PTEP

Prog. Theor. Exp. Phys. **2022** 013B02(35 pages)
DOI: 10.1093/ptep/ptab131

Anomaly and superconnection

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Received August 27, 2021; Accepted October 8, 2021; Published October 20, 2021

We study anomalies of fermions with spacetime-dependent mass. Using Fujikawa's method, it is found that the anomalies associated with the $U(N)_+ \times U(N)_-$ chiral symmetry and $U(N)$ flavor symmetry for even and odd dimensions, respectively, can be written in terms of superconnections. In particular, the anomaly for a vector-like $U(1)$ symmetry is given by the Chern character of the superconnection in both even- and odd-dimensional cases. It is also argued that the non-Abelian anomaly for a system in D-dimensional spacetime is characterized by a $(D+2)$-form part of the Chern character of the superconnection which generalizes the usual anomaly polynomial for the massless case. These results enable us to analyze anomalies in the systems with interfaces and spacetime boundaries in a unified way. Applications to index theorems, including the Atiyah–Patodi–Singer index theorem and a Callias-type index theorem, are also discussed. In addition, we give a natural string theory interpretation of these results.

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1. Introduction

Quantum anomaly is one of the most fascinating topics in quantum field theory. It implies important constraints to have a consistent gauge theory and provides powerful tools to investigate non-perturbative properties of quantum field theory. It has been used to discuss phase structures of strongly coupled systems and give non-trivial evidence of conjectured dualities. Another interesting aspect of the anomaly is its beautiful mathematical structures. In particular, the relations between the anomalies and various index theorems have attracted much attention and have been vigorously studied by both physicists and mathematicians.

In this paper we investigate perturbative anomalies in the systems with N Dirac fermions including spacetime-dependent mass as well as external gauge fields associated with $U(N)_+ \times U(N)_-$ chiral symmetry or $U(N)$ flavor symmetry for even- or odd-dimensional cases, respectively. The spacetime-dependent mass is equivalent to an external scalar field (Higgs field) that couples with the fermions through the Yukawa coupling. Although the masses of the quarks and leptons in nature are considered to be constant, spacetime-dependent mass naturally appears in the standard model and various other models when the value of the Higgs field is not constant. It also appears in hadron physics and condensed matter physics, because the effective mass of fermions can vary depending on some parameters of the environment, such as temperature, chemical potentials, magnetic field, strength of the interaction, etc., which can be spacetime dependent.

Funded by SCOAP³

Apart from possible applications to realistic systems, spacetime-dependent mass can be used as a theoretical tool to study quantum field theory. For example, it can be regarded as an external source coupled to a fermion bilinear operator. In particular, although the $U(N)_+ \times U(N)_-$ chiral symmetry is explicitly broken to a subgroup when the mass is non-zero, we can make the action invariant under the $U(N)_+ \times U(N)_-$ gauge transformation in Eq. (3.7) by promoting the mass to a spacetime-dependent external field. Then, we are allowed to discuss the anomaly for this symmetry even though the mass is non-zero. In this sense, the spacetime-dependent mass plays a similar role to the external gauge field, with which the action becomes gauge invariant. Furthermore, it can be used to study chiral fermions localized on an interface or fermions in a spacetime with boundaries. When we make the mass very large except for some regions in spacetime, the low-energy modes are trapped in the regions with small masses, which effectively induces a system with boundaries. If the mass profile has a zero locus of non-zero codimension, it represents an interface defined by the mass. As we review in Sect. 4.1, it is possible to realize Weyl fermions localized in such interfaces. This mechanism is widely used to construct theories with chiral fermions in lattice gauge theories, phenomenological models of elementary particles with extra dimensions, etc.

In fact, the anomaly for the fermions with spacetime-dependent mass (Higgs field) was analyzed in the 1980s in Refs. [1,2].[1] The conclusion of these papers was that the mass does not contribute to the anomaly at all. This is true in the case that the mass is bounded and fixed while the cut-off scale is sent to infinity. However, as we will demonstrate, the mass dependence of the anomaly survives when the mass is unbounded. Remarkably, we will also find that the anomaly exists even for odd-dimensional cases when the spacetime-dependent mass is introduced. Our discussion is closely related to that of Refs. [4,5], in which coupling constants including the masses are promoted to external scalar fields, and the anomalies are extended to include them. The systems with massive fermions were analyzed in Ref. [4], and it was found that the space of masses can be considered as a compact space with non-trivial topology by including $|m| \to \infty$, and anomalies in D-dimensional systems are characterized by a $(D+2)$-form, which is a generalization of the usual anomaly polynomial involving differential forms on the space of masses. This also shows that it is crucial to consider $|m| \to \infty$ to have a non-trivial anomaly that involves the masses.

The main goal of the first half of this paper (Sect. 3) is to show that the anomaly $(D+2)$-form and the anomaly associated with $U(1)_V$ symmetry are given by the Chern character written in terms of the superconnection introduced in Ref. [6]. This was also suggested in Ref. [4]. We will show this explicitly by using Fujikawa's method. Our formulas in Eqs. (3.56) and (3.58) can be used for both even- and odd-dimensional cases, provided that the superconnection of the even and odd types are used accordingly.

These results are probably not surprising for those who are familiar with the Chern–Simons (CS) terms including the tachyon field in unstable D-brane systems, which are written with the Chern character of the superconnection [7–10]. As we discuss in Sect. 5, the systems with Dirac fermions in various dimensions can be realized on a D-brane with unstable D9-branes. The mass of the fermion is proportional to the value of the tachyon field, and hence the spacetime-dependent mass can be naturally obtained by considering a varying tachyon field. The anomaly of the fermions is supposed to be canceled by the contribution from the CS term. Therefore,

[1] See also [3, Sect. 6.5.1].

string theory suggests that the superconnection appears in the formulas of anomaly, which is indeed what we find in the field theory analysis.

The rest of the paper (Sect> 4) is devoted to the applications of these formulas. We consider systems with interfaces and boundaries realized by the spacetime-dependent mass. Most of the discussion there consists of consistency checks and demonstration of our formulas in Eqs. (3.56) and (3.58). We show in several explicit examples that some known results can be consistently reproduced in a simple and unified way. The results of Sect. 4.2.2 are new. In this section, a system with a spacetime-dependent boundary condition is considered and the anomalies due to this boundary condition are obtained.

The paper is organized as follows. We start with a brief review of the superconnection in Sect. 2. In Sect. 3, we derive our main formulas for the anomaly with spacetime-dependent mass using Fujikawa's method. Applications of these formulas are given in Sect. 4. The cases with interfaces and boundaries are studied in Sects. 4.1 and 4.2, respectively, and implications for index theorems are discussed in Sect. 4.3. The systems with spacetime-dependent mass can be realized in string theory, and our results have natural interpretations in string theory as explained in Sect. 5. Finally, in Sect. 6 we summarize our results and make concluding remarks.

2. Superconnection

In this section we briefly review the superconnection introduced in Ref. [6] with physicist-friendly notations. Our description here is not as general as that given in the original paper, but restricted to the cases to be used in the following sections. See, e.g., Refs. [6,11] for more general and mathematically rigorous descriptions. A superconnection[2] $\mathcal{A}$ of the even type is a matrix-valued field composed of $U(N) \times U(N)$ gauge fields (A_+, A_-) and a bifundamental scalar field T as

$$\mathcal{A} = \begin{pmatrix} A_+ & iT^\dagger \\ iT & A_- \end{pmatrix} = A_+ e^+ + A_- e^- + iT^\dagger \sigma^+ + iT\sigma^-, \tag{2.1}$$

where

$$e^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.2}$$

In our notation, the gauge fields $A_\pm = A_{\pm\mu}(x)dx^\mu$ are one-forms that take values in anti-Hermitian $N \times N$ matrices. $\sigma^\pm$ in Eq. (2.1) and dx^μ are treated as fermions, i.e. they anti-commute with each other in the products. The field strength of the superconnection is defined as[3]

$$\mathcal{F} \equiv d\mathcal{A} + \mathcal{A}^2 = \begin{pmatrix} F_+ - T^\dagger T & iDT^\dagger \\ iDT & F_- - TT^\dagger \end{pmatrix}, \tag{2.3}$$

where

$$\begin{aligned} &F_\pm \equiv dA_\pm + A_\pm^2, \\ &DT \equiv dT + A_- T - TA_+, \qquad DT^\dagger \equiv dT^\dagger + A_+ T^\dagger - T^\dagger A_-. \end{aligned} \tag{2.4}$$

[2] In this paper, the word "superconnection" is used for the field $\mathcal{A}$ rather than the covariant derivative $d + \mathcal{A}$, which is often used in mathematical literature.

[3] The products of differential forms are the wedge product, though the symbol for the wedge product, $\wedge$, is omitted.

The Chern character is defined as

$$\mathrm{ch}(\mathcal{F}) \equiv \sum_{k\geq 0} \left(\frac{i}{2\pi}\right)^{k/2} \left[\mathrm{Str}\left(e^{\mathcal{F}}\right)\right]_k , \tag{2.5}$$

where $[\cdots]_k$ denotes the k-form part of the differential form in the square brackets, and Str is the supertrace[4] defined by

$$\mathrm{Str}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \mathrm{Tr}(a) - \mathrm{Tr}(d) \qquad \text{(even case)}. \tag{2.6}$$

Because of Eq. (2.6), only the even form part in Eq. (2.5) can be non-zero.

A useful formula for a one-parameter family of superconnections denoted as $\mathcal{A}_t$ with a parameter $t \in [0, 1]$ is

$$\mathrm{Str}\left(e^{\mathcal{F}_1}\right) - \mathrm{Str}\left(e^{\mathcal{F}_0}\right) = d\left(\int_0^1 dt\, \mathrm{Str}\left(e^{\mathcal{F}_t}\partial_t \mathcal{A}_t\right)\right), \tag{2.7}$$

where $\mathcal{F}_t = d\mathcal{A}_t + \mathcal{A}_t^2$. For $\mathcal{A}_t = \mathcal{A}|_{T\to tT} = \mathcal{A}_0 + t\mathcal{T}$ with $\mathcal{A}_0 = A_+e^+ + A_-e^-$ and $\mathcal{T} = iT^\dagger\sigma^+ + iT\sigma^-$, this formula implies

$$\mathrm{Str}\left(e^{\mathcal{F}}\right) = \mathrm{Tr}(e^{F_+}) - \mathrm{Tr}(e^{F_-}) + d\left(\int_0^1 dt\, \mathrm{Str}\left(e^{\mathcal{F}_t}\mathcal{T}\right)\right). \tag{2.8}$$

Since $\mathrm{Str}(e^{\mathcal{F}_t}\mathcal{T})$ is gauge invariant, Eq. (2.8) implies that $\mathrm{ch}(\mathcal{F})$ and $\mathrm{ch}(F_+) - \mathrm{ch}(F_-)$ are equivalent up to an exact form. For a trivial bundle (or, in a local patch) the formula in Eq. (2.7) with $\mathcal{A}_t = t\mathcal{A}$ implies[5]

$$\mathrm{Str}\left(e^{\mathcal{F}}\right) = d\left(\int_0^1 dt\, \mathrm{Str}\left(e^{td\mathcal{A}+t^2\mathcal{A}^2}\mathcal{A}\right)\right). \tag{2.9}$$

This implies that the Chern character can be expressed locally as

$$\mathrm{ch}(\mathcal{F}) = d\Omega, \tag{2.10}$$

where Ω is the CS form given by

$$\Omega = \sum_{k\geq 0} \left(\frac{i}{2\pi}\right)^{(k+1)/2} \left[\int_0^1 dt\, \mathrm{Str}\left(e^{td\mathcal{A}+t^2\mathcal{A}^2}\mathcal{A}\right)\right]_k . \tag{2.11}$$

This Ω is, in general, not gauge invariant.

The superconnection of the odd type is given by Eq. (2.1) with the restrictions $A_+ = A_-$ and $T = T^\dagger$:

$$\mathcal{A} = \begin{pmatrix} A & iT \\ iT & A \end{pmatrix} = A\,1_2 + iT\sigma_1, \tag{2.12}$$

where $1_2 = e^+ + e^-$ is the unit matrix of size 2 and $\sigma_1 = \sigma^+ + \sigma^- = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The field strength is

$$\mathcal{F} \equiv d\mathcal{A} + \mathcal{A}^2 = \begin{pmatrix} F - T^2 & iDT \\ iDT & F - T^2 \end{pmatrix}, \tag{2.13}$$

with $F \equiv dA + A^2$ and $DT \equiv dT + [A, T]$.

[4]In some literature, the symbol Str is used for the symmetrized trace, which should not be confused with the supertrace in this paper. For the symmetrized trace, we use $\mathrm{Tr}^{\mathrm{sym}}$.

[5]When the gauge group is $U(N_+) \times U(N_-)$ with $N_+ \neq N_-$, the right-hand side has an additional constant term $N_+ - N_-$.

The supertrace for the odd case is defined as

$$\mathrm{Str}\begin{pmatrix} a & b \\ b & a \end{pmatrix} \equiv \sqrt{2}\, i^{-3/2}\,\mathrm{Tr}(b) \qquad \text{(odd case)}. \tag{2.14}$$

The reason for putting the normalization factor $\sqrt{2}\, i^{-3/2}$ will become clear later.[6] We also define an analog of the Chern character for the odd case by the same formula as in Eq. (2.5). In this case, only the odd form part contributes. The formulas in Eqs. (2.7)–(2.11) also hold for the odd case. In particular, Eq, (2.8) with $A_+ = A_-$ and $T = T^\dagger$ gives

$$\mathrm{Str}\left(e^{\mathcal{F}}\right) = d\left(\int_0^1 dt\,\mathrm{Str}\left(e^{\mathcal{F}_t} iT\sigma_1\right)\right), \tag{2.15}$$

where $\mathcal{F}_t = (F - t^2T^2)1_2 + itDT\sigma_1$. Therefore, the Chern character can also be written as

$$\mathrm{ch}(\mathcal{F}) = d\Omega', \tag{2.16}$$

where

$$\Omega' = \sum_{k\geq 0}\left(\frac{i}{2\pi}\right)^{(k+1)/2}\left[\int_0^1 dt\,\mathrm{Str}\left(e^{\mathcal{F}_t} iT\sigma_1\right)\right]_k. \tag{2.17}$$

Unlike Ω in Eq. (2.11), this Ω' is gauge invariant.

3. Derivation of the anomaly

3.1. *Even-dimensional cases*

3.1.1. *Massive fermions and chiral anomaly.* In this section we consider a system with N Dirac fermions ψ in a $D = 2r$-dimensional flat Euclidean spacetime ($r \in \mathbb{Z}_{>0}$). We include external gauge fields $A = (A_+, A_-)$ associated with $U(N)_+ \times U(N)_-$ chiral symmetry and a spacetime-dependent mass m, which belongs to the bifundamental representation of $U(N)_+ \times U(N)_-$.[7] The action is

$$S = \int d^Dx\,\left(\overline{\psi}_+ \not{D}_+\psi_+ + \overline{\psi}_-\not{D}_-\psi_- + \overline{\psi}_- m\psi_+ + \overline{\psi}_+ m^\dagger\psi_-\right) = \int d^Dx\,\overline{\psi}\mathcal{D}\psi, \tag{3.1}$$

where[8]

$$\psi(x) \equiv \begin{pmatrix}\psi_+(x)\\ \psi_-(x)\end{pmatrix} \qquad \overline{\psi}(x) \equiv \left(\overline{\psi}_+(x), \overline{\psi}_-(x)\right), \tag{3.2}$$

and

$$\mathcal{D} \equiv \begin{pmatrix}\not{D}_+ & m^\dagger(x) \\ m(x) & \not{D}_-\end{pmatrix}, \qquad \not{D}_+ \equiv \sigma^{\mu\dagger}(\partial_\mu + A_{+\mu}). \qquad \not{D}_- \equiv \sigma^\mu(\partial_\mu + A_{-\mu}). \tag{3.3}$$

Here, σ^μ and $\sigma^{\mu\dagger}$ ($\mu = 1, 2, \ldots, D$) are $2^{r-1} \times 2^{r-1}$ matrices satisfying

$$\sigma^{\mu\dagger}\sigma^\nu + \sigma^{\nu\dagger}\sigma^\mu = \sigma^\nu\sigma^{\mu\dagger} + \sigma^\mu\sigma^{\nu\dagger} = 2\delta^{\mu\nu}, \tag{3.4}$$

so that

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^{\mu\dagger} & 0\end{pmatrix} \qquad (\mu = 1, 2, \ldots, 2r) \tag{3.5}$$

[6] The sign ambiguity of $i^{-3/2}$ is compensated by that of the $i^{k/2}$ factor in Eq. (2.5). Namely, the supertrace Str of the odd case always appears in the combination $i^{k/2}$Str with odd k in the anomaly, and $i^{k/2}\mathrm{Str}\left(\begin{smallmatrix} a & b \\ b & a\end{smallmatrix}\right) = \sqrt{2}\, i^{(k-3)/2}\mathrm{Tr}(b)$ has no ambiguity.

[7] Although we discuss N Dirac fermions, it is easy to get the results for $N_\pm$ positive/negative chirality Weyl fermions by considering a $U(N_+)_+ \times U(N_-)_-$ subgroup of $U(N)_+ \times U(N)_-$ with large enough N.

