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"Truth-telling Outcomes in a Reputational Cheap-talk Game with Binary Types"

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Truth-telling Outcomes in a Reputational Cheap-talk Game with Binary Types*

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Abstract

Experts with different abilities of information acquisition who receive multiple pieces of signals over time can choose the timing of recommendation and whether to be truthful in a later period, when a recommendation is made in an earlier period. Giving inconsistent recommendations may be seen as a sign of a poor information acquisition ability, but it can also work as a "safety net" that prevents the worst reputation. This study uses a simple binary-ability framework to capture this aspect and proposes equilibriums where all information is delivered truthfully on the path. I examine when such an equilibrium exists, and compare such equilibriums with those where only partial information is delivered; it is found that the former brings higher expected payoffs to the expert than the latter under a certain range of parameters when the utility function is strictly convex in the reputation.

JEL Classification: D72, L14
Key Words: truth-telling, reputation concerns, cheap talks

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1 Introduction

Information acquisition abilities of experts are often not known to the principal and thereby they want to be thought to have access to more precise information. One might naively assume, truthful report is the best response of the expert if the signal is informative about the state of the world because correct report implies the expert is more likely to have high ability. However, in dynamic circumstances it is not necessarily the case. Suppose an expert received one signal and reported it truthfully. When the expert received a second signal that is inconsistent with the first one, making another truthful report may harm the reputation as it indicates the ability is not high enough to produce two correct signals. Would it be always incentive compatible for the expert to report truthfully, if the second signal is consistent with the first one? Surprisingly, it might not be as well: as the expert may want to avoid the worst case – two incorrect signals – and settle for moderate, thus, safer reputation. As Grubb (2011) pointed out, when experts have concerns about reputation, they may withhold their information until the next period. They may want to choose the *timing* and *content* of their advice strategically, while principals want transparent information.

Timing, on one hand, was the main research question discussed in Tajika (2021). He focused on the equilibriums where information is only conveyed in one period; then compared those of early recommendation and of later recommendation and showed a sufficient condition that experts prefer earlier recommendation even though the accuracy of the signal increases over time, making the later recommendation more desirable. It reminds us of Guttman et al. (2014), who compared earlier and later disclosures and concluded later one is interpreted more favorably in equilibriums. The main difference between them are the *content* of the communication. In the former, information is soft, unverifiable, and about how well-informed the expert is. In the latter, information is hard, verifiable, and about how well-performed the expert did. This paper belongs to the first stem. However, it has been already known this is a challenging topic: Ottaviani and Sørensen (2006a,b) showed generic non-existence of equilibrium with full revelation, when the set of the signals is continuous.

Thereby Tajika (2021)'s work is built on discrete signal spaces and asked a question about when is preferable for the experts to give advice when signals arrive sequentially. Consequentially, the discussion is based on the equilibriums in which part of information is missing. In this paper I inherit the model to ask whether and when all the information is conveyed in a timely manner. It makes the incentive conditions in later periods more complicated: in one-shot-recommendation equilibriums, herding and anti-herding effects above-mentioned, and the incentive conflicts by them do not arise. These aspect must be taken into account because, in reality, we see many experts updating their stance over time. When experts makes another recommendation after one, should we expect it is truthful? If so, when is it? If not, why is that?

To answer these questions, I set up a two-period-two-state game with an expert who receives a series of noisy signals whose accuracy indicates the ability of the expert. The expert can choose when and what to tell, through cheap talks under a constraint that recommendations have to be made at least once. At the end of the game, the state is publicly revealed, and the expert gets evaluated and earns the corresponding payoff. The key results are as follows: first, there exists an non-empty set of distributions that sustain the strategy always telling the truth; second, the set of distributions that sustain truth-telling outcomes is strictly larger than the previous set; third, it is also beneficial for the expert with regard to ex ante payoffs if it is strictly convex in the reputation and if the change in accuracy across periods is sufficiently small.

The second result is the highlight of this paper. To explain this, I start with the difference between always truth-telling equilibriums and equilibriums with truth-telling outcomes. Having the expert always tell the truth requires having the expert tell the truth after lying in the previous period. This is more than needed to achieve on-path-truth-telling outcomes as, in such equilibrium, the histories where the expert lied in the first period never come, if the expert played along the strategy. Removing this incentive compatibility condition strictly enlarges the set of distributions that sustain truth-telling outcomes. In what follows I will refer to those equilibriums where only truthful reports are made on path, as truth-telling path equilibriums.

