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ON OBSTACLE PROBLEM FOR BRAKKE'S MEAN CURVATURE FLOW*

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Abstract. We consider the obstacle problem of the weak solution to the mean curvature flow, in the sense of Brakke's mean curvature flow. We prove the global existence of the weak solution with obstacles which have $C^{1,1}$ boundaries in two and three space dimensions. To obtain the weak solution, we use the Allen–Cahn equation with forcing term.

Key words. mean curvature flow, Allen–Cahn equation, phase-field method

AMS subject classifications. Primary, 35K93; Secondary, 53E10

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1. Introduction.

Let $T > 0$, and let $d \geq 2$ be an integer. Assume that $U_t \subset \mathbb{R}^d$ is a bounded open set and M_t is a smooth boundary of U_t for any $t \in [0, T)$. We call the family of the hypersurfaces $\{M_t\}_{t\in[0,T)}$ the mean curvature flow if

$$v = h \qquad \text{on } M_t,\ t \in (0, T). \tag{1.1}$$

Here, v and h are the normal velocity vector and the mean curvature vector of M_t, respectively. Brakke [5] proved the global existence of the multiphase weak solution to (1.1) called Brakke's mean curvature flow. However, since the flow is defined by an integral inequality, its solution may become an empty set after a certain time. Subsequently, Kim and Tonegawa [21] proved the global existence of nontrivial Brakke's mean curvature flow, by showing that each volume of the multiphase is continuous with respect to t. The phase-field method and the elliptic regularization by Ilmanen [17, 18] are known as another set of proofs of the global existence of the Brakke's mean curvature flow. Similar to the Brakke's mean curvature flow, the weak solution called L^2-flow was studied by Mugnai and Röger [28, 29]. In addition, the regularity of Brakke's mean curvature flow is studied by Brakke [5], White [44], Kasai and Tonegawa [20], and Tonegawa [42]. Concerning results for other types of weak solutions, the existence theorem of the viscosity solutions via the level set method was presented independently by Chen, Giga, and Goto [10] and Evans and Spruck [13] at the same period, and a weak solution using a variational method was studied by Almgren, Taylor, and Wang [3] and Luckhaus and Sturzenhecker [24].

Let O_+ and O_- be open sets with $\mathrm{dist}\,(O_+, O_-) > 0$. In this paper, we consider the weak solution to (1.1) with the obstacles O_+ and O_-, namely, a family of open sets $\{U_t\}_{t\in[0,T)}$ satisfies $O_+ \subset U_t$ and $U_t \cap O_- = \emptyset$ for any $t \in [0, T)$, and the boundary $M_t = \partial U_t$ satisfies (1.1) on $(\overline{O_+ \cup O_-})^c$, in the sense of Brakke's mean curvature flow. Since the mean curvature flow can be regarded as a simple model of the cell motility, it is natural to consider its obstacle problem (see [11, 27]). In addition, the

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obstacle problems for elliptic equations including the minimal surface equation have been studied over a long period of time (see [8, 30, 32, 35] and references therein).

About the obstacle problem for the mean curvature flow, Almeida, Chambolle, and Novaga [4] showed the global existence of weak solutions for $d \geq 2$ by a variational method. Moreover, they proved the short time existence and uniqueness of $C^{1,1}$ solutions for $d = 2$, when the obstacle has a compact $C^{1,1}$ boundary. Mercier and Novaga [26] extended the short time existence and uniqueness of $C^{1,1}$ solutions for $d \geq 2$, and they also proved the global existence and uniqueness of the graphical viscosity solutions if the boundaries of obstacles are also graphs. In the case of the viscosity solution with the level set method, Mercier [25] showed the global existence and uniqueness of continuous viscosity solutions to

$$u_t + F(\nabla u, \nabla^2 u) + k|\nabla u| = 0 \quad \text{on } \{u^- \leq u \leq u^+\},$$

where u^- and u^+ are given uniformly continuous functions with $u^- \leq u^+$, k is a given Lipschitz function, and the assumptions of F allow this equation to be the mean curvature flow with forcing term k, in the sense of the level set method. Ishii, Kamata, and Koike [19] proved the global existence and uniqueness of Lipschitz viscosity solutions when $k \equiv 0$ and $u^\pm$, $\partial_t u^\pm$, $\partial_{x_k} u^\pm$, $\partial_{x_k x_l} u^\pm \in L^\infty(\mathbb{T}^d \times [0,\infty))$ for any $1 \leq k, l \leq d$, where $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$. Giga, Tran, and Zhang [15] studied the large time behavior of viscosity solutions with constant driving force k.

Let $d = 2$ or 3, $\Omega := \mathbb{T}^d$, and the obstacles $O_+, O_- \subset \Omega$ have $C^{1,1}$ boundaries and satisfy $\operatorname{dist}(O_+, O_-) > 0$. In this paper, we prove the global existence of the weak solution to (1.1) with obstacles in the sense of Brakke (see Theorem 5.1). Note that the weak solution obtained in this paper has similar properties to the weak solution by the minimizing movement in [4, Theorem 4.6] (see Remark 5.3). However, since the uniqueness of the flow we obtain is not known, it is an open question whether Brakke's mean curvature flow coincides with the weak solution studied in [4].

To obtain the result, we use the phase-field method. Bretin and Perrier [6] studied the Allen–Cahn equation with a penalized double well potential depending on the obstacles. In contrast, roughly speaking, the Allen–Cahn equation considered in this paper (see (3.4)) is formally an approximation to the following:

$$v = h + gn \qquad \text{on } M_t,\ t \in (0,\infty), \tag{1.2}$$

where n is the outward unit normal vector of M_t and g is given by

$$g(x) = \begin{cases} \frac{d}{R_0} & \text{if } x \in \overline{O_+}, \\ -\frac{d}{R_0} & \text{if } x \in \overline{O_-}, \\ 0 & \text{otherwise}, \end{cases}$$

where R_0 is given in (2.2). If the solution M_t touches the obstacle at x, the absolute value of its mean curvature $|h(x,t)|$ is less than $\frac{d}{R_0}$, hence the solution cannot move into the obstacle. Note that this argument was used in Mercier and Novaga [26]. In order to use this argument in the phase-field method, we give an appropriate forcing term for the Allen–Cahn equation and show simple sub- and supersolutions that correspond to obstacles (see Lemma 4.1).

To obtain the convergence of the Allen–Cahn equation to the Brakke flow, we need to prove that the Radon measure given by the energy of the Allen–Cahn equation has good properties, such as that it converges to the mass measure of an integral varifold (see [17]). In the case of $d = 2$ or 3, Röger and Schätzle [33] proved the properties under the suitable assumptions for the energies of the Allen–Cahn equation (this results have been used in [23, 28, 29, 34, 39]). The assumption for d in the main result of this paper comes from the use of [33, 29] (see Remark 5.4).

The organization of this paper is as follows. In section 2, we set out basic definitions and assumptions about the obstacles and the initial data. In section 3 we introduce the Allen–Cahn equation we deal with in this paper. In addition we also show the standard estimates for the solution. In section 4 we give supersolutions and subsolutions to the Allen–Cahn equation that are necessary to show that the solutions to (1.1) do not intrude upon the obstacles. In section 5 we prove the global existence of the weak solution to (1.1) with obstacles, in the sense of Brakke.