[8] This notation is useful for our purpose, but is not a standard one. A more standard notation is obtained by replacing $\overline{\psi}$ and $\mathcal{D}$ with $\overline{\psi}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)\mathcal{D}$, respectively.

are D-dimensional gamma matrices in a chiral representation. We choose a representation of γ^μ such that

$$\gamma^1\gamma^2\cdots\gamma^{2r} = i^r \begin{pmatrix} 1_{2^{r-1}} & 0 \\ 0 & -1_{2^{r-1}} \end{pmatrix} \equiv i^r\gamma^{2r+1} \tag{3.6}$$

is satisfied, where γ^{2r+1} is the chirality operator.

The crucial point here is that we allow the mass parameter m to depend on the spacetime coordinate x^μ and regard it as an external scalar field, which is sometimes called a Higgs field in the literature, that plays a similar role as the external gauge fields A_+ and A_-. Then, the classical action is invariant under $U(N)_+ \times U(N)_-$ chiral gauge transformation that acts on the external fields as well as the dynamical fermions as

$$\begin{aligned}
&\psi_+ \to U_+\psi_+, \qquad \overline{\psi}_+ \to \overline{\psi}_+U_+^{-1}, \qquad \psi_- \to U_-\psi_-, \qquad \overline{\psi}_- \to \overline{\psi}_-U_-^{-1},\\
&A_+ \to U_+A_+U_+^{-1} + U_+dU_+^{-1}, \qquad A_- \to U_-A_-U_-^{-1} + U_-dU_-^{-1},\\
&m \to U_-m\,U_+^{-1}, \qquad m^\dagger \to U_+m^\dagger U_-^{-1},
\end{aligned} \tag{3.7}$$

with $(U_+(x), U_-(x)) \in U(N)_+ \times U(N)_-$.

As is well known, the chiral symmetry is anomalous in quantum theory. In fact, when the external fields are non-trivial, the partition function

$$Z[A, m] \equiv e^{-\Gamma[A,m]} \equiv \int [d\psi d\overline{\psi}]\, e^{-S(\psi,\overline{\psi},A,m)} \tag{3.8}$$

gets a non-trivial phase under the chiral gauge transformation in Eq. (3.7), even though the action is invariant.

Let us briefly review the explicit form of the anomaly for the massless case. Under an infinitesimal chiral gauge transformation ($U_+ = e^{-v_+}$, $U_- = e^{-v_-}$ with $v_+, v_- \ll 1$) with

$$\delta_v A_+ = dv_+ + [A_+, v_+], \qquad \delta_v A_- = dv_- + [A_-, v_-], \tag{3.9}$$

the effective action for the massless case $\Gamma[A] \equiv \Gamma[A, m = 0]$ defined in Eq. (3.8) transforms as $\Gamma \to \Gamma + \delta_v\Gamma$ with

$$\delta_v\Gamma[A] = \int I^1_{2r}(v, A), \tag{3.10}$$

where $I^1_{2r}(v, A)$ is a $2r$-form obtained as a solution of the descent equations[9]

$$dI^1_{2r} = \delta_v I^0_{2r+1}, \qquad dI^0_{2r+1} = I_{2r+2} \tag{3.11}$$

with

$$I_{2r+2}(A) = -2\pi i\,[\mathrm{ch}(F_+) - \mathrm{ch}(F_-)]_{2r+2}\,. \tag{3.12}$$

Here, $[\cdots]_{2r+2}$ denotes the $(2r+2)$-form part of the differential form in the square brackets, and $\mathrm{ch}(F_\pm) = \mathrm{Tr}\left(\exp\left\{\frac{i}{2\pi}F_\pm\right\}\right)$ is the Chern character. $I_{2r+2}(A)$ is called the anomaly polynomial, and $I^0_{2r+1}(A)$ is the CS $(2r+1)$-form.[10]

[9] See, e.g., Refs. [3,12–14] for reviews of the anomalies.

[10] In this paper we consider a flat spacetime. For curved spacetime, $\mathrm{ch}(F)$ should be replaced with $\mathrm{ch}(F)\widehat{A}(R)$, where $\widehat{A}(R)$ is the $\widehat{A}$ genus. Explicit expressions for $I^0_{2r+1}(A)$ and $I^1_{2r}(A)$ are

$$I^0_{2r+1}(A) = \left(\frac{i}{2\pi}\right)^r \frac{1}{r!}\int_0^1 dt\, \left(\mathrm{Tr}\left(A_+F^r_{t+}\right) - \mathrm{Tr}\left(A_-F^r_{t-}\right)\right),$$

$$I^1_{2r}(v, A) = \left(\frac{i}{2\pi}\right)^r \frac{1}{(r-1)!}\int_0^1 dt\,(1-t)\left(\mathrm{Tr}^{\mathrm{sym}}\left(v_+d\left(A_+F^{r-1}_{t+}\right)\right) - \mathrm{Tr}^{\mathrm{sym}}\left(v_-d\left(A_-F^{r-1}_{t-}\right)\right)\right)$$

As pioneered in Refs. [15,16], the chiral anomaly in Eq. (3.10) can be understood as a consequence of the fact that the path integral measure for the fermions is not invariant under the chiral transformation in Eq. (3.7). After a careful regularization, it can be shown that the fermion path integral measure transforms as

$$[d\psi d\overline{\psi}] \to [d\psi d\overline{\psi}]\,\mathcal{J}, \tag{3.13}$$

with the Jacobian $\mathcal{J}$ given by

$$\log \mathcal{J} = \int I^1_{2r}(v, A) \tag{3.14}$$

under the infinitesimal chiral transformation, reproducing the result in Eq. (3.10).

The form of the Jacobian $\mathcal{J}$ in Eq. (3.13) depends on the regularization. In Refs. [1,16], a manifestly gauge-covariant form of the anomaly with

$$\log \mathcal{J} = \int I^{1\,\mathrm{cov}}_{2r}(v, A), \tag{3.15}$$

where

$$I^{1\,\mathrm{cov}}_{2r}(v, A) = \left(\frac{i}{2\pi}\right)^r \frac{1}{r!}\left(\mathrm{Tr}\left(v_+ F^r_+\right) - \mathrm{Tr}\left(v_- F^r_-\right)\right), \tag{3.16}$$

is obtained with a covariant regularization (see Sect. 3.1.2). This form of the anomaly is called the covariant anomaly, while Eq. (3.10) is called the consistent anomaly. Unlike the consistent anomaly, the covariant anomaly does not satisfy the descent equations in Eq. (3.11) and cannot be written as the gauge variation of a well-defined effective action. The consistent and covariant anomalies are related by the addition of a Bardeen–Zumino counterterm in the associated currents [17] (see Appendix B).

We are particularly interested in the anomaly for the $U(1)_V$ transformation which corresponds to $v_+ = v_- = -i\alpha(x)\, 1_N$ with a function $\alpha(x)$ and the unit matrix 1_N.[11] In this case, Eq. (3.16) is

$$I^{1\,\mathrm{cov}}_{2r}(-i\alpha, A) = -i\alpha\,[\mathrm{ch}(F_+) - \mathrm{ch}(F_-)]_{2r} = \frac{\alpha}{2\pi} I_{2r}(A). \tag{3.17}$$

The main claim of this section is that, when the spacetime-dependent mass m is turned on, the Chern character $\mathrm{ch}(F_+) - \mathrm{ch}(F_-)$ appearing in Eqs. (3.12) and (3.17) is replaced with the Chern character written by the superconnection in Eq. (2.5). More explicitly, the anomaly polynomial $I_{2r+2}(A)$ in Eq. (3.12), the covariant anomaly $I^{1\,\mathrm{cov}}_{2r}(v, A)$ in Eq. (3.16), and the $U(1)_V$ anomaly $I^{1\,\mathrm{cov}}_{2r}(v, A)$ in Eq. (3.17) are replaced with

$$I_{2r+2}(A, \widetilde{m}) = -2\pi i\,[\mathrm{ch}(\mathcal{F})]_{2r+2}, \tag{3.18}$$

$$I^{1\,\mathrm{cov}}_{2r}(v, A, \widetilde{m}) = \left(\frac{i}{2\pi}\right)^r \left[\mathrm{Str}\left(v\, e^{\mathcal{F}}\right)\right]_{2r}, \tag{3.19}$$

$$I^{1\,\mathrm{cov}}_{2r}(-i\alpha, A, \widetilde{m}) = -i\alpha\,[\mathrm{ch}(\mathcal{F})]_{2r}, \tag{3.20}$$

up to closed forms and contribution from local counterterms, where $F_{t\pm} \equiv t dA_\pm + t^2 A^2_\pm$ and $\mathrm{Tr}^{\mathrm{sym}}$ stands for the symmetrized trace.

[11] More precisely, what we are concerned with here is the mixed anomaly between $U(1)_V$ and $SU(N)_+ \times SU(N)_- \times U(1)_A$.

respectively, where $v \equiv \mathrm{diag}(v_+,\ v_-)$, $\widetilde{m} \equiv m/\Lambda$ is the mass rescaled by the cut-off Λ (see Eq. (3.35) for the definition), and

$$\mathcal{F} = \begin{pmatrix} F_+ - \widetilde{m}^\dagger \widetilde{m} & iD\widetilde{m}^\dagger \\ iD\widetilde{m} & F_- - \widetilde{m}\widetilde{m}^\dagger \end{pmatrix} \tag{3.21}$$

is the field strength of the superconnection in Eq. (2.3) with $T = \widetilde{m}$. Equation (3.18) is related to $I^1_{2r}(v, A, \widetilde{m})$, which gives the consistent anomaly

$$\delta_v \Gamma[A, m] = \int I^1_{2r}(v, A, \widetilde{m}) \tag{3.22}$$

by the descent equation in Eq. (3.11).[12] Since $I_{2r+2}(A, \widetilde{m})$ is not a polynomial of the field strength $\mathcal{F}$, we refer to it as an anomaly $(2r+2)$-form following Ref. [4]. Equation (3.19) is the covariant anomaly related to the Jacobian $\mathcal{J}$ defined with a covariant regularization adopted in Sect. 3.1.2 by

$$\log \mathcal{J} = \int I^{1\,\mathrm{cov}}_{2r}(v, A, \widetilde{m}). \tag{3.23}$$

Equation (3.20) is obtained from Eq. (3.19) by setting $v_+ = v_- = -i\alpha 1_N$. In Eqs. (3.22) and (3.23), we take the $\Lambda \to \infty$ limit after the integration.

Note that when m is bounded, $\widetilde{m}$ vanishes in the limit $\Lambda \to \infty$ and the m dependence drops out [1,2]. However, there are some physically interesting systems in which the mass is of the order of the cut-off scale or unbounded, and the m dependence in the anomaly may survive. For example, a system with a boundary can be realized by setting $m \to \infty$ in a region of the spacetime. Another interesting example is a system with localized massless fermions on an interface (defect) with mass of the order of the cut-off scale in the bulk, such as the domain-wall fermions used in lattice quantum chromodynamics [18]. We consider such examples in Sect. 4.

Another related issue is that, as shown in Ref. [6], the de Rham cohomology class of Eq. (3.18) is independent of $\widetilde{m}$ because of the relation in Eq. (2.8), which would mean that the m-dependent part of the $(2r+2)$-form in Eq. (3.18) does not contribute to the anomaly. This is true in a compact spacetime. However, for an open space, the $\widetilde{m}$-dependent part of the anomaly $(2r+2)$-form can give a non-trivial element of the cohomology with compact support.[13] As discussed in Sect. 4.1, this non-trivial element is interpreted as the anomaly of the fermions localized on the interfaces located around the zero locus of the mass profile. The local counterterm that cancels this anomaly is the contribution from the anomaly inflow.

In Sect. 3.1.2 we will explicitly show Eqs. (3.19) and (3.20) using Fujikawa's method, following the prescription given in Ref. [1]. Our argument for Eq. (3.18) is more indirect. This is suggested as a consequence of the relation in Eq. (3.17) between the non-Abelian anomaly in $2(r-1)$ dimensions characterized by $I_{2r}(A)$ and the Abelian anomaly given by $I^{1\,\mathrm{cov}}_{2r}(-i\alpha, A)$ in $2r$ dimensions [19,20] generalized to the cases with spacetime-dependent mass. This issue is discussed in Sect. 3.1.3.

3.1.2. *Calculation of the Jacobian.* In order to show Eqs. (3.19) and (3.20), we evaluate the Jacobian $\mathcal{J}$ in Eq. (3.13) for the $U(N)_+ \times U(N)_-$ transformation in Eq. (3.7). In the following, we demonstrate the derivation of Eq. (3.20) in detail, focusing on the $U(1)_V$ transformation

[12] See Sect. 4.1 for more on the use of the anomaly $(D+2)$-form in Eq. (3.18).
[13] See Ref. [4] for more on this point.

that acts on the fermions as

$$\psi(x) \to e^{i\alpha(x)}\psi(x), \qquad \overline{\psi}(x) \to e^{-i\alpha(x)}\overline{\psi}(x), \tag{3.24}$$

which is a special case of the transformation in Eq. (3.7) with $U_+ = U_- = e^{i\alpha}1_N$. The generalization to general $U(N)_+ \times U(N)_-$ transformations that leads to Eq. (3.19) is straightforward.

Following Ref. [1], we expand the fermion fields ψ and $\overline{\psi}$ with respect to the eigenfunctions of the Hermitian operators $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$, respectively. Let n_ϕ and n_φ be the number of zero modes of $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$, respectively, and choose the eigenfunctions such that they satisfy the eigenequations[14]

$$\mathcal{D}^\dagger\mathcal{D}\varphi_n(x) = \lambda_n^2\varphi_n(x) \qquad (n \in \{k - n_\varphi \mid k = 1, 2, 3, \dots\}), \tag{3.25}$$

$$\mathcal{D}\mathcal{D}^\dagger\phi_n(x) = \lambda_n^2\phi_n(x) \qquad (n \in \{k - n_\phi \mid k = 1, 2, 3, \dots\}) \tag{3.26}$$

and the normalization conditions

$$\int d^Dx\, \varphi_m^\dagger(x)\varphi_n(x) = \delta_{m,n}, \qquad \int d^Dx\, \phi_m^\dagger(x)\phi_n(x) = \delta_{m,n}. \tag{3.27}$$

Here, the eigenvalues of $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$ are denoted as λ_n^2, because they are non-negative and can be written as the square of real numbers.[15] Without loss of generality, we assume $\lambda_n = 0$ for $n \le 0$ and $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$. Note that the eigenvalues for Eqs. (3.25) and (3.26) are the same, because the non-zero modes φ_n and ϕ_n with $n > 0$ are related by

$$\phi_n(x) = \frac{1}{\lambda_n}\mathcal{D}\varphi_n(x), \qquad \varphi_n(x) = \frac{1}{\lambda_n}\mathcal{D}^\dagger\phi_n(x) \qquad (\text{for } n > 0) \tag{3.28}$$

up to phase.

Then, fermions $\psi(x)$ and $\overline{\psi}(x)$ can be expanded as

$$\psi(x) = \sum_n a_n\varphi_n(x), \qquad \overline{\psi}(x) = \sum_n \overline{b}_n\phi_n^\dagger(x), \tag{3.29}$$

where a_n and $\overline{b}_n$ are Grassmann-odd coefficients, and the action in Eq. (3.1) becomes

$$S = \sum_n \lambda_n\overline{b}_n a_n. \tag{3.30}$$

The fermion path integral measure is formally defined as

$$[d\psi d\overline{\psi}] = \prod_x d\psi(x)d\overline{\psi}(x) = \det(\varphi_n(x))^{-1}\det\left(\phi_n^\dagger(x)\right)^{-1}\prod_n da_n\prod_m d\overline{b}_m, \tag{3.31}$$

where $\det(\varphi_n(x))^{-1}\det(\phi_n^\dagger(x))^{-1}$ is the Jacobian for the change of variables from $\{\psi(x), \overline{\psi}(x)\}$ to $\{a_n, \overline{b}_n\}$.

Under the $U(1)_V$ transformation in Eq. (3.24), a_n and $\overline{b}_n$ transform as

$$\begin{aligned} a_n \to a_n' &\equiv \int d^Dx\, \varphi_n^\dagger(x)e^{i\alpha(x)}\psi(x) \simeq \sum_m\left(\delta_{m,n} + i\int d^Dx\, \varphi_n^\dagger(x)\alpha(x)\varphi_m(x)\right)a_m, \\ \overline{b}_n \to \overline{b}_n' &\equiv \int d^Dx\, \overline{\psi}(x)e^{-i\alpha(x)}\phi_n(x) \simeq \sum_m \overline{b}_m\left(\delta_{m,n} - i\int d^Dx\, \phi_m^\dagger(x)\alpha(x)\phi_n(x)\right), \end{aligned} \tag{3.32}$$

[14]Here, we have assumed that the spectra of $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$ are discrete. Later, we will consider the cases with non-compact spacetime. In such cases, the asymptotic behavior of the mass and the gauge fields should be chosen appropriately to have discrete spectra.