This study is related to reputation-concerned communication, initiated by Holmstrom and Costa (1986) and Holmström (1999), analyzing investment decision by career-concerned managers. Scharfstein and Stein (1990) and Morris (2001) also made a contribution in reputational cheap talks. While Morris (2001) considered that experts care about reputation to make his recommendation more credible, models in Scharfstein and Stein (1990) consider reputation per se. In this vein, recent studies including Ottaviani and Sørensen (2006a,b), Levy (2004) and Tajika (2021) explored further, whereas the last is directly related to this study. This study shows how challenging it is to achieve truthful outcomes when it comes to dynamic, even under an extremely simple circumstance.

It also contributes to the literature that analyzes media behavior. Gentzkow and Shapiro (2006) showed that experts are likely to recommend dishonestly when the prior of the receiver and/or the political stance of the sender are biased. In this study, both sides are not biased as the states are equally likely a priori and the sender does not take any action than to evaluate the accuracy. Still, I have demonstrated that there are incentives to recommend dishonestly, to avoid extremely low reputation.

The implication of these results is clear. Principals who want expert to report truthfully should not expect it would be possible just because it was in a static circumstance. Dynamic circumstance brings herding and anti-herding incentives up. Additionally, rewarding mechanism, that is convex and designed to depend on reputation not only on whether the prediction was correct or not, may enable truthful outcomes that was not possible under a performance-based rewarding mechanism.

This paper is organized as follows: Section 2 describes a model, and Section 3 shows a benchmark case and truth-telling outcomes. Then, analysis will be presented in Section 4. This section includes the comparison with other equilibriums where information is conveyed in only some periods, and the impact of a monetary transfer to the expert after all the available information for the public has been revealed.

2 Model

This is a special case of Tajika (2021)'s model. In particular, I narrow our focus to binary-type cases where the payoff is solely determined by reputation[1]. Consider a two-period game where there exist an expert (he) and an evaluator (she). There are two states in this world, $\omega \in \Omega := \{x, y\}$, that are equally likely. At the start of the game, a state is drawn and it remains fixed throughout the game. It is not known to both players, but the expert privately receives a series of noisy signals about the state through time. The extent of the noise varies depending on his type. Formally, in each period, he receives one piece of information from a signal space, $S = \Omega$. Subscription $t = 1, 2$ is used to describe the period. The expert's type space is also binary, $\theta \in \Theta = \{\theta_H, \theta_L\}$, with $0.5 < \theta_L < \theta_H < 1$, which indicates the accuracy of the signals. Specifically, θ indicates the accuracy of the signal in the first period, $Pr(s_1 = \omega|\omega) = \theta$. The signal gets more precise in the second with the accuracy increased to $Pr(s_2 = \omega|\omega) = (1 + \alpha)\theta/(1 + \alpha\theta) \geq \theta$ with $\alpha \geq 0$. $Pr(s_2 = \omega|\omega)$ is weakly increasing in θ and α and is equal to θ if $\alpha = 0$. Although it is a specific form, it makes the model tractable. The probability of the expert being θ_H is denoted by $\pi \in (0, 1)$. All the information structure and the flow of the game are common knowledge among the players except the type, state, and the realized signals. To keep it simple, I assume that the expert does not know his own type[2].

The expert has a chance to make a recommendation in each period after receiving the signal. He has to give advice to the evaluator at least once. In other words, he can skip the first chance but has to make a recommendation in the final period. If he gave her advice in the first period, he can either make another recommendation or waive the chance to say further. Hence, the message space of the first period can be represented by $R_1 = \{x, y, \emptyset\}$. In the second period, the message space is represented by $R_2 = \{x, y, \emptyset\}$ if he made any informative recommendation[3] in the first stage, and $R_2 \setminus \{\emptyset\}$ otherwise.

[1] I will cover the payoff including a monetary transfer that is considered in Tajika (2021) in Section 4.
[2] This is not an essential assumption on results throughout this study. This will be stated again in Remark 2.
[3] By informative recommendation, I referred to the messages that are not $\emptyset$.

Then, strategy function is given by $r_1 : S \to R_1$, and given $r_1 \in R_1$, $r_2 : S^2 \times R_1 \to R_2 \setminus (\{r_1\} \cap \{\emptyset\})$.

