Stochastic Model Predictive Control Using Simplified Affine Disturbance Feedback for Chance-Constrained Systems

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***Abstract*—This letter covers the model predictive control of linear discrete-time systems subject to stochastic additive disturbances and chance constraints on their state and control input. We propose a simplified control parameterization under the framework of affine disturbance feedback, and we show that our method is equivalent to parameterization over the family of state feedback policies. Using our method, associated finite-horizon optimization can be computed efficiently, with a slight increase in conservativeness compared with conventional affine disturbance feedback parameterization.**

***Index Terms*—Stochastic systems, stochastic optimal control, constrained control.**

I. Introduction

MODEL predictive control (MPC) has become important for control applications in various fields because it can effectively cope with the complex dynamics of systems with multiple inputs and outputs and can intuitively consider input or state constraints [1], [2]. Furthermore, robust MPC has attracted the attention of many scholars because it enables the stability and performance of a system against worst-case perturbations analyzed under bounded uncertainties. However, the probability of real-world, worst-case perturbations can be very small [3], so control design based on worst-case perturbations would be too conservative.

In contrast to robust MPC, stochastic MPC (SMPC) finds a different way to solve MPC problems while considering uncertainties. SMPC utilizes probabilistic descriptions of objective values and constraint violations and allows accounting for acceptable levels of risk during system operation. SMPC with probabilistic constraints ("chance constraints") is for controlling the predicted probability distributions of system states optimally over a finite prediction horizon while ensuring the satisfaction of constraints at the desired probability level (see [4], [5], [6], [7], [8], and [9]).

Manuscript received August 29, 2020; revised November 1, 2020; accepted November 20, 2020. Date of publication December 2, 2020; date of current version December 24, 2020. This work was supported in part by the Japan Science and Technology Agency CREST under Grant JPMJCR15K2. Recommended by Senior Editor F. Zhang.
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Digital Object Identifier 10.1109/LCSYS.2020.3042085

The chance constraints of the SMPC problem are generally hard to deal with. However, when a disturbance has a known structure, many approximation methods can be found, and many probabilistic approximation techniques exist (see [10] and [11]). Additional complexity in SMPC comes from the choice of control parameterization. It is common knowledge that optimizing over open-loop control sequences directly leads to very conservative control performance and may be infeasible when a constraint exists [12]. Thus, the optimization should be done using families of feedback control policies. The core difficulty with this type of feedback policy is that optimizing the feedback policy over arbitrary nonlinear functions is extremely difficult. Proposals that take this approach are typically intractable (e.g., based on dynamic programming [13] or based on the generation of disturbance sequences from a certain set, as in [14]).

One solution is to parameterize the control policy in terms of the affine functions of the sequence of states. However, the set of constraint-admissible policies of this form is nonconvex [15]. Many approaches to solving this problem have been proposed. A widely adopted approach is to fix a stabilizing feedback gain over the prediction horizon [16], [17], [18]. Though tractable, this approach is problematic since the method of selecting the gain to minimize conservativeness is unclear.

Another solution is to parameterize the control policy as an affine function of the disturbance, called the "affine disturbance feedback control policy." This kind of parameterization is very old in stochastic programming [19] and has been shown to be equivalent to state feedback policies in [20]. The advantage of affine disturbance feedback control parameterization is that the set of its decision variables is guaranteed to be convex. However, the main disadvantage of this parameterization is that the number of decision variables grows quadratically with the prediction horizon, so the real-time calculation is disastrous when the prediction horizon grows.

In this letter, we intend to reduce the computation time while taking advantage of the affine disturbance feedback control policy. Many researchers have considered related works. In [21], a striped prediction scheme was proposed with numbers of variables and constraints that only grow linearly with the prediction horizon. In [22], the authors used a striped lower triangular control policy for disturbance compensation in stochastic MPC and further reduced the computation time by performing the computation of the disturbance compensation matrix offline. In [22], the control parameterization combined state feedback, feedforward, and disturbance feedback to guarantee feasibility and stability, making its structure rather complex.

