic resemblance is encouraging.

4.2.3 The dynamics of the model

In this section the dynamics of the model will be briefly reviewed, by citing the constraint algebra and equations of motion of the model.

In order to describe the algebra, it is useful to introduce some notation;

$$\mathcal{H}(\xi) \equiv \xi_a \mathcal{H}_a,$$
$$\mathcal{J}(\eta) \equiv \eta_{ab} \mathcal{J}_{ab},$$

where ξ and η are an arbitrary vector and respectively an anti-symmetric two-tensor. The constraint algebra of the canonical tensor model is then given by

$$\begin{aligned}\{\mathcal{H}(\xi^1),\mathcal{H}(\xi^2)\} &= \mathcal{J}\left([\tilde{\xi}^1,\tilde{\xi}^2]+2\lambda\xi^1\wedge\xi^2\right),\\ \{\mathcal{J}(\eta),\mathcal{H}(\xi)\} &= \mathcal{H}(\eta\xi),\\ \{\mathcal{J}(\eta^1),\mathcal{J}(\eta^2)\} &= \mathcal{J}([\eta^1,\eta^2]).\end{aligned} \tag{4.11}$$

Here $\tilde{\xi}_{ab} := P_{abc}\xi_c$, $(\xi^1\wedge\xi^2)_{ab} := \xi^1_a\xi^2_b - \xi^1_b\xi^2_a$ and $[\eta^1,\eta^2]_{ab} := \eta^1_{ac}\eta^2_{cb} - \eta^2_{ac}\eta^1_{cb}$. When comparing this algebra to the ADM algebra in Eq. (1.5), it can be seen that the algebra has a non-trivial dependence on one of the canonical variables, P_{abc}, on the right hand side of the first equation. This is in accord with the ADM formalism, as can be seen in Eq. (1.5), since it also has a non-trivial dependence on the spatial metric. It is a rather nontrivial fact that this dependence shows up in exactly the right place.

In the canonical tensor model, the tensors Q_{abc} and P_{abc} have no explicit time dependence and the Hamiltonian is treated as the generator of time translations

$$\begin{aligned}\dot{Q}_{abc} &\equiv \frac{dQ_{abc}}{dt} = \{Q_{abc},\mathbf{H}\},\\ \dot{P}_{abc} &\equiv \frac{dP_{abc}}{dt} = \{P_{abc},\mathbf{H}\}.\end{aligned} \tag{4.12}$$

Using the Hamiltonian of Eq. (4.9), one can calculate these equations of motion directly. The result is given by

$$\begin{aligned}\dot{Q}_{abc} &= \frac{1}{2}\sum_\sigma n_{\sigma_a}P_{\sigma_b gh}Q_{\sigma_c gh} + n_d P_{\sigma_a de}Q_{e\sigma_b\sigma_c} + n_{\sigma_a d}Q_{d\sigma_b\sigma_c},\\ \dot{P}_{abc} &= \frac{1}{2}\sum_\sigma -n_d P_{\sigma_a de}P_{e\sigma_b\sigma_c} + \lambda n_{\sigma_a}\delta_{\sigma_b\sigma_c} + n_{\sigma_a d}P_{d\sigma_b\sigma_c}.\end{aligned} \tag{4.13}$$

4.3 The quantum canonical tensor model

In this section, the quantum version of the canonical tensor model will be introduced, since the main objective of the model is to provide a quantum mechanically consistent model for gravity. In Section 4.3.1 the canonical quantisation procedure will be used in order to define the quantum mechanical system. After that, in Section 4.3.2, the physical Hilbert space will be examined, and some exact solutions to the Wheeler-de Witt equations will be given.

4.3.1 Canonical quantisation procedure

The quantum mechanical version of the canonical tensor model is defined using a canonical quantisation procedure. Since the configuration space of the canonical tensor model is $\mathbb{R}^{\mathcal{N}}$, this procedure, originally introded by Dirac [113], is well-known in quantum mechanical literature [114]. The quantisation of the canonical tensor model was performed originally in [115].

The canonical quantisation procedure commences in three steps. First, one defines a Hilbert space as a space for operators to act on. Second, the fundamental operators and their commutation relation is chosen in accord with their classical Poisson bracket. Lastly

the other relevant observables on the Phase space are mapped to quantum operators acting on the Hilbert space.

In the canonical tensor model, the most natural choice for the kinematical Hilbert space are the square integrable functions over the configuration space $\mathcal{C} \cong \mathbb{R}^{\mathcal{N}}$;[2]

$$\mathscr{H}_{kin} := L^2\left(\mathbb{R}^{\mathcal{N}}\right), \tag{4.14}$$

where $\mathcal{N} := \frac{1}{6}N(N+1)(N+2)$. The fundamental operators are chosen to be the self-adjoint operator versions of the canonical variables of the phase space

$$\begin{aligned}\hat{Q}_{abc} &: \mathscr{H}_{kin} \to \mathscr{H}_{kin},\\ \hat{P}_{abc} &: \mathscr{H}_{kin} \to \mathscr{H}_{kin}.\end{aligned}$$

The commutation relations between these operators are the quantum analogy of the Poisson bracket in Eq. (4.4), and are given by

$$\begin{aligned}[\hat{Q}_{abc},\, \hat{P}_{def}] &= i\sum_{\sigma} \delta_{a\sigma_d}\delta_{b\sigma_e}\delta_{c\sigma_f},\\ [\hat{Q}_{abc},\, \hat{Q}_{def}] &= 0,\\ [\hat{P}_{abc},\, \hat{P}_{def}] &= 0,\end{aligned} \tag{4.15}$$

where σ are the permutations of (def). A possible representation of this algebra is, for $\psi \in \mathscr{H}_{kin}$,

$$\hat{Q}_{abc}(\psi) = Q_{abc}\psi, \hat{P}_{abc}(\psi) \quad = i\frac{\partial}{\partial Q_{abc}}\psi.$$

Note that the operators on a Hilbert space are densely defined, so this last equation makes sense.

The main other relevant observable in the canonical tensor model is the Hamiltonian Eq. (4.9). The quantum versions of these observables are given by the operators

$$\begin{aligned}\hat{\mathcal{H}}_a &:= \frac{1}{2}(\hat{P}_{abc}\hat{P}_{bde}\hat{Q}_{cde} - \lambda\hat{Q}_{abb} + i\lambda_H\hat{P}_{abb}),\\ \hat{\mathcal{J}}_{ab} &:= \frac{1}{4}(\hat{Q}_{acd}\hat{P}_{bcd} - \hat{Q}_{bcd}\hat{P}_{acd}).\end{aligned} \tag{4.16}$$

Note that there is an extra term proportional to λ_H in the Hamiltonian constraint operator $\hat{\mathcal{H}}_a$ coming from the normal ordering. This constant λ_H may be fixed by requiring the operator to be self-adjoint (which is desired for proper observables)

$$\lambda_H = \frac{1}{2}(N+2)(N+3). \tag{4.17}$$

Lastly, for consistency one wants to confirm whether the constraint algebra is still closed. It turns out that the algebra remains of the same form, which is encouraging:

$$\begin{aligned}[\hat{\mathcal{H}}(\xi^1), \hat{\mathcal{H}}(\xi^2)] &= i\hat{\mathcal{J}}\left([\hat{\xi}^1, \hat{\xi}^2] + 2\lambda\xi^1 \wedge \xi^2\right),\\ [\hat{\mathcal{J}}(\eta), \hat{\mathcal{H}}(\xi)] &= i\hat{\mathcal{H}}(\eta\xi),\\ [\hat{\mathcal{J}}(\eta^1), \hat{\mathcal{J}}(\eta^2)] &= i\hat{\mathcal{J}}([\eta^1, \eta^2]).\end{aligned}$$

Here, $[,]$ denotes a commutator, $\xi^1 \wedge \xi^2_{ab} := \xi^1_a\xi^2_b - \xi^2_a\xi^1_b$, and $\hat{\xi}_{ab} := \hat{P}_{abc}\xi_c$.

[2] The Hilbert space is called the "kinematical Hilbert space" because the Hamiltonian constraints that will be dealt with in Section 4.3.2 in order to define the "physical Hilbert space".

4.3.2 The physical Hilbert space and some wave functions

Since the Hamiltonian of the canonical tensor model is fully constraint, just like the Hamiltonian of general relativity, one needs to implement this on a quantum level as well. The approach taken is the same as in other canonical quantum gravity approaches [54], whereone restricts the Hilbert space $\mathscr{H}_{kin}$ to the so-called *physical Hilbert space* $\mathscr{H}_{phys}$.

The physical Hilbert space is defined as the states that are annihilated by the constraints in Eq. (4.16):

$$\begin{aligned} \hat{\mathcal{H}}_a(\Psi) &\equiv 0, \\ \hat{\mathcal{J}}_{ab}(\Psi) &\equiv 0. \end{aligned} \tag{4.18}$$

The first of these equations is the canonical tensor model equivalent of the Wheeler-deWitt equation, and these equations are the direct equivalent of the requirements on the physical Hilbert space in canonical quantum gravity (see Eq. (1.7)).

A formal picture of the process of applying the constraints above is given by

$$\mathscr{H}_{kin} \xrightarrow{\hat{\mathcal{J}}_{ab}} \mathscr{H}_{sym} \xrightarrow{\hat{\mathcal{H}}_a} \mathscr{H}_{phys}.$$

Though the application of the $\mathcal{J}_{ab}$ constraint does not seem to pose any problems, it should be noted that currently the physical Hilbert space is not fully understood. It is a known fact that if the spectrum of the Hamiltonian constraint is continuous around zero, the elements of the physical Hilbert space will become distributions instead of simple functions [54, 116]. However, this might not be a problem in the canonical tensor model, and the existence of the solutions to Eq. (4.18) below are encouraging in that regard. However, it might still be possible that the standard inner product of $\mathscr{H}_{kin}$ is not the best suited anymore.

There are several general solutions to the constraint equations Eq. (4.18) found [115, 117, 118][3] and analysed [5, 4, 120, 121, 84, 1, 85], some of which are defined for general N. It is a very nontrivial fact and feature of the model, that explicit solutions to the constraint equations may actually be found. There are two types of functions known for general N, one in the Q-representation, and one in the P-representation. Both of them will be cited here. For a derivation, see [118].

In the Q-representation, the constraints have the following form

$$\begin{aligned} \hat{\mathcal{H}}_a &= \frac{1}{2}(Q_{cde}\partial^{(Q)}_{abc}\partial^{(Q)}_{bde} + i\lambda_H \partial^{(Q)}_{abb}), \\ \hat{\mathcal{J}}_{ab} &= \frac{1}{4}(Q_{acd}\partial^{(Q)}_{bcd} - Q_{bcd}\partial^{(Q)}_{acd}). \end{aligned}$$

where $\partial^{(Q)}_{acd} := \frac{\partial}{\partial Q_{acd}}$. By using the following Ansatz

$$\Psi(Q) = \int_{\mathbb{R}^N} \mathrm{d}\phi f(\phi^2)\, \mathrm{e}^{iQ\phi^3},$$

where $\pi \in \mathbb{R}^N$, one can find a solution to the constraint equations. Here, the following notations are used

$$\begin{aligned} \phi^2 &\equiv \phi_a\phi_a, \\ \mathrm{d}\phi &\equiv \prod_a \mathrm{d}\phi_a, \\ Q\phi^3 &\equiv Q_{abc}\phi_a\phi_b\phi_c. \end{aligned} \tag{4.19}$$

[3] Some of the functions were found because of analysis done for random tensor networks related to the canonical tensor model [117, 118, 119, 68].