At the end of the final period, the true state becomes public. Both players can observe the true state and whether the advice coincides with the state. The evaluator updates her beliefs, $\beta : R_1 \times R_2 \times \Omega \to \Delta\Theta$, about the type of the expert following Bayes' rule. Off-path beliefs will be given later when needed. The ex post expected accuracy, perceived by the evaluator, is denoted by $\theta_{r_1 r_2 \omega}$, where r_t is the message sent at period $t = 1, 2$. Sometimes superscription will be added to distinguish the belief system under which the expectation is formed. The payoff of the expert is now determined by

$$\Phi(\underbrace{E_\beta[\theta | r_1, r_2, \omega]}_{\theta^{(\beta)}_{r_1 r_2 \omega}}),$$

where Φ is an increasing differentiable function of expected ability. The expert receives higher reward if the evaluator considers him as competent, or, having higher information acquisition ability. The reputational payoffs, however, depends only on the expected ability calculated upon the ex post distribution. For example, two different ex post distributions with the same mean, give the expert the same reputational payoffs.

3 Truth-telling path equilibriums

In this section, equilibriums with truthful recommendations in both periods will be examined. With regard to information structure ordering of Blackwell (1953), this is the most socially desirable equilibrium. A trivial recommendation strategy that induces such outcomes is such that the expert reports the signal he received in the period every time he gets a new piece of information. In what follows, the fully truthful recommendation strategy will be presented (hereinafter, referred to as FT strategy; FT equilibrium refers to equilibriums where FT strategy is played) as a bench mark[4]. It refers to the strategy where the expert recommends honestly at any history. The evaluator forms her beliefs correspondingly. Formally, $r_1^T(s_1) = s_1$, $r_2^T(s_1, s_2; r_1) = s_2$ and

$$\begin{aligned}
\theta^T_{xxx} = \theta^T_{yyy} &= \frac{\theta_L^3(1+\alpha\theta_H)(1-\pi) + \theta_H^3(1+\alpha\theta_L)\pi}{\theta_L^2(1+\alpha\theta_H)(1-\pi) + \theta_H^2(1+\alpha\theta_L)\pi}, \\
\theta^T_{xyx} = \theta^T_{xyy} = \theta^T_{yxy} = \theta^T_{yxx} &= \frac{(1-\theta_L)\theta_L^2(1+\alpha\theta_H)(1-\pi) + (1-\theta_H)\theta_H^2(1+\alpha\theta_L)\pi}{(1-\theta_L)\theta_L(1+\alpha\theta_H)(1-\pi) + (1-\theta_H)\theta_H(1+\alpha\theta_L)\pi}, \\
\theta^T_{xxy} = \theta^T_{yyx} &= \frac{(1-\theta_L)^2\theta_L(1+\alpha\theta_H)(1-\pi) + (1-\theta_H)^2\theta_H(1+\alpha\theta_L)\pi}{(1-\theta_L)^2(1+\alpha\theta_H)(1-\pi) + (1-\theta_H)^2(1+\alpha\theta_L)\pi}
\end{aligned} \tag{1}$$

where $\theta^T_{r_1 r_2 \omega}$ is the Bayes updated expected accuracy under FT strategy after observing r_1, r_2, and ω. The out-of-equilibrium belief is to put 1 on θ_L.

Lemma 1. *For any $\alpha \geq 0$, fully truthful recommendation strategy and the corresponding beliefs form an equilibrium iff*

$$\frac{1}{1+\alpha} \leq \frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx})} \leq 1 + \alpha \tag{2}$$

Proposition 1. *For any $\alpha \geq 0$, and Φ, there exists a distribution of ability where the fully truthful recommendation and the corresponding beliefs form an equilibrium.*

To understand this proposition intuitively, note that $\theta^T_{xyx} = \theta^T_{yxx}$ and $\theta^T_{yxy} = \theta^T_{xyy}$. Inconsistent recommendations lead to the same reputational payoffs, and it does not matter in which period his recommendation matched the true state. Roughly speaking, a moderate, that is, not the best but not the worst, reputational payoff is reserved by simply recommending different messages from the previous period. Suppose the expert recommended truthfully in the first period and received the same signal in

[4]See Tajika (2021).

the second. Sending the truthful message in the second period may cause losing the fixed reputational payoffs if the signals were incorrect. Therefore, to make the expert report truthfully, the reputational payoff when the signals were correct has to be large enough to satisfy the right side of inequality (2), incentivising the expert to recommend truthfully. Suppose now he received the different signal in the second period. His incentive for sending different messages is aligned with recommending truthfully. However, if the payoff for two straight correct recommendation is too large, the expert may want to report consistently, exhibiting herding phenomenon. This corresponds to the left side of inequality (2).