2. Notation and assumptions. First we recall some notions and definitions from the geometric measure theory and refer to [2, 5, 14, 36, 43] for more details. Let d be a positive integer. For $y \in \mathbb{R}^d$ and $r > 0$, we define $B_r(y) := \{x \in \mathbb{R}^d \mid |x - y| < r\}$. We denote the space of bounded variation functions on $U \subset \mathbb{R}^d$ as $BV(U)$. For a function $\psi \in BV(U)$, we write the total variation measure of the distributional derivative $\nabla\psi$ by $\|\nabla\psi\|$. Let μ be a Radon measure on $\mathbb{R}^d$. We denote $\mu(\phi) = \int \phi \, d\mu$ for $\phi \in C_c(\mathbb{R}^d)$. We call μ k-rectifiable ($1 \le k \le d-1$) if μ is given by $\mu = \theta \mathscr{H}^k \lfloor_M$, where $M \subset \mathbb{R}^d$ is a $\mathscr{H}^k$-measurable countably k-rectifiable set and $\theta \in L^1_{loc}(\mathscr{H}^k \lfloor_M)$ is a positive function $\mathscr{H}^k$-a.e. on M. Especially, if θ is integer-valued $\mathscr{H}^k$-a.e. on M, then we say μ is k-integral. Note that if M is a countably k-rectifiable set with locally finite and $\mathscr{H}^k$-measurable, then there exists the approximate tangent space $T_x M$ for $\mathscr{H}^k$-a.e. $x \in M$. For k-dimensional subspace $S \subset \mathbb{R}^d$ and $g \in C^1_c(\mathbb{R}^d; \mathbb{R}^d)$, we denote $\operatorname{div}_S g := \sum_{i=1}^k \nu_i \cdot \nabla_{\nu_i} g$, where $\{\nu_1, \ldots, \nu_k\}$ is an orthonormal basis of S. For a rectifiable Radon measure $\mu = \theta \mathscr{H}^k \lfloor_M$, h is called a generalized mean curvature vector if

$$\int \operatorname{div}_{T_x M} g \, d\mu = -\int h \cdot g \, d\mu$$

for any $g \in C^1_c(\mathbb{R}^d; \mathbb{R}^d)$. The left-hand side is called the first variation of μ. The weak solution to the mean curvature flow considered in this paper is as follows.

Definition 2.1. *Let $U \subset \mathbb{R}^d$ be an open set. A family of Radon measures $\{\mu_t\}_{t \in [0,T)}$ on U is called Brakke's mean curvature flow if*

$$\int_U \phi \, d\mu_t \Big|_{t=t_1}^{t_2} \le \int_{t_1}^{t_2} \int_U \{(\nabla\phi - \phi h) \cdot h + \phi_t\} \, d\mu_t dt \tag{2.1}$$

for all $0 \le t_1 < t_2 < \infty$ and $\phi \in C^1_c(U \times [0,\infty); [0,\infty))$. Here h is the generalized mean curvature vector of μ_t. Note that (2.1) is called Brakke's inequality.

Next, we state assumptions for the initial data and the obstacles. Let $\Omega = \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$, $O_+ \subset \Omega$, and $O_- \subset \Omega$ be bounded open sets. We assume that there exist $R_0 > 0$ and $R_1 > 0$ such that

$$O_\pm = \bigcup_{B_{R_0}(x) \subset O_\pm} B_{R_0}(x) \qquad \text{(the interior ball condition)} \tag{2.2}$$

and $\operatorname{dist}(O_+, O_-) > R_1$. Note that if O_+ and O_- have $C^{1,1}$ boundaries, then (2.2) is satisfied for some $R_0 > 0$ (see [1]). Let $U_0 \subset \Omega$ be a bounded open set, and we denote $M_0 := \partial U_0$. Throughout this paper, we assume the following:

1. There exists $\delta_1 > 0$ such that $O_+ \subset U_0$ with $\operatorname{dist}(O_+, M_0) > \delta_1$ and $U_0 \subset (O_-)^c$ with $\operatorname{dist}(O_-, M_0) > \delta_1$.

2. There exist $D_0 > 0$ and $R_2 \in (0,1)$ such that

$$\sup_{x\in\Omega,\, 0<R<R_2} \frac{\mathscr{H}^{d-1}(M_0 \cap B_R(x))}{\omega_{d-1}R^{d-1}} \le D_0 \quad \text{(the upper bounds of the density)}. \tag{2.3}$$

Here ω_{d-1} is a $(d-1)$-dimensional volume of the unit ball in $\mathbb{R}^{d-1}$.
3. There exists a family of open sets $\{U_0^i\}_{i=1}^\infty$ such that U_0^i has a C^3 boundary M_0^i such that (U_0, M_0) be approximated strongly by $\{(U_0^i, M_0^i)\}_{i=1}^\infty$, that is,

$$\lim_{i\to\infty} \mathscr{L}^d(U_0 \triangle U_0^i) = 0 \quad \text{and} \quad \lim_{i\to\infty} \|\nabla \chi_{U_0^i}\| = \|\nabla\chi_{U_0}\| \ \text{ as measures.} \tag{2.4}$$

Moreover,

$$\operatorname{dist}(O_\pm, M_0^i) > \delta_1/2 \qquad \text{for any } i \in \mathbb{N}. \tag{2.5}$$

Remark 2.2. For example, if U_0 is a Caccioppoli set, then (2.4) is satisfied (see [16, Theorem 1.24]). In addition, if M_0 is C^1, then (2.3) with $D_0 = 1 + o(R_2)$ and (2.4) hold.

3. Allen–Cahn equation with forcing term. In this section, we consider the Allen–Cahn equation with forcing term and give basic energy estimates for the solution.

Set $W(s) = (1-s^2)^2/2$ and $q^\varepsilon(r) := \tanh(\frac{r}{\varepsilon})$ for $r \in \mathbb{R}$ and $\varepsilon > 0$. Then q^ε is a solution to

$$\frac{\varepsilon(q_r^\varepsilon)^2}{2} = \frac{W(q^\varepsilon)}{\varepsilon} \qquad \text{and} \qquad q_{rr}^\varepsilon = \frac{W'(q^\varepsilon)}{\varepsilon^2} \tag{3.1}$$

with $q^\varepsilon(0) = 0$, $q^\varepsilon(\pm\infty) = \pm 1$, and $q_r^\varepsilon(r) > 0$ for any $r \in \mathbb{R}$.

Let $d \ge 2$, and let $\{\varepsilon_i\}_{i=1}^\infty$ be a positive sequence with $\varepsilon_i \downarrow 0$ as $i \to \infty$ and $\varepsilon_i \in (0,1)$ for any $i \in \mathbb{N}$ (we often write ε_i as ε for simplicity). For $U_0^i \subset \Omega$ we define a periodic function $r_0^{\varepsilon_i}$ by

$$r_0^{\varepsilon_i}(x) = \begin{cases} \operatorname{dist}(x, M_0^i) & \text{if } x \in U_0^i, \\ -\operatorname{dist}(x, M_0^i) & \text{if } x \notin U_0^i. \end{cases}$$

We remark that $|\nabla r_0^{\varepsilon_i}| \le 1$ a.e. $x \in \Omega$ and $r_0^{\varepsilon_i}$ is smooth near M_0^i. Let $\tilde{r}_0^{\varepsilon_i} \in C^3(\Omega)$ be a smoothing of $r_0^{\varepsilon_i}$ with $|\nabla \tilde{r}_0^{\varepsilon_i}| \le 1$ and $|\nabla^2 \tilde{r}_0^{\varepsilon_i}| \le \varepsilon_i^{-1}$ in Ω, and $\tilde{r}_0^{\varepsilon_i} = r_0^{\varepsilon_i}$ near M_0^i.