The final solution is given by

$$\Psi(Q) = \int \mathrm{d}\phi (\phi^2)^\alpha \, \mathrm{e}^{iQ\phi^3} \, . \tag{4.20}$$

In the P-representation, one might expect solutions to be found as well, since the the Hamiltonian constraint becomes a first-order system of partial differential equations. In this representation it is given by

$$\hat{H}_a = \frac{i}{2}(P_{abc}P_{bde}\partial_{cde} + \lambda_H P_{abb} - \lambda \partial_{abb}).$$

In order to solve this, one can use a similar Ansatz as was used for the Q-representation

$$\Psi(P) = \int \mathrm{d}\phi f(\phi^2) e^{iP\phi^3}.$$

For a fixed value of λ_H, there is a solution given by

$$\Psi_{\lambda_H}(P) = (\Psi(P))^{\lambda_H/2}, \tag{4.21}$$

where

$$\Psi(P) = \int \mathrm{d}\phi \mathrm{d}\tilde{\phi} \, \mathrm{e}^{i\left(P\phi^3 + \phi^2\tilde{\phi} - \frac{4}{27\lambda}\tilde{\phi}^3\right)} \, . \tag{4.22}$$

Here, again $\phi \in \mathbb{R}^N$ and there is a new integration variable $\tilde{\phi} \in \mathbb{R}$ used. This integration variable comes from the Airy function, which is used to solve part of the differential equation

$$Ai(x) = \int_{-\infty}^{\infty} \mathrm{d}t \;\; \mathrm{e}^{i(xt + \frac{1}{3}t^3)} \, .$$

4.4 The current state of the model

Since its introduction in [64], many aspects of the model have been analysed as a serious candidate for quantum gravity. In this section, some of the results will be explained, in order to understand the potential of this model. There are three main areas of development: The classical relationship to general relativity, analysis of the quantum model and wave functions, and the interpretation of the model.

4.4.1 Classical results

When constructing a theory for quantum gravity, it is important that the results are consistent with the currently most prominent theory for classical gravity available: General relativity. As already alluded to in Chapter 1, general relativity has been tested to extremely high precision, and any "classical limit" of a quantum theory of gravity should at least reproduce the same results.

As explained in Section 4.2, the phase space of the model has been chosen as symmetric tensors because they may be used to represent spatial slices according to Chapter 3. The Hamiltonian is then built on first principles, to resemble the ADM-formalism of general relativity as closely as possible while keeping computations manageable. In that sense the model is expected to be the simplest nontrivial candidate theory, and it is encouraging that the model has a very rich structure already.

There are two main approaches that have been tried to tie the model to general relativity, which may be considered opposites. Firstly, the $N = 1$ case has been examined [66]. In this case, the dynamics of the model simplifies drastically. By doing a Legendre transform of the Hamiltonian in this case, the following action is found

$$S_{N=1\,\mathrm{CTM}} := \int dt \left(\frac{\dot{Q}^2(t)}{4n(t)Q(t)} + \lambda n(t) Q(t) \right),$$

where $Q \equiv \frac{1}{3} Q_{111}$ is the only variable in the configuration space, and $n(t)$ is the lapse function. This exactly corresponds to the minisuperspace approximation given by the action of the Friedmann-Robertson-Walker universe [53, 122]. This is very encouraging, as this is exactly what one would hope for.

The second case that has been examined is a formal continuum limit [67, 68]. The limit is formal in the sense that, instead of showing the emergence of a continuous space, it was assumed to be the case. This emergence of a space was done by replacing the labels a by elements of a D-dimensional space

$$a \to x \in \mathbb{R}^D,$$

and the sum over elements is replaced by an integral

$$\sum_a \to \int \mathrm{d}^D x.$$

Furthermore, the tensor, now written as P_{xyz} was assumed to be almost local. For a local tensor, one would expect P_{xyz} to only be non-zero for $x \sim y \sim z$. Such a tensor may be written as

$$P^0_{xyz} = \int_{\mathbb{R}^D} \mathrm{d}\omega \beta(\omega) \delta^D(x-\omega) \delta^D(y-\omega) \delta^D(z-\omega),$$

which, being strictly local, would not have any expected dynamics, but by considering a derivative expansion of a tensor P_{xyz} around this local tensor, one could understand the dynamics.

In [67] it was shown that the constraint algebra of general relativity is recovered from the canonical tensor model constraint algebra of Eq. (4.11) when taking such a continuum limit. Moreover, in [68] it was then shown that, when one considers the equations of motion of the canonical tensor model in Eq. (4.13) in the Hamilton-Jacobi formalism, the continuum limit introduced here corresponds to general relativity coupled to a scalar field.[4]

It is remarkable that a tensor model that is defined from first principles, has such clear connections to general relativity. There is still work to be done, since the continuum limit discussed here is rather formal. Note that the interpretation here seems to differ from the algebraic interpretation introduced in Chapter 3, but this is mainly an artifact of the overcountably infinite-dimensional vectorspace of smooth functions $C^\infty(\mathbb{R}^D)$. Note that this approach seems to correspond to a basis of functions where the functions $f_{x_0}(x)$ are localised around x_0. Therefore, the interpretation seems consistent, but the mathematics for dealing with these cases has yet to be developed.

[4] This study motivated the introduction of the "mother canonical tensor model", which derives the canonical tensor model through the Hamilton-Jacobi procedure [123].

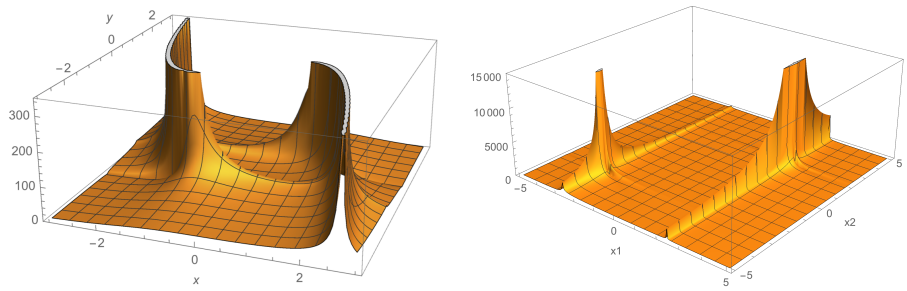

Figure 4.1: Some examples of the calculations performed in [5] in order to demonstrate the symmetry highlighting in the canonical tensor model. On the left: The size of $|\Psi|^2$ in terms of the parameterisation of the $O(2)$ symmetric tensor in x, y. The high peaks follow the curve of Eq. (4.23), which are exactly the configurations that lead to an $O(1,2)$ symmetry. On the right: The symmetry highlighting mechanism for $N = 7$. In this parameterisation, the line $x_1 = \pm 2.5$ represents an $SO(4)$ symmetric tensor, while $x_2 = \pm 2.5$ represents $O(2)$. At the intersection of these lines, for $x_2 = x_1$ the symmetry is enhanced to $O(6)$ and at $x_2 = -x_2$ to $O(4) \times O(2)$.

4.4.2 Interesting properties of the quantum model

The canonical tensor model, having a quite interesting classical structure as explained in Section 4.4.1, also has interesting quantum mechanical properties. In this section, some of the results in that area will be discussed.

Interestingly, as has already been explained in Section 4.3.2, the canonical tensor model has known exact solutions to the constraint equations. This is highly nontrivial and remarkable for any theory of quantum gravity. Some of the wave-functions have been analysed in the past, and some of the results will be presented here.

The main function analysed was the one in the P-representation, in Eq. (4.22). The original analysis was done by using a mechanism introduced in [5]. The mechanism consists of a conjecture that quantities of the form

$$\Psi(Q) := \int_{\mathcal{C}} \mathrm{d}\phi e^{iS(Q,\phi)},$$

where Q is some configuration variable and ϕ an integration variable, and $S(Q,\phi)$ is invariant under transformations of some Lie-group G, will exhibit highlighting of symmetries. This means that the function Ψ will be peaked at configurations Q which are invariant under some subgroup $H \subset G$. In [5] the discussion was still quite general, and a relatively simple toy-model was evaluated and the mechanism was shown to work.

In [4] the mechanism was subsequently applied to the canonical tensor model wave function of Eq. (4.22). The reason for choosing this wave function is that the exponent $S(P_{abc}, \phi)$ is sufficiently complex such that there are nontrivial fixed points to be expected. For $N = 3$, there is only one possible Lie-subgroup $O(2) \subset O(3)$. This case was shown to be highlighted for specific configurations, where the real symmetry is actually $O(1,2)$; a symmetry which suggests the existence of a time symmetry in the model as well. In some ways this does make sense, since the wave function was obtained using the Hamiltonian constraint - which generates time translations - as well. The $SO(2)$ symmetry is parameterised by two parameters, here called x and y. In Fig. 4.1 an example is shown how the curves symmetric under $O(1,2)$ are highlighted. These

curves are given by

$$y = x - \frac{4}{27}x^3 \tag{4.23}$$

Another example for $N = 7$ is given, where the biggest subgroup possible is $O(6)$, and smaller symmetry groups have considerably smaller peaks.

The emergence of these symmetries is an important indication that the canonical tensor model might show the emergence of macroscopic spaces, as these kind of spaces usually have some symmetric properties. With the algebraic tensor model interpretation, this becomes even clearer because a single point would generate a whole continuous set of points under this Lie-group symmetry, as explained around Eq. (3.29).

Further analysis of this wave function was done in [120, 121, 2] for the case of a negative cosmological constant and a toy model in [124]. These works suggest that for the emergence of macroscopic spacetimes, a positive cosmological constant is needed. Stronger evidence for this was found in [84].

4.4.3 Extracting geometry and the interpretation of the model

In this section, two more topics will be discussed. Firstly, the extraction of geometrical and topological properties from a tensor using techniques from data analysis will be discussed. To finalise this chapter, some closing remarks on the interpretation of the model as an algebraic tensor model will be made.

In [6], important groundwork was laid for the interpretation of the canonical tensor model as a model describing dynamical spacetime. Though the relationship to general relativity discussed in Section 4.4.1 is encouraging, an explicit interpretation of the tensor relating it to Riemannian geometry was not found yet. The work in [6] explained how one could regain topological and geometric data from a tensor P_{abc}. Two crucial elements in this setup were the tensor rank decomposition and persistent homology.

The tensor rank decomposition used in this work is defined in Section 3.2.2 as the minimal tensor rank decomposition, denoted as

$$P_{abc} = \sum_{i=1}^{R} \phi_a^i \phi_b^i \phi_c^i.$$

The elements ϕ^i were then treated as the "set of points" corresponding to the tensor. Then, an inner product was introduced on the set of points

$$\phi^i \cdot \phi^j := \sum_{a=1}^{N} \phi_a^i \phi_a^j.$$

It was then argued that, if this inner product is large, points are to be treated as close. If one then picks a cutoff value ϵ, one can construct a graph, where points that have an inner product larger than ϵ are connected. The graph distance can then be used as a parameter for persistent homology. Using this technique, the authors were able to show that for a wide variety of tensors that were constructed using almost the same inner product definition as Eqs. (3.35) and (3.36), the homology groups could be recovered. Moreover, by using a discrete version of the heat kernel and a virtual diffusion process, a distance measure on the set of points was defined, implying that the geometric data of a Riemannian manifold could be recovered.

Note that in the above, it was not claimed that a rigorous understanding of the interpretation of tensors had been found. But, the work was very important to show that topological and geometric data was clearly encoded in the tensor, and could partly be recovered (albeit with some margins for error).

The original interpretation of the canonical tensor model, was as a tensor model of fuzzy spaces [64].[5] This came from a tensor P_{abc} that was interpreted as the structure constants of some algebra of functions

$$f_a \cdot f_b = P_{abc} f_c,$$

with inner product structure

$$P_{abc} = \langle f_c |\, f_a \cdot f_b \rangle \,.$$

This interpretation was, however, difficult to maintain, since for a finite tensor an associative algebra always corresponds to a discrete non-connected set of points. Therefore, in order for the space to be non-connected, one needs to accept non-associative algebras, making the interpretation as an algebra of functions over a manifold difficult.