Remark 1. *When $\alpha = 0$, the necessary and sufficient condition for truthful recommendation equilibrium is $\Phi(\theta^T_{xxx}) + \Phi(\theta^T_{xxy}) - 2\Phi(\theta^T_{xyx}) = 0$. Fix α and θ_L. As θ_H approaches 1, θ^T_{xxy} and θ^T_{xyx} approach θ_L. Thus, π must approach 0 for the condition to be satisfied. If θ_H approaches θ_L, the post beliefs must converge to the same value. In this case, any π meet the condition; say $\pi = 0$. By continuity of Φ, for any $\theta_H \in [\theta_L, 1]$ there exists $\pi \in [0, 1]$ that there exists FT equilibrium. When $\alpha > 0$, the set of (θ_H, π) that establish truthful recommendation equilibrium has a positive measure in general.*

Remark 2. *Although equilibriums may still exist even when the expert precisely knows own ability, it becomes more difficult to satisfy the condition for the existence. After either a truthful or dishonest recommendation, the expert with a high ability will be more confident about the signal and have a stronger incentive to make a truthful recommendation, while the expert with a low ability has a relatively weaker incentive to be truthful and is more likely to deviate. The incentive conditions for the low ability expert makes the equilibrium hard to exist.*

3.1 On-Path Truth-telling Recommendation

The FT equilibrium, however, is more demanding than it actually needs to be to lead to an outcome where the expert recommends truthfully in both periods. This is because under the strategy the expert always tell the truth even off the path. As a result, when $\alpha = 0$, it requires $\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx})$ precisely equals to $\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy})$ to be FT equilibrium. In this circumstance, those equilibriums should not be expected to exist in general. Even when $\alpha > 0$, full truthfulness severely restricts the set of distributions that sustain FT equilibrium.

For this reason, in this section, on-path truth-telling recommendation strategy is introduced (hereinafter, referred to as PT; PT equilibrium refers to equilibriums where PT strategy is played). It is the same with truthful recommendation in the first period. If the expert recommended truthfully in the first period, the continuation strategy is the same as that of the truthful recommendation strategy. It is different only when the expert reported untruthfully in the first period. Formally, $r^*_1(s_1) = s_1$, $r^*_2(s_1, s_2; s_1) = s_2$, and

$$r^*_2(x, x; y) = \begin{cases} x & \text{if } \frac{1-p_{xx}}{p_{xx}} \leq \frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx})} \\ y & \text{otherwise} \end{cases} \tag{3}$$

$$r^*_2(x, y; y) = \begin{cases} y & \text{if } \frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx})} \leq 1 + \alpha \\ x & \text{otherwise} \end{cases} \tag{4}$$

$$r^*_2(s_1, s_2; \emptyset) = s_2 \quad \text{for } s_1, s_2 \in \{x, y\}.$$

$r^*_2(y, x; x)$ and $r^*_2(y, y; x)$ are defined analogously. The corresponding beliefs would be (1). If this consists of an equilibrium, the equilibrium path is the same as the one of FT equilibrium.

Proposition 2. *For any $\alpha \geq 0$, and Φ, there exists a distribution of ability where the on-path truthful recommendation and the corresponding beliefs form an equilibrium. A distribution sustains PT equilibriums if and only if*

$$\frac{1}{1+\alpha} \leq \frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx})} \leq \frac{p_{xx}}{1 - p_{xx}}, \tag{5}$$

where $p_{xx} := Pr(\omega = x | s_1 = s_2 = x)$ is the probability of, given the expert having received the same two signals, those signals being correct. Moreover, $1 + \alpha < \frac{p_{xx}}{1-p_{xx}}$.