Our control policy proposed in this letter combines the related simplification techniques of affine disturbance feedback to create a concise controller form; it is shown to be equivalent to state feedback control policies. This equivalence allows us, in principle, to directly apply the stability results discussed in research related to state feedback control policies (e.g., discussion on systems with bounded disturbances [7], [23] or unbounded disturbances [24], [25]). As shown in a later section, our control policy admits a feasible domain that is the same as a corresponding state feedback control policy but whose set of feasible decision variables is convex. Compared with the original affine disturbances control law, this approach significantly reduces the number of decision variables with stability results similar to those of state feedback parameterization.

The rest of this letter is organized as follows. Section II discusses the system class we consider throughout this letter. Section III discusses our simplified affine disturbance feedback policy and the corresponding deterministic equivalent MPC framework under this parameterization. Section IV discusses the properties of our method. Section V presents our simulation results. Section VI concludes this letter.

II. Problem Formulation

In this letter, we consider a linear discrete-time system with an additive disturbance:

$$x_{t+1} = Ax_t + Bu_t + Ew_t, \tag{1}$$

where t is the discrete time, $x_t \in \mathbb{R}^n$ denotes the state, $u_t \in \mathbb{R}^m$ denotes the control input, and $w_t \in \mathbb{R}^r$ is a random disturbance. For convenience, the prediction of the system's behavior over a finite horizon N is described as

$$\mathbf{x}_t = \mathbf{A}x_{0|t} + \mathbf{B}\mathbf{u}_t + \mathbf{E}\mathbf{w}_t, \tag{2}$$

where $\mathbf{x}_t = [x_{0|t}^{\mathrm{T}}, x_{1|t}^{\mathrm{T}}, \ldots, x_{N|t}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(N+1)n}$ denotes a sequence of system states, $\mathbf{u}_t = [u_{0|t}^{\mathrm{T}}, \ldots, u_{N-1|t}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nm}$ denotes a sequence of control inputs, and $\mathbf{w}_t = [w_{0|t}^{\mathrm{T}}, \ldots, w_{N-1|t}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nr}$ is a sequence of stochastic disturbances. Matrices $\mathbf{A} \in \mathbb{R}^{(N+1)n\times n}$, $\mathbf{B} \in \mathbb{R}^{(N+1)n\times Nm}$, and $\mathbf{E} \in \mathbb{R}^{(N+1)n\times Nr}$ are given as follows:

$$\mathbf{A} := \begin{bmatrix} I \\ A \\ \vdots \\ A^N \end{bmatrix}, \quad \mathbf{B} := \begin{bmatrix} O & O & \cdots & O \\ B & O & \cdots & O \\ AB & B & \cdots & O \\ \vdots & \ddots & \ddots & \vdots \\ A^{N-1}B & \cdots & AB & B \end{bmatrix},$$

$$\mathbf{E} := \begin{bmatrix} O & O & \cdots & O \\ E & O & \cdots & O \\ AE & E & \cdots & O \\ \vdots & \ddots & \ddots & \vdots \\ A^{N-1}E & \cdots & AE & E \end{bmatrix}.$$

We make the following assumptions throughout this letter:

Assumption 1: A measurement of all states is available at each sample instant.

Assumption 2: Matrix E is column full rank.

Assumption 3: The disturbances are assumed to be independent and identically normally distributed random variables (i.e., $\mathbf{w}_t \sim \mathcal{N}(0, I)$).

Besides the dynamics, the system has probabilistic constraints on state and control input

$$\mathbb{P}(\mathbf{G}\mathbf{x}_t \leq \mathbf{g}) \geq 1 - \alpha_x, \tag{3}$$

$$\mathbb{P}(\mathbf{S}\mathbf{u}_t \leq \mathbf{s}) \geq 1 - \alpha_u, \tag{4}$$

where $\mathbf{G} \in \mathbb{R}^{l\times(N+1)n}$, $\mathbf{g} \in \mathbb{R}^l$, $\mathbf{S} \in \mathbb{R}^{q\times Nm}$, and $\mathbf{s} \in \mathbb{R}^q$, $\mathbb{P}$ denotes the probability, α_x and α_u denote the probability level of constraint violation, and they are restricted to be in $(0, 0.5]$. The choice of α_x and α_u generally depends on the desired controller performance. In general, the choice of α_u will be very small (e.g., 0.005) to ensure as few violations of input constraints as possible.

Assuming the system states can be measured at all times, the finite-horizon SMPC problem with chance constraints is stated as follows.