[15]Be aware that λ_n is not the eigenvalue of $\mathcal{D}$. $\mathcal{D}$ is not Hermitian and its eigenvalues are not real in general.

where we have assumed $\alpha(x) \ll 1$. Then, the Jacobian in Eq. (3.13) is

$$\log \mathcal{J} = -i \int d^D x\, \alpha(x)\mathcal{I}(x), \tag{3.33}$$

where

$$\mathcal{I}(x) \equiv \sum_n \left(\varphi_n^\dagger(x)\varphi_n(x) - \phi_n^\dagger(x)\phi_n(x)\right). \tag{3.34}$$

$\mathcal{I}(x)$ can be regularized by introducing a UV cut-off Λ as

$$\begin{aligned}
\mathcal{I}(x) &= \lim_{\Lambda\to\infty} \sum_n e^{-\frac{\lambda_n^2}{\Lambda^2}} \left(\varphi_n^\dagger(x)\varphi_n(x) - \phi_n^\dagger(x)\phi_n(x)\right) \\
&= \lim_{\Lambda\to\infty} \sum_n \left(\varphi_n^\dagger(x) e^{-\frac{1}{\Lambda^2}\mathcal{D}^\dagger\mathcal{D}}\varphi_n(x) - \phi_n^\dagger(x) e^{-\frac{1}{\Lambda^2}\mathcal{D}\mathcal{D}^\dagger}\phi_n(x)\right) \\
&= \lim_{\Lambda\to\infty} \int \frac{d^D k}{(2\pi)^D} e^{-ikx}\, \mathrm{Tr}_s \left(e^{-\frac{1}{\Lambda^2}\mathcal{D}^\dagger\mathcal{D}} - e^{-\frac{1}{\Lambda^2}\mathcal{D}\mathcal{D}^\dagger}\right) e^{ikx},
\end{aligned} \tag{3.35}$$

where Tr_s is the trace over both flavor and spinor indices. The cut-off Λ will be sent to infinity at the end of the calculation.[16]

To evaluate Eq. (3.35), note that $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$ are written as

$$\mathcal{D}^\dagger\mathcal{D} = -D_\mu^2 - \Lambda^2\widehat{\mathcal{F}}, \qquad \mathcal{D}\mathcal{D}^\dagger = -D_\mu^2 - \Lambda^2\widehat{\mathcal{F}}', \tag{3.36}$$

where

$$D_\mu = \begin{pmatrix} \partial_\mu + A_{+\mu} & 0 \\ 0 & \partial_\mu + A_{-\mu} \end{pmatrix} \tag{3.37}$$

and

$$\widehat{\mathcal{F}} = \begin{pmatrix} \frac{1}{2\Lambda^2}\sigma^\mu\sigma^{\nu\dagger}F_{+\mu\nu} - \widetilde{m}^\dagger\widetilde{m} & \frac{1}{\Lambda}\sigma^\mu D_\mu\widetilde{m}^\dagger \\ \frac{1}{\Lambda}\sigma^{\mu\dagger}D_\mu\widetilde{m} & \frac{1}{2\Lambda^2}\sigma^{\mu\dagger}\sigma^\nu F_{-\mu\nu} - \widetilde{m}\widetilde{m}^\dagger \end{pmatrix}, \tag{3.38}$$

$$\widehat{\mathcal{F}}' = \begin{pmatrix} \frac{1}{2\Lambda^2}\sigma^{\mu\dagger}\sigma^{\nu}F_{+\mu\nu} - \widetilde{m}^\dagger\widetilde{m} & -\frac{1}{\Lambda}\sigma^{\mu\dagger} D_\mu\widetilde{m}^\dagger \\ -\frac{1}{\Lambda}\sigma^{\mu}D_\mu\widetilde{m} & \frac{1}{2\Lambda^2}\sigma^{\mu}\sigma^{\nu\dagger} F_{-\mu\nu} - \widetilde{m}\widetilde{m}^\dagger \end{pmatrix}, \tag{3.39}$$

with $\widetilde{m} \equiv m/\Lambda$. Then, Eq. (3.35) becomes

$$\begin{aligned}
\mathcal{I}(x) &= \lim_{\Lambda\to\infty} \int \frac{d^D k}{(2\pi)^D} \mathrm{Tr}_s \left(e^{\frac{1}{\Lambda^2}(ik_\mu + D_\mu)^2 + \widehat{\mathcal{F}}} - e^{\frac{1}{\Lambda^2}(ik_\mu + D_\mu)^2 + \widehat{\mathcal{F}}'}\right) \\
&= \lim_{\Lambda\to\infty} \Lambda^D \int \frac{d^D \widetilde{k}}{(2\pi)^D} e^{-\widetilde{k}_\mu^2}\, \mathrm{Tr}_s \left(e^{\frac{1}{\Lambda^2}D_\mu^2 + \frac{2i}{\Lambda}\widetilde{k}^\mu D_\mu + \widehat{\mathcal{F}}} - e^{\frac{1}{\Lambda^2}D_\mu^2 + \frac{2i}{\Lambda}\widetilde{k}^\mu D_\mu + \widehat{\mathcal{F}}'}\right),
\end{aligned} \tag{3.40}$$

where $\widetilde{k}_\mu \equiv k_\mu/\Lambda$. Using the formula

$$\mathrm{tr}\left(\sigma^{\mu_1}\sigma^{\mu_2\dagger}\cdots\sigma^{\mu_{2k-1}}\sigma^{\mu_{2k}\dagger} - \sigma^{\mu_1\dagger}\sigma^{\mu_2}\cdots\sigma^{\mu_{2k-1}\dagger}\sigma^{\mu_{2k}}\right) = \begin{cases} 0 & (k < r), \\ (2i)^r\epsilon^{\mu_1\cdots\mu_{2r}} & (k = r), \end{cases} \tag{3.41}$$

where $\epsilon^{\mu_1\cdots\mu_{2r}}$ is the Levi–Civita symbol with $\epsilon^{1,2,\ldots,D} = 1$, and assuming that the gauge field, $\widetilde{m}$, and $\widetilde{k}_\mu$ as well as their derivatives are all of $\mathcal{O}(1)$ in the $1/\Lambda$ expansion,[17] it is easy to verify

[16] The explicit form of the anomaly actually depends on the choice of the regularization scheme. We adopt this heat kernel regularization in a covariant form.

[17] In Sect. 4 we consider the cases with m being a linear function of x^μ. One may wonder whether $\widetilde{m}$ can be regarded as an $\mathcal{O}(1)$ parameter, even though $\widetilde{m}$ diverges at $|x| \to \infty$. In that case, our treatment here can be understood as the evaluation of the $\Lambda \to \infty$ limit of the integration $\int d^D x\, \alpha(x)\mathcal{I}(x)$ by using rescaled coordinates $\widetilde{x}^\mu = x^\mu/\Lambda$.

that

$$\mathcal{I}(x) = \lim_{\Lambda\to\infty} \Lambda^D \int \frac{d^D\widetilde{k}}{(2\pi)^D} e^{-\widetilde{k}_\mu^2}\,\mathrm{Tr}_s\left(e^{\widehat{\mathcal{F}}} - e^{\widehat{\mathcal{F}}'}\right) = \lim_{\Lambda\to\infty} \frac{\Lambda^D}{2^D\pi^{D/2}}\mathrm{Tr}_s\left(e^{\widehat{\mathcal{F}}} - e^{\widehat{\mathcal{F}}'}\right) \tag{3.42}$$

and

$$\mathrm{Tr}_s\left(e^{\widehat{\mathcal{F}}} - e^{\widehat{\mathcal{F}}'}\right) d^{2r}x = \Lambda^{-2r}(2i)^r \left[\mathrm{Str}\left(e^{\mathcal{F}}\right)\right]_{2r} + \mathcal{O}(\Lambda^{-2r-1}), \tag{3.43}$$

where $d^{2r}x = dx^1 \cdots dx^{2r}$ and $\mathcal{F}$ is the superconnection defined in Eq. (3.21). Neglecting the $\mathcal{O}(\Lambda^{-1})$ terms, this implies that[18]

$$\mathcal{I}(x)d^{2r}x = \left(\frac{i}{2\pi}\right)^r \left[\mathrm{Str}\left(e^{\mathcal{F}}\right)\right]_{2r} = [\mathrm{ch}(\mathcal{F})]_{2r}\,, \tag{3.44}$$

and hence we obtain

$$\log\mathcal{J} = -i\int \alpha(x)\,[\mathrm{ch}(\mathcal{F})]_D\,, \tag{3.45}$$

which is the desired result, Eq. (3.20).

In Sect. 4 we consider the cases with $A_+ = A_-$ and the mass given by a scalar matrix as

$$m = \mu(x)1_N, \tag{3.46}$$

where $\mu(x)$ is a complex function and 1_N is the unit matrix of size N. In this case, we have

$$\mathrm{ch}(\mathcal{F}) = \frac{i}{2\pi} d\widetilde{\mu}^\dagger d\widetilde{\mu}\, e^{-|\widetilde{\mu}|^2}\mathrm{ch}(F), \tag{3.47}$$

with $F \equiv F_+ = F_-$ and $\widetilde{\mu} \equiv \mu/\Lambda$, and the Jacobian in Eq. (3.45) becomes

$$\log\mathcal{J} = \frac{1}{2\pi}\int d\widetilde{\mu}^\dagger d\widetilde{\mu}\, e^{-|\widetilde{\mu}|^2}\alpha(x)\,[\mathrm{ch}(F)]_{D-2}\,. \tag{3.48}$$

3.1.3. *Anomaly $(D+2)$-form.* In this subsection we give a simple derivation of the anomaly $(D+2)$-form in Eq. (3.18) using the result in Eq. (3.20) for the $U(1)_V$ anomaly. Although the description here is for the even-dimensional case, the argument is applicable to the odd-dimensional case as well.

We decompose the $U(N)_+ \times U(N)_-$ gauge fields into the $U(1)_V$ gauge field V and the rest, and write the Chern character as

$$\mathrm{ch}(\mathcal{F}) = \exp\left\{\frac{i}{2\pi}f^V\right\}\mathrm{ch}(\mathcal{F}_0), \tag{3.49}$$

where $f^V \equiv dV$ is the field strength of the $U(1)_V$ gauge field and $\mathcal{F}_0 \equiv \mathcal{F}|_{f^V=0} = \mathcal{F} - f^V 1_{2N}$. First, we try to show Eq. (3.18) for the case with $f^V = 0$. To this end, let us consider the $U(1)_V$ anomaly in Eq. (3.20) with $f^V = 0$ in a $(D+2)$-dimensional system:

$$I^{1\,\mathrm{cov}}_{D+2}(-i\alpha, A, \widetilde{m})|_{f^V=0} = -i\alpha[\mathrm{ch}(\mathcal{F}_0)]_{D+2}. \tag{3.50}$$

Note that for this component of the anomaly there is no difference between the covariant and consistent anomalies[19] (see Appendix B.1). The anomaly $(D+4)$-form for the $(D+2)$-dimensional system that reproduces Eq. (3.50) via the descent equations in Eq. (3.11) is

$$f^V[\mathrm{ch}(\mathcal{F}_0)]_{D+2}. \tag{3.51}$$

Now, consider a $(D+2)$-dimensional spacetime of the form $S^2 \times M_D$, where M_D is a D-dimensional manifold. We assume that f^V has a flux with $\int_{S^2} f^V = -2\pi i$, and $\mathcal{F}_0$ is independent

[18] This formula (in the $\Lambda \to \infty$ limit with $\widetilde{m}$ kept fixed) corresponds to the local index theorem proved in Ref. [21]. See also Ref. [22].

[19] We thank Y. Tanizaki for the discussion on this point.

of the coordinates on S^2. In this case, each fermion in the $(D+2)$-dimensional system has one zero mode on S^2, and hence we get a D-dimensional system with N Dirac fermions in the limit that the radius of S^2 becomes zero. The anomaly $(D+2)$-form for this D-dimensional system is given by integrating Eq. (3.51) over S^2, yielding

$$I_{D+2}(A, \widetilde{m})|_{f^V=0} = \int_{S^2} f^V [\mathrm{ch}(\mathcal{F}_0)]_{D+2} = -2\pi i \, [\mathrm{ch}(\mathcal{F}_0)]_{D+2}, \tag{3.52}$$

which is Eq. (3.18) for the $f^V = 0$ case.

The f^V dependence of the anomaly $(D+2)$-form can easily be recovered by replacing $\mathcal{F}_0$ with $\mathcal{F}$, which completes the derivation of Eq. (3.18).

3.2. *Odd-dimensional cases*

3.2.1. *Anomaly in odd dimensions.* In this section we consider a system with N Dirac fermions ψ in a $D = (2r+1)$-dimensional flat Euclidean spacetime ($r \in \mathbb{Z}_{\geq 0}$). The flavor symmetry is $U(N)$ and the associated external gauge field is denoted as A. We include a spacetime-dependent mass m, which is a Hermitian matrix of size N and belongs to the adjoint representation of $U(N)$. The action is

$$S = \int d^D x \, \overline{\psi} \, (\not{D} + m) \, \psi = \int d^D x \, \overline{\psi} \mathcal{D} \psi, \tag{3.53}$$

where

$$\not{D} \equiv \gamma^\mu (\partial_\mu + A_\mu), \qquad \mathcal{D} \equiv \not{D} + m, \tag{3.54}$$

and γ^μ ($\mu = 1, 2, \ldots, 2r+1$) are gamma matrices satisfying $\gamma^{\mu\dagger} = \gamma^\mu$ and $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$. For explicit computation, we choose γ^μ to be of the form in Eq. (3.5) for $\mu = 1, \ldots, 2r$ and γ^{2r+1} in Eq. (3.6) for $\mu = 2r+1$. This action is invariant under the $U(N)$ flavor symmetry:

$$\psi \to U\psi, \qquad \overline{\psi} \to \overline{\psi} U^{-1}, \qquad A \to UAU^{-1} + UdU^{-1}, \qquad m \to UmU^{-1}, \tag{3.55}$$

with $U(x) \in U(N)$.

Our claim is that the formulas analogous to Eqs. (3.18), (3.19), and (3.20),

$$I_{D+2}(A, \widetilde{m}) = -2\pi i \, [\mathrm{ch}(\mathcal{F})]_{D+2}, \tag{3.56}$$

$$I_D^{1\,\mathrm{cov}}(v, A, \widetilde{m}) = \left(\frac{i}{2\pi}\right)^{D/2} \left[\mathrm{Str}\left(v \, e^{\mathcal{F}}\right)\right]_D, \tag{3.57}$$

$$I_D^{1\,\mathrm{cov}}(-i\alpha, A, \widetilde{m}) = -i\alpha \, [\mathrm{ch}(\mathcal{F})]_D, \tag{3.58}$$

hold even for the odd-dimensional cases, using the odd-dimensional analog of the Chern character in Eq. (2.5) defined by the supertrace for the odd case in Eq. (2.14). Unlike the even-dimensional cases discussed in Sect. 3.1, both Eqs. (3.56) and (3.58) vanish when the mass m vanishes. The anomaly appears only when m is turned on.

We show in Sect. 3.2.2 that the formula in Eq. (3.45) for the $U(1)_V$ transformation in Eq. (3.24) also holds for the odd-dimensional cases by examining the Jacobian of the fermion path integral measure using Fujikawa's method. This implies Eq. (3.58). The derivation can be easily generalized to Eq. (3.57). Equation (3.56) follows from Eq. (3.58) by an indirect argument given in Sect. 3.1.3.

The meaning of Eq. (3.56) is somewhat more ambiguous, because, for odd D, we can find a gauge-invariant $(D+1)$-form $I^0_{D+1}(A, \widetilde{m})$ satisfying $I_{D+2}(A, \widetilde{m}) = dI^0_{D+1}(A, \widetilde{m})$ (see Eq. (2.16)).

Then, the odd-dimensional analogue of the descent equations in Eq. (3.11),

$$dI^1_D = \delta_v I^0_{D+1}, \qquad dI^0_{D+1} = I_{D+2}, \tag{3.59}$$

would imply that the anomaly I^1_D simply vanishes. However, as we will see in Sect. 4.1.1, $I^0_{D+1}(A,\widetilde{m})$ is non-vanishing at infinity in our examples with non-trivial interfaces, and I_{D+2} can be a non-trivial element of the cohomology with compact support.[20] We will argue that the anomaly of the fermions on the interfaces can be extracted from Eq. (3.45).