The work in [3], explained in Chapter 3, aimed to combine the spirit of the original interpretation of the canonical tensor model with the results of [6]. This led to the interpretation that the tensor P_{abc} does not correspond to the structure constants of the whole algebra, but the structure constants of the generators of the algebra. This combined with conjecture Conjecture 3.1 could merge these two pictures. One of the nice things of this interpretation is that it gives a very clear view of the $N = 1$ case above. The $N = 1$ case would in this picture correspond to only taking into account the constant function of the algebra, and therefore only describes the homogeneous and isotropic part of the universe.

[5] Note that the original definition was as a complex tensor model symmetric under a generalised Hermiticity condition. However, here only the real version will be treated.

Chapter 5

A matrix model from a simplified wave function

In this chapter, the results of [1] will be explained. For this, Section 5.1 first explains the model that is analysed and the previously obtained results in [120, 121]. Section 5.2 then discussed the numerical analysis performed, and finally its results and implications for the canonical tensor model are discussed in Section 5.3.

5.1 Pairwise contracted matrix model

In this section, the matrix model under consideration will be introduced. The matrix model turns up in a simplication of one of the wave functions of the canonical tensor model. In Section 5.1.1, the matrix model will be defined. Then, in Section 5.1.2, a brief review will be given of a perturbation theory analysis that was done in [120]. Section 5.1.3 will then discuss the main results from initial numerical analysis done in [121].

5.1.1 Simplifying a wave function

The model under consideration is defined from a simplification of the wave function in Eq. (4.22), given by

$$\psi_\epsilon(P) = \int_{\mathbb{R}^N} \mathrm{d}\phi \, \mathrm{e}^{i\left(P\phi^3+\phi^2\right)-\epsilon\phi^2}, \tag{5.1}$$

where the same notation as in Eq. (4.19) was used. This function is actually the same as the toy model analysed in [5]. The parameter $\epsilon > 0$ is introduced to ensure the convergence of the function, and in the end the limit

$$\psi(P) \equiv \lim_{\epsilon\to 0} \psi_\epsilon(P),$$

is supposed to be taken. Note that in the case of the canonical tensor model, the full wave function is given by a quantity of the form $\psi(P)^R$.

An interesting quantum-mechanical question is whether the wave function is actually normalised, i.e. if it is part of the kinematical Hilbert space. To answer this question, one can study the following object with $\alpha > 0$

$$g(N,R,\kappa) := \int_{\mathbb{R}^{\mathcal{N}}} \mathrm{d}P \; \mathrm{e}^{-\alpha P^2} \, \psi_\epsilon(P)^R, \tag{5.2}$$

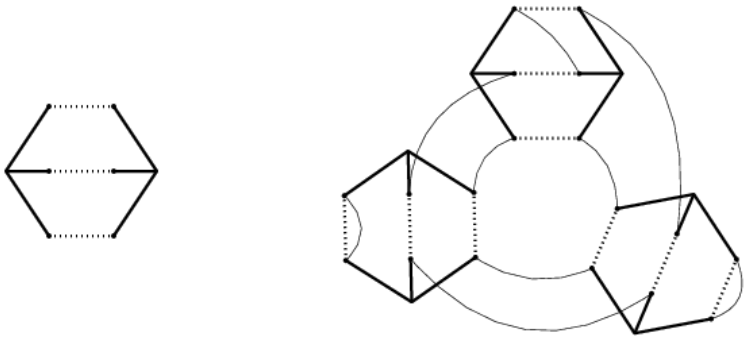

Figure 5.1: Examples of Feynmann graphs appearing in this model taken from [120]. On the left, an interaction $\sum_{i,j=1}^{R}\sum_{a,b,c=1}^{N}\phi_a^i\phi_b^i\phi_c^i\phi_a^j\phi_b^j\phi_c^j$ is shown. The ϕ_a^i are located at the points, whereas the solid lines represent contractions of i and j, and the dashed lines represent contractions of a, b, c. On the right, an example of a connected Feynmann diagram with three vertices. The thin lines represent Wick contractions.

where $\mathcal{N} = \frac{1}{6}N(N+1)(N+2)$ denotes the amount of degrees of freedom of the tensor P. If, for $\alpha \to 0$ this quantity converges, the function is square integrable. Define

$$Z_{N,R}(\lambda, k) := \int_{\mathbb{R}^{NR}} \mathrm{d}\phi \ \mathrm{e}^{-\lambda U(\phi) - k\,\mathrm{Tr}[\phi\phi^t]}, \tag{5.3}$$

where the R copies of ϕ_a are now treated as matrices ϕ_a^i with $a \in \{1, \ldots, R\}$. $Z_{N,R}(\lambda, k)$ is the partition function of the matrix model under consideration, for the interaction term

$$U(\phi) := \sum_{a,b,c=1}^{N}\sum_{i,j=1}^{R}\phi_a^i\phi_b^i\phi_c^i\phi_a^j\phi_b^j\phi_c^j,$$

where

$$U_{ij} := \sum_{a,b,c=1}^{N}\phi_a^i\phi_b^i\phi_c^i\phi_a^j\phi_b^j\phi_c^j.$$

One can now show that

$$g(N, R, \alpha) = \left(\frac{\pi}{\alpha}\right)^{\frac{\mathcal{N}}{2}} Z_{N,R}\left(\frac{1}{4\alpha}, -i + \epsilon\right).$$

Note that the matrix model in Eq. (5.3) has some unusual properties. Firstly, it is a rectangular matrix model, since N and R do not need to be the same. Secondly, while the lower indices are always contracted pairwise, as is usual in matrix models, the upper indices are contracted pairwise due to the interaction term. Due to this, the model does not have the full $O(R) \times O(N)$ symmetry that is usually the symmetry of this kind of model. However, the lower indices still respect the $O(N)$ symmetry and the upper indices are now only symmetric under permutations. The correct symmetry group of the model is thus $S_R \times O(N)$.

5.1.2 Analytical results

In [120], several analytical calculations were performed for the matrix model above in Eq. (5.3). This was done for both a Feynmann graph expansion, and through the analysis of a convergent series.

The Feynmann expansion analysis is useful for exploring which graphs dominate in what kind of regime. To this end, one uses a formal expansion of the partition function above as

$$Z_{N,R}(\lambda, k) \propto \sum_{n \leq 1} \frac{(-\lambda)^n}{n!} \langle U(\phi)^n \rangle_0 \,,$$

where

$$\langle U(\phi)^n \rangle_0 \equiv \int_{\mathbb{R}^{NR}} \mathrm{d}\phi \, U(\phi)^n \, \mathrm{e}^{-k \, \mathrm{Tr}[\phi\phi^t]} \,.$$

Some examples of generated Feynmann diagrams are shown in Fig. 5.1. One can consider several $N \to \infty$ limits of these models, where one takes

$$R \sim N^\alpha.$$

For $0 < \alpha \leq 1$, the so-called tree-like graphs dominate, while for $\alpha > 2$ the so-called star-like graphs dominate. $1 < \alpha \leq 2$ is a mixed region, where small graphs tend to be tree-like, and large graphs tend to be star-like.[1]

After the analysis above, the authors described the model as a convergent series. To do this, they split up the variables ϕ^i_a into radial and angular components. Since $\pi^i_a \in \mathbb{R}^{NR}$, this means a decomposition into $\mathbb{R}_+ \times S^{NR-1}$. The angular part may then be written as a convergent series, whereas the radial part can be integrated explicitly. This leads to

$$Z_{N,R} = V_{NR-1} \int_0^\infty \mathrm{d}r \, r^{NR-1} f_{N,R}(\lambda r^6) \, \mathrm{e}^{-kr^2},$$

with

$$f_{N,R}(t) \equiv \frac{1}{V_{NR-1}} \int_{S^{NR-1}} \mathrm{d}\tilde{\phi} \, \mathrm{e}^{-tU(\tilde{\phi})},$$

and $V_{NR-1} = \int_{S^{NR-1}} \mathrm{d}\tilde{\phi}$ is the volume of a $NR-1$-sphere. $\tilde{\phi} \equiv \phi/r$ is the angular part of ϕ, and $\mathrm{d}\tilde{\phi}$ denotes the standard measure on the $NR-1$-sphere.

Up to leading order this leads to the conclusion that there seem to be two phases for the limit $\kappa \to 0_+$ of $g(N, R, \kappa)$. If $R < \frac{1}{2}(N+1)(N+2)$, the function $g(N, R, \kappa)$ diverges with a powerlaw, whereas the function converges for $R > \frac{1}{2}(N+1)(N+2)$. For $R = R_c \equiv \frac{1}{2}(N+1)(N+2)$ the function has a logarithmic divergence. Combined with the discussion above, this is an interesting result. Since the critical value R_c goes like

$$R_c \equiv \frac{1}{2}(N+1)(N+2) \sim N^2, \tag{5.4}$$

this means that this critical value is exactly at the border of the star-like graph domination. Moreover, this value, to leading order, is almost the same as the value required by the canonical tensor model wavefunction Eq. (4.22), namely

$$R = \frac{1}{2}(N+2)(N+3).$$

However, since this is only up to leading order, it is hard to draw strong conclusions for this.

[1] For the discussion here, the precise nature of the star-like and tree-like graphs is of less importance. For this one can refer to [120].

5.1.3 Monte carlo analysis

In [121], the model above was numerically analysed. Consider some observable $\mathcal{O}$ which is observable under the $S_R \times O(N)$ symmetry mentioned above. The expectation value of this operator is given by

$$\langle \mathcal{O} \rangle = \frac{1}{Z_{N,R}(\lambda, k)} \int_{\mathbb{R}^{NR}} \mathrm{d}\phi \mathcal{O}(\phi)\, \mathrm{e}^{-\lambda U(\phi) - k\,\mathrm{Tr}[\phi\phi^t]}\,.$$

The expectation value of several observables, namely $\mathrm{Tr}[\phi\phi^t], U(\phi)$, and a third observable

$$U_d(\phi) \equiv \sum_{i=1}^{R} U_{ii}(\phi), \tag{5.5}$$

were then numerically calculated using the famous metropolis algorithm [125]. This led to some interesting results, for instance that there seems to be a transition region at $k \to 0$ for $R \sim \frac{1}{2}(N+1)(N+2) = R_c$. This is interestingly in line with the predictions of [120]. However, due to drastic slowing down of the simulations for this region in the radial direction, it could not be established whether this was a phase transition or just a cross-over.

5.2 Numerical analysis

In this section, the numerical analysis performed in [1] is explained. While there was already some numerical analysis done in [121], as explained in Section 5.1.3, the results were not conclusive enough yet due to the slowing down of the metropolis algorithm. In [1], this was largely solved by first integrating out the troublesome radial direction, and using an extension of the metropolis algorithm, called "Hamiltonian Monte Carlo".

It should be noted that the model under consideration in [1] differs slightly from the original in Eq. (5.3), namely

$$Z_{N,R}(\lambda, k) := \int_{\mathbb{R}^{NR}} \mathrm{d}\phi\ \mathrm{e}^{-\lambda U(\phi) - k\, U_d(\phi)}\,. \tag{5.6}$$

The reason for this is that it allows to integrate out the radial direction, which was the main reason for the slowdown in the simulation as explained in Section 5.1.3. Note that for $k > 0$, the partition function above is guaranteed to be convergent, but for $k = 0$ this is not necessarily the case. One of the main questions under investigation in [1] is whether a sensible $\frac{k}{\lambda} \to 0$ limit exists. The answer will be positive, where there is a critical value R_c such that for $R < R_c$ the function converges.