Problem 1 (SMPC):

$$\begin{aligned} \min_{\mathbf{u}_t} \quad & \mathbb{E}\left[\sum_{k=0}^{N-1}(x_{k|t}^{\mathrm{T}}Qx_{k|t} + u_{k|t}^{\mathrm{T}}Ru_{k|t}) + x_{N|t}^{\mathrm{T}}Q_N x_{N|t}\right] \\ \text{s.t.} \quad & \mathbf{x}_t = \mathbf{A}x_{0|t} + \mathbf{B}\mathbf{u}_t + \mathbf{E}\mathbf{w}_t \\ & \mathbb{P}(\mathbf{G}\mathbf{x}_t \leq \mathbf{g}) \geq 1 - \alpha_x \\ & \mathbb{P}(\mathbf{S}\mathbf{u}_t \leq \mathbf{s}) \geq 1 - \alpha_u \\ & \mathbf{w}_t \sim \mathcal{N}(0, I), \end{aligned}$$

where $Q > 0$, $Q_N > 0$, and $R > 0$ are given symmetric matrices of appropriate dimensions.

For the linear system (2), Problem 1 considers joint chance constraints on the states and control inputs. Regarding implementation, although joint chance constraints can ensure that the constraint as a whole is satisfied to a certain confidence level, it is incredibly difficult to solve, even numerically. Therefore, we consider another way of dealing with chance constraints under the following assumption:

Assumption 4: The chance constraints on states and control inputs are assumed to consist of individual constraints, each a linear function of one state variable. Hence, the chance constraints in Problem 1 take the form

$$\mathbb{P}(\mathbf{G}_i\mathbf{x}_t \leq \mathbf{g}_i) \geq 1 - \alpha_{x,i}, \quad (i = 1, \ldots, a), \tag{5}$$

$$\mathbb{P}(\mathbf{S}_i\mathbf{u}_t \leq \mathbf{s}_i) \geq 1 - \alpha_{u,i}, \quad (i = 1, \ldots, b), \tag{6}$$

where $\mathbf{G}_i$ and $\mathbf{S}_i$ denote each row of the matrix $\mathbf{G}$ and $\mathbf{S}$, $\mathbf{g}_i$ denotes each component of the vector $\mathbf{g}$, $\mathbf{s}_i$ denotes each component of the vector $\mathbf{s}$, a denotes the number of rows of matrix $\mathbf{G}$, and b denotes the number of rows of matrix $\mathbf{S}$.

Linear individual chance constraints in the form (5) have been shown to be equivalent to a second-order cone constraint in [4]. In our setup, $\mathbf{w}_t \sim \mathcal{N}(0, I)$, the individual chance constraint (5) can be transformed into a deterministic constraint with the help of a standard Gaussian cumulative distribution function (CDF). By standardizing the distribution, we have

$$\mathbb{P}\left(\frac{\mathbf{G}_i\mathbf{E}\mathbf{w}_t}{\|\mathbf{G}_i\mathbf{E}\|} \leq \frac{-\mathbf{G}_i(\mathbf{A}x_{0|t}+\mathbf{B}\mathbf{u}_t)+\mathbf{g}_i}{\|\mathbf{G}_i\mathbf{E}\|}\right) \geq 1-\alpha_{x,i}, \quad (7)$$

where $\frac{\mathbf{G}_i\mathbf{E}\mathbf{w}_t}{\|\mathbf{G}_i\mathbf{E}\|}$ is a standard Gaussian distribution variable.

Given the definition of CDF, this equation can be rewritten as

$$\Phi\left(\frac{-\mathbf{G}_i(\mathbf{A}x_{0|t}+\mathbf{B}\mathbf{u}_t)+\mathbf{g}_i}{\|\mathbf{G}_i\mathbf{E}\|}\right) \geq 1-\alpha_{x,i}, \quad (8)$$

where Φ is the standard Gaussian CDF. Thus, the individual chance constraint (5) can be transformed into a deterministic one by using the inverse of standard Gaussian CDF as

$$\mathbf{G}_i(\mathbf{A}x_{0|t}+\mathbf{B}\mathbf{u}_t)+\Phi^{-1}(1-\alpha_{x,i})\|\mathbf{G}_i\mathbf{E}\| \leq \mathbf{g}_i, \quad (9)$$

and so can (6).