Under PT strategy, the inequality of the right side of (2) is replaced with a weaker condition maintaining the same outcomes. Hence the set of distributions that sustain truth-telling outcome becomes strictly larger when using PT strategy than using FT. This is especially striking when α approaches to 0. The interval characterized by (2) shrinks to $\{1\}$, which requires $\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx}) = \Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy})$ to establish an FT equilibrium. On the other hand, the interval characterized by (5) has a positive length. In the following section, it is assumed that α is sufficiently small. From the above-mentioned reasons the claims about FT equilibriums built on the assumption might be considered weak because the set of the distributions that sustains FT equilibriums converges to a null set. This concern can be solved by adopting PT equilibrium.

Example 1. *Suppose that $\theta_H = 0.8$, $\theta_L = 0.5$ and $\pi = 0.9$. Suppose $\Phi(\theta) = \theta^2$ and set $\alpha = 1$. Then, there exists a PT but not an FT equilibrium.*

4 Comparisons

4.1 Payoffs

There are other equilibriums with different equilibrium paths. While truth-telling path equilibriums give the society the maximum information, it may not be the best option for experts who care the reputation. Choosing the one with maximum ex ante payoff to the expert seems to be a reasonable equilibrium selection rule. In this section I compare the ex ante payoffs of expert between truth-telling path and the paths where truth-telling occurs in only one period, and messages are ignored in the other period. The beliefs of the evaluator are assumed to be consistent with the expert's strategy.

When Φ is linear, the ex ante payoffs under truth-telling equilibriums and ones of the other equilibriums are all the same. This is obvious because the ex ante distribution of accuracy is merely a mean preserving spread of ex post accuracy. In this section, Φ is assumed to be convex and two strategies where informative messages are sent less than twice will be compared: equilibriums by waiting strategy and equilibriums by consistent strategy. Under waiting strategy, the expert maintains silence in the first period, and waits for a more accurate signal. Formally, $r_1^W(s_1) = \emptyset$ and $r_2^W(r_1, s_1, s_2) = s_2$. A waiting equilibrium is an equilibrium in which waiting strategy is played. It can be readily shown that there certainly exists such an equilibrium: the evaluator updates her beliefs, $\theta^W_{r_1 r_2 \omega}$, defined correspondingly, unless the expert does not send $\emptyset$ in the first period, in which case it is believed that $\theta = \theta_L$. r_2^W is a unique best response to this beliefs.

Proposition 3. *Suppose α is sufficiently small. Then the payoffs of FT (or PT) strategy are greater than those under waiting strategy if Φ is strictly convex.*

Under consistent strategy, the expert makes a truthful recommendation in the first period and repeats the first signal in the second period. In consistent strategy, $r_1^C(s_1) = s_1$ and $r_2^C(s_1, s_2; r_1) = s_1$. Consistent equilibrium is defined and is shown to exist in a similar way.

Proposition 4. *Suppose α is sufficiently small. Then the payoffs of FT (or PT) strategy are greater than those under consistent strategy if Φ is strictly convex.*

4.2 Monetary Transfer

So far, it has been assumed that the expert yields payoffs that are solely determined by reputation. In many cases, however, experts are employed by firms and paid for their expertise. In what follows, a monetary reward is given to the expert if the last recommendation matches the true state. The payoff will be described as

$$K\mathbb{1}(r_2 = \omega) + \Phi(E_\beta[\theta | r_1, r_2, \omega]),$$

where the first term indicates the monetary rewards with $K \geq 0$. The reward only depends on the final recommendation and the true state regardless of whether he changed his words; although the model does not specifically depict the evaluator's action, she might make an action after receiving all the recommendation provided, which gives a positive payoff to the evaluator if the action matches the true state. Given that those recommendations are believed to be honest, she will follow the final recommendation as the second signal is more precise than the first. In such cases, K can be seen as a monetary transfer from her.

Denote by q_{cc}, q_{ci}, q_{ic}, and q_{ii},

$$\begin{aligned} q_{cc} &= Pr(\omega = s_2 = s_1 | s_1) = \pi \frac{\theta_H^2(1+\alpha)}{1+\alpha\theta_H} + (1-\pi)\frac{\theta_L^2(1+\alpha)}{1+\alpha\theta_L} \\ q_{ci} &= Pr(\omega = s_1 \neq s_2 | s_1) = \pi \frac{\theta_H(1-\theta_H)}{1+\alpha\theta_H} + (1-\pi)\frac{\theta_L(1-\theta_L)}{1+\alpha\theta_L} \\ q_{ii} &= Pr(s_2 = s_1 \neq \omega | s_1) = \pi \frac{(1-\theta_H)^2}{1+\alpha\theta_H} + (1-\pi)\frac{(1-\theta_L)^2}{1+\alpha\theta_L}, \text{ and} \\ q_{ic} &= Pr(\omega = s_2 \neq s_1 | s_1) = \pi \frac{\theta_H(1-\theta_H)(1+\alpha)}{1+\alpha\theta_H} + (1-\pi)\frac{\theta_L(1-\theta_L)(1+\alpha)}{1+\alpha\theta_L} \equiv (1+\alpha)q_{ci}, \end{aligned} \tag{6}$$