Note that from a physical point of view, the replacement of the term $-k\,\mathrm{Tr}[\phi\phi^t] = -k\sum_{i=1}^{R}\sum_{a=1}^{N}\phi_a^i\phi_a^i$ by $-kU_d(\phi) = \sum_{i=1}^{R}(\sum_{a=1}^{N}\phi_a^i\phi_a^i)^3$ may be interpreted by considering the wave function Eq. (4.22). The wave function in question was given by

$$\Psi(P) \propto \left[\int_{\mathbb{R}^N} \mathrm{d}\phi\, \mathrm{e}^{iP\phi^3}\, \mathrm{Ai}(\kappa\phi^2)\right]^{\lambda_H/2}, \tag{5.7}$$

where the same notation as in Eq. (4.19) was used, and $\kappa = -(9\lambda/4)^{1/3}\frac{4}{9\lambda}$. If one now wants to probe the square integrability, as in Eq. (5.2), one finds

$$\begin{aligned}\langle\Psi|\ \mathrm{e}^{-\alpha P^2}\ |\Psi\rangle &:= \int_{\mathbb{R}^{\mathcal{N}}} \mathrm{d}P\ \mathrm{e}^{-\alpha\, P^2}\ |\Psi(P)|^2, \\ &\propto \alpha^{-\mathcal{N}/2} \int_{\mathbb{R}^{NR}} \mathrm{d}\phi\, \mathrm{e}^{-\frac{1}{4\alpha}\sum_{i,j=1}^{R} U_{ij}(\phi)} \prod_{i=1}^{R} \mathrm{Ai}(\kappa\phi^2).\end{aligned} \tag{5.8}$$

For $\kappa > 0$, the Airy function has the properties of a damping function. An example of a simplification in this region would thus be given by replacing the Airy function by a damping function which is easier to handle, such as

$$\mathrm{e}^{-k(\phi^2)^3} = \mathrm{e}^{-k\, U_d(\phi)},$$

where k is some constant.

The integration of the radial part and angular expectation values of operators will be explained in Section 5.2.1. Section 5.2.2 will then explain the Hamiltonian Monte Carlo method, and in Section 5.2.3 the results of this analysis will be discussed.

5.2.1 Integrating out the radial direction

A useful trick when dealing with numerical analysis of multi-dimensional non-compact integrals like Eq. (5.6) is try to integrate out some variables such that one is left with a compact integral, which is much easier to deal with when performing Monte Carlo simulation. This is easier to do for Eq. (5.6) than for Eq. (5.3), and this is one of the reasons for considering a different matrix model.

If one divides ϕ_a^i up into a radial part r and an angular part $\tilde{\phi}_a^i$ such that

$$\phi_a^i = r\tilde{\phi}_a^i,$$

one can do a variable transformation to hyperspherical variables. When doing this one can now integrate out the r variable:

$$\begin{aligned}Z_{N,R}(\lambda,k) &= \int_{S^{NR-1}} \mathrm{d}\tilde{\phi} \int_0^\infty \mathrm{d}r\, r^{NR-1}\, \mathrm{e}\left(-\left(\lambda\sum_{i,j=1}^{R} U_{ij}(\tilde{\phi}) + k\sum_{i=1}^{R} U_{ii}(\tilde{\phi})\right) r^6\right), \\ &= \frac{1}{6}\,\Gamma\left(\frac{NR}{6}\right) \int_{S^{NR-1}} \mathrm{d}\tilde{\phi} \left(\lambda\sum_{i,j=1}^{R} U_{ij}(\tilde{\phi}) + k\sum_{i=1}^{R} U_{ii}(\tilde{\phi})\right)^{-\frac{NR}{6}},\end{aligned}$$

where S^{NR-1} denotes $NR-1$ dimensional sphere, $d\tilde{\phi}$ is the volume element on S^{NR-1}, and $\Gamma(\cdot)$ is the gamma function. The benefit is that now one has a compact integral left, which drastically simplifies numerical computations.

Let us define the angular part of the partition function as

$$z_{N,R}(\lambda,k) := \int_{S^{NR-1}} \mathrm{d}\tilde{\phi} \left(\lambda\sum_{i,j=1}^{R} U_{ij}(\tilde{\phi}) + k\sum_{i=1}^{R} U_{ii}(\tilde{\phi})\right)^{-\frac{NR}{6}}, \tag{5.9}$$

and the angular expectation value of an observable $\mathcal{O}$ as

$$\langle\mathcal{O}(\tilde{\phi})\rangle := \frac{1}{z_{N,R}(\lambda,k)} \int_{S^{NR-1}} \mathrm{d}\tilde{\phi}\, \mathcal{O}(\tilde{\phi}) \left(\lambda\sum_{i,j=1}^{R} U_{ij}(\tilde{\phi}) + k\sum_{i=1}^{R} U_{ii}(\tilde{\phi})\right)^{-\frac{NR}{6}}.$$

If one now has an operator on all of ϕ with weight w, meaning

$$\mathcal{O}(\phi) = r^w \mathcal{O}(\tilde{\phi}),$$

the expectation value of this operator can be found to be (see [1] for a derivation)

$$\langle \mathcal{O}(\phi) \rangle = \frac{\Gamma[\frac{NR+w}{6}]}{\Gamma[\frac{NR}{6}]} \left\langle \mathcal{O}(\tilde{\phi}) \left[\lambda \sum_{i,j=1}^{R} U_{ij}(\tilde{\phi}) + k \right]^{-w/6} \right\rangle, \tag{5.10}$$

where $\sum_{i=1}^{R} U_{ii}(\tilde{\phi}) = 1$ was used. This relation may now be used to relate observables of the original matrix model Eq. (5.6) to Eq. (5.9). The numerics are now done for Eq. (5.9).

5.2.2 Hamiltonian Monte Carlo

In this section, the Hamiltonian Monte Carlo numerical method will be introduced. For this, it is good to first recall how the standard Metropolis algorithm works. A comprehensive overview of the topic may be found in [126].

The Metropolis algorithm, originally introduced in 1953 [127], is a method used for obtaining random samples from some probability distribution $P(x)$ with $x \in X$, where X denotes the configuration space of some integral. The algorithm works as follows. One needs to define an arbitrary probability density $g(x|\, y)$, which gives a probability for y given the current value x. This probability density needs to satisfy the requirement for *detailed balance*, i.e.

$$g(x|\, y) = g(y|\, x).$$

An example of such a probability, which is also used in [121, 1], density would be a normal distribution around x. To initialise the algorithm one chooses a (random) point to start with, x_1. Say now, one is at a point x_n. Choose a candidate point y with probability $g(x_n|\, y)$. Then one defines the acceptance ratio

$$R := \min\left(\frac{P(y)}{P(x_n)}, 1\right).$$

Then, one accepts this candidate point as the new point $x_{n+1} = y$ with probability R. Otherwise, the next point remains unchanged $x_{n+1} = x_n$.

The algorithm above may be shown to have the distribution $P(x)$ as a fixed point, i.e.

$$\lim_{n\to\infty} p_n(x) = P(x),$$

where p_n denotes the probability of the random walker being at point x. This means that if, for some probability density $P(x)$, one wants to calculate expectation values of observables $\mathcal{O}(x)$ as

$$\langle \mathcal{O} \rangle := \int_X \mathrm{d}x\, \mathcal{O}(x)\, P(x) \approx \frac{1}{M} \sum_{k=1}^{M} O(q_k),$$

where q_k is a list of M points created by taking some of the points x_n above.[2] In particular, one can use it to calculate expectation values of a partition function.

[2] This is the place where it becomes a bit case-specific. Since the points x_{n+1} and x_n are not independent, one needs to do several iterations before adding a new point to the list q_k. How many depends on the acceptance rate above.

One of the downsides of the Metropolis algorithm is that in practice it can get "stuck" around local maxima. For this reason, an extension of the Metropolis algorithm was proposed in [128], nowadays called *Hamiltonian Monte Carlo* [126]. The goal is to use Hamilton dynamics to propose candidate points for the Metropolis algorithm that may be further away. For this, one first creates a phase space from the configuration space X, by introducing canonical conjugate variables p_i to x_i. Then one defines the Hamiltonian of the system by

$$H(x,p) = U(p) + V(x),$$

where $U(p) = \sum_{i=1}^{d} \frac{p_i^2}{2}$, where d is the dimension of the space, and $V(x) = -\log P(x)$. Solving the equations of motion is a difficult exercise in general, but one can use a discrete approximation to do this. The discrete approximation steps do needs to satisfy two criteria exactly: Time-reversal symmetry and the conservation of phase-space volume.

The *leap frog method* is a method that satisfies the above properties by executing a discrete jump in two steps, firstly by doing a free (geodesic) motion in the configuration variables x_i and subsequently updating the momentum variables p_i. For a flat phase space, these two subsequent steps are given by

$$\text{(1) } \delta x_i = \epsilon\, p_i,\ \delta p_i = 0,$$
$$\text{(2) } \delta x_i = 0,\ \delta p_i = -\epsilon \frac{\partial V(x)}{\partial x_i}.$$

It can be easily checked that these steps both satisfy the criteria. For the phase space volume one needs to check if

$$\begin{vmatrix} \frac{\partial}{\partial x_i}(x_i + \delta x_i) & \frac{\partial}{\partial p_i}(x_i + \delta x_i) \\ \frac{\partial}{\partial x_i}(p_i + \delta p_i) & \frac{\partial}{\partial p_i}(p_i + \delta p_i) \end{vmatrix} = 1,$$

for both of the steps individually, which is clearly the case. For the time-reversal symmetry, one has to confirm if

$$x_i + \delta x_i \to x_i, p_i + \delta p_i \quad \to p_i,$$

whenever one takes $\epsilon \to -\epsilon$. This is also clearly the case for the steps above.

In the matrix model discussed here, the configuration space is a unit hypersphere, yielding the constraints

$$\sum_{i=1}^{d} x_i^2 = 1,$$
$$\sum_{i=1}^{d} x_i p_i = 0.$$

It may be checked that the following two steps will satisfy these constraints and the time-reversal and phase-space volume requirements. For the first (geodesic motion) step:

$$\text{(1')} \begin{pmatrix} x_i' \\ p_i' \end{pmatrix} = \begin{pmatrix} \cos\theta & \frac{\sin\theta}{|p|} \\ -|p|\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_i \\ p_i \end{pmatrix}, \tag{5.11}$$

where $|p| = \sqrt{\sum_i p_i^2}$ and $\theta = \epsilon\,|p|$. The second step to update the momentum variable is then given by

$$\text{(2')}\ \delta x_i = 0,\ \delta p_i = -\epsilon \frac{\partial V(x)}{\partial x_i} + \epsilon x_i \sum_j x_j \frac{\partial V(x)}{\partial x_j}. \tag{5.12}$$

In Section 5.2.3, this method will be applied to the model discussed in Section 5.2.1.

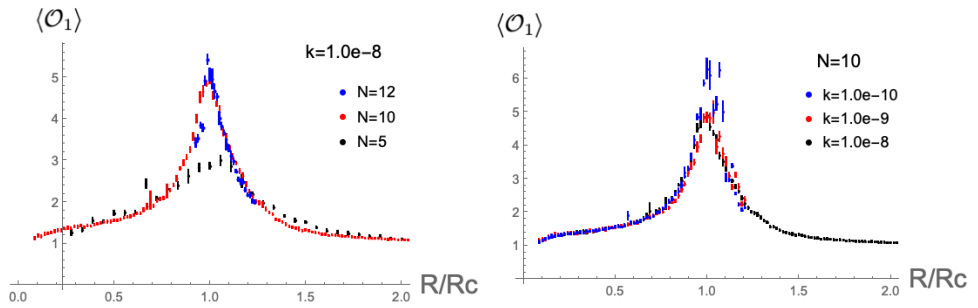

Figure 5.2: The expectation values of $\mathcal{O}_1$ from Eq. (5.14). On the left, a single value $k = 10^{-8}$ was chosen and N was varied for $N = 5, 8, 10$. On the right, a single value $N = 10$ was chosen and k was varied for $k = 10^{-8}, 10^{-9}, 10^{-10}$. The horizontal axis R/R_c was normalised according to the definition of R_c in Eq. (5.13).