III. Control Parameterization

At each time instant t, we are interested in solving Problem 1 over the class of state feedback policy. The affine disturbance feedback control policy was proposed by [15] to get an equivalent convex optimization problem, described as

$$u_{i|t} = \sum_{j=0}^{i-1} M_{i,j|t}w_{j|t} + h_{i|t}, \quad (10)$$

where $M_{i,j|t} \in \mathbb{R}^{m\times r}$ and $h_{i|t} \in \mathbb{R}^m$ are decision variables. For convenience, we define matrix $\mathbf{M}'_t \in \mathbb{R}^{Nm\times Nr}$ and vector $\mathbf{h}'_t \in \mathbb{R}^{Nm}$ in such a way that

$$\mathbf{h}'_t := \left[h_{0|t}^{\mathrm{T}}, \ldots, h_{N-1|t}^{\mathrm{T}}\right]^{\mathrm{T}},$$
$$\mathbf{M}'_t := \begin{bmatrix} O & \cdots & \cdots & O \\ M_{1,0|t} & O & \cdots & O \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0|t} & \cdots & M_{N-1,N-2|t} & O \end{bmatrix}. \quad (11)$$

The affine disturbance feedback control policy (10) contains $N(N-1)mr/2+m$ different decision variables, which means it grows quadratically with the prediction horizon, making it difficult to use in real-time calculations. Regarding calculation time, we consider a kind of affine disturbance feedback that contains fewer decision variables, called "simplified affine disturbance feedback" (SADF), which takes the form of:

$$u_{i|t} = \sum_{k=0}^{i-1} M_{i-k|t}w_{k|t} + h_{i|t}, \quad (12)$$

where $u_{i|t}$ is an affine function of the i disturbances preceding time $t{+}i$, $M_{i-k|t} \in \mathbb{R}^{m\times r}$ is the matrix of coefficients associated with the past k-step disturbances, and $w_{k|t}$ is a disturbance that was realized at the last step. For convenience, we also define matrix $\mathbf{M}_t \in \mathbb{R}^{Nm\times Nr}$ and vector $\mathbf{h}_t \in \mathbb{R}^{Nm}$ in such a way that

$$\mathbf{h}_t := \left[h_{0|t}^{\mathrm{T}}, \ldots, h_{N-1|t}^{\mathrm{T}}\right]^{\mathrm{T}},$$
$$\mathbf{M}_t := \begin{bmatrix} O & \cdots & \cdots & O \\ M_{1|t} & O & \cdots & O \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1|t} & \cdots & M_{1|t} & O \end{bmatrix}. \quad (13)$$

This parameterization is a simplified version of the original affine disturbance feedback control policy (10). Note that M_t has a strictly lower triangular Toeplitz structure with a constant block in each descending diagonal from left to right and only contains just $(N-1)mr$ different decision variables, that grow linearly with the prediction horizon.

Using the approach mentioned above, the chance constraints under the SADF control parameterization can be transformed into deterministic ones as

$$\mathbf{G}_i(\mathbf{A}x_0+\mathbf{B}\mathbf{h}_t)+\Phi^{-1}(1-\alpha_{x,i})\|\mathbf{G}_i(\mathbf{B}\mathbf{M}_t+\mathbf{E})\| \leq \mathbf{g}_i. \quad (14)$$

Similarly, chance constraints on control input can also be transformed into deterministic ones as

$$\mathbf{S}_i\mathbf{h}_t+\Phi^{-1}(1-\alpha_{u,i})\|\mathbf{S}_i\mathbf{M}_t\| \leq \mathbf{s}_i. \quad (15)$$

These constraints are second-order cone constraints that concern the decision variable $(\mathbf{M}_t, \mathbf{h}_t)$, and deterministic equivalent chance constraints on the state (14) and control input (15) can be rewritten as $(\mathbf{x}, \mathbf{u}) \in Z$ for convenience.