respectively. These are the probabilities of the signals being correct conditional on the first signal; the subscript i stands for "incorrect," and c stands for "correct." Also note that because the states are equally likely, the realization of the first signal does not change the ex ante probability of the second signal being correct. In other words, q_{cc} is also equal to $Pr(\omega = s_2 = s_1)$ and the others are analogous. Additionally, let $q_c := q_{cc} + q_{ci} \equiv Pr(\omega = s_1|s_1)$ and $q_i := q_{ic} + q_{ii} \equiv Pr(\omega \neq s_1|s_1)$. These are the probabilities that the first signal is correct and incorrect, conditional on it, respectively.

Proposition 5. *For any $\alpha \geq 0$, fully truthful recommendation strategy and the corresponding beliefs form an equilibrium iff*

$$\frac{1}{1+\alpha} \leq \frac{\Phi(\theta_{xyx}^T) - \Phi(\theta_{xxy}^T) + K}{\Phi(\theta_{xxx}^T) - \Phi(\theta_{xyx}^T) + K} \leq 1+\alpha. \tag{7}$$

Proposition 6. *For any $\alpha, K \geq 0$, on-path truthful recommendation strategy and the corresponding beliefs form an equilibrium iff*

$$\frac{1}{1+\alpha} \leq \frac{\Phi(\theta_{xyx}^T) - \Phi(\theta_{xxy}^T) + K}{\Phi(\theta_{xxx}^T) - \Phi(\theta_{xyx}^T) + K} \leq \frac{p_{xx}}{1-p_{xx}} - \frac{q_c - q_i}{q_{ii}} \frac{K}{\Phi(\theta_{xxx}^T) - \Phi(\theta_{xyx}^T) + K}. \tag{8}$$

Furthermore,

$$1+\alpha < \frac{p_{xx}}{1-p_{xx}} - \frac{q_c - q_i}{q_{ii}} \frac{K}{\Phi(\theta_{xxx}^T) - \Phi(\theta_{xyx}^T) + K}. \tag{9}$$

The impact of $K \geq 0$ in FT equilibriums is straight forward. As K increases, the (LHS) of the right inequality of (7) in Proposition 5 monotonically approaches to 1. Although Proposition 6 seems more complicated, the right inequality of (8) is equivalent to

$$q_{cc}(\Phi(\theta_{xxx}^T) - \Phi(\theta_{xyx}^T)) + q_{ii}(\Phi(\theta_{xxy}^T) - \Phi(\theta_{xyx}^T)) + \underbrace{(q_{ic} - q_{ci})}_{\geq 0} K \geq 0, \tag{10}$$

which holds for any $K \geq 0$ if it does under $K = 0$. Hence both (7) and (9) hold if K becomes sufficiently large: it is consistent with the intuition. The second claim of Proposition 6 corresponds with that of Proposition 2. It says that the inclusion relation between the sets that sustain truth-telling outcomes under two different strategies is robust to the monetary transfer.

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A Proofs

Proof of Lemma 1. Suppose the expert played according to r_1. Then, the expected payoffs of playing r_2 are given as

$$\begin{aligned} &p_{xx}\Phi(\theta^T_{xxx}) + (1-p_{xx})\Phi(\theta^T_{xxy}) \quad \text{if } s_1 = s_2,\\ &p_{yx}\Phi(\theta^T_{xyy}) + (1-p_{yx})\Phi(\theta^T_{xyx}) \quad \text{if } s_1 \neq s_2,\end{aligned}$$