5.2.3 Phase structure of the model

In this section, the results of applying the methods of Sections 5.2.1 and 5.2.2 to the model of Eq. (5.9) will be explained, as were reported in [1].

In this case, the variables $\tilde{\phi}^i_a$ can be described as NR-dimensional vectors constrained to the sphere where

$$|\tilde{\phi}|^2 = \sum_{i=1}^{R}\sum_{a=1}^{N} \tilde{\phi}^i_a \tilde{\phi}^i_a = 1.$$

Therefore, the leap frog steps of Eqs. (5.11) and (5.12) can be used. In this case, the potential $V(\tilde{\phi})$ is given by

$$V(\tilde{\phi}) = \frac{NR}{6}\log\left(\lambda\sum_{i,j=1}^{R} U_{ij}(\tilde{\phi}) + k\sum_{i=1}^{R} U_{ii}(\tilde{\phi})\right).$$

One can now calculate the partial derivatives of this potential to be

$$\frac{\partial V(\phi)}{\partial \phi^i_a} = \frac{NR/6}{\lambda\sum_{j,k=1}^{R}(\phi^j\cdot\phi^k)^3 + k|\phi|^{3/2}}\left(6\lambda\sum_{j=1}^{R}\phi^j_a(\phi^i\cdot\phi^j)^2 + 2k\phi^i_a\phi^2\right),$$

so that the second step of the leapfrog from Eq. (5.12) for the momentum variable π^i_a on the constraint surface is given by

$$\begin{aligned}\delta\pi^i_a &= -\epsilon\frac{\partial V}{\partial\phi^i_a} + \epsilon\phi^i_a\sum_{j=1}^{R}\sum_{a=1}^{N}\phi^j_a\frac{\partial V}{\partial\phi^j_a},\\ &= -\frac{\epsilon\lambda NR}{\lambda\sum_{j,k=1}^{R}(\phi^j\cdot\phi^k)^3 + k}\left(\sum_{j=1}^{R}\phi^j_a(\phi^i\cdot\phi^j)^2 - \phi^i_a\sum_{j,k=1}^{R}(\phi^j\cdot\phi^k)^3\right),\end{aligned}$$

where the constraint $|\phi|^2 = 1$ was used.

Below, the results of the simulations will be discussed. Here, $\lambda = 1$ is taken in all simulations.

The first major result is to find the critical value R_c for which there might be a phase transition (or a cross-over region). In the analytic and numerical analysis before, as explained in Section 5.1, the critical value was found to be about $R_c = \frac{1}{2}(N+1)(N+2)$. In [1] it was found to actually be more about

$$R_c = \frac{1}{2}(N+1)(N+2) - N + 2. \tag{5.13}$$

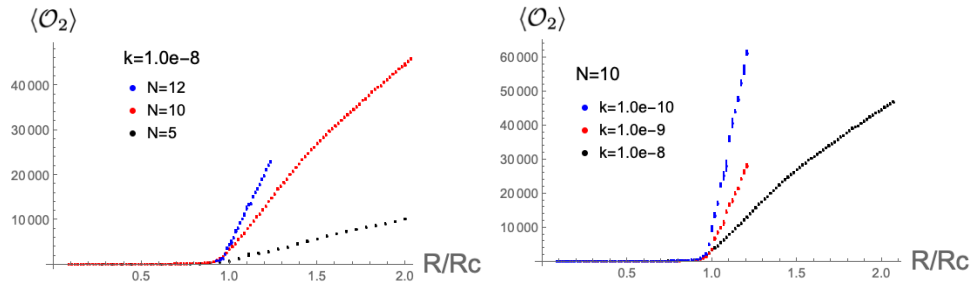

Figure 5.3: The expectation values of $\mathcal{O}_2$ from Eq. (5.15). On the left, a single value $k = 10^{-8}$ was chosen and N was varied for $N = 5, 8, 10$. On the right, a single value $N = 10$ was chosen and k was varied for $k = 10^{-8}, 10^{-9}, 10^{-10}$. The horizontal axis R/R_c was normalised according to the definition of R_c in Eq. (5.13).

In order to investigate this, one looks at the expectation values of certain observables. A useful observable to look at is

$$\mathcal{O}_1 := N \sum_{i \neq j} (\phi^i \cdot \phi^j)^2. \tag{5.14}$$

This operator may be interpreted as a measure for the correlation between ϕ^is, since for a completely uncorrelated phase

$$\langle \mathcal{O}_1 \rangle \sim N \sum_{i \neq j} \langle \phi^i_a \phi^i_b \rangle \langle \phi^j_a \phi^j_b \rangle \sim 1.$$

Figure 5.2 shows the results of the simulation for several values of N and k. It is clear that for all cases, the observable was almost 1 whenever R and R_c had very different values. But for $R \sim R_c$ the correlation between the elements becomes larger, and seems to become larger and larger for higher N. This implies that there is an actual phase transition point at the thermodynamic limit.

Another interesting observable is given by

$$\mathcal{O}_2 := \phi^2 = r^2. \tag{5.15}$$

This observable has a weight of $w = 2$ and has therefore an expectation value of (see Eq. (5.10))

$$\langle \mathcal{O}_2 \rangle := \frac{\Gamma[\frac{NR+2}{6}]}{\Gamma[\frac{NR}{6}]} \left\langle \lambda \sum_{i,j=1}^{R} (\phi^i \cdot \phi^j)^3 + k \right\rangle.$$

The results of the simulation of this operator are given in Fig. 5.3. It is clear that for $R < R_c$, $\langle \mathcal{O}_2 \rangle \sim 0$, but for $R > R_c$ the value $\langle \mathcal{O}_2 \rangle > 0$. Therefore, it seems like a good candidate of an order parameter to characterise the phase transition. As may be seen in Fig. 5.3, for larger N the transition point becomes more and more sharp but continuous, which again is evidence for point $R = R_c$ to actually be a continuous phase transition in the thermodynamic limit.

A last operator considered is $U_d(\phi)$ as defined in Eq. (5.5). The numerical results for $N = 10$ are shown in Fig. 5.4. Firstly, note that the R-dependence again shows that there seems to be a phase transition at $R = R_c$. The operator U_d is important in the analysis of the wave function renormalisability in Section 5.3. First, note that for the free energy

$$F_{N,R}(\lambda, k) := -\log Z_{N,R}(\lambda, k), \tag{5.16}$$

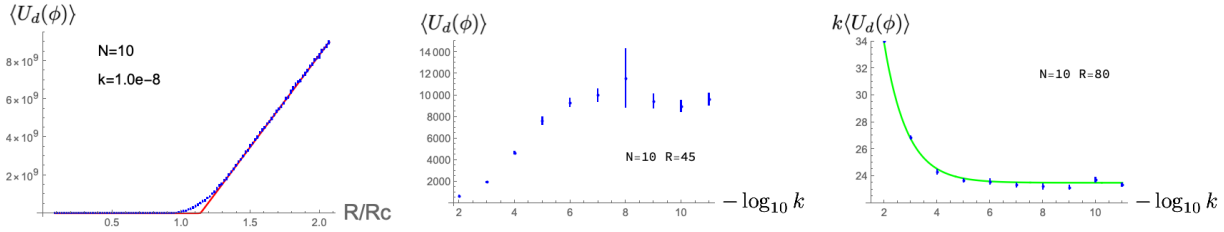

Figure 5.4: The expectation values of U_d as defined in Eq. (5.5) for $N = 10$, taken from [1]. On the left the dependence of the expectation value on R. The blue dots are the measured values and the red line is the result of the perturbative calculation. In the middle, the k dependence is shown for $R = 45 < R_c$ which reaches a constant value for $k \to 0$, where on the right the k dependence is shown for $R = 80 > R_c$ with a fit of $k \langle U_d(\phi) \rangle \approx 23.3 + 107\sqrt{k}$.

one can see from the definition of the partition function $Z_{N,R}(\lambda, k)$ in Eq. (5.6) that

$$\frac{\partial}{\partial k} F_{N,R}(\lambda, k) = \langle U_d(\phi) \rangle \,.$$

This means that if one wants to compute the free energy, and thus indirectly the the partition function through Eq. (5.16), as a function of k, one can use the expectation value above and perform the integral

$$F_{N,R}(\lambda, k) = \int_{k_0}^{k} \mathrm{d}k \; \langle U_d(\phi) \rangle + F_0,$$

with F_0 some integration constant. In particular, $\lim_{k\to 0} \langle U_d \rangle$ determines $\lim_{k\to 0} F_{N,R}(\lambda, k)$. Note that one may also write

$$F_{N,R}(\lambda, k) = F_{N,R}(1, \lambda/k) + \frac{NR}{6} \log \lambda \tag{5.17}$$

by performing a re-scaling $\phi_a^i \to \lambda^{-1/6} \phi_a^i$.

From Fig. 5.4 it seems clear that there are two main cases for the behaviour of $\langle U_d(\phi) \rangle$, namely for $R < R_c$ and $R > R_c$. For $R < R_c$, Fig. 5.4 shows that $\langle U_d(\phi) \rangle$ converges to a constant, which will be called

$$U_d^0 := \lim_{k \to 0_+} \langle U_d(\phi) \rangle \,.$$

This then gives for the free energy

$$F_{N,R}(\lambda, k) = U_d^0 \frac{k}{\lambda} + \frac{NR}{6} \log \lambda + p_{N,R}(\frac{k}{\lambda}),$$

with $p_{N,R}(\frac{k}{\lambda})$ smaller than $\frac{k}{\lambda}$ in order with a finite limit. For $R > R_c$, Fig. 5.4 shows that the dependence goes like $\langle U_d(\phi) \rangle \sim \frac{1}{k}$. Therefore, the free energy will be of the form

$$F_{N,R}(\lambda, k) = \tilde{U}_d^0 \log(\frac{k}{\lambda}) + \frac{NR}{6} \log \lambda + \tilde{p}_{N,R}(\frac{k}{\lambda}), \tag{5.18}$$

where

$$\tilde{U}_d^0 := \lim_{k \to 0_+} k \, \langle U_d(\phi) \rangle \,,$$

and $\tilde{p}_{N,R}(\frac{k}{\lambda})$ is of lower order. Curiously, the numerical results seem to suggest that up to leading order, $\tilde{U}_d^0$ is given by

$$\tilde{U}_d^0 = \frac{NR}{6} - \frac{\mathcal{N}}{2} + \delta \tilde{U}_d^0,$$

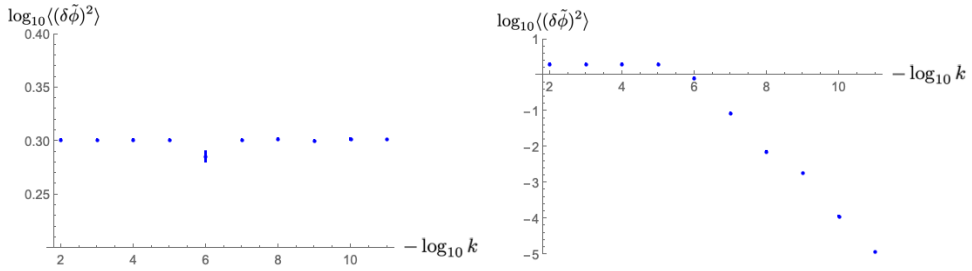

Figure 5.5: Two examples of the change in $\langle(\delta\tilde{\phi})^2\rangle$ of Eq. (5.19) for $N = 10$, taken from [1]. On the left $R = 45$ is taken, and on the right the value is $R = 80$. A clear decrease in simulation speed for $R = 80$ is shown when $k < 10^{-6}$.

with $\mathcal{N} = \frac{1}{6}N(N+1)(N+2)$ is the amount of degrees of freedom of a symmetric tensor of order three, and $\delta\tilde{U}_d^0$ a sub-leading term. Note that for $R_c > R > \frac{1}{2}(N+1)(N+2)/2$ this value is positive, since $\tilde{U}_d^0$ is positive, but for $R > \frac{1}{2}(N+1)(N+2)/2$ it might be either positive or negative.