3.2.2. *Calculation of the Jacobian.* The Jacobian of the fermion path integral measure for the $U(1)_V$ transformation in Eq. (3.24) in the odd-dimensional case can be calculated in a similar way to the even-dimensional case in Sect. 3.1.2. In particular, Eqs. (3.33), (3.35), and (3.40) can be used for the $D = (2r+1)$ case, with $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{F}}'$ defined as

$$\widehat{\mathcal{F}} = \frac{1}{2\Lambda^2}\gamma^\mu\gamma^\nu F_{\mu\nu} + \frac{1}{\Lambda}\gamma^\mu D_\mu\widetilde{m} - \widetilde{m}^2, \qquad \widehat{\mathcal{F}}' = \frac{1}{2\Lambda^2}\gamma^\mu\gamma^\nu F_{\mu\nu} - \frac{1}{\Lambda}\gamma^\mu D_\mu\widetilde{m} - \widetilde{m}^2. \tag{3.60}$$

Note that $\widehat{\mathcal{F}}'$ is obtained by replacing γ^μ with $-\gamma^\mu$ in $\widehat{\mathcal{F}}$. Therefore, when the matrix in the trace in Eq. (3.40) is expanded with respect to γ^μ, only the terms with odd numbers of γ^μ can contribute. Furthermore, using the relation[21]

$$\mathrm{tr}\,(\gamma^{\mu_1}\cdots\gamma^{\mu_{2k+1}}) = \begin{cases} 0 & (k<r), \\ (2i)^r\epsilon^{\mu_1\cdots\mu_{2r+1}} & (k=r), \end{cases} \tag{3.61}$$

we find that Eq. (3.42) also holds for the odd-dimensional case, and

$$\mathrm{Tr}_s\left(e^{\widehat{\mathcal{F}}} - e^{\widehat{\mathcal{F}}'}\right) d^{2r+1}x = \Lambda^{-(2r+1)}(2i)^{r+1/2}\left[\mathrm{Str}\left(e^{\mathcal{F}}\right)\right]_{2r+1} + \mathcal{O}(\Lambda^{-(2r+1)-1}), \tag{3.62}$$

where $\mathcal{F}$ is the superconnection of the odd type given by Eq. (2.13) with $T = \widetilde{m} = m/\Lambda$:

$$\mathcal{F} = \begin{pmatrix} F - \widetilde{m}^2 & iD\widetilde{m} \\ iD\widetilde{m} & F - \widetilde{m}^2 \end{pmatrix}. \tag{3.63}$$

Note that we have taken into account the $\sqrt{2}i^{-3/2}$ factor in the definition of the supertrace Str for the odd case in Eq. (2.14). Then, we obtain

$$\mathcal{I}(x)d^{2r+1}x = \left(\frac{i}{2\pi}\right)^{(2r+1)/2}\left[\mathrm{Str}\left(e^{\mathcal{F}}\right)\right]_{2r+1} = [\mathrm{ch}(\mathcal{F})]_{2r+1}\,. \tag{3.64}$$

This implies that

$$\log\mathcal{J} = -i\int\alpha(x)[\mathrm{ch}(\mathcal{F})]_{2r+1}, \tag{3.65}$$

which takes the same form as Eq. (3.45) for $D = 2r+1$.

In particular, when the mass is a scalar matrix given by

$$m = \mu(x)1_N, \tag{3.66}$$

with a real function $\mu(x)$, we have

$$\mathrm{ch}(\mathcal{F}) = \frac{1}{\sqrt{\pi}}\,d\widetilde{\mu}\,e^{-\widetilde{\mu}^2}\mathrm{ch}(F) \tag{3.67}$$

[20] A similar statement holds for the mass-dependent part of the Chern character $I_{2r}(A,\widetilde{m})$ for the even-dimensional case, as mentioned in Sect. 3.1.1 and demonstrated in Sec. 4.1.2.

[21] Here, γ^{2r+1} is chosen to be the same as in Eq. (3.6).

and

$$\log \mathcal{J} = -\frac{i}{\sqrt{\pi}} \int d\widetilde{\mu}\, e^{-\widetilde{\mu}^2} \alpha(x)\, [\mathrm{ch}(F)]_{2r}\,, \tag{3.68}$$

where $\widetilde{\mu} \equiv \mu/\Lambda$.

4. Applications

4.1. *Anomalies on interfaces*

In this section we consider mass profiles with isolated zero loci, which we call interfaces, and show that the anomaly carried by the fermions localized on the interfaces can be easily extracted by the formulas obtained in Sect. 3. As pointed out in Refs. [4,23], the anomaly of the localized modes implies the existence of a diabolical point in the space of parameters of the theory, as mentioned at the end of Sect. 4.1.3.

4.1.1. *Kink (codimension 1 interface).* We consider a $D = (2r + 1)$-dimensional system given by Eq. (3.53) with a kink-like mass profile as

$$m = \mu(y) 1_N = uy 1_N, \tag{4.1}$$

where $y \equiv x^{2r+1}$ is one of the spatial coordinates and u is a real parameter. Since the mass m diverges at $|y| \to \infty$, the operators $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$ have discrete spectra as required in Sect. 3.1.2.

To simplify the discussion, we assume that the gauge field as well as $\alpha(x)$ are independent of y. Then, the integration over y in Eq. (3.68) can be done, and we obtain

$$\log \mathcal{J} = -i\, \mathrm{sgn}(u) \int \alpha(x)\, [\mathrm{ch}(F)]_{2r}\,, \tag{4.2}$$

where $\mathrm{sgn}(u) = u/|u|$ is a sign function and the integration is taken over the $2r$-dimensional space along $x^{1 \sim 2r}$ directions. Note that this result is independent of the cut-off Λ, and hence it survives in the $\Lambda \to \infty$ limit. The dependence on the parameter u is only through its sign. Knowing this fact, it may be convenient for some purposes to take the $|u| \to \infty$ limit as

$$\lim_{|u|\to\infty} \mathrm{ch}(\mathcal{F}) = \mathrm{sgn}(u)\delta(y) dy\, \mathrm{ch}(F). \tag{4.3}$$

In fact, Eq. (4.2) does not depend on the detail of the profile in Eq. (4.1). As is clear from Eq. (3.68), we get the same result, Eq. (4.2), for any function $\mu(y)$ satisfying $\mu(y) \to \pm\infty$ (or $\mu(y) \to \mp\infty$) as $y \to \pm\infty$.

The expression in Eq. (4.2) agrees with the anomaly for Weyl fermions in a $2r$-dimensional spacetime. In fact, Eq. (4.2) is identical to Eq. (3.15) with Eq. (3.17), provided we identify $(F_+, F_-) = (F, 0)$ for $u > 0$ or $(F_+, F_-) = (0, F)$ for $u < 0$. We interpret this as the anomaly contribution from the Weyl fermions localized on the interface at $y = 0$. As a check, it is easy to show that there exist positive- or negative-chirality Weyl fermions at the interface as the zero modes of the operator $\mathcal{D} = \not{D} + m$ with $u > 0$ or $u < 0$, respectively [24]. To see this, let us consider the Dirac equation $\mathcal{D}\psi = 0$, where $\mathcal{D}$ is defined in Eq. (3.54). Working in the $A_{2r+1} = 0$ gauge, this equation can be written as

$$\not{D}^{(2r)}\psi + \gamma^{2r+1}\partial_y \psi + \mu(y)\psi = 0, \tag{4.4}$$

where

$$\not{D}^{(2r)} = \sum_{\mu=1}^{2r} \gamma^\mu (\partial_\mu + A_\mu) \tag{4.5}$$

is the Dirac operator in the $2r$-dimensional space. Then, we find a solution localized around $y = 0$:

$$\psi(\vec{x}, y) = e^{-\frac{1}{2}|u|y^2}\psi^{(2r)}(\vec{x}), \tag{4.6}$$

where $\vec{x} = (x^1, \ldots, x^{2r})$ and $\psi^{(2r)}(\vec{x})$ is the $2r$-dimensional Weyl fermion at the interface satisfying

$$\not{D}^{(2r)}\psi^{(2r)} = 0, \qquad \gamma^{2r+1}\psi^{(2r)} = \mathrm{sgn}(u)\psi^{(2r)}. \tag{4.7}$$

Note, however, that the anomaly contribution of the localized Weyl fermions are known to be canceled by the contribution from the bulk via the anomaly inflow mechanism [25]. Outside the region with $\mu(x) = 0$, the one-loop effective action contains a term with the CS $(2r + 1)$-form, whose gauge variation precisely cancels the anomaly of the localized fermions. Our result in Eq. (4.2) can be interpreted in two ways. One is that the variation of the CS term simply vanishes when the gauge field and the gauge variation are independent of y, and Eq. (4.2) is the contribution of the localized fermion. The other is that the anomaly of the localized fermion at $y = 0$ is canceled by the contribution from the CS term, but the variation of the CS term also produces the same amount of anomaly at $y = \pm\infty$, which gives Eq. (4.2). We will make more comments on the relation to the anomaly inflow below.

Let us next discuss the anomaly $(D + 2)$-form in Eq. (3.56). Inserting Eq. (4.1) into Eq. (3.56), we obtain

$$I_{2r+3}(A, \widetilde{m}) = -2\sqrt{\pi}\, i\, e^{-\widetilde{m}^2} d\widetilde{m}\,[\mathrm{ch}(F)]_{2r+2} = df(\widetilde{m})\, I_{2r+2}(A), \tag{4.8}$$

where $I_{2r+2}(A) \equiv -2\pi i[\mathrm{ch}(F)]_{2r+2}$ and

$$f(x) \equiv \frac{1}{\sqrt{\pi}}\int_0^x e^{-y^2}dy = \frac{1}{2}\mathrm{erf}(x). \tag{4.9}$$

A possible choice of I^0_{2r+2} satisfying the relation $I_{D+2} = dI^0_{D+1}$ in Eq. (3.59) with $D = 2r + 1$ is

$$I^0_{2r+2}(A, \widetilde{m}) = f(\widetilde{m})I_{2r+2}(A). \tag{4.10}$$

Since this is invariant under the $U(N)$ transformation, we have $\delta_v I^0_{2r+2}(A, \widetilde{m}) = 0$ and the anomaly I^1_{2r+1} related to I^0_{2r+2} by the decent relation in Eq. (3.59) vanishes. However, this does not mean the m-dependent anomaly $(D + 2)$-form $I_{2r+3}(A, \widetilde{m})$ is useless. In fact, we can extract the information of the anomaly from the fermions localized at the interface from Eq. (4.8) as follows.

The point is that the factor $f(\widetilde{m})$ in Eq. (4.10) does not vanish but approaches $\pm\frac{1}{2}\mathrm{sgn}(u)$ at $y = \pm\infty$. Therefore, the relation $I_{2r+3} = dI^0_{2r+2}$ with a gauge-invariant $(2r + 2)$-form I^0_{2r+2} does not imply that I_{2r+3} is trivial as an element of a cohomology with compact support. To find the anomaly for the localized modes, we decompose I^0_{2r+2} in Eq. (4.10) into a local part that vanishes at $y \to \pm\infty$ and a closed form that does not contribute in the relation $I_{2r+3} = dI^0_{2r+2}$ as

$$I^0_{2r+2}(A, \widetilde{m}) = I^{0\,\mathrm{local}}_{2r+2}(A, \widetilde{m}) + d\omega_{2r+1}(A, \widetilde{m}), \tag{4.11}$$

with

$$I^{0\,\mathrm{local}}_{2r+2}(A, \widetilde{m}) \equiv -df(\widetilde{m})I^0_{2r+1}(A), \qquad \omega_{2r+1}(A, \widetilde{m}) \equiv f(\widetilde{m})I^0_{2r+1}(A), \tag{4.12}$$

where $I^0_{2r+1}(A)$ is the CS $(2r + 1)$-form satisfying $I_{2r+2}(A) = dI^0_{2r+1}(A)$.

We interpret $I^{0\,\mathrm{local}}_{2r+2}(A, \widetilde{m})$ as the part that gives the anomaly localized at the interface. Integrating $I^{0\,\mathrm{local}}_{2r+2}(A, \widetilde{m})$ over the y direction, one obtains the CS $(2r+1)$-form

$$I^{0\,\mathrm{local}}_{2r+1}(A) \equiv -\int_{\{y\}} I^{0\,\mathrm{local}}_{2r+2}(A, \widetilde{m}) = \mathrm{sgn}(u) I^{0}_{2r+1}(A), \tag{4.13}$$

which is related to the anomaly $I^{1\,\mathrm{local}}_{2r}(v, A)$ for the Weyl fermions localized at the interface by the descent relation $\delta_v I^{0\,\mathrm{local}}_{2r+1}(A) = dI^{1\,\mathrm{local}}_{2r}(v, A)$ in Eq. (3.59). Here, $\int_{\{y\}}$ denotes the integral over y. The anomaly $(2r+2)$-form for the localized fermions is given by

$$I^{\mathrm{local}}_{2r+2}(A) \equiv \int_{\{y\}} I_{2r+3}(A, \widetilde{m}) = \mathrm{sgn}(u) I_{2r+2}(A). \tag{4.14}$$

The second term in Eq. (4.11) corresponds to the anomaly contribution from the bulk that cancels the anomaly localized at the interface around $y = 0$ through the anomaly inflow [25]. To see this explicitly, it is convenient to take the $|u| \to \infty$ limit, in which $f(\widetilde{m})$ and $df(\widetilde{m})$ approach a step function and a delta function one-form with support at $y = 0$, respectively:

$$f(\widetilde{m}) \to \frac{1}{2}\mathrm{sgn}(u)\,\mathrm{sgn}(y), \qquad df(\widetilde{m}) \to \mathrm{sgn}(u)\delta(y)dy. \tag{4.15}$$

Then, $I^{0\,\mathrm{local}}_{2r+2}(A, \widetilde{m})$ is completely localized at $y = 0$ and ω_{2r+1} becomes

$$\omega_{2r+1}(A, \widetilde{m}) \to -\frac{1}{2}\mathrm{sgn}(\widetilde{m}) I^{0}_{2r+1}(A), \tag{4.16}$$

which can be interpreted as the CS term in the bulk induced from the path integral of the massive fermions, which precisely cancels the anomaly localized at the interface.

4.1.2. *Vortex (codimension 2 interface).* Next, consider a $D = (2r+2)$-dimensional system in Eq. (3.1) with a vortex-type mass profile given by

$$m = \mu(z) 1_N = uz 1_N, \tag{4.17}$$

where $z = x^{2r+1} - ix^{2r+2}$ and u is a complex parameter. Here, we assume that the gauge fields as well as the parameter α are independent of z, and satisfy $A_+ = A_- \equiv A$ and $A_{2r+1} = A_{2r+2} = 0$, for simplicity.

Then, Eq. (3.48) implies that

$$\log \mathcal{J} = -i\int \alpha(x)\,[\mathrm{ch}(F)]_{2r}\,. \tag{4.18}$$

This agrees with the anomaly of a $2r$-dimensional system with Weyl fermions, and is interpreted as the anomaly contribution from the Weyl fermion localized on the interface at $z = \bar{z} = 0$.

Again, we can explicitly find localized Weyl fermions as follows [25–27]. For this purpose, it is convenient to choose $\sigma^\mu = \gamma^\mu_{(2r)}$ ($\mu = 1, \ldots, 2r+1$) and $\sigma^{2r+2} = -i1_{2^r}$, where $\gamma^\mu_{(2r)}$ ($\mu = 1, \ldots, 2r$) are gamma matrices in $2r$ dimensions and $\gamma^{2r+1}_{(2r)}$ is the chirality operator for them. In this case, the Dirac equation $\mathcal{D}\psi = 0$ can be written as

$$\not{D}^{(2r)}\psi_+ + 2(P_+\partial_z - P_-\partial_{\bar{z}})\psi_+ + \overline{uz}\psi_- = 0, \tag{4.19}$$

$$\not{D}^{(2r)}\psi_- + 2(P_+\partial_{\bar{z}} - P_-\partial_z)\psi_- + uz\psi_+ = 0, \tag{4.20}$$

where $\not{D}^{(2r)}$ is defined in Eq. (4.5) and $P_\pm \equiv \frac{1}{2}(1_{2^r} \pm \gamma^{2r+1}_{(2r)})$ is a projection operator that projects to positive-/negative-chirality spinors in $2r$-dimensions. Then, we find a solution localized around $z = 0$:

$$\psi_+(\vec{x}, z, \bar{z}) = \psi_-(\vec{x}, z, \bar{z}) = e^{-\frac{1}{2}u|z|^2}\psi^{(2r)}(\vec{x}), \tag{4.21}$$

where we have assumed u to be real and positive without loss of generality, and $\psi^{(2r)}$ is a positive-chirality massless Weyl fermion in $2r$-dimensions.[22]

The role of the anomaly $(D+2)$-form in Eq. (3.18) can be discussed in a similar way as the codimension 1 interface considered in Sect. 4.1.1. For the mass profile in Eq. (4.17), the anomaly $(D+2)$-form (with $D = 2r+2$) becomes

$$I_{2r+4}(A, \widetilde{m}) = df_1(\widetilde{m})\, I_{2r+2}(A), \tag{4.22}$$

where $I_{2r+2}(A) \equiv -2\pi i[\mathrm{ch}(F)]_{2r+2}$ is the anomaly polynomial for a Weyl fermion in $2r$ dimensions and f_1 is a one-form given by

$$f_1(\widetilde{m}) \equiv \frac{i}{4\pi}\left(1 - e^{-|\widetilde{m}|^2}\right)\left(d\log\widetilde{m} - d\log\widetilde{m}^\dagger\right). \tag{4.23}$$

Note that f_1 is non-vanishing at $|z| \to \infty$, while its derivative

$$df_1(\widetilde{m}) = \frac{i}{2\pi} d\widetilde{m}^\dagger d\widetilde{m}\, e^{-|\widetilde{m}|^2} \tag{4.24}$$

decays exponentially as $|z| \to \infty$, and approaches a delta function two-form with support at $z = \bar{z} = 0$ in the $u \to \infty$ limit. The integral of df_1 over the z-plane is normalized as

$$\int df_1 = 1. \tag{4.25}$$

The CS-form $I^0_{2r+3}(A, \widetilde{m})$ satisfying $I_{2r+4}(A, \widetilde{m}) = dI^0_{2r+3}(A, \widetilde{m})$ can be chosen as

$$I^0_{2r+3}(A, \widetilde{m}) = f_1(\widetilde{m}) I_{2r+2}(A) = I^{0\,\mathrm{local}}_{2r+3}(A, \widetilde{m}) + d\omega_{2r+2}(A, \widetilde{m}), \tag{4.26}$$

where

$$I^{0\,\mathrm{local}}_{2r+3}(A, \widetilde{m}) \equiv df_1(\widetilde{m}) I^0_{2r+1}(A), \qquad \omega_{2r+2}(A, \widetilde{m}) \equiv -f_1(\widetilde{m}) I^0_{2r+1}(A). \tag{4.27}$$

Here, $I^0_{2r+1}(A)$ is the CS-form satisfying $I_{2r+2}(A) = dI^0_{2r+1}(A)$.