Thus, a reformulation of Problem 1 using SADF (12) is stated as follows:

Problem 2 (SMPC With SADF):

$$\begin{aligned} \min_{\mathbf{u}_t}\ & \mathbb{E}\left[\sum_{k=0}^{N-1}(x_{k|t}^{\mathrm{T}}Qx_{k|t}+u_{k|t}^{\mathrm{T}}Ru_{k|t})+x_{N|t}^{\mathrm{T}}Q_Nx_{N|t}\right] \\ \text{s.t.}\ & \mathbf{x}_t = \mathbf{A}x_{0|t}+\mathbf{B}\mathbf{u}_t+\mathbf{E}\mathbf{w}_t \\ & \mathbf{u}_t = \mathbf{M}_t\mathbf{w}_t+\mathbf{h}_t \\ & (\mathbf{x}_t, \mathbf{u}_t) \in Z \\ & \mathbf{w}_t \sim \mathcal{N}(0, I). \end{aligned}$$

The expectation $\mathbb{E}$ in the objective function can be obtained by substituting the expectation and variance information of disturbance $w_{k|t}$. Thus, Problem 2 can be treated as a deterministic MPC with nonlinear constraints, and it can be solved in principle by general nonlinear MPC solvers such as CasADi [26], C/GMRES [27], etc. Notably, although Problem 2 inherits the characteristics of the convex optimization problem of affine disturbance feedback parameterization, there is still a question of whether SADF is the reasonable feedback control policy we want. So, in the next section, we will discuss the relationship between SADF and the state feedback control policy further.

IV. Properties of SADF

The set of admissible pair $(\mathbf{M}, \mathbf{h})$ for SADF is defined as

$$\Xi^{\mathrm{sadf}}(x) := \left\{ (\mathbf{M}, \mathbf{h}) \,\middle|\, \begin{array}{l} (\mathbf{M}, \mathbf{h}) \text{ satisfies (13)}, \\ \mathbf{x} = \mathbf{A}x+\mathbf{B}\mathbf{u}+\mathbf{E}\mathbf{w}, \\ \mathbf{u} = \mathbf{M}\mathbf{w}+\mathbf{h}, \\ (\mathbf{x}, \mathbf{u}) \in Z, \mathbf{w} \sim \mathcal{N}(0, I), \end{array} \right\} \quad (16)$$

and the set of initial states x for which an admissible control law of the form (12) exists as

$$X^{\mathrm{sadf}} := \{x \in \mathbb{R}^n | \Xi^{\mathrm{sadf}} \neq \emptyset\}. \quad (17)$$

Remark 1: Note that the original affine disturbance feedback law (10) subsumes the proposed SADF (12). Therefore, (12) is a more conservative approximation to (10).

The original affine disturbance feedback control law (10) was proven to be equivalent to an affine state feedback control law in [20]. Thus, the important question is whether the proposed SADF (12) is also an equivalent and tractable formulation of a certain state feedback law. In other words, is this a reasonable approximation?

Consider a state feedback control law:

$$u_{i|t} = \sum_{k=0}^{i} K_{i-k|t} x_{k|t} + v_{i|t}. \tag{18}$$

We also define the augmented matrix $\mathbf{K}_t \in \mathbb{R}^{Nm\times(N+1)n}$ and vector $\mathbf{v}_t \in \mathbb{R}^{Nm}$ as

$$\mathbf{v}_t := \left[v_{0|t}^{\mathrm{T}}, \ldots, v_{N-1|t}^{\mathrm{T}}\right]^{\mathrm{T}},$$
$$\mathbf{K}_t := \begin{bmatrix} K_{0|t} & O & \cdots & O & O \\ K_{1|t} & K_{0|t} & \cdots & O & O \\ \vdots & \ddots & \ddots & \vdots & O \\ K_{N-1|t} & \cdots & K_{1|t} & K_{0|t} & O \end{bmatrix}. \tag{19}$$

Notably, the state feedback control law with structure (18) subsumes the well-known state feedback control law $u_{i|t} = K_t x_{i|t} + v_{i|t}$ [15], with knowledge of prior states. Thus, it contains more degrees of freedom.

In a manner similar to (16), we can also define the set of admissible $(\mathbf{K}, \mathbf{v})$ as

$$\Xi^{\mathrm{sf}}(x) := \left\{ (\mathbf{K}, \mathbf{v}) \middle| \begin{array}{l} (\mathbf{K}, \mathbf{v}) \text{ satisfies (19)}, \\ \mathbf{x} = \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w}, \\ \mathbf{u} = \mathbf{K}\mathbf{x} + \mathbf{v}, \\ (\mathbf{x}, \mathbf{u}) \in Z, \mathbf{w} \sim \mathcal{N}(0, I) \end{array} \right\} \tag{20}$$

and the same for the set of initial states x for which an admissible control law of the form (18) is

$$X^{\mathrm{sf}} := \{x \in \mathbb{R}^n | \Xi^{\mathrm{sf}} \neq \emptyset\}. \tag{21}$$

Next, we show that the state feedback control law (18) and the SADF control law (12) are equivalent.