Define

$$\varphi_0^{\varepsilon_i}(x) := q^{\varepsilon_i}(\tilde{r}_0^{\varepsilon_i}(x)), \quad i \ge 1. \tag{3.2}$$

Let $g^{\varepsilon_i} \in C^\infty(\Omega)$ be a smooth function such that

$$g^{\varepsilon_i}(x) = \begin{cases} \frac{d}{R_0} & \text{if } \operatorname{dist}(x, O_+) \le \sqrt{\varepsilon_i}, \\ -\frac{d}{R_0} & \text{if } \operatorname{dist}(x, O_-) \le \sqrt{\varepsilon_i}, \\ 0 & \text{if } \min\{\operatorname{dist}(x, O_+), \operatorname{dist}(x, O_-)\} \ge 2\sqrt{\varepsilon_i}, \end{cases} \tag{3.3}$$

with $\max_{x\in\Omega} |g^{\varepsilon_i}(x)| \le \frac{d}{R_0}$, $\max_{x\in\Omega} |\nabla g^{\varepsilon_i}(x)| \le M\varepsilon_i^{-1}$, and $\max_{x\in\Omega} |\nabla^2 g^{\varepsilon_i}(x)| \le M\varepsilon_i^{-2}$ for any $i \in \mathbb{N}$, where $M > 0$ is independent of i. To define g^{ε_i}, we may assume that $2\sqrt{\varepsilon_i} \le \frac{R_1}{3}$ for any $i \in \mathbb{N}$, if necessary.

In this paper, we consider the following Allen–Cahn equation:

$$\begin{cases} \varepsilon_i \varphi_t^{\varepsilon_i} = \varepsilon_i \Delta \varphi^{\varepsilon_i} - \dfrac{W'(\varphi^{\varepsilon_i})}{\varepsilon_i} + g^{\varepsilon_i} \sqrt{2W(\varphi^{\varepsilon_i})}, & (x,t) \in \Omega \times (0,\infty), \\ \varphi^{\varepsilon_i}(x,0) = \varphi_0^{\varepsilon_i}(x), & x \in \Omega. \end{cases} \tag{3.4}$$

Remark 3.1. The definition of the initial data (3.2) implies $\max_{x\in\Omega} |\varphi_0^\varepsilon(x)| < 1$. Therefore, we have $\sup_{x\in\Omega, t\in[0,T)} |\varphi^\varepsilon(x,t)| < 1$ for the solution φ^ε to (3.4) and $T > 0$ by the maximum principle (see [31]). We give the proof in Proposition 3.3. Note that the function $\sqrt{2W(\varphi^\varepsilon)}$ is important in the proof. By $|\varphi^\varepsilon| < 1$, we can define $r^\varepsilon = r^\varepsilon(x,t)$ by $\varphi^\varepsilon(x,t) = q^\varepsilon(r^\varepsilon(x,t))$, that is, $r^\varepsilon(x,t) = (q^\varepsilon)^{-1}(\varphi^\varepsilon(x,t))$.

Remark 3.2. Equation (3.4) corresponds to the mean curvature flow with forcing term (1.2) (see [29, 37, 39]). Not only for $|\varphi^\varepsilon| < 1$, we also need $\sqrt{2W(\varphi^\varepsilon)}$ to simplify the forcing term when we rewrite (3.4) as an PDE of r^ε (see (4.1)). Furthermore, if we adopt g^ε instead of $g^\varepsilon\sqrt{2W(\varphi^\varepsilon)}$ in (3.4), then the calculation of (5.2) below will fail. In the case of $g^\varepsilon \equiv 0$, the convergence of (3.4) to the mean curvature flow with no obstacles is well known (see [7, 9, 12, 17]).

Here we give the standard pointwise estimate for the solution to (3.4).

PROPOSITION 3.3. *Let φ^ε be a solution to* (3.4). *Then* $\sup_{x\in\Omega, t\in[0,T)} |\varphi^\varepsilon(x,t)| < 1$ *for any $T > 0$.*

Proof. We only show $\sup_{x\in\Omega, t\in[0,T)} \varphi^\varepsilon(x,t) < 1$, because we can obtain $\varphi^\varepsilon > -1$ similarly. By (3.2), we have $\sup_{x\in\Omega} |\varphi_0^\varepsilon(x)| < 1$. Assume that $\{t \in [0,T) \,|\, \exists x \in \Omega \text{ s.t. } \varphi^\varepsilon(x,t) = 1\} \neq \emptyset$ and set $t_0 := \inf\{t \in [0,T) \,|\, \exists x \in \Omega \text{ s.t. } \varphi^\varepsilon(x,t) = 1\}$. Then $t_0 \in (0,T)$ by $\sup_{x\in\Omega} |\varphi_0^\varepsilon(x)| < 1$ and φ^ε satisfies

$$\begin{aligned} \varphi^\varepsilon \le& \Delta\varphi^\varepsilon + \left| \frac{2\varphi^\varepsilon}{\varepsilon^2}(1+\varphi^\varepsilon) \right| (1-\varphi^\varepsilon) + \left| \frac{g}{\varepsilon}(1+\varphi^\varepsilon) \right| (1-\varphi^\varepsilon) \\ \le& \Delta\varphi^\varepsilon + M(1-\varphi^\varepsilon) \end{aligned} \tag{3.5}$$

for any $(x,t) \in \Omega \times (0,t_0)$, where

$$M = \max_{x\in\Omega, t\in[0,T]} \left\{ \left| \frac{2\varphi^\varepsilon}{\varepsilon^2}(1+\varphi^\varepsilon) \right| + \left| \frac{g}{\varepsilon}(1+\varphi^\varepsilon) \right| \right\}$$

and we used $1 - \varphi^\varepsilon \ge 0$ in $\Omega \times (0,t_0)$. We denote $\alpha := \max_{x\in\Omega} \varphi_0^\varepsilon(x)$. Note that $\alpha < 1$. Set $\overline{\varphi}(t) := 1 - (1-\alpha)e^{-Mt}$. Then $\overline{\varphi}$ is monotone increasing and satisfies

$$\overline{\varphi}_t = \Delta\overline{\varphi} + M(1-\overline{\varphi}) \tag{3.6}$$

and $\overline{\varphi}(0) = \alpha \ge \varphi_0^\varepsilon(x)$ for any $x \in \Omega$. We remark that φ^ε is a subsolution to (3.6) in $\Omega \times (0,t_0)$ by (3.5). Therefore, the comparison principle implies $\varphi^\varepsilon(x,t) \le \overline{\varphi}(t) \le \overline{\varphi}(T) < 1$ for any $(x,t) \in \Omega \times [0,t_0)$. Then we would have a contradiction from $\varphi^\varepsilon(x,t_0) = 1$ for some $x \in \Omega$. Therefore, $\varphi^\varepsilon(x,t) < 1$ for any $(x,t) \in \Omega \times [0,T)$ and φ^ε satisfies (3.5) in $\Omega \times (0,T)$. Using the comparison principle again, we obtain $\sup_{x\in\Omega, t\in[0,T)} |\varphi^\varepsilon(x,t)| \le \overline{\varphi}(T) < 1$. ∎

Next, we define the measures that correspond to the surface M_t in section 1.

DEFINITION 3.4. *Set $\sigma := \int_{-1}^1 \sqrt{2W(s)}\, ds$. Assume that φ^{ε_i} is a solution to* (3.4). *We denote Radon measures $\mu_t^{\varepsilon_i}$, $\tilde{\mu}_t^{\varepsilon_i}$, and $\hat{\mu}_t^{\varepsilon_i}$ by*

$$\mu_t^{\varepsilon_i}(\phi) := \frac{1}{\sigma} \int_\Omega \phi(x) \Big(\frac{\varepsilon_i |\nabla\varphi^{\varepsilon_i}(x,t)|^2}{2} + \frac{W(\varphi^{\varepsilon_i}(x,t))}{\varepsilon_i} \Big) dx, \tag{3.7}$$

$$\tilde{\mu}_t^{\varepsilon_i}(\phi) := \frac{1}{\sigma}\int_\Omega \phi(x)\varepsilon_i|\nabla\varphi^{\varepsilon_i}(x,t)|^2 dx \quad \textit{and} \quad \hat{\mu}_t^{\varepsilon_i}(\phi) := \frac{1}{\sigma}\int_\Omega \phi(x)\frac{2W(\varphi^{\varepsilon_i}(x,t))}{\varepsilon_i}dx$$

for any $\phi \in C_c(\Omega)$.