where

$$\begin{aligned} p_{xx} &= \frac{\theta_H^2(1+\alpha)}{\theta_H^2(1+\alpha)+(1-\theta_H)^2}\frac{\left(\frac{\theta_H^2(1+\alpha)}{1+\alpha\theta_H}+\frac{(1-\theta_H)^2}{1+\alpha\theta_H}\right)\pi}{\left(\frac{\theta_H^2(1+\alpha)}{1+\alpha\theta_H}+\frac{(1-\theta_H)^2}{1+\alpha\theta_H}\right)\pi+\left(\frac{\theta_L^2(1+\alpha)}{1+\alpha\theta_L}+\frac{(1-\theta_L)^2}{1+\alpha\theta_L}\right)(1-\pi)}\\ &\quad+\frac{\theta_L^2(1+\alpha)}{\theta_L^2(1+\alpha)+(1-\theta_L)^2}\frac{\left(\frac{\theta_L^2(1+\alpha)}{1+\alpha\theta_L}+\frac{(1-\theta_L)^2}{1+\alpha\theta_L}\right)(1-\pi)}{\left(\frac{\theta_H^2(1+\alpha)}{1+\alpha\theta_H}+\frac{(1-\theta_H)^2}{1+\alpha\theta_H}\right)\pi+\left(\frac{\theta_L^2(1+\alpha)}{1+\alpha\theta_L}+\frac{(1-\theta_L)^2}{1+\alpha\theta_L}\right)(1-\pi)},\\ &= \frac{\frac{\theta_H^2(1+\alpha)}{1+\alpha\theta_H}\pi+\frac{\theta_L^2(1+\alpha)}{1+\alpha\theta_L}(1-\pi)}{\left(\frac{\theta_H^2(1+\alpha)}{1+\alpha\theta_H}+\frac{(1-\theta_H)^2}{1+\alpha\theta_H}\right)\pi+\left(\frac{\theta_L^2(1+\alpha)}{1+\alpha\theta_L}+\frac{(1-\theta_L)^2}{1+\alpha\theta_L}\right)(1-\pi)}, \qquad (11)\\ p_{yy} &= \frac{\frac{(1-\theta_H)^2}{1+\alpha\theta_H}\pi+\frac{(1-\theta_L)^2}{1+\alpha\theta_L}(1-\pi)}{\left(\frac{\theta_H^2(1+\alpha)}{1+\alpha\theta_H}+\frac{(1-\theta_H)^2}{1+\alpha\theta_H}\right)\pi+\left(\frac{\theta_L^2(1+\alpha)}{1+\alpha\theta_L}+\frac{(1-\theta_L)^2}{1+\alpha\theta_L}\right)(1-\pi)} = 1-p_{xx}, \qquad (12)\\ p_{xy} &= \frac{1}{2+\alpha}, \qquad (13)\\ p_{yx} &= \frac{1+\alpha}{2+\alpha} = 1-p_{xy}, \qquad (14)\end{aligned}$$

$p_{s_1 s_2} := Pr(\omega = x|s_1, s_2)$ is the probability of the state being x after receiving s_1 and s_2 as the first and the second signal respectively. Note that $p_{xx} > p_{yx} \geq p_{xy} > p_{yy}$.

If the expert lies in the second period, that is, $r_2 \neq s_1$ the expected payoffs are

$$\begin{aligned} &p_{xx}\Phi(\theta^T_{xyx}) + (1-p_{xx})\Phi(\theta^T_{xyy}) \quad \text{if } s_1 = s_2, \\ &p_{yx}\Phi(\theta^T_{xxy}) + (1-p_{yx})\Phi(\theta^T_{xxx}) \quad \text{if } s_1 \neq s_2 \end{aligned} \tag{15}$$

For the continuation strategy to recommend truthfully in the second period to be optimal, the following has to hold:

$$\frac{p_{xx}}{1-p_{xx}} \geq \frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx})} \geq \frac{1-p_{yx}}{p_{yx}} = \frac{1}{1+\alpha} \tag{16}$$

Now suppose the expert recommended the opposite signal from what he received in the first period. For the expert to recommend truthfully in the second period, the following two inequalities are required:

$$\begin{aligned} p_{xx}\Phi(\theta^T_{xyy}) + (1-p_{xx})\Phi(\theta^T_{xyx}) \geq p_{xx}\Phi(\theta^T_{xxy}) + (1-p_{xx})\Phi(\theta^T_{xxx}) \\ p_{yx}\Phi(\theta^T_{xxx}) + (1-p_{yx})\Phi(\theta^T_{xxy}) \geq p_{yx}\Phi(\theta^T_{xyx}) + (1-p_{yx})\Phi(\theta^T_{xyy}) \end{aligned} \tag{17}$$