One of the issues in the numerical analysis in Section 5.1.3 was that the Monte Carlo simulation slowed down considerably around $R \gtrsim R_c$ for small k. With the approach in this section, this is improved drastically. While a slowdown still occurs if k is taken too small, the value for which this occurs is several orders of magnitude lower than in the previous analysis. A useful measure for the speed of updates in the algorithm is given by

$$\langle(\delta\tilde{\phi})^2\rangle := \frac{1}{M}\sum_{m=1}^{M}\left|\tilde{\phi}(m+1) - \tilde{\phi}(m)\right|^2, \tag{5.19}$$

as this gives an indication of how quickly points change throughout the simulation. The results of this for an example with $N = 10$ and $R = 45, 80$ may be found in Fig. 5.5. These values are chosen such that one is clearly below R_c in Eq. (5.13) and one clearly above. It is clear that below R_c, the simulation speed is fine when taking $k \to 0$. For $R > R_c$, the speed decreases a lot when taking $k < 10^{-6}$. Note, however, that this is a drastic improvement over the previous numerical analysis in [121], where the smallest value probed was of the order of $k \sim 10^{-1}$.

5.3 Implications for the canonical tensor model

As already explained in Section 5.2, the matrix model investigated here has a close connection to a wave function of the canonical tensor model in Eq. (5.7). In this section, the implications to the canonical tensor model will be discussed.

Note that the link to the tensor model relies on the assumption that[3]

$$\Psi_{simple}(P) := \left[\int_{\mathbb{R}^N} \mathrm{d}\phi\ \mathrm{e}^{iP\phi^3 - k(\phi^2)^3}\right]^{R/2}, \tag{5.20}$$

is a good approximation for $\Psi(P)$ in the sense that

$$\begin{aligned}\langle\Psi|\ \mathrm{e}^{-\alpha P^2}\ |\Psi\rangle &\sim \langle\Psi_{simple}|\ \mathrm{e}^{-\alpha P^2}\ |\Psi_{simple}\rangle\,, \\ &\propto \alpha^{-\mathcal{N}/2} Z_{N,R}(\frac{1}{4\alpha}, k).\end{aligned} \tag{5.21}$$

[3]Here, ϕ_a corresponds to a single vector. If one then takes R copies of ϕ, as is done when calculating Eq. (5.21), one gets a rectangular matrix ϕ_a^i.

This corresponds to replacing the Airy function with $\mathrm{Ai}(\kappa\phi^2) \to \mathrm{e}^{-k(\phi^2)^3}$.

For the consistency of the canonical tensor model, the value R has to be a specific value, namely

$$R = \lambda_H,$$

where $\lambda_H = \frac{1}{2}(N+2)(N+3)$ as in Eq. (4.17). One of the major findings in [1], as shown in Section 5.2.3, is that there is a critical value

$$R_c = \frac{1}{2}(N+2)(N+3) - N + 2,$$

at which there seems to be a phase transition. This means that the consistency condition of the canonical tensor model, up to leading order, puts the canonical tensor model right on the continuous phase transition point. This is very encouraging from a physical point of view, since for the emergence of macroscopic spaces, the expectation is that one needs to take a continuum limit around a continuous phase transition. The $N \to \infty$ limit with Eq. (4.17) seems to do exactly that.

To answer the question whether the physical Hilbert space is indeed a proper subset of the kinematical Hilbert space $L^2(\mathbb{R}^{\mathcal{N}})$, one needs to show if the solutions to the constraint equations Eq. (4.16) are square integrable. In the current case, the simplification of the wave function is considered. Note that for the canonical tensor model, $R > R_c$, and thus one can use Eq. (5.18) to find

$$\alpha^{\mathcal{N}/2} Z_{N,R}(\frac{1}{4\alpha}, k) \sim \alpha^{-\delta \tilde{U}_d^0},$$

as the dominant contribution in the $\alpha \to 0$ limit. As already explained in Section 5.2.3, for $R_c < R < \frac{1}{2}(N+1)(N+2)$ one has $\tilde{U}_d^0 > 0$. Therefore, this would be a divergent property. However, for the canonical tensor model, $R > \frac{1}{2}(N+1)(N+2)$, and the value for $\tilde{U}_d^0$ is unrestricted in that region. More research in this area is required to fully say whether or not this function will converge. Note that even if the function diverges, this might just point to the fact that the inner product on the physical Hilbert space is different from the inner product on the kinematical Hilbert space.

Chapter 6

Counting tensor rank decompositions

In this chapter, the work in [2] will be summarised. In this work, a configuration space of tensor rank decompositions was defined and studied. In particular, the behaviour of volumes in this configuration space, giving a measure for the amount of tensor rank decompositions.

Studying tensor rank decompositions is interesting in the context of the canonical tensor model, since the tensor rank decomposition appears to play a crucial role in the theory [6, 3]. The essential question answered in [2] is "how can one quantify the amount of tensors close to a given tensor Q?". The study was an extension of the work in Chapter 5, since the quantities studied have a relationship with the partition function of Eq. (5.6), which will be pointed out below.

Note that the work discusses a more general scenario than usually used in the canonical tensor model. Firstly, in the canonical tensor model only tensors of degree three are considered. In this chapter, the degree of the tensor is taken to be arbitrary. Secondly, there are two cases considered; namely symmetric tensors and generic (non-symmetric) tensors. In the canonical tensor model, only symmetric tensors are considered.

In Section 6.1 the configration space of tensor rank decompositions will be introduced. The main definitions of the objects that are studied will also be given in that section, including the connection to the partition function in Eq. (5.6). In Section 6.2, a derivation of the volume law will be given, and the conditions under which this is valid according to numerical analysis will be discussed. Note that the notation here differs from Chapter 5.

6.1 The configuration space of tensor rank decompositions

In this section the configuration space of tensor rank decompositions will be introduced, and the volume quantities that will be analysed will be defined. Two types of real tensor spaces are considered in [2]; the symmetric tensors of degree K, $\mathrm{Sym}^K(\mathbb{R}^N)$, and the space of generic tensors of degree K, $\mathbb{R}^{N\otimes K}$. In this section, only the symmetric tensor case will be discussed. The case for generic tensors is very similar and similar results

hold. For the generic case, refer to [2].

For a given rank R, the general form for a symmetric tensor rank decomposition is given by

$$\Phi_{a_1 \dots a_K} = \sum_{i=1}^{R} \lambda_i \varphi^i_{a_1} \dots \varphi^i_{a_K}, \tag{6.1}$$

where $\varphi^i_{a_k} \in S^{N-1}_+$ lies on the upper-hemisphere of the $N-1$-dimensional unit sphere and $\lambda_i \in \mathbb{R}$. Note that, just like in Section 3.2.2, the rank R is not necessarily the minimal rank. From this definition, one can deduce the configuration space of tensor rank decompositions as

$$\mathcal{F}_R := \mathbb{R}^R \times \underbrace{S^{N-1}_+ \times \dots \times S^{N-1}_+}_{R \text{ times}} = \mathbb{R}^R \times S^{N-1 \times R}_+. \tag{6.2}$$

Equation (6.1) links a given tensor rank decomposition in the space $\mathcal{F}_R$ to a tensor in the tensor space $\mathrm{Sym}^K(\mathbb{R}^N)$. However, the main objects of interest are the tensor rank decompositions, and not their corresponding tensors. Some structure from the tensor space is useful to consider, for instance an inner product for $Q, P \in \mathrm{Sym}^K(\mathbb{R}^N)$

$$Q \cdot P = \sum_{a_1 \dots a_K = 1}^{N} Q_{a_1 \dots a_K} P_{a_1 \dots a_K},$$

which also introduces a norm to the space $\|Q\|^2 := Q \cdot Q = \sum_{a_1 \dots a_K = 1}^{N} |Q_{a_1 \dots a_K}|^2 \equiv Q^2$.

On the configuration space $\mathcal{F}_R$, one can introduce a measure by the infinitesimal element

$$\mathrm{d}\Phi_w = \prod_{i=1}^{R} |\lambda_i|^{w-1} \mathrm{d}\lambda_i \, \mathrm{d}\varphi^i, \tag{6.3}$$

where $\mathrm{d}\lambda_i$ is the usual line-element of the real numbers, and $\mathrm{d}\varphi^i$ is the usual volume element on the $N-1$-dimensional unit-sphere. $w \geq 1$ is introduced for generality. For $R < R'$ it holds that $\mathcal{F}_R \subset \mathcal{F}_{R'}$, so the spaces $\mathcal{F}_R$ form an increasing sequence of spaces, which limits to the whole symmetric tensor space of tensors of degree K:

$$\mathcal{F}_R \uparrow_{R \to \infty} \mathrm{Sym}^K(\mathbb{R}^N) \cong \mathbb{R}^{N_Q},$$

where $N_Q := \binom{N+K-1}{K}$ counts the degrees of freedom of the tensor space.

The following quantity introduces a way of counting the amount of tensor rank decompositions which are close (within square distance Δ) of a given tensor Q

$$\mathcal{V}^\epsilon_R(Q, \Delta) := \int_{\mathcal{F}_R} \mathrm{d}\Phi_w \, \Theta(\Delta - \|Q - \Phi\|^2) \, \mathrm{e}^{-\epsilon \sum_{i=1}^{R} \lambda_i^2}, \tag{6.4}$$

where ϵ is a (small) positive parameter to ensure convergence of the quantity. In order to be able to make general statements for average tensors, it is useful to integrate this quantity over normalised tensors $\tilde{Q}_{a_1 \dots a_K}$ (such that $\|\tilde{Q}\|^2 = 1$):

$$\mathcal{Z}_R(\Delta; \epsilon) := \frac{1}{V_{\|Q\|=1}} \int_{\|Q\|=1} \mathrm{d}\tilde{Q} \, \mathcal{V}^\epsilon_R(\tilde{Q}, \Delta), \tag{6.5}$$

where $V_{\|Q\|=1} := \int_{\|Q\|=1} \mathrm{d}\tilde{Q} = \frac{2\pi^{N_Q/2}}{\Gamma(N_Q/2)}$. A good question to ask is whether or not the limit

$$\mathcal{Z}_R(\Delta) := \lim_{\epsilon \to 0^+} \mathcal{Z}_R(\Delta; \epsilon), \tag{6.6}$$

exists and is finite. As it turns out, the quantity does exist under the condition that the following quantity exists and is finite

$$G_R := \lim_{\epsilon \to 0^+} G_R(\epsilon) := \lim_{\epsilon \to 0^+} \int_{\mathcal{F}_R} \mathrm{d}\Phi_w \, \mathrm{e}^{-\Phi^2 - \epsilon \sum_{i=1}^R \lambda_i^2}. \tag{6.7}$$

Note that this may be rewritten as

$$G_R(\epsilon) = \int_{\mathcal{F}_R} \prod_{i=1}^R \mathrm{d}\lambda_i |\lambda_i|^{w-1} \mathrm{d}\varphi^i \mathrm{e}^{-\sum_{i,j=1}^R \lambda_i (\varphi^i \cdot \varphi^j)^K \lambda_j - \epsilon \sum_{i=1}^R \lambda_i^2}, \tag{6.8}$$

with an inner product

$$\varphi^i \cdot \varphi^j = \sum_{a=1}^N \varphi_a^i \varphi_a^j.$$

This relates to the matrix model in Eq. (5.6). The partition function Eq. (5.6) may be rewritten in hyperspherical coordinates for every $\phi^i \to (r_i, \varphi^i)$ with the convention that $\varphi \in S_+^{N-1}$ and $r_i \in \mathbb{R}$,

$$\begin{aligned} Z_{N,R}(\lambda = 1, k) &= \int \prod_{i=1}^R |r_i|^{N-1} \mathrm{d}r_i \mathrm{d}\varphi^i \, \mathrm{e}^{-\sum_{i,j=1}^R (r_i (\varphi^i \cdot \varphi^j) r_j)^3 - k \sum_{i=1}^R r_i^6}, \\ &= const. \int_{\mathcal{F}_R} \prod_{i=1}^R |\lambda_i|^{\frac{N-3}{3}} \mathrm{d}\lambda_i \mathrm{d}\varphi^i \mathrm{e}^{-\sum_{i,j=1}^R \lambda_i (\varphi^i \cdot \varphi^j)^3 \lambda_j - k \sum_{i=1}^R \lambda_i^2}, \end{aligned}$$

where the substitution $\lambda_i = r_i^3$ was used and $const.$ is an irrelevant numerical factor. Therefore, curiously, $G_R(\epsilon)$ seems to be a generalisation of the partition function, where $\epsilon = k$, $K = 3$ and $w = \frac{N}{K}$.