Remark 3.5. If there exist $t \geq 0$ and a Radon measure μ_t on Ω such that

$$\int_\Omega \left| \frac{\varepsilon|\nabla\varphi^\varepsilon(x,t)|^2}{2} - \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon} \right| dx \to 0 \tag{3.8}$$

and $\mu_t^\varepsilon \to \mu_t$ as Radon measures, namely,

$$\int_\Omega \phi\, d\mu_t^\varepsilon \to \int_\Omega \phi\, d\mu_t \qquad \text{for any } \phi \in C_c(\Omega),$$

then $\tilde{\mu}_t^\varepsilon$ and $\hat{\mu}_t^\varepsilon$ also converge to μ_t as Radon measures.

By the definition of the initial data φ_0^ε, we obtain the following proposition.

Proposition 3.6 (see Proposition 1.4 of [17]). *We see that* $\sup_{i\in\mathbb{N}} \mu_0^{\varepsilon_i}(\Omega) < \infty$. *Moreover,* $\mu_0^{\varepsilon_i} \to \mathscr{H}^{d-1}\lfloor_{M_0}$ *as Radon measures.*

Set $D_1 = \sup_{i\in\mathbb{N}} \mu_0^{\varepsilon_i}(\Omega)$. Proposition 3.6 implies $D_1 < \infty$. The integration by parts implies the following standard estimates.

Proposition 3.7. *Let* φ^ε *be a solution to* (3.4). *Then we have*

$$\frac{d}{dt}\int_\Omega \frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx + \frac{1}{2}\int_\Omega \varepsilon\left(-\Delta\varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\right)^2 dx \leq \frac{d^2}{R_0^2}\int_\Omega \frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx, \tag{3.9}$$

$$\frac{d}{dt}\int_\Omega \frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx + \frac{1}{2}\int_\Omega \varepsilon(\varphi_t^\varepsilon)^2\,dx \leq \frac{d^2}{R_0^2}\int_\Omega \frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx, \tag{3.10}$$

and

$$\int_\Omega \frac{\varepsilon|\nabla\varphi^\varepsilon(x,t)|^2}{2} + \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon}\,dx \leq D_1 e^{\frac{d^2}{R_0^2}t}. \tag{3.11}$$

Proof. By the integration by parts and Young's inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx = \int_\Omega \varepsilon\left(-\Delta\varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\right)\varphi_t^\varepsilon\,dx\\
&= \int_\Omega \varepsilon\left(-\Delta\varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\right)\left(\Delta\varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon^2} + g^\varepsilon\frac{\sqrt{2W(\varphi^\varepsilon)}}{\varepsilon}\right)dx\\
&= -\int_\Omega \varepsilon\left(-\Delta\varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\right)^2 dx + \int_\Omega \varepsilon\left(-\Delta\varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\right)g^\varepsilon\frac{\sqrt{2W(\varphi^\varepsilon)}}{\varepsilon}\,dx\\
&\leq -\frac{1}{2}\int_\Omega \varepsilon\left(-\Delta\varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\right)^2 dx + \int_\Omega (g^\varepsilon)^2\frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx.
\end{aligned}$$

By this and $\sup_{x\in\Omega}|g| \leq \frac{d}{R_0}$ we obtain (3.9) and (3.11). Similarly, we can obtain (3.10). ∎

Next, we show the monotonicity formula. Set

$$\rho_{y,s}(x,t) = \frac{1}{(4\pi(s-t))^{\frac{d-1}{2}}}e^{-\frac{|x-y|^2}{4(s-t)}}, \qquad x,y\in\mathbb{R}^d,\ 0\leq t<s<\infty.$$

Similar to the proof in [38, p. 2028], we obtain the following monotonicity formula.

PROPOSITION 3.8. *Let φ^{ε_i} be a solution to* (3.4) *with initial data φ^{ε_i} which satisfies* (3.2)*, and let $\mu_t^{\varepsilon_i}$ be a Radon measure defined in* (3.7)*. Then, we have*

$$\frac{d}{dt}\int_{\mathbb{R}^d}\rho_{y,s}(x,t)\,d\mu_t^\varepsilon(x) \le \frac{1}{2(s-t)}\int_{\mathbb{R}^d}\rho_{y,s}(x,t)\left(\frac{\varepsilon|\nabla\varphi^\varepsilon(x,t)|^2}{2} - \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon}\right)dx + \frac{d^2}{2R_0^2}\int_{\mathbb{R}^d}\rho_{y,s}(x,t)\,d\mu_t^\varepsilon(x). \tag{3.12}$$

Here, μ_t^ε is extended periodically to $\mathbb{R}^d$.

For the solution φ^ε to (3.4), under the parabolic change of variables $\tilde{x} = \frac{x}{\varepsilon}$ and $\tilde{t} = \frac{t}{\varepsilon^2}$, we have

$$\tilde{\varphi}^\varepsilon_{\tilde{t}} = \Delta_{\tilde{x}}\tilde{\varphi}^\varepsilon - W'(\tilde{\varphi}^\varepsilon) + \varepsilon\tilde{g}^\varepsilon\sqrt{2W(\tilde{\varphi}^\varepsilon)}, \quad \tilde{x}\in\Omega_\varepsilon,\ \tilde{t}>0, \tag{3.13}$$

where $\Omega_\varepsilon = (\mathbb{R}/\varepsilon^{-1}\mathbb{Z})^d$, $\tilde{\varphi}^\varepsilon(\tilde{x},\tilde{t}) = \varphi^\varepsilon(x,t)$, $\tilde{g}^\varepsilon(\tilde{x},\tilde{t}) = g^\varepsilon(x,t)$, and

$$\|\tilde{g}^\varepsilon\|_{L^\infty} \le \frac{d}{R_0}, \quad \|\nabla_{\tilde{x}}\tilde{g}^\varepsilon\|_{L^\infty} \le M, \quad \|\nabla^2_{\tilde{x}}\tilde{g}^\varepsilon\|_{L^\infty} \le M. \tag{3.14}$$

Therefore, the external force term $\varepsilon\tilde{g}^\varepsilon\sqrt{2W(\tilde{\varphi}^\varepsilon)}$ can be regarded as a small perturbation, and we can obtain the following lemma.

LEMMA 3.9. *For the solution φ^ε to* (3.4)*, there exists a constant $c>0$ depending only on d, M, D_1, and T such that*

$$\sup_{\Omega\times[\varepsilon^2,T)}\varepsilon|\nabla\varphi^\varepsilon| \le c \quad \textit{and} \quad \sup_{\Omega\times[\varepsilon^2,T)}\varepsilon^2|\nabla^2\varphi^\varepsilon| \le c \tag{3.15}$$

for any $\varepsilon>0$.