Theorem 1: The sets of admissible states for control policy, (12) and (18), are identical (i.e., $X^{\mathrm{adf}} = X^{\mathrm{sf}}$ holds). Moreover, given any $x \in X^{\mathrm{adf}}$, for any admissible $(\mathbf{M}, \mathbf{h})$, there exists a pair of $(\mathbf{K}, \mathbf{v})$ yields the same state and input sequence for the same disturbance sequence $\mathbf{w}$, and vice versa.

Proof: $X^{\mathrm{sf}} \subseteq X^{\mathrm{adf}}$: Based on the definition of an admissible set for any given $x \in X^{\mathrm{sf}}$, there exists a pair of $(\mathbf{K}, \mathbf{v})$ satisfies the constraints in (20). Therefore, for a given disturbance sequence $\mathbf{w}$, we can eliminate $\mathbf{x}$ for $\mathbf{u}$ through simultaneous equations (2) and (18) as

$$\mathbf{u} = \mathbf{K}(I - \mathbf{B}\mathbf{K})^{-1}(\mathbf{A}x_0 + \mathbf{B}\mathbf{v} + \mathbf{E}\mathbf{w}) + \mathbf{v}. \tag{22}$$

Notably, $(I - \mathbf{B}\mathbf{K})$ is always nonsingular since $\mathbf{B}\mathbf{K}$ is strictly lower block triangular. The control input can be divided into terms related to disturbance $\mathbf{w}$ and terms not related to $\mathbf{w}$ as

$$\mathbf{u} = \mathbf{K}(I - \mathbf{B}\mathbf{K})^{-1}\mathbf{E}\mathbf{w} + (\mathbf{K}(I - \mathbf{B}\mathbf{K})^{-1}(\mathbf{A}x_0 + \mathbf{B}\mathbf{v}) + \mathbf{v}). \tag{23}$$

Choose $(\mathbf{M}, \mathbf{h})$ as

$$\mathbf{M} = \mathbf{K}(I - \mathbf{B}\mathbf{K})^{-1}\mathbf{E}, \tag{24a}$$
$$\mathbf{h} = \mathbf{K}(I - \mathbf{B}\mathbf{K})^{-1}(\mathbf{A}x_0 + \mathbf{B}\mathbf{v}) + \mathbf{v}. \tag{24b}$$

By applying the results in [28], the product of two lower triangular Toeplitz matrices, $\mathbf{B}$ and $\mathbf{K}$, is again a lower triangular Toeplitz matrix. Moreover, if a lower triangular Toeplitz matrix is invertible, then its inverse is also Toeplitz. One can easily verify that $\mathbf{M}$ has a strictly lower block triangular Toeplitz structure, meaning the pair of $(\mathbf{M}, \mathbf{h})$ satisfies (16) and can yield the same input sequence as the given pair of $(\mathbf{K}, \mathbf{v})$. Thus $x \in X^{\mathrm{sf}}$ implies $x \in X^{\mathrm{adf}}$.

$X^{\mathrm{adf}} \subseteq X^{\mathrm{sf}}$: For any given $x \in X^{\mathrm{adf}}$, there exists a pair of $(\mathbf{M}, \mathbf{h})$ that satisfies the constraints in (16). For a given disturbance sequence $\mathbf{w}$, we can also eliminate $\mathbf{w}$ for $\mathbf{u}$ through simultaneous equations (2) and (12) as

$$\mathbf{u} = (I + \mathbf{M}\mathbf{E}^{-}\mathbf{B})^{-1}(\mathbf{M}\mathbf{E}^{-}(\mathbf{x} - \mathbf{A}x_0) + \mathbf{h}), \tag{25}$$

where $\mathbf{E}^{-} \in \mathbb{R}^{Nr\times Nn}$ denotes the left inverse of $\mathbf{E}$, so that $\mathbf{E}^{-}\mathbf{E} = I$, and it is given explicitly as

$$\mathbf{E}^{-} := \begin{bmatrix} O & E^{-} & O & \cdots & O \\ O & -E^{-}A & E^{-} & \cdots & O \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O & O & \cdots & -E^{-}A & E^{-} \end{bmatrix},$$