In [2], this quantity was compared to another quantity

$$\begin{aligned} \mathcal{C}_R(\Delta) &:= \int_{\mathcal{F}_R} \mathrm{d}\Phi_w \, \Theta\left(\Delta - \|\Phi\|^2\right), \\ &= \frac{G_R}{\Gamma\left[\frac{wR}{2} + 1\right]} \Delta^{\frac{wR}{2}}. \end{aligned}$$

If one now takes the ratio $\mathcal{Z}_R / \mathcal{C}_R$, the G_R dependence actually drops out. For more discussion, please refer to [2].

6.2 A volume law of tensor rank decompositions

In this section, the quantity defined in Eq. (6.6) will be calculated. In Section 6.2.1, it will be shown that under the condition that Eq. (6.7) is finite, Eq. (6.6) may be calculated exactly, with the dependence on Δ completely fixed. The only unknown in the solution is the constant G_R. In Section 6.2.2 will then be discussed when this condition is actually satisfied by numerical analysis.

6.2.1 Calculating the average volume

The general strategy for the deriviation is as follows, all assuming that the quantity G_R is finite. First, the Laplace transform of Eq. (6.6) will be taken. Then, Proposition 6.1 will give the solution for the Laplace transform. Lastly, in Proposition 6.2 the inverse Laplace transform will be performed. Some of the mathematical details are omitted, and for those is referred to [2].

The Laplace transform of Eq. (6.6) is given by:

$$\begin{aligned}\bar{\mathcal{Z}}_R(\gamma) &= \int_0^\infty \mathrm{d}\Delta\, \mathcal{Z}_R(\Delta)\, \mathrm{e}^{-\gamma\Delta}, \\ &= \frac{1}{V_{\|Q\|=1}} \lim_{\epsilon\to 0^+} \int_{\|Q\|=1} \mathrm{d}\tilde{Q} \int_{\mathcal{F}_R} \mathrm{d}\Phi_w \int_{\|\tilde{Q}-\Phi\|^2}^\infty \mathrm{d}\Delta\ \mathrm{e}^{-\gamma\Delta-\epsilon\sum_{i=1}^R \lambda_i^2}, \\ &= \frac{1}{\gamma V_{\|Q\|=1}} \lim_{\epsilon\to 0^+} \int_{\|Q\|=1} \mathrm{d}\tilde{Q} \int_{\mathcal{F}_R} \mathrm{d}\Phi_w\ \mathrm{e}^{-\gamma(\tilde{Q}-\Phi)^2-\epsilon\sum_{i=1}^R \lambda_i^2}.\end{aligned}$$

As will be clear later, it is useful to multiply this quantity by γ:

$$\bar{Z}_R(\gamma) := \gamma\bar{\mathcal{Z}}_R(\gamma) = \frac{1}{V_{\|Q\|=1}} \lim_{\epsilon\to 0^+} \int_{\|Q\|=1} \mathrm{d}\tilde{Q} \int_{\mathcal{F}_R} \mathrm{d}\Phi_w\ \mathrm{e}^{-\gamma(\tilde{Q}-\Phi)^2-\epsilon\sum_{i=1}^R \lambda_i^2}. \tag{6.9}$$

This will be undone again at a later stage. For later use it is convenient to also define this quantity as a function of ϵ without taking the limit:

$$\bar{Z}_R(\gamma;\epsilon) := \frac{1}{V_{\|Q\|=1}} \int_{\|Q\|=1} \mathrm{d}\tilde{Q} \int_{\mathcal{F}_R} \mathrm{d}\Phi_w\ \mathrm{e}^{-\gamma(\tilde{Q}-\Phi)^2-\epsilon\sum_{i=1}^R \lambda_i^2}. \tag{6.10}$$

Because, for the Laplace transform, a multiplication by γ corresponds to taking a derivative in Δ-space, this effectively means that the above quantity corresponds to the Laplace transform of the distributive quantity

$$Z_R(\Delta;\epsilon) := \int_{\|Q\|=1} \mathrm{d}\tilde{Q}\, \mathcal{D}V_R^\epsilon(\tilde{Q},\Delta) := \int_{\|Q\|=1} \mathrm{d}\tilde{Q} \int_{\mathcal{F}_R} \mathrm{d}\Phi_w\, \delta(\Delta-\|\tilde{Q}-\Phi\|^2)\ \mathrm{e}^{-\epsilon\sum_{i=1}^R \lambda_i^2},$$

where $\delta(x)$ ($x\in\mathbb{R}$) is the delta distribution, assuming that (6.9) is well-defined.

An important ingredient is now the following.

Proposition 6.1. *Given that Eq.* (6.7) *is finite, Eq.* (6.9) *is finite and given by*

$$\bar{Z}_R(\gamma) = G_R\, \gamma^{-\frac{w\,R}{2}}\, {}_1F_1\left(\frac{N_Q - w\,R}{2}, \frac{N_Q}{2}, -\gamma\right).$$

Proof. Below only a sketch of the proof will be given. For the full proof with the full mathematical details and some steps in between, please refer to [2].

The general proof consists of two steps. The first step is showing that if G_R is finite, $\bar{Z}_R$ is finite as well. By using the reverse triangle inequality, one can show that

$$\begin{aligned}\bar{Z}_R(\gamma;\epsilon) &\le \frac{1}{V_{\|Q\|=1}} \int_{\|Q\|=1} \mathrm{d}\tilde{Q}\ \mathrm{e}^{\gamma\frac{A}{1-A}\tilde{Q}^2} \int_{\mathcal{F}_R} \mathrm{d}\Phi_w\ \mathrm{e}^{-\gamma A\Phi^2-\epsilon\sum_{i=1}^R \lambda_i^2}, \\ &= (\gamma A)^{-\frac{w\,R}{2}}\, \mathrm{e}^{\gamma\frac{A}{1-A}}\, G_R\left(\frac{\epsilon}{\gamma A}\right).\end{aligned}$$

This means that, as long as $G_R = \lim_{\epsilon\to 0^+} G_R(\epsilon)$ is finite, $\bar{Z}_R(\gamma) = \lim_{\epsilon\to 0^+} \bar{Z}_R(\gamma;\epsilon)$ is finite.

The second step is showing the exact form of $\bar{Z}_R(\gamma)$ under the assumption that G_R and thus $\bar{Z}_R$ is finite. For this, an extra quantity is defined

$$Y(\alpha,\gamma) := \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^{N_Q}} \mathrm{d}Q \int_{\mathcal{F}_R} \mathrm{d}\Phi_w \; \mathrm{e}^{-\alpha Q^2 - \gamma(Q-\Phi)^2 - \epsilon\sum_{i=1}^R \lambda_i^2},$$

which is finite under the same assumption that G_R is finite. Note that Q is now integrated over the whole tensor space $\mathbb{R}^{N_Q}$. On one hand, by solving it as a Gaussian integral, one can show that

$$Y(\alpha,\gamma) = \pi^{N_Q/2}\gamma^{-\frac{N_Q+w\,R}{2}}\,(1+t)^{-\frac{N_Q-w\,R}{2}}\,t^{-\frac{w\,R}{2}}G_R,$$

and on the other hand, by using hyperspherical coordinates $(|Q|,\tilde{Q})$,

$$\frac{1}{2}V_{\|Q\|=1}\gamma^{-\frac{N_Q+w\,R}{2}}\int_0^\infty \mathrm{d}x\, x^{\frac{N_Q+w\,R}{2}-1}\bar{Z}_R(x)\,\mathrm{e}^{-t\,x}\,.$$

Therefore, one can see that

$$\int_0^\infty \mathrm{d}x\, x^{\frac{N_Q+w\,R}{2}-1}\bar{Z}_R(x)\,\mathrm{e}^{-t\,x} = \Gamma[N_Q/2]\,G_R\,(1+t)^{-\frac{N_Q-w\,R}{2}}t^{-\frac{w\,R}{2}}.$$

The left-hand side is nothing but the definition of the Laplace transform of the funciton $x^{\frac{N_Q+w\,R}{2}-1}\bar{Z}_R(x)$. Therefore, by taking the inverse Laplace transform one can show

$$\bar{Z}_R(x) = G_R\,x^{-\frac{w\,R}{2}}\,{}_1F_1\left(\frac{N_Q-w\,R}{2},\frac{N_Q}{2},-x\right).$$

□

The multiplication by γ of Eq. (6.9) can now be undone:

$$\bar{\mathcal{Z}}_R(\gamma) = G_R\gamma^{-\frac{w\,R}{2}-1}\,{}_1F_1\left(\frac{N_Q-w\,R}{2},\frac{N_Q}{2},-\gamma\right). \tag{6.11}$$

Proposition 6.2. *Given that G_R in Eq.* (6.7) *is finite, $\mathcal{Z}_R(\Delta)$, as defined in Eq.* (6.6)*, is given by*

$$\mathcal{Z}_R(\Delta) = \frac{2G_R}{\Gamma\left[\frac{w\,R}{2}\right]}\cdot\begin{cases}\frac{1}{N_Q}\Delta^{\frac{N_Q}{2}}\,{}_2F_1\left(1-\frac{w\,R}{2},\frac{N_Q-w\,R}{2},1+\frac{N_Q}{2},\Delta\right), & \Delta\le 1,\\ \frac{1}{w\,R}\Delta^{\frac{w\,R}{2}}\,{}_2F_1\left(-\frac{w\,R}{2},\frac{N_Q-w\,R}{2},\frac{N_Q}{2},1/\Delta\right), & \Delta\ge 1,\end{cases} \tag{6.12}$$

Proof. Below only a sketch of the proof will be given. For the full proof with the full mathematical details and some steps in between, please refer to [2].