Proof. For the rescaled solution to (3.13) with (3.14), the standard parabolic argument implies the interior estimates of $|\nabla_{\tilde{x}}\tilde{\varphi}^\varepsilon|$ and $|\nabla^2_{\tilde{x}}\tilde{\varphi}^\varepsilon|$ (see [22] and [40, Lemma 4.1]). Hence we obtain (3.15). ∎

4. Subsolution and supersolution. We construct simple subsolutions and supersolutions to (3.4) that represent obstacles. In this section, we extend Ω, $O_\pm$, and the solution φ^ε periodically to $\mathbb{R}^d$. Set $\underline{r_y}(x) = \frac{1}{2R_0}(R_0^2 - |x-y|^2)$, $\underline{\varphi_y^\varepsilon}(x) = q^\varepsilon(\underline{r_y}(x))$ and $\overline{\varphi_y^\varepsilon}(x) = -q^\varepsilon(\underline{r_y}(x)) = q^\varepsilon(-\underline{r_y}(x))$ on $\mathbb{R}^d$.

LEMMA 4.1. *Assume that $B_{R_0}(y)\subset O_+$. Then there exists $\epsilon_1 = \epsilon_1(d,R_0)>0$ such that $\underline{\varphi_y^\varepsilon}$ is a subsolution to* (3.4) *with $\mathbb{R}^d$ instead of Ω for any $\varepsilon\in(0,\epsilon_1)$.*

Proof. Without loss of generality we assume $y=0$. Let φ^ε be a solution to (3.4) with $\mathbb{R}^d$ instead of Ω. By Remark 3.1 and (3.1), we have

$$\begin{aligned}\sqrt{2W(q^\varepsilon)}r_t^\varepsilon = \varepsilon q_r^\varepsilon r_t^\varepsilon &= \varepsilon q_r^\varepsilon\Delta r^\varepsilon + \varepsilon q_{rr}^\varepsilon|\nabla r^\varepsilon|^2 - \frac{W'(q^\varepsilon)}{\varepsilon} + g^\varepsilon\sqrt{2W(q^\varepsilon)}\\ &= \sqrt{2W(q^\varepsilon)}\Delta r^\varepsilon + \frac{W'(q^\varepsilon)}{\varepsilon}(|\nabla r^\varepsilon|^2-1) + g^\varepsilon\sqrt{2W(q^\varepsilon)}.\end{aligned}$$

Thus the first equation (3.4) with $\mathbb{R}^d$ instead of Ω is equivalent to

$$r_t^\varepsilon - \Delta r^\varepsilon + \frac{2q^\varepsilon(r^\varepsilon)}{\varepsilon}(|\nabla r^\varepsilon|^2-1) - g^\varepsilon = 0, \qquad (x,t)\in\mathbb{R}^d\times(0,\infty), \tag{4.1}$$

where we used $W'(q^\varepsilon)/\sqrt{2W(q^\varepsilon)} = -2q^\varepsilon$. Therefore, we need only prove

$$-\Delta \underline{r_0} + \frac{2q^\varepsilon(\underline{r_0})}{\varepsilon}(|\nabla \underline{r_0}|^2 - 1) - g^\varepsilon \le 0, \qquad (x,t) \in \mathbb{R}^d \times (0,\infty) \tag{4.2}$$

for sufficiently small $\varepsilon > 0$. We compute that

$$-\Delta \underline{r_0} + \frac{2q^\varepsilon(\underline{r_0})}{\varepsilon}(|\nabla \underline{r_0}|^2 - 1) - g^\varepsilon = \frac{d}{R_0} + \frac{2q^\varepsilon(\underline{r_0})}{\varepsilon}\left(\frac{|x|^2}{R_0^2} - 1\right) - g^\varepsilon. \tag{4.3}$$

First, we consider the case of $|x| \le R_0$. We compute that

$$\frac{d}{R_0} + \frac{2q^\varepsilon(\underline{r_0})}{\varepsilon}\left(\frac{|x|^2}{R_0^2} - 1\right) - g^\varepsilon \le \frac{d}{R_0} - \frac{d}{R_0} = 0, \tag{4.4}$$

where we used $\frac{2q^\varepsilon(\underline{r_0})}{\varepsilon}\left(\frac{|x|^2}{R_0^2} - 1\right) \le 0$ and $g^\varepsilon(x) = \frac{d}{R_0}$. Therefore, (4.3) and (4.4) imply (4.2) if $|x| \le R_0$.

Next, we consider the case of $R_0 \le |x| \le R_0 + \sqrt{\varepsilon}$. Note that $\frac{2q^\varepsilon(\underline{r_0})}{\varepsilon}\left(\frac{|x|^2}{R_0^2} - 1\right) \le 0$ and $g^\varepsilon(x) = \frac{d}{R_0}$ also hold in this case. Hence we obtain (4.2) if $R_0 \le |x| \le R_0 + \sqrt{\varepsilon}$.

Finally, we consider the case of $R_0 + \sqrt{\varepsilon} \le |x|$. We compute

$$\begin{aligned}\frac{2q^\varepsilon(\underline{r_0})}{\varepsilon}\left(\frac{|x|^2}{R_0^2} - 1\right) &\le \frac{2}{\varepsilon}\tanh\left(-\frac{2R_0\sqrt{\varepsilon}+\varepsilon}{2R_0\varepsilon}\right)\frac{2R_0\sqrt{\varepsilon}+\varepsilon}{R_0^2} \\ &\le -\frac{4}{\sqrt{\varepsilon}R_0}\tanh\left(\frac{1}{\sqrt{\varepsilon}}\right).\end{aligned}$$

Therefore, we have

$$\frac{d}{R_0} + \frac{2q^\varepsilon(\underline{r_0})}{\varepsilon}\left(\frac{|x|^2}{R_0^2} - 1\right) - g^\varepsilon \le 2\frac{d}{R_0} - \frac{4}{\sqrt{\varepsilon}R_0}\tanh\left(\frac{1}{\sqrt{\varepsilon}}\right),$$

where we used $\max_{x\in\mathbb{R}^d}|g^\varepsilon(x)| \le \frac{d}{R_0}$. Thus there exists $\epsilon_1 = \epsilon_1(d, R_0) > 0$ such that (4.2) holds for any $\varepsilon \in (0, \epsilon_1)$. ∎

Similarly, we obtain the following lemma.

LEMMA 4.2. *Assume that* $B_{R_0}(y) \subset O_-$. *Then* $\overline{\varphi_y^\varepsilon}$ *is a supersolution to* (3.4) *with* $\mathbb{R}^d$ *instead of* Ω *for any* $\varepsilon \in (0, \epsilon_1)$, *where* ϵ_1 *is as in Lemma* 4.1.

In order to use the comparison principle, we need the following estimates for the initial data.

LEMMA 4.3. *Assume that* $B_{R_0}(y) \subset O_+$ *and* $B_{R_0}(z) \subset O_-$. *Then*

$$\underline{\varphi_y^{\varepsilon_i}}(x) \le \varphi_0^{\varepsilon_i}(x) \qquad \textit{and} \qquad \overline{\varphi_z^{\varepsilon_i}}(x) \ge \varphi_0^{\varepsilon_i}(x), \qquad x \in \mathbb{R}^d, \tag{4.5}$$

for sufficiently large $i \ge 1$.

Proof. To show the first inequality of (4.5), we need only prove that

$$\underline{r_y}(x) \le \tilde{r}_0^{\varepsilon_i}(x), \qquad x \in \mathbb{R}^d, \tag{4.6}$$

for sufficiently large $i \ge 1$.