The anomaly contribution of the fermions localized at the interface, denoted as $I^{1\,\mathrm{local}}_{2r}(A)$, is related to

$$I^{0\,\mathrm{local}}_{2r+1}(A) \equiv \int_{\{z,\bar{z}\}} I^{0\,\mathrm{local}}_{2r+3}(A, \widetilde{m}) = I^0_{2r+1}(A), \tag{4.28}$$

where $\int_{\{z,\bar{z}\}}$ denotes the integral over the z-plane, by the descent relation $dI^{1\,\mathrm{local}}_{2r} = \delta_v I^{0\,\mathrm{local}}_{2r+1}$. In other words, it is characterized by the anomaly polynomial

$$I^{\mathrm{local}}_{2r+2}(A) \equiv \int_{\{z,\bar{z}\}} I_{2r+4}(A, \widetilde{m}) = I_{2r+2}(A). \tag{4.29}$$

On the other hand, $\omega_{2r+2}(A, \widetilde{m})$ gives the bulk contribution of the anomaly that cancels the anomaly on the interface.

4.1.3. *Interfaces of higher codimension.* The discussion in Sects. 4.1.1 and 4.1.2 can be generalized to the cases with interfaces of higher codimensions. We are interested in the interfaces with Weyl fermions on them.

A codimension n interface in $D = (2r+n)$-dimensional spacetime can be constructed by giving a mass of the form

$$m(x) = u \sum_{I=1}^{n} \Gamma^I x^I, \tag{4.30}$$

[22] A negative-chirality mode is obtained when the mass is $m = u\bar{z}\, 1_N$, which represents an anti-vortex.

where Γ^I ($I = 1, 2, \ldots, n$) are matrices of size $N = 2^{[(n-1)/2]}$ related to n-dimensional gamma matrices $\widehat{\gamma}^I$ by

$$\widehat{\gamma}^I = \Gamma^I \quad \text{(for odd } n \text{ and } D), \qquad \widehat{\gamma}^I = \begin{pmatrix} & \Gamma^I \\ \Gamma^{I\dagger} & \end{pmatrix} \quad \text{(for even } n \text{ and } D). \tag{4.31}$$

In this case, it can be shown that there is a Weyl fermion on the interface at $x^1 = \cdots = x^n = 0$ obtained as a localized fermion zero mode, as we have seen this explicitly in Sects. 4.1.1 and 4.1.2 for $n = 1, 2$. We give an indirect argument for this fact for general n in connection to index theorems in Sect. 4.3.2 and string theory interpretation in Sect. 5.

It is also possible to get k Weyl fermions by replacing Γ^I in Eq. (4.30) by $1_k \otimes \Gamma^I$ as

$$m(x) = u \sum_{I=1}^{n} 1_k \otimes \Gamma^I x^I. \tag{4.32}$$

In this case, the gauge group is $U(kN)$ or $U(kN)_+ \times U(kN)_-$ for odd or even D, respectively, and the vector-like $U(k)$ subgroup of the form $g \otimes 1_N$ with $g \in U(k)$ is unbroken. Then, k Weyl fermions coupled with the $U(k)$ gauge field a can be obtained by setting the $U(kN)$ gauge field A as

$$A = a \otimes 1_N. \tag{4.33}$$

It is straightforward to check that the anomaly for these Weyl fermions on the interface can be obtained by inserting the mass profile Eq. (4.32) and the gauge field Eq. (4.33) into our formulas in Eqs. (3.18)–(3.20) and (3.56)–(3.58). In particular, the expressions in Eqs. (4.14) and (4.29) of the anomaly $(2r + 2)$-form for the localized fermions are generalized as

$$I_{2r+2}^{\text{local}}(a) \equiv \int_n I_{2r+n+2}(A, \widetilde{m}), \tag{4.34}$$

where $\int_n$ denotes the integral over x^I ($I = 1, 2, \ldots, n$). This agrees with the anomaly polynomial for $2r$-dimensional Weyl fermions coupled to the $U(k)$ gauge field a.

As discussed in Refs. [4,23], the anomaly contributions from fermion zero modes localized on the interfaces implies that there is at least one point in the space of parameters of the theory, called a diabolical point, at which the theory is not trivially gapped. In our examples, it is of course clear that the massless point $m = 0$ is the diabolical point. However, since the anomaly takes a discrete value, the existence of the diabolical point is robust against continuous deformations of the theory. In fact, as we have seen, the anomaly depends only on the asymptotic behavior of the mass profile. The existence of the diabolical point can be shown without examining the theory at the massless point. This point is more explicit in the Callias-type index theorem in Eq. (4.74) discussed in Sect. 4.3.2.

4.2. *Anomaly in spacetime with boundaries*

Since the fermions cannot propagate in a region with infinite mass, it is possible to realize a spacetime with boundaries by considering a spacetime-dependent mass that blows up in some regions. In this subsection we discuss the anomaly driven by the boundary condition imposed on the fermions, using our formulas obtained in Sect. 3.

4.2.1. *Odd-dimensional cases.* Let us first consider a $D = (2r + 1)$-dimensional system of N Dirac fermions with $y \equiv x^{2r+1}$-dependent mass given by

$$m(y) = \mu(y)1_N = \begin{cases} (m_0 + u'(y - L))1_N & (L < y), \\ m_0 1_N & (0 \leq y \leq L), \\ (m_0 + uy)1_N & (y < 0), \end{cases} \tag{4.35}$$

where u, u', and m_0 are real parameters.[23] We assume that the gauge field is independent of y in the $y < 0$ and $L < y$ regions.

When $|u|$ and $|u'|$ are large enough, this system can be regarded as that of N Dirac fermions with mass m_0 living in an interval $0 \leq y \leq L$ with boundaries at $y = 0$ and $y = L$. The boundary conditions for the fermion fields follow from the requirement that they do not blow up at $y \to \pm\infty$. The discussion around Eqs. (4.4)–(4.7) implies that the corresponding boundary conditions are

$$\left(\gamma^{2r+1}\psi - \mathrm{sgn}(u)\psi\right)|_{y=0} = 0, \qquad \left(\gamma^{2r+1}\psi - \mathrm{sgn}(u')\psi\right)|_{y=L} = 0, \tag{4.36}$$

which are equivalent to one of the boundary conditions considered in Ref. [28].

In this setup, the formula in Eq. (3.68) implies that the Jacobian is

$$\log \mathcal{J} = i\kappa_- \int_{y=0} \alpha\,[\mathrm{ch}(F)]_{2r} + i\kappa_+ \int_{y=L} \alpha\,[\mathrm{ch}(F)]_{2r}\,, \tag{4.37}$$

with

$$\kappa_- = \frac{1}{2}\mathrm{sgn}(u) + f(\widetilde{m}_0), \qquad \kappa_+ = \frac{1}{2}\mathrm{sgn}(u') - f(\widetilde{m}_0), \tag{4.38}$$

where $\widetilde{m}_0 \equiv m_0/\Lambda, f(z)$ is the function defined in Eq. (4.9), and α is assumed to be independent of y in the $y < 0$ and $L < y$ regions. When the cut-off Λ is sent to infinity, while keeping m_0 finite, $f(\widetilde{m}_0)$ simply vanishes and we get

$$\kappa_- = \frac{1}{2}\mathrm{sgn}(u), \qquad \kappa_+ = \frac{1}{2}\mathrm{sgn}(u'). \tag{4.39}$$

Note that each term in Eq. (4.37) with Eq. (4.39) is proportional to the anomaly contribution from a Weyl fermion in $2r$ dimensions. However, since the coefficients $\kappa_\pm$ are not integers, it is not possible to interpret this result as the contribution from the Weyl fermions localized at the boundaries. This is because the wave function of the fermions is not completely localized at the boundary in our setup, unless we take the $|\widetilde{m}_0| \to \infty$ limit. One way to understand Eq. (4.39) is to use the anomaly inflow argument given in Sect. 4.1.1. Namely, the anomaly contributions from the modes localized at $y = 0$ and/or $y = L$ are canceled by the bulk CS terms, but the gauge variation of the (half-level) CS terms implies non-vanishing surface terms at $y = \pm\infty$, which gives Eq. (4.37) with Eq. (4.39) as α and F are independent of y for $y < 0$ and $L < y$. On the other hand, one can argue that $\kappa_\pm$ can be shifted as $\kappa_\pm \to \kappa_\pm \pm \beta$ by adding a local counterterm of the form

$$S_{\mathrm{c.t.}} = \beta \int V[\mathrm{ch}(F)]_{2r}, \tag{4.40}$$

where V is the $U(1)$ gauge field, and including its gauge variation in the Jacobian, Eq. (4.37). Therefore, only the combination $\kappa_+ + \kappa_- = \frac{1}{2}(\mathrm{sgn}(u) + \mathrm{sgn}(u'))$ is free from this ambiguity.

It is nonetheless useful to find the anomaly contribution of the localized fermionic zero modes. Assuming that m_0 is very large and the y dependence of the gauge field is negligible,

[23] Strictly speaking, since $\partial_y^2 m$ has delta function singularities at $y = 0$, L, the assumption that we made above Eq. (3.42) is not satisfied. However, it can be shown that these singularities do not contribute and the result is unchanged. Alternatively, one could replace $\mu(y)$ with a smooth function with the same asymptotic behavior as Eq. (4.35), which also gives the same result.

the solutions of the Dirac equation in Eq. (4.4) in the region $0 \leq y < L$ are approximately a linear combination of exponentially increasing and decreasing modes as

$$\psi(\vec{x}, y) \simeq e^{-m_0 y}\psi_+^{(2r)}(\vec{x}) + e^{m_0 y}\psi_-^{(2r)}(\vec{x}), \tag{4.41}$$

where $\psi_\pm^{(2r)}$ satisfies

$$\not{D}^{(2r)}\psi_\pm^{(2r)} = 0, \qquad \gamma^{2r+1}\psi_\pm^{(2r)} = \pm\psi_\pm^{(2r)}. \tag{4.42}$$

Then, the boundary conditions in Eq. (4.36) imply that there are Weyl fermions localized near the boundary with chirality $\mathrm{sgn}(u)$ and $\mathrm{sgn}(u')$ localized around $y = 0$ and $y = L$, if $\mathrm{sgn}(m_0) = \mathrm{sgn}(u)$ and $\mathrm{sgn}(m_0) = -\mathrm{sgn}(u')$, respectively. The anomaly contributions of these localized modes are obtained by formally taking the limit $|\widetilde{m}_0| \to \infty$ in Eq. (4.38),[24] in which we have

$$\kappa_- = \frac{1}{2}(\mathrm{sgn}(u) + \mathrm{sgn}(m_0)), \qquad \kappa_+ = \frac{1}{2}(\mathrm{sgn}(u') - \mathrm{sgn}(m_0)). \tag{4.43}$$

4.2.2. *Even-dimensional cases.* In this subsection we consider a $D = 2r$-dimensional spacetime with boundaries realized by the mass profile

$$m(x) = \mu(y)g(x) = \begin{cases} u'(y-L)g(x) & (L < y), \\ 0 & (0 \leq y \leq L), \\ uyg(x) & (y < 0), \end{cases} \tag{4.44}$$

where $y \equiv x^{2r}$, $g(x) \in U(N)$, and $u, u' \in \mathbb{C}$. Since the phases of u and u' can be absorbed in $g(x)$, we assume $u, u' > 0$ without loss of generality. We take a gauge with $A_{+y} = A_{-y} = 0$ and assume that the gauge fields (A_+, A_-) and $g(x)$ are independent of y in the $y \leq 0$ and $L \leq y$ regions. Since $\mu(y)$ vanishes in the region $0 < y < L$, the $g(x)$ dependence in this region drops out. Therefore, we can choose $g(x)$ to be discontinuous in the region $\epsilon < y < L - \epsilon$ with $0 < \epsilon \ll L$, and the configuration of $g(x)$ at $y = 0$ and $y = L$ can be topologically different.

As discussed around Eq. (4.36) for the odd-dimensional case, by the requirement that the fermion fields do not blow up at $y \to \pm\infty$, the boundary conditions corresponding to the mass profile in Eq. (4.44) are obtained as

$$\left(\gamma^{2r}\psi^g - \psi^g\right)|_{y=0,L} = 0, \tag{4.45}$$

where $\psi^g \equiv \begin{pmatrix} g & \\ & 1 \end{pmatrix}\psi = \begin{pmatrix} g\psi_+ \\ \psi_- \end{pmatrix}$.[25] Therefore, this system can be regarded as that of massless N Dirac fermions on the interval $0 \leq y \leq L$ with a boundary condition as in Eq. (4.45). Note that this boundary condition depends on the spacetime coordinates through $g(x)$. With fixed $g(x)$, the boundary condition in Eq. (4.45) breaks the $U(N)_+ \times U(N)_-$ gauge symmetry down to the $U(N)$ subgroup that consists of elements $(U_+, U_-) \in U(N)_+ \times U(N)_-$ with $U_- = gU_+g^{-1}$. However, as is evident from our construction, the boundary condition in Eq. (4.45) is invariant under the gauge transformation

$$A_+ \to A_+^{U_+}, \qquad A_- \to A_-^{U_-}, \qquad g \to U_-\, g\, U_+^{-1}, \tag{4.46}$$

and it makes sense to consider the anomaly with respect to $U(N)_+ \times U(N)_-$ even at the boundaries.

[24] In this limit, only the localized zero modes are expected to contribute, since the modes with energy greater than Λ are suppressed by the heat kernel regularization in Eq. (3.35).

[25] This type of boundary condition with constant g was introduced in the bag model of hadrons [29,30]. The cases with $g = 1$ or $g = i$ were considered recently in Refs. [28,31–33].

For this configuration, the field strength of the superconnection in Eq. (3.21) becomes

$$\begin{aligned}\mathcal{F} &= \begin{pmatrix} g^{-1} & \\ & 1_N \end{pmatrix} \begin{pmatrix} F_+^g - \widetilde{\mu}^2 1_N & i\left(d\widetilde{\mu}1_N - \left(A_- - A_+^g\right)\widetilde{\mu}\right) \\ i\left(d\widetilde{\mu}1_N + \left(A_- - A_+^g\right)\widetilde{\mu}\right) & F_- - \widetilde{\mu}^2 1_N \end{pmatrix} \begin{pmatrix} g & \\ & 1_N \end{pmatrix} \\ &= \begin{pmatrix} g^{-1} & \\ & 1_N \end{pmatrix} \left(-\widetilde{\mu}^2 1_{2N} + F_+^g e^+ + F_- e^- + id\widetilde{\mu}\sigma_1 + \widetilde{\mu}\left(A_- - A_+^g\right)\sigma_2\right) \begin{pmatrix} g & \\ & 1_N \end{pmatrix},\end{aligned} \tag{4.47}$$

where $\widetilde{\mu} \equiv \mu/\Lambda$, $A_+^g \equiv gA_+g^{-1} + gdg^{-1}$, and $F_+^g \equiv gF_+g^{-1}$. The second line of Eq. (4.47) is written in the notation introduced in Eq. (2.1) with $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$. Then, we obtain

$$\mathrm{Str}(e^{\mathcal{F}}) = e^{-\widetilde{\mu}^2}\mathrm{Str}\left(e^{F_+^g e^+ + F_- e^- + \widetilde{\mu}(A_- - A_+^g)\sigma_2}\left(1 + id\widetilde{\mu}\sigma_1\right)\right), \tag{4.48}$$

and hence the Jacobian in Eq. (3.45) becomes

$$\log\mathcal{J} = -i\int_{0<y<L}\alpha\left[\mathrm{ch}(F_+) - \mathrm{ch}(F_-)\right]_{2r} - i\int_{y=L}\alpha[\omega]_{2r-1} + i\int_{y=0}\alpha[\omega]_{2r-1}, \tag{4.49}$$

where we have assumed that α is independent of y in the $y < 0$ and $L < y$ regions, and defined

$$\omega \equiv i\sum_{r\geq 1}\left(\frac{i}{2\pi}\right)^r \int_0^\infty dt\, e^{-t^2}\left[\mathrm{Str}\left(e^{F_+^g e^+ + F_- e^- + t(A_- - A_+^g)\sigma_2}\sigma_1\right)\right]_{2r-1}. \tag{4.50}$$

This ω is a formal sum of differential forms on the boundaries (i.e. the $y = 0$ and $y = L$ planes). The one-form and three-form components of ω are

$$[\omega]_1 = \frac{i}{2\pi}\mathrm{Tr}\left(A_- - A_+^g\right), \tag{4.51}$$

$$[\omega]_3 = -\frac{1}{8\pi^2}\mathrm{Tr}\left(\left(A_- - A_+^g\right)\left(F_- + F_+^g\right) - \frac{1}{3}\left(A_- - A_+^g\right)^3\right). \tag{4.52}$$

One can show that this is a generalization of CS-forms satisfying

$$d\omega|_{y=0,L} = \left(\mathrm{ch}(F_-) - \mathrm{ch}(F_+)\right)|_{y=0,L}, \tag{4.53}$$

and it is manifestly invariant under the gauge transformation in Eq. (4.46). To show Eq. (4.53), consider the $L \leq y$ region and note that ω at $y = L$ can be written as

$$\omega|_{y=L} = \int_{\{L\leq y\}}\mathrm{ch}(e^{\mathcal{F}}), \tag{4.54}$$

where $\int_{\{L \leq y\}}$ denotes the integration over y with $L \leq y$. Then, applying the exterior derivative $d = d_x + d_y$, where $d_x \equiv \sum_{\mu=1}^{2r-1} dx^\mu\partial_\mu$ and $d_y \equiv dy\,\partial_y$, and using the fact that $\mathrm{ch}(e^{\mathcal{F}})$ is a closed form, we obtain

$$d\omega|_{y=L} = \int_{\{L\leq y\}} d_x\mathrm{ch}(e^{\mathcal{F}}) = -\int_{\{L\leq y\}} d_y\mathrm{ch}(e^{\mathcal{F}}) = -\mathrm{ch}(e^{\mathcal{F}})|_{y=L}, \tag{4.55}$$

which implies Eq. (4.53).