Since G_R is finite, it follows from Proposition 6.1 that Eq. (6.11) exists and is finite. Therefore the main task is to perform the inverse Laplace transform. Taking this inverse Laplace transform is a bit technical in nature. One has to rewrite Eq. (6.11) in terms of the Whittaker functions $M_{\mu,\nu}(\gamma)$ as

$$\bar{\mathcal{Z}}_R(\gamma) = G_R\,\gamma^{-\frac{w\,R}{2}-\frac{N_Q}{4}-1}\,\mathrm{e}^{-\frac{\gamma}{2}}\;M_{\frac{N_Q}{4}-\frac{w\,R}{2},\frac{N_Q}{4}-\frac{1}{2}}(\gamma).$$

If one rewrites this as

$$\bar{\mathcal{Z}}_R(\gamma) = G_R\,\underbrace{\gamma^{-\frac{N_Q}{4}}\,\mathrm{e}^{-\frac{\gamma}{2}}\;M_{\frac{N_Q}{4}-\frac{w\,R}{2},\frac{N_Q}{4}-\frac{1}{2}}(\gamma)}_{L[f]}\;\underbrace{\gamma^{-\frac{w\,R}{2}-1}}_{L[g]},$$

one can first find the inverse Laplace transforms of $L[f]$ and $L[g]$ (where $L[f]$ and $L[g]$ denote the Laplace transforms of the functions f and g respectively). These inverse Laplace transforms are given by (see [2])

$$g(t) = \frac{t^{\frac{w\,R}{2}}}{\Gamma\left[\frac{w\,R}{2}+1\right]},$$
$$f(t) = \begin{cases} \beta\left(\frac{w\,R}{2}, \frac{N_Q - w\,R}{2}\right)^{-1} t^{\frac{N_Q - w\,R}{2}-1}(1-t)^{\frac{w\,R}{2}-1}, & 0<t<1, \\ 0, & \text{otherwise.} \end{cases}$$

The convolution theorem states that for the convolution of two functions

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau,$$

the Laplace transform behaves as

$$L(f*g) = L(f)L(g).$$

This leads to

$$\mathcal{Z}_R(\Delta) = \begin{cases} c_R \int_0^{\Delta} q^{\frac{N_Q - w\,R}{2}-1}(1-q)^{\frac{w\,R}{2}-1}(\Delta-q)^{\frac{w\,R}{2}}\,\mathrm{d}q, & \Delta \leq 1, \\ c_R \int_0^{1} q^{\frac{N_Q - w\,R}{2}-1}(1-q)^{\frac{w\,R}{2}-1}(\Delta-q)^{\frac{w\,R}{2}}\,\mathrm{d}q, & \Delta \geq 1, \end{cases}$$

which evaluates to the final result. □

6.2.2 Finiteness of the volume law

In this section, it will be investigated for which cases the quantity G_R is finite. In order to simplify the analysis, only the case of $w = 1$ will be discussed. In order to find out if the quantity G_R in Eq. (6.7) exists, it helps to, just like in Section 5.2, first integrate out the radial components. For this, rewrite $G_R(\epsilon)$ as

$$\begin{aligned} G_R(\epsilon) &:= \int_{\mathcal{F}_R} \mathrm{d}\Phi_{w=1}\ \mathrm{e}^{-\Phi^2 - \epsilon\sum_{i=1}^R \lambda_i^2}, \\ &= \int_{\mathcal{F}_R} \prod_{i=1}^R \mathrm{d}\lambda_i\,\mathrm{d}\varphi_i\ \mathrm{e}^{-\sum_{i,j=1}^R \lambda_i\left(\left(\varphi^i\cdot\varphi^j\right)^K + \epsilon\delta^{ij}\right)\lambda_j}. \end{aligned}$$

This is just a multi-dimensional Gaussian integral for the radial components, and may be solved

$$G_R(\epsilon) = (\pi)^{R/2} \int_{S_+^{N-1}\times R} \prod_{i=1}^R \mathrm{d}\varphi^i\ \frac{1}{\sqrt{\det\left[\left(\varphi^i\cdot\varphi^j\right)^K + \epsilon\delta^{ij}\right]}},$$

which is a compact finite (for $\epsilon > 0$) integral.

The numerical analysis was done using Monte Carlo methods with the following algorithm:

1. Construct R, N-dimensional random normalised vectors using Gaussian sampling.
2. Generate the matrix M^{ij} by taking inner products (and adding ϵ to the diagonal elements).

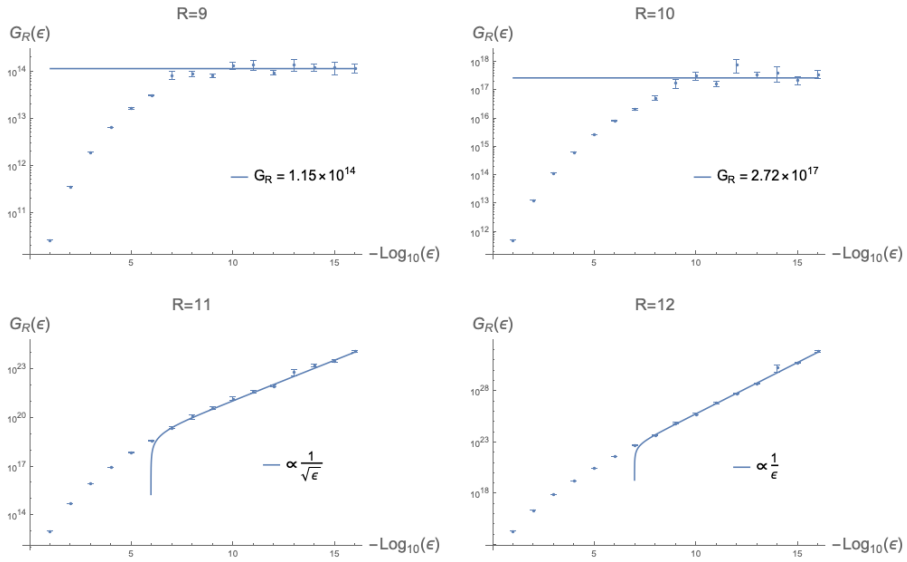

Figure 6.1: An example of the verification of R_c and the determination of the numerical value of G_R for $K = 3$ and $N = 3$, taken from [2]. The dots (with error-bars) represent the measurements, and the fitted curves are $C*\epsilon^{-\frac{R-R_c}{2}}+const.$ for $R > R_c$ as Eq. (6.13), and the constant value G_R for $R \leq R_c$ as in Eq. (6.7). In this case, $R_c = 10$.

3. Calculate the determinant of M^{ij} and evaluate the integrand.

4. Repeat this process M times.

Here, the matrix M^{ij} is given by

$$M_\epsilon^{ij} := \left(\varphi^i \cdot \varphi^j\right)^K + \epsilon\delta^{ij}.$$

What divergent behaviour to expect can be explained as follows. Let us take the limit of $\lim_{\epsilon\to 0^+} M_\epsilon^{ij} =: M^{ij}$. It is clear that this integral diverges whenever the matrix is degenerate. Assume now that M^{ij} has rank r, meaning that the matrix M^{ij} in diagonalised form has $R - r$ zero-entries. Thus, adding a small but positive ϵ to the diagonal entries results in the following expansion One can describe the expected convergent or divergent behaviour of G_R as follows. Consider the limit

$$\lim_{\epsilon\to 0^+} M_\epsilon^{ij} =: M^{ij}.$$

The integral above clearly diverges whenever the matrix is degenerate. If the matrix M^{ij} has rank r, meaning that the matrix M^{ij} in diagonalised form has $R - r$ zero-entries, then in terms of a small ϵ the determinant has the following expansion

$$\det M_\epsilon = A\,\epsilon^{R-r} + \mathcal{O}(\epsilon^{R-r-1}),$$

leading to leading order for the integrand

$$\frac{1}{\sqrt{\det M_\epsilon}} \sim \epsilon^{-\frac{R-r}{2}}.$$

Thus, if there is a set with measure nonzero in the integration region with $r < R$, the final ϵ-dependence for small epsilon is expected to be

$$G_R(\epsilon) \approx C\,\epsilon^{-\frac{R-r}{2}} + \mathcal{O}(\epsilon^{-\frac{R-r-1}{2}}) \tag{6.13}$$

where the constant factor C is the measure of the divergent set, and the other factor is due to non-leading order non-zero measure integration regions. Define the critical rank R_c to be the last value of R for which the integral convergent. Then $r = R_c$ gives the leading order contribution in Eq. (6.13) by definition. By the definition of R_c, for $R \leq R_c$, $G_R(\epsilon)$ should converge to a constant value. In [2], the method above was used to calculate numerically the ϵ-dependence of $G_R(\epsilon)$. An example of this is given in Fig. 6.1.

In [2], the value R_c was determined for many cases. From this analysis, a formula for the critical rank was conjectured

$$R_c \approx \frac{N_Q}{w},$$

which should be noted, is also the leading-order contribution found in the matrix model of Chapter 5, which is related to G_R as explained in Section 6.1.

Chapter 7

Conclusion

This dissertation focuses on the development of the canonical tensor model, both from a fundamental point of view and the analysis of specific wave functions. Section 7.1 will briefly review the main points, and explain their relevance in the research done for the canonical tensor model. Section 7.2 will then conclude the chapter, by giving some interesting future research that could be done.

7.1 Summary of the thesis

In this thesis, the main goal was to describe the work done in [1, 2] and put it in the context of the canonical tensor model. To this end, in order to fully appreciate the context of these works, the canonical tensor model was thoroughly explained, and a new potential algebraic interpretation introduced in [3] was reviewed.

In Chapters 2 and 3 an extensive overview of the mathematical background of algebraic tensor models was given. First, some knowledge from algebraic geometry was reviewed, showing the equivalence of a compact Riemannian manifold to a smooth algebra together with the spectrum of some operator. This opens up the door to an attractive description of gravity; not only is the algebraic description manifestly diffeomorphism invariant, but it also reduces the complexity of the description to a countably infinite-dimensional configuration space. After this introduction to some mathematical concepts, the real groundwork was laid for the algebraic tensor models. The main two achievements here were the development of the associative closure, which links even finite-dimensional tensors to infinite-dimensional algebras, and the realisation that the positive minimal tensor rank decomposition appears to generate potential homomorphisms, and thus may be used to construct the associative closure.

Chapter 4 then serves as an introduction to the canonical tensor model. Both the classical model and the quantum model are explained, and some of the recent research is cited. One of the important parts is to introduce the wave function Eq. (4.22), which is the physical motivation behind the matrix model studied in Chapter 5. Chapter 5 then explains the main parts of the analysis of the matrix model in [1]. The main numerical results were explained and put into the context of the canonical tensor model, with the main result being that the model does seem to indeed have a continuous phase transition in the thermodynamic limit, and that the canonical tensor model places itself on it

naturally through a consistency condition.

In the investigation of the canonical tensor model, the data-analytic tool tensor rank decomposition keeps making an appearance. On one hand it seems crucial in the interpretation of the model as a geometric space, and it even makes an appearance when studying the wave-function. This motivated the research in [2], where the configuration space of tensor rank decompositions was further investigated. This work was briefly reviewed in Chapter 6, where an explicit formula was found for the average volume of this space around normalised tensors and the convergence of this notion was investigated.

7.2 Future research directions

All of the research done above has interesting future directions. Here some future research directions following Chapters 3, 4 and 6 will be given [1, 2, 3].

For the work on algebraic tensor models, there are several aspects that could be considered. Firstly, some fundamental questions are left open to consider. For instance the conjecture in Conjecture 3.1 might be proven, or limited in scope. Also some interesting questions on the associative closure come up, for instance if there are certain requirements on the tensor P_{abc} that, if satisfied, will lead to smooth manifolds. Secondly, it would be interesting to investigate the possibility to derive a Hamiltonian from general relativity directly for a given operator, though the mathematics might not be straightforward. Lastly, the application to the canonical tensor model might be further looked at. For instance, one could see if tensors that represent continuous spaces - for instance the five-dimensional circle - are located at peaks of the wave-function [5, 4].

For the research on the matrix model in Chapter 5, there are also several things that would be interesting to investigate further. Firstly, while the numerical understanding of the model has improved, a proper analytical understanding is still lacking. Secondly, the matrix model was introduced as a simplification of the wave function of the canonical tensor model in Eq. (4.22), but it only really holds for the case where the cosmological constant $\lambda < 0$, since the Airy function may be seen as a damping function in this case. It would be interesting to see if a similar model could be obtained for the $\lambda > 0$ case, and if some of the characteristics found would still hold.

The analysis of the configuration space of tensor rank decompositions might also still be improved. Firstly, there was a conjectured formula for the maximum rank R_m for which the volume law is convergent based on numerical results. It would be interesting to see if one could prove this conjecture. Secondly, dividing Z_R by the quantity C_R made the dependence on the divergent factor G_R drop out. It would be interesting to see if Z_R/C_R has some interesting interpretation, even for $R > R_m$.

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