where $E^{-} \in \mathbb{R}^{r\times n}$ denotes the left inverse of E. Note that E^{-} exists because E is a full column rank matrix by Assumption 2. Since $\mathbf{M}\mathbf{E}^{-}\mathbf{B}$ is strictly lower block triangular, $I + \mathbf{M}\mathbf{E}^{-}\mathbf{B}$ is always nonsingular. So, we can choose $(\mathbf{K}, \mathbf{v})$ as

$$\mathbf{K} = (I + \mathbf{M}\mathbf{E}^{-}\mathbf{B})^{-1}\mathbf{M}\mathbf{E}^{-}, \tag{26a}$$
$$\mathbf{v} = (I + \mathbf{M}\mathbf{E}^{-}\mathbf{B})^{-1}(\mathbf{h} - \mathbf{M}\mathbf{E}^{-}\mathbf{A}x_0). \tag{26b}$$

It is easy to confirm $\mathbf{M}\mathbf{E}^{-}$ and $\mathbf{M}\mathbf{E}^{-}\mathbf{B}$ as being strictly lower block triangular Toeplitz. Through the properties of Toeplitz matrix multiplication, we see that $\mathbf{K}$ has a strictly lower triangular Toeplitz structure, meaning the pair of $(\mathbf{K}, \mathbf{v})$ satisfies (20) and can yield the same input sequence as the given pair of $(\mathbf{M}, \mathbf{h})$. Thus, $x \in X^{\mathrm{adf}}$ implies $x \in X^{\mathrm{sf}}$. ■

Theorem 1 tells us that the optimal solution $(\mathbf{M}^{*}, \mathbf{h}^{*})$ obtained by SADF is actually equivalent to a state feedback control policy $(\mathbf{K}^{*}, \mathbf{v}^{*})$. In MPC implementation, the first optimal control input is applied; i.e.,

$$\pi(x_t) = K_{0|t}^{*}(x_t)x_t + v_{0|t}^{*}(x_t). \tag{27}$$

Thus, the closed-loop system is given by

$$x_{t+1} = Ax_t + B\pi(x_t) + Ew_t. \tag{28}$$

In addition, if we suppose the disturbances are bounded with an unknown bound, and if the MPC control is defined as in (27), we can guarantee the closed-loop convergence and feasibility. Although the assumption of bounded disturbances contravenes the original assumption that the disturbance w_t has infinite support, it is meaningful in practice.

Proof of this conclusion relies on the concepts of invariance with probability p [29] and associated methods. Similar proof can be found in [25] (omitted for brevity).

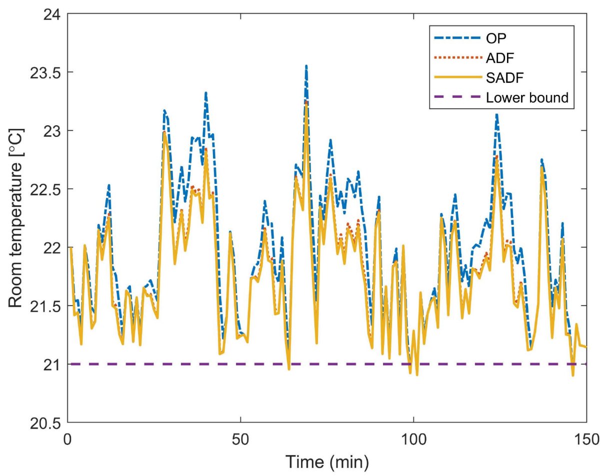

Fig. 1. Room temperature profile [°C].

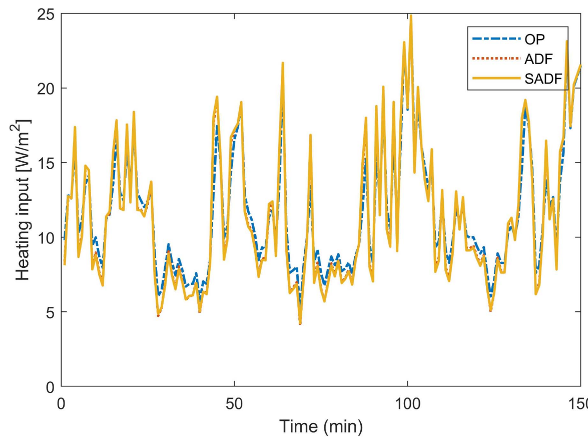

Fig. 2. Heating [W/m^2].