We assume that for any $N \geq 1$ there exist $i \geq N$ and $x' \in \Omega$ such that $\underline{r_y}(x') > \tilde{r}_0^{\varepsilon_i}(x')$ with $\underline{r_y}(x') \geq 0$. In addition, we may assume that $\sup_x |r_0^{\varepsilon_i}(x) - \tilde{r}_0^{\varepsilon_i}(x)| < \frac{\delta_1}{4}$. Then $x' \in \overline{B_{R_0}(y)}$ and

$$0 \leq \underline{r_y}(x') \leq \operatorname{dist}(x', \partial B_{R_0}(y)) \tag{4.7}$$

by $\max_{x\in\overline{B_{R_0}(y)}} |\nabla \underline{r_y}(x)| \leq 1$ and $\underline{r_y} = 0$ on $\partial B_{R_0}(y)$. The assumptions $B_{R_0}(y) \subset O_+ \subset U_0^i$ and (2.5) imply

$$\operatorname{dist}(x', \partial B_{R_0}(y)) + \frac{\delta_1}{2} \leq \operatorname{dist}(x', M_0^i). \tag{4.8}$$

Then (4.7) and (4.8) imply

$$\begin{aligned}\underline{r_y}(x') &\leq \operatorname{dist}(x', \partial B_{R_0}(y)) \leq \operatorname{dist}(x', M_0^i) - \frac{\delta_1}{2}\\ &= r_0^{\varepsilon_i}(x') - \frac{\delta_1}{2} < \tilde{r}_0^{\varepsilon_i}(x').\end{aligned}$$

This is a contradiction to $\underline{r_y}(x') > \tilde{r}_0^{\varepsilon_i}(x')$. In the case of $\underline{r_y}(x') < 0$, we may obtain a contradiction similarly by using $\min_{x\in(B_{R_0}(y))^c} |\nabla \underline{r_y}(x)| = 1$. Therefore, we obtain (4.6). We can show the second inequality of (4.5) by the similar argument. ◻

By Lemmas 4.1, 4.2, and 4.3, we have the following proposition.

PROPOSITION 4.4. *Assume that φ^ε is a solution to* (3.4)*, $B_{R_0}(y) \subset O_+$ and $B_{R_0}(z) \subset O_-$. Then*

$$\underline{\varphi_y^\varepsilon}(x) \leq \varphi^\varepsilon(x,t) \leq \overline{\varphi_z^\varepsilon}(x) \quad \text{for any } (x,t) \in \mathbb{R}^d \times [0,\infty) \tag{4.9}$$

for sufficiently small $\varepsilon > 0$.

For the proof, we need only use the standard comparison principle. Therefore, we omit it.

5. Existence of weak solution to mean curvature flow with obstacles. In this section, we prove the global existence of the weak solution to the mean curvature flow with obstacles in the sense of Brakke's mean curvature flow.

THEOREM 5.1. *Let $d = 2$ or 3, and let $\{\varepsilon_i\}_{i=1}^\infty$ be a positive sequence that converges to 0. Assume that M_0 and $O_\pm$ satisfy all the assumptions in section 2. Let φ^{ε_i} be a solution to* (3.4) *with initial data $\varphi_0^{\varepsilon_i}$ which satisfies* (3.2)*, and let $\mu_t^{\varepsilon_i}$ be a Radon measure defined in* (3.7)*. Then there exist a subsequence $\{\varepsilon_{i_j}\}_{j=1}^\infty$, a family of Radon measures $\{\mu_t\}_{t\in[0,\infty)}$, and $\psi \in BV_{loc}(\Omega\times[0,\infty) \cap C^{\frac{1}{2}}_{loc}([0,\infty); L^1(\Omega))$ such that the following hold:*

(1) $\mu_0 = \mathscr{H}^{d-1}\lfloor_{M_0}$.
(2) *For any $t \in [0,\infty)$, $\mu_t^{\varepsilon_{i_j}}$ converges to μ_t as Radon measures.*
(3) *For a.e. $t \geq 0$, μ_t is $(d-1)$-integral.*
(4) *$\psi = 0$ or 1 a.e. on $\Omega\times[0,\infty)$, and $\varphi^{\varepsilon_{i_j}} \to 2\psi - 1$ in $L^1_{loc}(\Omega\times(0,\infty))$, and a.e. pointwise. In addition, $\psi(\cdot,0) = \chi_{U_0}$ a.e. on Ω and $\|\nabla\psi(\cdot,t)\|(\phi) \leq \mu_t(\phi)$ for all $t \in [0,\infty)$ and $\phi \in C_c(\Omega;[0,\infty))$.*
(5) *$\operatorname{spt}\mu_t \cap O_\pm = \emptyset$ for any $t \geq 0$, and $\psi = 1$ a.e. on $O_+ \times [0,\infty)$, and $\psi = 0$ a.e. on $O_- \times [0,\infty)$.*

(6) $\{\mu_t\}_{t\in[0,\infty)}$ *is a Brakke's mean curvature flow on* $\Omega\setminus\overline{O_+\cup O_-}$*, that is,*

$$\int_\Omega \phi\, d\mu_t\Big|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2}\int_\Omega \{(\nabla\phi-\phi h)\cdot h+\phi_t\}\, d\mu_t dt \tag{5.1}$$

for all $0\leq t_1<t_2<\infty$ *and* $\phi\in C_c^1(\Omega\setminus\overline{O_+\cup O_-}\times[0,\infty);[0,\infty))$.

(7) *As an additional assumption, suppose that* D_0 *used in* (2.3) *satisfies* $D_0<2$. *Then there exists* $T_1>0$ *such that* $\|\nabla\psi(\cdot,t)\|=\mu_t$ *for a.e.* $t\in[0,T_1)$.

Remark 5.2. If M_0 is C^1, then the additional assumption of (7) holds (see Remark 2.2).

Remark 5.3. In a weak sense, $\tilde U(t):=\{x\in\Omega\,|\,\psi(x,t)=1\}$ corresponds to U_t in section 1 when $\|\nabla\psi(\cdot,t)\|=\mu_t$. This $\tilde U(t)$ has similar properties to the weak solution treated in [4, Theorem 4.6]. More precisely, $\tilde U(t)$ is a Caccioppoli set for any $t\geq 0$ by $\|\nabla\psi(\cdot,t)\|(\Omega)\leq\mu_t(\Omega)<\infty$ and for any $T>0$ there exists $C_2>0$ defined below such that $|\tilde U(t_2)\triangle\tilde U(t_1)|\leq C_2\sqrt{t_2-t_1}$ for any $0\leq t_1<t_2<T$ by $\psi\in C^{\frac12}_{loc}([0,\infty);L^1(\Omega))$ and $\psi=0$ or 1 a.e. on $\Omega\times[0,\infty)$.

Remark 5.4. In the case of $d=2$ or 3, thanks to [33, 29], we can prove the integrality of μ_t and the Brakke's inequality by the standard energy estimates (see [34]). In contrast, considering [40, 41], the pointwise estimate of $\big(\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2}-\frac{W(\varphi^\varepsilon)}{\varepsilon}\big)_+$ and the parabolic monotonicity formula seem to be important when $d\geq 4$.

Proof. The first statement (1) holds by Proposition 3.6. In the case of $d=2$ or 3, Propositions 4.3 and 4.4 in [29] imply that if

$$\sup_{i\in\mathbb{N}}\int_0^T\int_\Omega \frac{1}{\varepsilon_i}\left(g^{\varepsilon_i}\sqrt{2W(\varphi^{\varepsilon_i})}\right)^2 dxdt<\infty, \tag{5.2}$$

then there exist a subsequence $\varepsilon_i\to 0$ (denoted by the same index) and a family of Radon measures $\{\mu_t\}_{t\in[0,\infty)}$ such that (2) and (3) hold. From (3.11) we have

$$\int_0^T\int_\Omega \frac{1}{\varepsilon_i}\left(g^{\varepsilon_i}\sqrt{2W(\varphi^{\varepsilon_i})}\right)^2 dxdt\leq \frac{2D_1e^{\frac{d^2}{R_0^2}T}d^2T}{R_0^2},\qquad i\geq 1.$$

Therefore, (5.2) holds and we obtain (2) and (3).