An important observation is that even if the gauge fields are set to zero, Eq. (4.50) can be non-vanishing. In fact, for $A_+ = A_- = 0$, we obtain

$$[\omega]_{2r-1} = \left(\frac{-i}{2\pi}\right)^r\frac{(r-1)!}{(2r-1)!}\mathrm{Tr}((gdg^{-1})^{2r-1}). \tag{4.56}$$

When the spacetime is of the form $S^{2r-1} \times \{y\}$, the integral of this form over S^{2r-1} gives a winding number in $\pi_{2r-1}(U(N))$ represented by the map g: $S^{2r-1} \to U(N)$. If the winding number at $y = 0$ and $y = L$ are the same, a function g: $S^{2r-1} \times \{y\} \to U(N)$ that interpolates the configuration of g at $y = 0$ and $y = L$ can be found and the Jacobian in Eq. (4.49) can be canceled by the gauge variation of a local counterterm,

$$S_{\text{c.t.}} = -\int_{0<y<L} V[\omega]_{2r-1}, \tag{4.57}$$

where V is the $U(1)_V$ gauge field and $[\omega]_{2r-1}$ is given by Eq. (4.56). However, when the winding numbers at $y = 0$ and $y = L$ are different, this is not allowed and there is an anomaly.

Another interesting situation is the case with $g(x) = e^{i\phi(x)} 1_N$ and $A \equiv A_+ = A_-$. In this case, the formula in Eq. (4.50) implies

$$\omega = -\frac{d\phi}{2\pi}\text{ch}(F). \tag{4.58}$$

Therefore, when the spacetime is of the form $S^1 \times S^{2r-2} \times \{y\}$ and the winding number of $e^{i\phi}$ on S^1 for $y = 0$ and $y = L$ are different, there is an anomaly for the $U(1)_V$ symmetry in the presence of a non-vanishing background vector-like gauge field on S^{2r-2}.

4.3. *Index theorems*

From Eq. (3.27) and the first expression in Eq. (3.35), we find that the integral of $\mathcal{I}(x)$ gives the index of operator $\mathcal{D}$:

$$\int d^D x\, \mathcal{I}(x) = n_\varphi - n_\phi = \dim\ker\mathcal{D} - \dim\ker\mathcal{D}^\dagger \equiv \text{Ind}(\mathcal{D}), \tag{4.59}$$

and the result in Eq. (3.44) implies an index theorem written in terms of the superconnection:[26]

$$\text{Ind}(\mathcal{D}) = \int [\text{ch}(\mathcal{F})]_D. \tag{4.60}$$

When we set $m = 0$ and $A_- = 0$ in an even-dimensional case, this formula reduces to a more familiar form of the Atiyah–Singer (AS) index theorem: $\text{Ind}(\not{D}) = \int \text{ch}(F_+)$. Thus, Eq. (4.60) is a generalization of the AS index theorem that which includes spacetime-dependent mass and is supposed to hold even when the spacetime manifold is odd-dimensional and/or non-compact, provided that the spectra of $\mathcal{D}\mathcal{D}^\dagger$ and $\mathcal{D}^\dagger\mathcal{D}$ are discrete.

Here, we discuss some of the implications of this formula. We will not try to make the statements mathematically rigorous.[27] Nevertheless, we hope they are useful and worth mentioning.

4.3.1. *Atiyah–Patodi–Singer index theorem.*

The Atiyah–Patodi–Singer (APS) index theorem [34–36] is an index theorem for a Dirac operator on an even-dimensional manifold N with boundary, stated as

$$\text{Ind}(\not{D}) = \int \text{ch}(F)\widehat{A}(R) - \frac{1}{2}\eta(i\not{D}_b), \tag{4.61}$$

where $\not{D}$ is a Dirac operator on N, and $\eta(i\not{D}_b)$ is the eta invariant of a Dirac operator on the boundary denoted as $\not{D}_b$ (see Eq. (4.65)).[28]

In this subsection we first generalize Eq. (4.61) to include the spacetime-dependent mass m and then apply it to the system considered in Sect. 4.2.2. Let us consider a system as in Sect. 3.1 with $D = 2r$-dimensional spacetime of the form $N = M \times I$, where M is a $(2r-1)$-dimensional

[26] A quick way to get the expression of the index from the results of the Jacobian in the previous sections is to set $\alpha = i$ in $\log\mathcal{J}$, as $\text{Ind}(\mathcal{D}) = \log\mathcal{J}|_{\alpha=i}$.

[27] See, e.g., Ref. [22] for a mathematically rigorous description of index theorems using the superconnection.

[28] See Refs. [37–45] for recent physicist-friendly formulations and derivations. See also Ref. [46] and Appendix A.

manifold with coordinates x^μ ($\mu = 1, 2, \ldots, 2r-1$) and $I = [y_-, y_+] \subset \mathbb{R}$ is an interval parameterized by $y \equiv x^{2r} \in I$. For simplicity, as in the previous sections, we assume M to be flat and the $\widehat{A}$ genus is omitted.

It is convenient to choose σ^μ in Eq. (3.3) such that $\sigma^{2r} = 1_{2^{r-1}}$ and $\sigma^\mu = i\gamma^\mu$ ($\mu = 1, 2, \ldots, 2r - 1$) with γ^μ being the $(2r-1)$-dimensional gamma matrices. Then, the operator $\mathcal{D}$ defined in Eq. (3.3) and its conjugate $\mathcal{D}^\dagger$ can be written as

$$\mathcal{D} = \partial_y + H_y, \qquad \mathcal{D}^\dagger = -\partial_y + H_y \tag{4.62}$$

in the $A_{+y} = A_{-y} = 0$ gauge, where

$$H_y \equiv \begin{pmatrix} -i\not{D}_+^{(2r-1)} & m^\dagger \\ m & i\not{D}_-^{(2r-1)} \end{pmatrix}, \tag{4.63}$$

$$\not{D}_+^{(2r-1)} = \sum_{\mu=1}^{2r-1} \gamma^\mu(\partial_\mu + A_{+\mu}), \qquad \not{D}_-^{(2r-1)} = \sum_{\mu=1}^{2r-1} \gamma^\mu(\partial_\mu + A_{-\mu}). \tag{4.64}$$

Note that although H_y is y-dependent, it does not contain the derivative with respect to y and it can be regarded as a Hermitian operator acting on spinors on M. Here, the mass m can depend on both x^μ and y. When M is non-compact, the mass should diverge at infinity, as in the examples considered in s Sects. 4.1 and 4.2, so that H_y has a discrete spectrum.

The eta invariant of a Hermitian operator H is defined as

$$\eta(H) \equiv \lim_{s\to 0} \eta(s, H), \qquad \eta(s, H) \equiv \frac{2}{\Gamma((s+1)/2)} \int_0^\infty dt\, t^s\, \mathrm{Tr}_{\mathcal{H}} \left(H e^{-t^2 H^2} \right), \tag{4.65}$$

where the trace $\mathrm{Tr}_{\mathcal{H}}$ is over the Hilbert space $\mathcal{H}$ on which the operator H is acting, and the $s \to 0$ limit is taken after analytic continuation of $\eta(s, H)$ on the complex s-plane [34–36]. $\eta(s, H)$ can be written as a sum over eigenvalues λ of H as

$$\eta(s, H) = \sum_\lambda \mathrm{sgn}(\lambda)|\lambda|^{-s}. \tag{4.66}$$

Here and in the following, we assume that H does not have a zero eigenvalue, whenever it is used in $\eta(H)$ or $\eta(s, H)$. For the massless case, the eta invariant of H_y reduces to the difference of the eta invariant of the Dirac operators $i\not{D}_+^{(2r-1)}$ and $i\not{D}_-^{(2r-1)}$:

$$\eta(H_y)|_{m=0} = -\eta\left(i\not{D}_+^{(2r-1)}\right) + \eta\left(i\not{D}_-^{(2r-1)}\right). \tag{4.67}$$

Then, as explained in Appendix A, the index of $\mathcal{D}$ is given by

$$\mathrm{Ind}(\mathcal{D}|_I) = \lim_{\Lambda\to\infty} \int_{y_- < y < y_+} [\mathrm{ch}(\mathcal{F})]_{2r} + \frac{1}{2}\left[\eta(H_y)\right]_{y=y_-}^{y=y_+}, \tag{4.68}$$

where $\mathcal{F}$ is the field strength of the superconnection in Eq. (3.21) with $\Lambda \to \infty$ taken after the integration, $[f(y)]_{y=y_-}^{y=y_+} \equiv f(y_+) - f(y_-)$, and $\mathrm{Ind}(\mathcal{D}|_I)$ denotes the index of the operator $\mathcal{D}$ acting on spinors on $M \times I$ with the following APS boundary conditions. For the operator $\mathcal{D}$, when the wave function at $y = y_\pm$ is expanded with respect to eigenfunctions of $H_{y_\pm}$, the components with the negative (for $y = y_+$) or positive (for $y = y_-$) eigenvalues of $H_{y_\pm}$ have to vanish. The conditions for the operator $\mathcal{D}^\dagger$ are the same as $\mathcal{D}$ with the replacement $H_{y_\pm} \to -H_{y_\pm}$. These boundary conditions follow from the requirement that the wave function of the fermion does not blow up at $y \to \pm\infty$, when the system is extended to the $y < y_-$ and $y_+ < y$ regions with a y-independent configuration for $y \le y_-$ and $y_+ \le y$ (see Appendix A).

Let us apply Eq. (4.68) to the system considered in Sect. 4.2.2. Using Eq. (4.67), the formula in Eq. (4.68) with $[y_-, y_+] = [0, L]$ becomes

$$\mathrm{Ind}(\mathcal{D}|_{[0,L]}) = \int_{0<y<L} [\mathrm{ch}(F_+) - \mathrm{ch}(F_-)]_{2r} - \frac{1}{2}\left[\eta\left(i\not{D}_+^{(2r-1)}\right) - \eta\left(i\not{D}_-^{(2r-1)}\right)\right]_{y=0}^{y=L}, \quad (4.69)$$

which is the APS index theorem for the massless Dirac operator defined by $\mathcal{D}|_{m=0}$ with the APS boundary conditions. On the other hand, for $[y_-, y_+] = [-\infty, +\infty]$ Eq. (4.60) can be used, and from Eq. (4.49) we obtain

$$\mathrm{Ind}(\mathcal{D}) = \int_{0<y<L} [\mathrm{ch}(F_+) - \mathrm{ch}(F_-)]_{2r} + \int_{y=L} [\omega]_{2r-1} - \int_{y=0} [\omega]_{2r-1}. \quad (4.70)$$

This is interpreted as the index theorem for the massless fermions in the interval $[0, L]$ with the boundary condition given by Eq. (4.45).

The difference between Eqs. (4.69) and (4.70) can be evaluated by applying Eq. (4.68) to the cases with $[y_-, y_+] = [L, +\infty]$ and $[-\infty, 0]$ (more precisely, Eqs. (A.12) and (A.13) with $\eta_0 = 0$):

$$\mathrm{Ind}(\mathcal{D}|_{[L,+\infty]}) = \int_{y=L} [\omega]_{2r-1} + \frac{1}{2}\left(\eta\left(i\not{D}_+^{(2r-1)}\right) - \eta\left(i\not{D}_-^{(2r-1)}\right)\right)\Big|_{y=L},$$

$$\mathrm{Ind}(\mathcal{D}|_{[-\infty,0]}) = -\int_{y=0} [\omega]_{2r-1} - \frac{1}{2}\left(\eta\left(i\not{D}_+^{(2r-1)}\right) - \eta\left(i\not{D}_-^{(2r-1)}\right)\right)\Big|_{y=0}. \quad (4.71)$$

In particular, it implies a well-known relation between eta invariant of a Dirac operator and the CS-form ω defined by Eq. (4.50):

$$\int [\omega]_{2r-1} = \frac{1}{2}\left(\eta\left(i\not{D}_-^{(2r-1)}\right) - \eta\left(i\not{D}_+^{(2r-1)}\right)\right) \qquad (\mathrm{mod}\ \mathbb{Z}). \quad (4.72)$$

4.3.2. *Callias-type index theorem.* To illustrate the importance of the mass parameter (or the Higgs field) in the formula in Eq. (4.60), let us consider the case where the gauge fields are turned off. The spacetime manifold is chosen to be a D-dimensional plane $\mathbb{R}^D$, where D can be either even or odd. In order to have a discrete spectrum, we assume that the mass diverges at infinity. To be specific, the asymptotic behavior of the mass is assumed to be

$$\widetilde{m} \to rg(x) \qquad (\text{as } r \to \infty), \quad (4.73)$$

where $r = \sqrt{x_\mu x^\mu}$ is the radial coordinate of $\mathbb{R}^D$ and $g(x) \in U(N)$ is a unitary matrix that only depends on the angular coordinates of $\mathbb{R}^D$. For odd D, $g(x)$ is also required to be Hermitian.[29]

Then, the right-hand side of Eq. (4.60) can be easily evaluated by using Eq. (2.8). The result is

$$\mathrm{Ind}(\mathcal{D}) = \int \mathrm{ch}(\mathcal{F}) = \begin{cases} \left(\dfrac{-i}{2\pi}\right)^{\frac{D}{2}} \dfrac{\left(\frac{D}{2}-1\right)!}{(D-1)!} \displaystyle\int_{S^{D-1}} \mathrm{Tr}\left((gdg^{-1})^{D-1}\right) & (\text{for even } D), \\ \left(\dfrac{i}{8\pi}\right)^{\frac{D-1}{2}} \dfrac{1}{2\left(\frac{D-1}{2}\right)!} \displaystyle\int_{S^{D-1}} \mathrm{Tr}\left((dg)^{D-1}g\right) & (\text{for odd } D), \end{cases} \quad (4.74)$$

where S^{D-1} is the sphere at $r \to \infty$. The former (even D case) is the same as the integral of Eq. (4.56) over S^{D-1}, and the latter (odd D case) agrees with expression of the index for Callias's index theorem [47].

[29] Here, we assume $g(x)$ to be unitary for computational simplicity. However, this condition can be relaxed to $g(x) \in GL(N, \mathbb{C})$, as an invertible matrix (or invertible Hermitian matrix) can be continuously deformed to a unitary matrix (or unitary Hermitian matrix, respectively), keeping the invertibility.

We can apply these formulas for the configuration given by Eq. (4.30), in which $g(x)$ is given by

$$g(x) = \frac{1}{r}\sum_{I=1}^{n} \Gamma^I x^I. \tag{4.75}$$

Inserting this into Eq. (4.74), we obtain $\mathrm{Ind}(\mathcal{D}) = (-1)^{[\frac{D-1}{2}]}$, which is consistent with the fact that there is a fermionic zero mode as suggested in Sect. 4.1.3 from the existence of the anomaly.

5. Relation to string theory

Many of our results have natural interpretations in string theory. In fact, it is well known that the CS terms for unstable D-brane systems (D-brane–anti-D-brane systems and non-Bogomol'nyi–Prasad–Sommerfield (BPS) D-branes) can be written by using superconnections[30] [7–10] as

$$S_{\rm CS}^{\rm D9} = \int C\, \mathrm{ch}(\mathcal{F}), \tag{5.1}$$

where C is a formal sum of Ramond–Ramond (RR) n-form fields (n is even or odd for type IIA or type IIB string theory, respectively) and $\mathcal{F}$ is the field strength of the superconnection for the gauge field and tachyon field on them;[31] it is natural to anticipate the appearance of the superconnection in anomaly analysis of quantum field theory counterparts.

An easy way to realize even-dimensional systems having fermions with manifest chiral symmetry is to consider a Dp-brane ($p = -1, 1, 3, 5, 7$) with D9-branes and $\overline{\rm D9}$ -branes in type IIB string theory [49].[32] On the Dp-brane world-volume, $(p+1)$-dimensional fermions are obtained in the spectrum of p–9 strings and p–$\overline{9}$ strings. Here, a p–p' string is an open string stretched between a Dp-brane and a Dp'-brane, and $\overline{p}$ corresponds to a $\overline{{\rm D}p}$ -brane. It can be shown that p–9 strings and p–$\overline{9}$ strings create positive- and negative-chirality Weyl fermions, respectively. When we have N D9–$\overline{\rm D9}$ pairs, there are N flavors of fermions and the $U(N) \times U(N)$ gauge symmetry associated with the D9–$\overline{\rm D9}$ pairs corresponds to the $U(N)_+ \times U(N)_-$ chiral symmetry for the $(p + 1)$-dimensional system realized on the Dp-brane. The CS term of the D9–$\overline{\rm D9}$ system is written as in Eq. (5.1) with $\mathcal{F}$ being the field strength of the superconnection of the even type, Eq. (2.3), in which A_+ and A_- are the $U(N) \times U(N)$ gauge fields given by 9–9 strings and $\overline{9}$–$\overline{9}$ strings, respectively, and T is the tachyon field obtained by 9–$\overline{9}$ strings. The tachyon field T is in the bifundamental representation of the $U(N) \times U(N)$ symmetry. It couples with the fermions with Yukawa interaction, and the value of the tachyon field plays the role of the mass of the fermions. When $|T| \to \infty$ the fermions decouple, which corresponds to the annihilation of the D9–$\overline{\rm D9}$ pairs.