TABLE I
COMPARISON OF OBJECTIVE FUNCTION VALUES FOR DIFFERENT STATES UNDER THE SAME SETTINGS ($N = 30$)

	OP	ADF	SADF
state 1: $[28, 28, 21]^{\mathrm{T}}$	1778	1242	1256
state 2: $[22, 18, 15]^{\mathrm{T}}$	835	505	567
state 3: $[25, 25, 15]^{\mathrm{T}}$	2004	1430	1448

TABLE II
AVERAGE COMPUTATION TIME PER UPDATE (MS)

	OP	ADF	SADF
$N = 5$	9	19	13
$N = 15$	16	697	150
$N = 30$	31	21320	1420

V. NUMERICAL EXAMPLE

We tested our method with the control problem of room temperature [30]. The basic control target was keeping the room temperature higher than a certain level in the presence of external disturbances. The system contained three states: let x_1 be the room temperature, let x_2 be the temperature in the wall connected to another room, and let x_3 be the temperature in the wall connected to the outside. The system was subject to three disturbances in which w_1 denoted the outside temperature, w_2 denoted the solar radiation, and w_3 denoted the internal heat gains (e.g., people, electronic devices, etc.) and all the external disturbance subjects to $\mathcal{N}(0, I)$. The only control input u was the heating, which was constrained to $0 \leq u \leq 45$ [W/m^2]. The control objective was keeping the room temperature above 21°C with minimum energy; thus, the weighting matrix is $Q = 0$, and the state constraint is treated like a chance constraint $\mathbb{P}(x \geq 21) \geq 1 - \alpha$. The parameters of the system are taken from [31] as follows:

$$A := \begin{bmatrix} 0.8511 & 0.0541 & 0.0707 \\ 0.1293 & 0.8635 & 0.0055 \\ 0.0989 & 0.0032 & 0.7541 \end{bmatrix},$$

$$E = 10^{-3} \cdot \begin{bmatrix} 22.2170 & 1.7912 & 42.2123 \\ 1.5376 & 0.6944 & 2.9214 \\ 103.1813 & 0.1032 & 196.0444 \end{bmatrix},$$

$$B := \begin{bmatrix} 0.0035 \\ 0.0003 \\ 0.0002 \end{bmatrix}, \quad R := 1.$$

The following three control policies are compared:

1. SADF control policy (12),
2. Original affine disturbance feedback (ADF) control policy (10), and
3. Open-loop prediction (OP) control policy, i.e., $\mathbf{M} = 0$.

We carried out the simulation on a laptop computer with a 2.60 GHz Intel Core i7-6700HQ and CasADi toolkit [26] in MATLAB 2020a. In all the simulations, we subjected the system to the same disturbance realizations and constraint violation degree $\alpha_{x,i} = 0.1$, $\alpha_{u,i} = 0.005$. Figures 1 and 2 show a 150-minute simulation with the initial state $[22, 22, 15]^{\mathrm{T}}$ and the prediction horizon $N = 6$. The results show that an OP control policy led to conservative control behavior; the room temperature was almost always higher than the room temperature of the other two control policies. In contrast, ADF and SADF led to relatively less conservative control behavior.

To compare the three controllers more intuitively, we considered the objective function value for the same optimization problem (i.e., only a single optimization starting from the same state within a prediction horizon $N = 30$). Table I compares three states. The results show that the OP controller consumes the most energy for an identical optimization problem (i.e., most conservative), while the control SADF performance is slightly more conservative than ADF.

Table II shows the computation times of the three controllers. The table shows that the computation times of OP and SADF did not grow as quickly with respect to the length of the prediction horizon, while the computation time of ADF grew very quickly. Our method provides an acceptable computation time in this example.

VI. CONCLUSION

We proposed a simplified affine disturbance feedback control law for SMPC. The decision variable number decreased to $\mathcal{O}(N)$ compared with $\mathcal{O}(N^2)$ of the original affine disturbance feedback control law, resulting in a preferable trade-off between real-time calculation and control performance. This parameterization is shown to be equivalent to a state feedback control law, and the closed-loop stability of the SMPC problem can also be guaranteed under mild assumptions.

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