Next we prove (4). Note that the proof is almost the same as that in [40, Proposition 8.3]. Set

$$\Phi(s)=\sigma^{-1}\int_{-1}^s\sqrt{2W(a)}\,da\quad\text{and}\quad w^{\varepsilon_i}=\Phi\circ\varphi^{\varepsilon_i}.$$

We remark that $\Phi(1)=1$ and $\Phi(-1)=0$. We compute

$$|\nabla w^{\varepsilon_i}|=\sigma^{-1}|\nabla\varphi^{\varepsilon_i}|\sqrt{2W(\varphi^{\varepsilon_i})}\leq\sigma^{-1}\left(\frac{\varepsilon_i|\nabla\varphi^{\varepsilon_i}|^2}{2}+\frac{W(\varphi^{\varepsilon_i})}{\varepsilon_i}\right) \tag{5.3}$$

and

$$|w_t^{\varepsilon_i}|\leq\sigma^{-1}\left(\frac{\varepsilon_i|\varphi_t^{\varepsilon_i}|^2}{2}+\frac{W(\varphi^{\varepsilon_i})}{\varepsilon_i}\right).$$

Thus, by (3.10) and (3.11), there exists $C_1=C_1(d,R_0,D_1,W,T)>0$ such that

$$\max_{0\leq t\leq T}\int_\Omega|\nabla w^{\varepsilon_i}(x,t)|\,dx+\int_0^T\int_\Omega|w_t^{\varepsilon_i}|\,dxdt\leq C_1$$

for any $i \geq 1$. Therefore, $\{w^{\varepsilon_i}\}_{i=1}^{\infty}$ is bounded in $BV_{loc}(\Omega \times [0,T])$. The compactness theorem and a diagonal argument imply that there exist a subsequence (denoted by the same index) and $w \in BV_{loc}(\Omega \times [0,\infty))$ such that

$$w^{\varepsilon_i} \to w \qquad \text{strongly in } L^1_{loc}(\Omega \times [0,\infty))$$

and a.e. pointwise. We define $\psi = (1 + \Phi^{-1} \circ w)/2$. Then

$$\varphi^{\varepsilon_i} \to 2\psi - 1 \qquad \text{strongly in } L^1_{loc}(\Omega \times [0,\infty))$$

and a.e. pointwise. Note that by $\sup_i \int_\Omega \frac{W(\varphi^{\varepsilon_i})}{\varepsilon_i} < \infty$ for any $t \geq 0$, $\varphi^{\varepsilon_i} \to \pm 1$ for a.e. (x,t), and hence $\psi = 1$ or 0 for a.e. (x,t). Note that we can easily check that $\psi = w$ on $\Omega \times [0,\infty)$ and $\psi \in BV_{loc}(\Omega \times [0,\infty))$. For $0 \leq t_1 < t_2 < T$, there exists $C_2 = C_2(d, R_0, D_1, W, T) > 0$ such that

(5.4)
$$\begin{aligned}\int_\Omega |w^{\varepsilon_i}(x,t_2) - w^{\varepsilon_i}(x,t_1)|\, dx \leq & \int_\Omega \int_{t_1}^{t_2} |w_t^{\varepsilon_i}|\, dtdx \\ \leq & \sigma^{-1} \int_\Omega \int_{t_1}^{t_2} \left(\frac{\varepsilon_i |\varphi_t^{\varepsilon_i}|^2}{2} \sqrt{t_2 - t} + \frac{W(\varphi^{\varepsilon_i})}{\varepsilon_i \sqrt{t_2 - t}} \right) dtdx \\ \leq & C_2 \sqrt{t_2 - t_1}.\end{aligned}$$

By (5.4) and

$$\lim_{i \to \infty} \int_\Omega |w^{\varepsilon_i}(x,t_2) - w^{\varepsilon_i}(x,t_1)|\, dx = \int_\Omega |\psi(x,t_2) - \psi(x,t_1)|\, dx,$$

we obtain $\psi \in C^{\frac{1}{2}}_{loc}([0,\infty); L^1(\Omega))$. In addition, [17, Proposition 1.4] yields $\psi(\cdot,0) = \chi_{U_0}$ a.e. on Ω, and (5.3) and $\|\nabla\psi(\cdot,t)\| = \|\nabla w(\cdot,t)\|$ imply $\|\nabla\psi(\cdot,t)\|(\phi) \leq \mu_t(\phi)$ for all $t \in [0,\infty)$ and $\phi \in C_c(\Omega; [0,\infty))$. Therefore, we obtain (4).

Now we show (5). In order to obtain $\operatorname{spt} \mu_t \cap O_+ = \emptyset$, we need only prove that

$$\mu_t^{\varepsilon_i}(\overline{B_r(y)}) = \frac{1}{2}\tilde{\mu}_t^{\varepsilon_i}(\overline{B_r(y)}) + \frac{1}{2}\hat{\mu}_t^{\varepsilon_i}(\overline{B_r(y)}) \to 0 \qquad \text{as } i \to \infty \tag{5.5}$$

for any $\overline{B_r(y)} \subset O_+$ with $0 < r < R_0$. Assume that $\overline{B_r(y)} \subset O_+$. First, we show $\hat{\mu}_t^{\varepsilon_i}(\overline{B_r(y)}) \to 0$. Let $z \in \mathbb{R}^d$ satisfy $\overline{B_r(y)} \subset B_{R_0}(z)$. Then $\underline{\varphi_z^{\varepsilon_i}} \to 1$ uniformly on $\overline{B_r(y)}$, since $\min_{x \in \overline{B_r(y)}} \underline{r_z}(x) > 0$. In addition, Proposition 3.3 and (4.9) imply $\underline{\varphi_z^{\varepsilon_i}} \leq \varphi^{\varepsilon_i} \leq 1$. Therefore, $\hat{\mu}_t^{\varepsilon_i}(\overline{B_r(y)}) \to 0$.

To prove $\tilde{\mu}_t^{\varepsilon_i}(\overline{B_r(y)}) \to 0$, we suppose that $t > 0$ (in the case of $t = 0$, the claim is obvious). Let $\phi \in C_c^\infty(O_+)$ be a nonnegative test function. It is enough to show $\tilde{\mu}_t^{\varepsilon_i}(\phi) \to 0$. We may assume that $\operatorname{spt}\phi \subset B_{R_0}(z)$ for some $z \in O_+$. By the integration by parts, we have

$$\begin{aligned}\tilde{\mu}_t^{\varepsilon_i}(\phi) = & \frac{\varepsilon_i}{\sigma} \int_{\operatorname{spt}\phi} \phi \nabla(\varphi^{\varepsilon_i} - 1) \cdot \nabla\varphi^{\varepsilon_i}\, dx \\ = & -\frac{\varepsilon_i}{\sigma} \int_{\operatorname{spt}\phi} \left(\phi(\varphi^{\varepsilon_i} - 1)\Delta\varphi^{\varepsilon_i} + (\varphi^{\varepsilon_i} - 1)\nabla\phi \cdot \nabla\varphi^{\varepsilon_i} \right) dx.\end{aligned} \tag{5.6}$$

By (4.9), Proposition 3.3, and $\min_{x \in \operatorname{spt}\phi} \underline{r_z}(x) > 0$, there exists $C_3 > 0$ such that

$$\tanh(C_3/\varepsilon_i) \leq \varphi^{\varepsilon_i}(x,t) < 1, \qquad x \in \operatorname{spt}\phi.$$

Therefore,

$$|\varphi^{\varepsilon_i}(x,t)-1| \le 1-\tanh(C_3/\varepsilon_i) \le \varepsilon_i^2, \qquad x \in \operatorname{spt}\phi \tag{5.7}$$

for sufficiently large $i \ge 1$. By (3.15), (5.6), and (5.7), we obtain $\tilde{\mu}_t^{\varepsilon_i}(\phi) \to 0$. Hence we obtain (5.5), and consequently, $\operatorname{spt}\mu_t \cap O_+ = \emptyset$. The other case ($\operatorname{spt}\mu_t \cap O_- = \emptyset$) and the remained claims may be proved similarly.