Similarly, odd-dimensional systems with N Dirac fermions can be obtained by placing a Dp-brane ($p = 0, 2, 4, 6, 8$) with N non-BPS D9-branes in type IIA string theory. In this case, the CS term for the non-BPS D9-branes is given by Eq. (5.1), where $\mathcal{F}$ is the odd type given by Eq. (2.13). Here, A and T in $\mathcal{F}$ are the $U(N)$ gauge field and the tachyon field, respectively, on the non-BPS D9-branes. The tachyon field T is a Hermitian matrix of size N and transforms as the adjoint representation of the $U(N)$ symmetry. There are N Dirac fermions in the spectrum of

[30] As in the previous sections, we omit the terms with curvature represented by the $\widehat{A}$ genus.

[31] See Ref. [48] for a generalization.

[32] A T-dual version (N_c D4-branes with N_f D8–$\overline{\rm D8}$ pairs) is used in Ref. [50] to realize quantum chromodynamics in string theory.

p–9 strings, which are in the fundamental representation of $U(N)$, and the value of the tachyon field corresponds to the mass of the fermions.

Although the CS term in Eq. (5.1) for the unstable D-brane system was originally derived by the computation of the interaction with the RR fields, it can be determined by the requirement of the anomaly cancellation, as argued in Refs. [51–56]. For the brane configuration above, the standard argument shows that the anomaly contribution from the CS term for the unstable D9-branes, Eq. (5.1), and the Dp-brane,

$$S_{\mathrm{CS}}^{\mathrm{D}p} = \int_M C\,\mathrm{ch}(f), \tag{5.2}$$

where M is the $D = p + 1$-dimensional Dp-brane world-volume and $\mathrm{ch}(f) = \exp\left(\frac{i}{2\pi} f\right)$ is the Chern character for the $U(1)$ gauge field on it, is given by the anomaly $(D + 2)$-form of the form[33]

$$2\pi i\,[\mathrm{ch}(\mathcal{F})\mathrm{ch}(f)]_{D+2}\,. \tag{5.3}$$

Note that Eq. (5.3) can be written as $2\pi i\,[\mathrm{ch}(\mathcal{F})]_{D+2}$ by absorbing the $U(1)$ gauge field on the Dp-brane into the $U(1)_V$ part of the gauge field of the unstable D9-brane system. This contribution is supposed to cancel the anomaly contribution from the fermions, which is indeed the case with our proposal in Eqs. (3.18) and (3.56), provided that the tachyon field is identified with the mass as $T = \widetilde{m}$. From dimensional analysis, the cut-off Λ is of the order of the string scale, though the precise relation between Λ and the string length l_s is not clear.

The argument above suggests that the anomaly is characterized by the anomaly $(D + 2)$-form written in terms of the Chern character of the superconnection. However, as discussed in Sects. 4.1.1 and 4.1.2, since the T-dependent part of the anomaly $(D + 2)$-form drops out in the naive use of the anomaly descent relation, it is important to have more evidence for this statement. To this end, let us show that the analysis in Sect. 4.1 is consistent with the D-brane descent relation [57–61].[34]

It is known that a Dq-brane (q is even/odd for type IIA/IIB) localized at $x^I = 0$ ($I = 1, 2, \ldots, 9 - q$) can be realized as a soliton in the unstable D9-brane system by choosing the tachyon field as in Eq. (4.30) with $n \equiv 9 - q$ and $u \to \infty$ [60,61]. In fact, the tachyon configuration with Eq. (4.30) is related to the generator of K-groups $K(\mathbb{R}^n) \simeq \mathbb{Z}$ or $K^1(\mathbb{R}^n) \simeq \mathbb{Z}$ for even or odd n, respectively, given by the Atiyah–Bott–Shapiro construction [64], and these K-groups correspond to the Dq-brane charge. When we have the Dp-brane extended along $x^\mu = 0$ ($\mu = 0, 1, \ldots, p$) with $9 - q \le p$, the Dq-brane corresponds to the codimension $(9 - q)$ interface considered in Sect. 4.1.3 ($q = 8$ and $q = 7$ correspond to the kink and vortex considered in s Sects. 4.1.1 and 4.1.2, respectively).

For this intersecting Dp–Dq system, it can be shown that there is a Weyl fermion localized at the $(p + q - 8)$-dimensional intersection in the spectrum of p–q strings, obtained by quantization of the open string. This is consistent with the analysis of the localized fermionic zero modes in Sect. 4.1.

Furthermore, we can obtain k Dq-branes with $U(k)$ gauge field a on them by choosing the tachyon and gauge fields as in Eqs. (4.32) and (4.33). Then, one can show that the CS term for the Dq-brane is reproduced from Eq. (5.1) by inserting Eqs. (4.32) and (4.33) into Eq. (5.1)

[33] To be more precise, we should consider an anomaly 12-form of the form $2\pi i\left[\mathrm{ch}(\mathcal{F})\mathrm{ch}(f)\delta_{9-p}\right]_{12}$, where δ_{9-p} is a delta function $(9 - p)$-form supported on M.

[34] See, e.g., Refs. [62,63] for reviews.

and integrating over the transverse space [65] (see also Ref. [63]), which corresponds to the procedure in Ref. (4.34). As the anomaly contribution from the CS terms for the Dp-brane and Dq-branes precisely cancels that of the Weyl fermions created by the p–q strings, the anomaly polynomial for these Weyl fermions is given by Eq. (4.34), which is completely parallel to the discussion in Sect. 4.1 for the localized fermionic zero modes.

6. Conclusion

We have investigated the anomaly of fermions with spacetime-dependent mass. It was found in Sect. 3 that the $U(1)_V$ anomaly and the anomaly $(D+2)$-form are written with the Chern character of the superconnection in both even- and odd-dimensional cases as Eqs. (3.56), (3.58), (3.56), and (3.58). Applications of these formulas were discussed in Sect. 4. In Sect. 4.1, we considered the interfaces made by the spacetime-dependent mass on which Weyl fermions are localized and confirmed that our formulas can be used to extract the anomaly of these Weyl fermions. The boundaries of spacetime realized by making the mass large in some regions were studied in Sect. 4.2. A notable example was a system with the spacetime-dependent boundary conditions in Eq. (4.45) considered in Sect. 4.2.2. It was found that there are contributions to the anomaly from the boundaries, even when the gauge fields are turned off. Implications for the index theorems were discussed in Sect. 4.3, in which the AS and APS index theorems for the operator $\mathcal{D}$ defined in Eqs. (3.3) and (3.54) were given, and the application to the Callias-type index theorems was briefly described. Finally, in Sect. 5 we pointed out that the system of fermions with spacetime-dependent mass can be realized in string theory, and our formulas of anomaly are consistent with the anomaly cancellation via the anomaly inflow from the CS term of the unstable D9-brane systems.

We considered here complex Dirac fermions. An obvious interesting problem would be to generalize our discussion to systems with real or pseudo-real fermions, for which there are eight families of theories. For this purpose, the concept of real superconnections and their realization on unstable D-brane systems considered in Ref. [65] would be useful.

Although we have seen that the formulas for the anomaly with the superconnection are quite useful in some applications, we have not explored much on the significance of the superalgebra acting on it. It would be interesting if a deeper meaning behind this structure could be uncovered.[35]

Acknowledgments

We would like to thank S. Aoki, H. Fukaya, M. Honda, H. Shimada, Y. Tanizaki, S. Terashima, and K. Yonekura for useful discussions. We are especially grateful to H. Fukaya and K. Yonekura for valuable comments on a draft of this paper. We also acknowledge useful discussions during the YITP workshop "Topological Phase and Quantum Anomaly 2021" (YITP-T-21-03). In particular, we thank K. Ohmori and M. Yamashita for letting us know about a relevant paper [22] and helpful discussion on index theorems. The work of SS was supported by JSPS KAKENHI (Grant-in-Aid for Scientific Research (B)) grant number JP19H01897. The work of HK was supported by the establishment of university fellowships towards the creation of science technology innovation.

Funding

Open Access funding: SCOAP^3.

[35] See, e.g., Refs. [66–68] for works in this direction.

Appendix A. The APS index theorem

In this appendix we give a heuristic derivation of Eq. (4.68) following the argument given in the appendix of Ref. [46]. The setup is the same as that of Sect. 4.3.1. As mentioned below Eq. (4.68), we extend the system to $-\infty < y < +\infty$ by choosing a y-independent configuration in the regions $y \le y_-$ and $y_+ \le y$.

First, we derive one of the key relations:

$$\int_M d^{2r-1}x\,\mathcal{I}(x) = \frac{1}{\sqrt{\pi}} \lim_{\Lambda\to\infty} \frac{1}{\Lambda} \mathrm{Tr}_{\mathcal{H}} \left((\partial_y H_y)\, e^{-\frac{1}{\Lambda^2}H_y^2} \right). \tag{A.1}$$

Inserting

$$\mathcal{D}^\dagger\mathcal{D} = H_y^2 - \partial_y^2 - \partial_y H_y, \qquad \mathcal{D}\mathcal{D}^\dagger = H_y^2 - \partial_y^2 + \partial_y H_y \tag{A.2}$$

into Eq. (3.35), we obtain

$$\int_M d^{2r-1}x\,\mathcal{I}(x) = \lim_{\Lambda\to\infty} \Lambda \int \frac{d\widetilde{k}}{2\pi}\, e^{-\widetilde{k}^2} \mathrm{Tr}_{\mathcal{H}} \left(e^{\frac{1}{\Lambda^2}\partial_y^2 + \frac{2i}{\Lambda}\widetilde{k}\partial_y - \frac{1}{\Lambda^2}(H_y^2 - \partial_y H_y)} - e^{\frac{1}{\Lambda^2}\partial_y^2 + \frac{2i}{\Lambda}\widetilde{k}\partial_y - \frac{1}{\Lambda^2}(H_y^2 + \partial_y H_y)} \right), \tag{A.3}$$

where $\widetilde{k} = k_{2r}/\Lambda$. As we did around Eq. (3.40), we expand the right-hand side with respect to $1/\Lambda$, regarding $\widetilde{k}$ and H_y/Λ to be of $\mathcal{O}(1)$. The leading term in the $1/\Lambda$ expansion gives Eq. (A.1).

On the other hand, Eq. (4.65) implies

$$\begin{aligned} \partial_y \eta(H_y) &= \frac{2}{\sqrt{\pi}} \int_0^\infty dt\, \mathrm{Tr}_{\mathcal{H}} \left((\partial_y H_y) \left(1 - 2t^2 H_y^2 \right) e^{-t^2 H_y^2} \right) \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty dt\, \partial_t \mathrm{Tr}_{\mathcal{H}} \left(t (\partial_y H_y)\, e^{-t^2 H_y^2} \right) \\ &= -\frac{2}{\sqrt{\pi}} \lim_{\epsilon\to 0} \mathrm{Tr}_{\mathcal{H}} \left(\epsilon (\partial_y H_y)\, e^{-\epsilon^2 H_y^2} \right). \end{aligned} \tag{A.4}$$

Combining this with Eq. (A.1), we obtain

$$-\frac{1}{2}\partial_y \eta(H_y) = \int_M d^{2r-1}x\,\mathcal{I}(x), \tag{A.5}$$

which can be used when H_y does not have a zero eigenvalue.

Let us assume that H_y has zero eigenvalues at finite values of y denoted as y_i ($i = 1, 2, \ldots, k$) with $y_- < y_1 < y_2 < \cdots < y_k < y_+$. From the expression in Eq. (4.66), we see that the value of $\eta(H_y)$ jumps by $+2$ or -2 at $y = y_i$ when one of the eigenvalues of H_y crosses zero from below or above, respectively, while increasing y from $y = y_i - \epsilon$ to $y = y_i + \epsilon$ with a positive small parameter $0 < \epsilon \ll 1$. It is known that the index of the operator $\mathcal{D}$ is equal to half of the sum over these jumps [34–36]:[36]

$$\begin{aligned}\mathrm{Ind}(\mathcal{D}|_I) &= \frac{1}{2}\sum_{i=1}^{k}\big(\eta(H_{y_i+\epsilon})-\eta(H_{y_i-\epsilon})\big)\\ &= \frac{1}{2}\big(\eta(H_{y_+})-\eta(H_{y_-})\big)-\frac{1}{2}\sum_{i=0}^{k}\big(\eta(H_{y_{i+1}-\epsilon})-\eta(H_{y_i+\epsilon})\big)\\ &= \frac{1}{2}\big(\eta(H_{y_+})-\eta(H_{y_-})\big)-\frac{1}{2}\sum_{i=0}^{k}\int_{y_i}^{y_{i+1}} dy\,\partial_y\eta(H_y),\end{aligned} \tag{A.6}$$

where $y_0 \equiv y_-$ and $y_{k+1} \equiv y_+$. Using Eqs. (A.5) and (3.44), we obtain the desired result, Eq. (4.68):

$$\mathrm{Ind}(\mathcal{D}|_I) = \frac{1}{2}\big(\eta(H_{y_+})-\eta(H_{y_-})\big)+\lim_{\Lambda\to\infty}\int_{y_-<y<y_+}[\mathrm{ch}(\mathcal{F})]_{2r}. \tag{A.7}$$

Here, the boundary conditions for the fermions are such that the wave function does not blow up at $y \to \pm\infty$. In these regions, the Dirac equation $\mathcal{D}\psi = 0$ with Eq. (4.62) can be solved by

$$\psi = e^{-\lambda_\pm y}\psi_{\lambda_\pm}, \tag{A.8}$$

where $\psi_{\lambda_\pm}$ is an eigenfunction of $H_{y_\pm}$ with the eigenvalue $\lambda_\pm$. Therefore, the modes with $\lambda_+ < 0$ and $\lambda_- > 0$ are discarded, which gives the APS boundary conditions.

Note that formula Eq. (4.68) is valid only for the finite interval $I = [y_-, y_+]$. When one wishes to apply it for the cases with $y_- \to -\infty$ and/or $y_+ \to +\infty$, one should be careful about the order of the limit $y_\pm \to \pm\infty$ and $\Lambda \to \infty$, because they do not commute when the mass diverges at $y \to \pm\infty$, as we have seen in many examples in Sect. 4. Let us consider a system defined on $M \times \mathbb{R}$ with mass diverging at $y \to \pm\infty$. Suppose $|y_\pm|$ are large enough that H_y does not have a zero eigenvalue for any y satisfying $y < y_-$ or $y_+ < y$. Then, Eq. (A.6) implies that the index $\mathrm{Ind}(\mathcal{D}|_I)$ is the same as that for $I = \mathbb{R}$. Therefore, in this case, comparing Eqs. (4.60) and (4.68), we obtain

$$\frac{1}{2}\eta(H_{y_+})-\lim_{\Lambda\to\infty}\int_{y_+<y}[\mathrm{ch}(\mathcal{F})]_{2r} = \frac{1}{2}\eta(H_{y_-})+\lim_{\Lambda\to\infty}\int_{y<y_-}[\mathrm{ch}(\mathcal{F})]_{2r}. \tag{A.9}$$

Since the field configurations of the left-hand and right-hand sides are independent, we find

$$\eta(H_{y_+}) = 2\lim_{\Lambda\to\infty}\int_{y_+<y}[\mathrm{ch}(\mathcal{F})]_{2r}+\eta_0, \tag{A.10}$$

$$\eta(H_{y_-}) = -2\lim_{\Lambda\to\infty}\int_{y<y_-}[\mathrm{ch}(\mathcal{F})]_{2r}+\eta_0, \tag{A.11}$$

with a field-independent constant η_0. Using these relations, we obtain

$$\mathrm{Ind}(\mathcal{D}|_{[y_-,+\infty]}) = \frac{1}{2}\big(\eta_0-\eta(H_{y_-})\big)+\lim_{\Lambda\to\infty}\int_{y_-<y}[\mathrm{ch}(\mathcal{F})]_{2r}, \tag{A.12}$$

$$\mathrm{Ind}(\mathcal{D}|_{[-\infty,y_+]}) = \frac{1}{2}\big(\eta(H_{y_+})-\eta_0\big)+\lim_{\Lambda\to\infty}\int_{y<y_+}[\mathrm{ch}(\mathcal{F})]_{2r}. \tag{A.13}$$

[36] This fact can be easily understood in the adiabatic limit [69–71], in which H_y is slowly varying with respect to y. In such cases, the Dirac equation $\mathcal{D}\psi = 0$ has an approximate solution of the form $\psi = e^{-\int^y dy\,\lambda}\psi_\lambda$, where ψ_λ is an eigenfunction of H_y with eigenvalue $\lambda(y)$. This solution is normalizable when $\lambda > 0$ and $\lambda < 0$ as $y \to +\infty$ and $y \to -\infty$, respectively. Similarly, a normalizable approximate solution of $\mathcal{D}^\dagger\psi = 0$ is given by $\psi = e^{+\int^y dy\,\lambda}\psi_\lambda$ with $\lambda < 0$ and $\lambda > 0$ as $y \to +\infty$ and $y \to -\infty$, respectively. Therefore, the index is given by the difference of the number of eigenvalues that cross zero from below and above when y is increased from y_- to y_+.