Next, we show (6). Given arbitrary open set $U \subset \Omega \setminus \overline{O_+ \cup O_-}$, φ^{ε_i} is a solution to

$$\varepsilon_i \varphi_t^{\varepsilon_i} = \varepsilon_i \Delta \varphi^{\varepsilon_i} - \frac{W'(\varphi^{\varepsilon_i})}{\varepsilon_i}$$

on U. Then [23, Proposition 4.5] (with transport term $u \equiv 0$) tells us that μ_t satisfies Brakke's inequality (5.1) on U (see also [17, 28, 29, 34]). Nevertheless, for the convenience of the reader, we prove (6) here. By the integration by parts, we have

$$\begin{aligned}&\mu_{t_2}^{\varepsilon_i}(\phi) - \mu_{t_1}^{\varepsilon_i}(\phi)\\ &= \int_{t_1}^{t_2} \left(\frac{1}{\sigma} \int_U (-\phi \varepsilon_i^{-1} (w^{\varepsilon_i})^2 + \nabla\phi \cdot \nabla\varphi^{\varepsilon_i} w^{\varepsilon_i})\, dx + \mu_t^{\varepsilon_i}(\phi_t) \right) dt\end{aligned}$$

for $\phi \in C_c^1(U \times [0,\infty); [0,\infty))$, where $w^{\varepsilon_i} = -\varepsilon_i \Delta\varphi^{\varepsilon_i} + \dfrac{W'(\varphi^{\varepsilon_i})}{\varepsilon_i}$. By [28, Theorem 4.3],

$$\int_{s_1}^{s_2} \int_V |h|^2 \, d\mu_t dt \le \liminf_{i\to\infty} \int_{s_1}^{s_2} \int_V \varepsilon_i^{-1} (w^{\varepsilon_i})^2 \, dxdt$$

holds for any open set $V \times (s_1, s_2) \subset U \times [t_1, t_2]$. Therefore, we have

$$\int_{t_1}^{t_2} \int_U \phi |h|^2 \, d\mu_t dt \le \liminf_{i\to\infty} \int_{t_2}^{t_2} \int_U \phi \varepsilon_i^{-1} (w^{\varepsilon_i})^2 \, dxdt. \tag{5.8}$$

In addition, [28, Lemma 7.1] implies

$$\int_{t_1}^{t_2} \int_U \nabla\phi \cdot h \, d\mu_t dt = \lim_{i\to\infty} \int_{t_1}^{t_2} \int_U \nabla\phi \cdot \nabla\varphi^{\varepsilon_i} w^{\varepsilon_i} \, dxdt. \tag{5.9}$$

By (5.8), (5.9), and (2), (5.1) holds on U. Therefore, $\{\mu_t\}_{t\in[0,\infty)}$ is a Brakke's mean curvature flow on $\Omega \setminus \overline{O_+ \cup O_-}$.

Finally, we prove (7). From (3), for a.e. $t \ge 0$, there exists a $(d-1)$-rectifiable set M_t and $\theta_t : M_t \to \mathbb{N}$ such that $\mu_t = \theta_t \mathscr{H}^{d-1} \lfloor_{M_t}$. We need only prove that $\{\theta_t \ge 2\}$ has measure zero for a.e. $t \in [0, T_1)$ for a suitable $T_1 > 0$ (see [23, p. 275] and [40, p. 926]). We will determine T_1 in the following. By Proposition 3.7, we have

$$\sup_{i\in\mathbb{N}} \mu_t^{\varepsilon_i}(\Omega) < \infty \quad \text{and} \quad \sup_{i\in\mathbb{N}} \int_0^T \int_\Omega \varepsilon_i \left(-\Delta\varphi^{\varepsilon_i} + \frac{W'(\varphi^{\varepsilon_i})}{\varepsilon_i^2} \right)^2 dxdt < \infty$$

for any $t \ge 0$ and $T > 0$. Hence Proposition 6.1 in [28] implies that there exists a subsequence $\varepsilon_i \to 0$ (denoted by the same index) such that

$$\lim_{i\to\infty} \int_0^T \int_\Omega \left| \frac{\varepsilon_i |\nabla\varphi^{\varepsilon_i}|^2}{2} - \frac{W(\varphi^{\varepsilon_i})}{\varepsilon_i} \right| dxdt = 0 \tag{5.10}$$

for $2 \leq d \leq 3$. By (3.12) and (5.10), we have

$$\int_{\mathbb{R}^d} \rho_{y,s}(x,t)\, d\mu_t(x) \leq e^{\frac{d^2}{2R_0^2}t} \int_{\mathbb{R}^d} \rho_{y,s}(x,0)\, d\mu_0(x) \tag{5.11}$$

for any $0 \leq t < s$ and $y \in \mathbb{R}^d$. Assume that there exists x_0, t_0, and $N \geq 2$ such that $\theta_{t_0}(x_0) = N$ and $\lim_{r\to 0} \dfrac{\mu_{t_0}(B_r(x_0))}{\omega_{d-1} r^{d-1}} = N$. For $r > 0$ and $a > 0$, we compute

$$\begin{aligned}
&\int_{B_{ar}(x_0)} \rho_{x_0, t_0+r^2}(x, t_0)\, d\mu_{t_0}(x) \\
&= \frac{1}{(4\pi r^2)^{\frac{d-1}{2}}} \int_{B_{ar}(x_0)} e^{-\frac{|x-x_0|^2}{4r^2}}\, d\mu_{t_0}(x) \\
&= \frac{1}{(4\pi r^2)^{\frac{d-1}{2}}} \int_0^1 \mu_{t_0}\left(\left\{ x \in B_{ar}(x_0) \,|\, e^{-\frac{|x-x_0|^2}{4r^2}} > k \right\}\right) dk \\
&= \frac{1}{(4\pi r^2)^{\frac{d-1}{2}}} \int_{e^{-\frac{a^2}{4}}}^1 \mu_{t_0}\left(B_{\sqrt{4r^2 \log\frac{1}{k}}}(x_0)\right) dk \\
&\to \frac{N\omega_{d-1}}{\pi^{\frac{d-1}{2}}} \int_{e^{-\frac{a^2}{4}}}^1 \left(\log \frac{1}{k}\right)^{\frac{d-1}{2}} dk \quad \text{as } r \to \infty.
\end{aligned}$$

Note that $\int_0^1 \left(\log\frac{1}{k}\right)^{\frac{d-1}{2}} dk = \Gamma(\frac{d-1}{2}+1) = \pi^{\frac{d-1}{2}}/\omega_{d-1}$. Therefore,

$$\lim_{r\to 0} \int_{\mathbb{R}^d} \rho_{x_0,t_0+r^2}(x,t_0)\, d\mu_{t_0}(x) \geq N. \tag{5.12}$$

By (2.3) with $D_0 < 2$, there exists $T_1 \in (0,1)$ depending only on M_0 such that

$$\int_{\mathbb{R}^d} \rho_{y,s}(x,0)\, d\mu_0(x) < 2 \tag{5.13}$$

for any $(y,s) \in \mathbb{R}^d \times (0, T_1]$. Then we would have a contradiction from (5.11), (5.12), and (5.13). Therefore, we obtain (7). ∎

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