These formulas are formally the same as Eq. (4.68) with $[y_-, y_+]$ replaced with $[y_-, +\infty]$ or $[-\infty, y_+]$, and $\eta(H_{\pm\infty})$ replaced with η_0. Note that the second term on the right-hand side of Eqs. (A.12) and (A.13) is the generalized (gauge-invariant) CS-form given in Eq. (4.54) integrated over M.

For example, let us consider the case with compact M. As a simple field configuration, we choose $A_- = A_+ = 0$ and $m = uy1_N$ with a real non-zero constant u. In this case, we have

$$H_y = \begin{pmatrix} -i\gamma^\mu\partial_\mu & uy \\ uy & i\gamma^\mu\partial_\mu \end{pmatrix}, \qquad H_y^2 = \begin{pmatrix} -\partial^2 + (uy)^2 & 0 \\ 0 & -\partial^2 + (uy)^2 \end{pmatrix}, \tag{A.14}$$

and $\eta(H_y)$ is trivially zero for any $y \neq 0$. This implies $\eta_0 = 0$.

Appendix B. Consistent vs. covariant anomalies

For the massless cases, it is well known that the consistent and covariant anomalies are related by the Bardeen–Zumino counterterm [17]. In this appendix we review the relation between consistent and covariant anomalies, and sketch the derivation of the Bardeen–Zumino counterterms for the cases with spacetime-dependent mass in the covariant anomaly for completeness. Our strategy is to find a counterterm to be added to the covariant anomaly so that it satisfies the Wess–Zumino consistency condition. Note, however, that this approach is not powerful enough to fix the mass dependence of the anomaly $(D+2)$-form for the consistent anomaly. We also point out that anomalous violation of current conservation laws can be written in terms of supermatrix-valued currents.

B.1. Wess–Zumino consistency condition

Let us first introduce the notations for the consistent and covariant anomalies:

$$G(v) \equiv \delta_v \Gamma[A, m], \tag{B.1}$$

$$G^{\mathrm{cov}}(v) \equiv \int_M I_D^{1\,\mathrm{cov}}(v, A, \widetilde{m}), \tag{B.2}$$

where $\Gamma[A, m]$ is the effective action defined in Eq. (3.8), M is the D-dimensional spacetime, and $I_D^{1\,\mathrm{cov}}$ is given in Eqs. (3.19) and (3.57). By definition, the consistent anomaly $G(v)$ satisfies the Wess–Zumino consistency condition [72]

$$\delta_{v_1} G(v_2) - \delta_{v_2} G(v_1) = G([v_1, v_2]). \tag{B.3}$$

On the other hand, it is easy to check from the explicit expression that the covariant anomaly satisfies

$$\delta_{v_1} G^{\mathrm{cov}}(v_2) = G^{\mathrm{cov}}([v_1, v_2]), \tag{B.4}$$

which implies that

$$\delta_{v_1} G^{\mathrm{cov}}(v_2) - \delta_{v_2} G^{\mathrm{cov}}(v_1) = 2G^{\mathrm{cov}}([v_1, v_2]) \tag{B.5}$$

and hence the Wess–Zumino consistency condition is not satisfied.

The claim is that $G(v)$ and $G^{\mathrm{cov}}(v)$ are related (up to surface terms and the gauge variation of local counterterms) by

$$G(v) = G^{\mathrm{cov}}(v) + \alpha(v), \tag{B.6}$$

with

$$\alpha(v) \equiv \left(\frac{i}{2\pi}\right)^{D/2} \int_M \int_0^1 dt\, t \left[\mathrm{Str}^{\mathrm{sym}}\left(\mathscr{D} v\, e^{td\mathcal{A} + t^2\mathcal{A}^2} \mathcal{A}\right)\right]_D, \tag{B.7}$$

where $\mathcal{A}$ is the superconnection in Eqs. (2.1) or (2.12) for even or odd dimensions, respectively, with $T = \widetilde{m} = m/\Lambda$ and

$$\mathscr{D}v \equiv dv + [\mathcal{A}, v] = \delta_v \mathcal{A}. \tag{B.8}$$

Here, $\mathrm{Str}^{\mathrm{sym}}$ denotes the symmetrized supertrace, in which $\mathscr{D}v$, $td\mathcal{A} + t^2\mathcal{A}^2$, and $\mathcal{A}$ are symmetrized (taking into account the sign flip when the odd elements such as $\mathscr{D}v$ and $\mathcal{A}$ are exchanged) before taking the supertrace.

Let us show that the right-hand side of Eq. (B.6) satisfies the Wess–Zumino consistency condition oin Eq. (B.3). For this purpose, it is convenient to rewrite $\alpha(v)$ as

$$\alpha(v) = -\left(\frac{i}{2\pi}\right)^{D/2} \int_{M\times I} \left[\mathrm{Str}\left(\delta_v \widetilde{\mathcal{A}}\, e^{\widetilde{\mathcal{F}}}\right)\right]_{D+1}, \tag{B.9}$$

where $I \equiv [0, 1] \ni t$ and

$$\widetilde{\mathcal{A}} \equiv t\mathcal{A}, \qquad \widetilde{\mathcal{F}} \equiv \widetilde{d}\widetilde{\mathcal{A}} + \widetilde{\mathcal{A}}^2 = td\mathcal{A} + t^2\mathcal{A}^2 + dt\mathcal{A}, \qquad \widetilde{d} \equiv d + dt\frac{\partial}{\partial t}. \tag{B.10}$$

We also define covariant derivatives $\mathscr{D}$ and $\widetilde{\mathscr{D}}$ as

$$\mathscr{D}\eta \equiv d\eta + \mathcal{A}\,\eta - (-1)^{|\eta|}\eta\,\mathcal{A}, \qquad \widetilde{\mathscr{D}}\widetilde{\eta} \equiv \widetilde{d}\widetilde{\eta} + \widetilde{\mathcal{A}}\,\widetilde{\eta} - (-1)^{|\widetilde{\eta}|}\widetilde{\eta}\,\widetilde{\mathcal{A}}, \tag{B.11}$$

where η and $\widetilde{\eta}$ are supermatrix-valued fields in M and $M \times I$, respectively, and $|\eta|$ and $|\widetilde{\eta}|$ denote their fermion numbers (mod 2).[37]

Using the relations

$$\delta_{v_1}\delta_{v_2}\mathcal{A} - \delta_{v_1}\delta_{v_2}\mathcal{A} = \delta_{[v_1, v_2]}\mathcal{A}, \tag{B.12}$$

$$\delta_v\widetilde{\mathcal{F}} = \widetilde{d}\delta_v\widetilde{\mathcal{A}} + \widetilde{\mathcal{A}}\delta_v\widetilde{\mathcal{A}} + \delta_v\widetilde{\mathcal{A}}\widetilde{\mathcal{A}} = \widetilde{\mathscr{D}}\delta_v\widetilde{\mathcal{A}}, \tag{B.13}$$

and the Bianchi identity

$$\widetilde{\mathscr{D}}\widetilde{\mathcal{F}} = \widetilde{d}\widetilde{\mathcal{F}} + \widetilde{\mathcal{A}}\widetilde{\mathcal{F}} - \widetilde{\mathcal{F}}\widetilde{\mathcal{A}} = 0, \tag{B.14}$$

one can show that

$$\begin{aligned}\delta_{v_1}\alpha(v_2) - \delta_{v_2}\alpha(v_1) - \alpha([v_1, v_2]) &= -\left(\frac{i}{2\pi}\right)^{D/2} \int_{M\times I} \mathrm{Str}^{\mathrm{sym}}\left(\widetilde{\mathscr{D}}\left(\delta_{v_1}\widetilde{\mathcal{A}}\,\delta_{v_2}\widetilde{\mathcal{A}}\,e^{\widetilde{\mathcal{F}}}\right)\right)\\ &= -\left(\frac{i}{2\pi}\right)^{D/2} \int_{M\times I} \widetilde{d}\,\mathrm{Str}^{\mathrm{sym}}\left(\delta_{v_1}\widetilde{\mathcal{A}}\,\delta_{v_2}\widetilde{\mathcal{A}}\,e^{\widetilde{\mathcal{F}}}\right).\end{aligned} \tag{B.15}$$

Using Stokes' theorem and dropping the surface terms on the boundary of M,[38] the right-hand side of Eq. (B.15) is evaluated as

$$\begin{aligned}\int_{M\times I} \widetilde{d}\,\mathrm{Str}^{\mathrm{sym}}\left(\delta_{v_1}\widetilde{\mathcal{A}}\,\delta_{v_2}\widetilde{\mathcal{A}}\,e^{\widetilde{\mathcal{F}}}\right) &= \int_M \mathrm{Str}^{\mathrm{sym}}\left(\delta_{v_1}\mathcal{A}\,\delta_{v_2}\mathcal{A}\,e^{\mathcal{F}}\right)\\ &= \int_M \mathrm{Str}^{\mathrm{sym}}\left(\mathscr{D}v_1\mathscr{D}v_2\,e^{\mathcal{F}}\right)\\ &= \int_M \left(d\,\mathrm{Str}^{\mathrm{sym}}\left(v_1\mathscr{D}v_2\,e^{\mathcal{F}}\right) - \mathrm{Str}^{\mathrm{sym}}\left(v_1\mathscr{D}^2v_2\,e^{\mathcal{F}}\right)\right)\\ &= \int_M \mathrm{Str}^{\mathrm{sym}}\left(v_1[v_2, \mathcal{F}]\,e^{\mathcal{F}}\right)\\ &= \int_M \mathrm{Str}\left([v_1, v_2]\,e^{\mathcal{F}}\right),\end{aligned} \tag{B.16}$$

[37] Recall that the differential form dx^μ and $\sigma^\pm$ are treated as fermions. See Sect. 2.

[38] We only keep the parts that contribute to the anomaly $(D+2)$-form for the consistent anomaly.

where we have used

$$D\mathcal{F} = d\mathcal{F} + \mathcal{A}\mathcal{F} - \mathcal{F}\mathcal{A} = 0, \qquad \mathscr{D}^2 v = d\mathscr{D}v + \mathcal{A}\mathscr{D}v + \mathscr{D}v\mathcal{A} = [\mathcal{F}, v]. \tag{B.17}$$

Therefore, we get

$$\delta_{v_1}\alpha(v_2) - \delta_{v_2}\alpha(v_1) - \alpha([v_1, v_2]) = -G^{\rm cov}([v_1, v_2]), \tag{B.18}$$

which implies that the right-hand side of Eq. (B.6) satisfies the Wess–Zumino consistency condition in Eq. (B.3).

In Sect. 3.1.3 we used the fact that there is no difference between the consistent and covariant anomalies for the $U(1)_V$ transformation when the background $U(1)_V$ gauge field V is turned off. This fact can be easily seen from the expression of $\alpha(v)$ in Eq. (B.7). When v is proportional to the unit matrix and the $U(1)_V$ gauge field V is set to zero, $\alpha(v)$ in Eq. (B.7) can be written as

$$\alpha(v) = \int_M \delta_v V \beta(\mathcal{A}_0) = \int_M \delta_v (V\beta(\mathcal{A}_0)), \tag{B.19}$$

where $\mathcal{A}_0 \equiv \mathcal{A}|_{V=0}$ and

$$\beta(\mathcal{A}_0) \equiv \left(\frac{i}{2\pi}\right)^{D/2} \int_0^1 dt\, t \left[\mathrm{Str}^{\rm sym}\left(e^{td\mathcal{A}_0 + t^2\mathcal{A}_0^2}\mathcal{A}_0\right)\right]_{D-1}. \tag{B.20}$$

Therefore, this part can be canceled by the gauge variation of a local counterterm.

B.2. Currents and the Bardeen–Zumino counterterm

The gauge variation of the effective action $\Gamma[A, m]$ can be written as

$$\delta_v \Gamma[A, m] = \int d^D x \left((\mathscr{D}_\mu v)^a J_a^\mu + (\mathscr{D} v)^\alpha J_\alpha\right), \tag{B.21}$$

where

$$J_a^\mu(x) \equiv \frac{\delta\Gamma[A, m]}{\delta A_\mu^a(x)}, \qquad J_\alpha(x) \equiv \frac{\delta\Gamma[A, m]}{\delta \widetilde{m}^\alpha(x)}. \tag{B.22}$$

Here, A_μ^a and $\widetilde{m}^\alpha = m^\alpha/\Lambda$ are the components of the gauge field and the mass rescaled by a constant Λ, and $(\mathscr{D}_\mu v)^a = (\delta_v A_\mu)^a$ and $(\mathscr{D} v)^\alpha = (\delta_v \widetilde{m})^\alpha$ are their infinitesimal gauge variations; (S see Eq. (B.8). J_a^μ and J_α in Eq. (B.22) are the vacuum expectation values of the currents $\delta S/\delta A_a^\mu$ and the fermion bilinear operators $\delta S/\delta \widetilde{m}^\alpha$, respectively. Note that Λ here is just an arbitrary parameter. In fact, Eq. (B.21) does not depend on Λ.

J_a^μ and J_α can be considered as components of a supermatrix-valued current analogous to the superconnection in Eq. (2.1). To see this explicitly, we choose a basis of the supermatrices $\{T_a, T_\alpha\}$ such that the superconnection can be written as $\mathcal{A} = A_\mu^a dx^\mu T_a + \widetilde{m}^\alpha T_\alpha$, and introduce a dual basis $\{T^a, T^\alpha\}$ satisfying

$$\mathrm{Str}(T_a T^b) = \delta_a^b, \qquad \mathrm{Str}(T_\alpha T^\beta) = \delta_\alpha^\beta, \qquad \mathrm{Str}(T_a T^\beta) = 0, \qquad \mathrm{Str}(T_\alpha T^b) = 0. \tag{B.23}$$

A supermatrix-valued current is defined as

$$\mathscr{J}(x) \equiv *J_a^{(1)}(x)\, T^a + *J_\alpha^{(0)}(x)\, T^\alpha, \tag{B.24}$$

where $*$ is the Hodge star operator:

$$*J_a^{(1)}(x) \equiv \frac{1}{(D-1)!}\epsilon_{\mu_1\cdots\mu_D} J_a^{\mu_1}(x)\, dx^{\mu_2}\cdots dx^{\mu_D}, \tag{B.25}$$

$$*J_\alpha^{(0)}(x) \equiv J_\alpha(x)\, dx^1 \cdots dx^D. \tag{B.26}$$

Using this, Eq. (B.21) can be written as

$$\delta_v \Gamma[A, m] = \int \mathrm{Str}(\mathscr{D}v\, \mathscr{J}), \tag{B.27}$$

and the anomaly equation, obtained as the functional derivative of Eq. (B.1) with respect to $v(x)$, becomes

$$*(\mathscr{D}\mathscr{J})_a = -\frac{\delta G(v)}{\delta v^a}, \tag{B.28}$$

which shows that the consistent anomaly $G(v)$ represents the anomalous violation of the current conservation law. For example, for the axial $U(1)$ symmetry (with $v_+ = -v_- = -i\alpha 1_N$) in four dimensions, the left-hand side of Eq. (B.28) becomes

$$*(\mathscr{D}\mathscr{J})_{U(1)_A} = \partial_\mu \left\langle \overline{\psi}\gamma^\mu\gamma^5\psi \right\rangle + 2im\left\langle \overline{\psi}\gamma^5\psi \right\rangle \tag{B.29}$$

and, together with the right-hand side obtained from Eq. (3.22),[39] Eq. (B.28) reduces to the well-known formula for the axial $U(1)$ anomaly.

From the expression in Eq. (B.7), we find that $\alpha(v)$ can be written in the form

$$\alpha(v) = \int_M d^D x \left((\mathscr{D}_\mu v)^a P_a^\mu + (\mathscr{D}v)^\alpha P_\alpha\right) = \int \mathrm{Str}(\mathscr{D}v\,\mathscr{P}), \tag{B.30}$$

where P_a^μ and P_α are local functions of the gauge field and the mass, and $\mathscr{P} \equiv *P_a T^a + *P_\alpha T^\alpha$. Then, the relation in Eq. (B.6) implies that the covariant anomaly is understood as the anomalous violation of conservation laws,

$$*(\mathscr{D}\mathscr{J}^{\mathrm{cov}})_a = -\frac{\delta G^{\mathrm{cov}}(v)}{\delta v^a}, \tag{B.31}$$

for the covariant currents defined by

$$J_a^{\mathrm{cov}\,\mu}(x) \equiv J_a^\mu(x) - P_a^\mu(x), \quad J_\alpha^{\mathrm{cov}}(x) \equiv J_\alpha(x) - P_\alpha(x), \quad \mathscr{J}^{\mathrm{cov}}(x) \equiv \mathscr{J}(x) - \mathscr{P}(x). \tag{B.32}$$

These P_a^μ, P_α, and $\mathscr{P}$ are the Bardeen–Zumino counterterms generalized to include the spacetime-dependent mass.

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