It is now equal to

$$\frac{p_{xx}}{1-p_{xx}} \geq \left(\frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx})}\right)^{-1} \geq \frac{1}{1+\alpha} \tag{18}$$

Combining (16) and (18), it has to be

$$\min\left\{\frac{p_{xx}}{1-p_{xx}}, 1+\alpha\right\} \geq \frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx})} \geq \max\left\{\frac{1-p_{xx}}{p_{xx}}, \frac{1}{1+\alpha}\right\} \tag{19}$$

Considering $1+\alpha = p_{xy}/(1-p_{xy})$ and $p_{xx} > p_{xy} = 1 - p_{yx}$, (19) results in (2).

It remains to show that, given $r^T_2(s_1, s_2; r_1) = s_2$, the optimal behavior for the expert in the first period is to recommend truthfully. As the monetary rewards are only determined by the second recommendation, only the reputational payoffs will be taken into account. It is obvious that there is no incentive to send $r_1 = \emptyset$, because the evaluator believes the expert is of type θ_L with probability 1.

For truthful recommendation to be the optimal behavior given the second period strategy, r^T_2, it needs to be satisfied that

$$\begin{aligned} &q_{cc}\Phi(\theta^T_{xxx}) + q_{ii}\Phi(\theta^T_{xxy}) + q_{ci}\Phi(\theta^T_{xyx}) + q_{ic}\Phi(\theta^T_{xyx}) \\ &\geq q_{cc}\Phi(\theta^T_{xyx}) + q_{ii}\Phi(\theta^T_{xyx}) + q_{ci}\Phi(\theta^T_{xxy}) + q_{ic}\Phi(\theta^T_{xxx}). \end{aligned} \tag{20}$$

It is summarized as

$$\begin{aligned} &\underbrace{(1+\alpha)\left\{\pi\frac{\theta_H(2\theta_H - 1)}{1+\alpha\theta_H} + (1-\pi)\frac{\theta_L(2\theta_L-1)}{1+\alpha\theta_L}\right\}}_{>0}(\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyy})) \\ &\geq \underbrace{\left\{\pi\frac{(1-\theta_H)(1-2\theta_H)}{1+\alpha\theta_H} + (1-\pi)\frac{(1-\theta_L)(1-2\theta_L)}{1+\alpha\theta_L}\right\}}_{<0}(\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy})). \end{aligned} \tag{21}$$

It always holds that, as $1 > \theta_H > \theta_L > 0.5$, the (LHS) is always positive and the (RHS) is always negative. □

Proof of Proposition 1. Lemma 1 shows that a sufficient condition for a truthful equilibrium to hold is

$$\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx}) = \Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy}) \tag{22}$$

It is obvious that, when there is only one type of accuracy, the ex post belief would not change. In other words, for any fixed θ_H and θ_L, if $\pi = 0$ or $\pi = 1$, $\theta^T_{xxx} = \theta^T_{xyx} = \theta^T_{xxy}$; hence, $\Phi(\theta^T_{xxx}) + \Phi(\theta^T_{xxy}) - 2\Phi(\theta^T_{xyx}) = 0$. On the other hand, as Φ is a differentiable increasing function, the derivatives of $\Phi(\theta^T_{xxx}) + \Phi(\theta^T_{xxy}) - 2\Phi(\theta^T_{xyx})$ at $\pi = 0$ and $\pi = 1$ is positive if and only if,

$$\left.\frac{\mathrm{d}\,\theta^T_{xxx}}{\mathrm{d}\,\pi}\right|_{\pi=0} + \left.\frac{\mathrm{d}\,\theta^T_{xxy}}{\mathrm{d}\,\pi}\right|_{\pi=0} - 2\left.\frac{\mathrm{d}\,\theta^T_{xyx}}{\mathrm{d}\,\pi}\right|_{\pi=0} = \frac{(\theta_H - \theta_L)^3(1+\alpha\theta_L)}{(1-\theta_L)^2\theta_L(1+\alpha\theta_H)} > 0$$
$$\left.\frac{\mathrm{d}\,\theta^T_{xxx}}{\mathrm{d}\,\pi}\right|_{\pi=1} + \left.\frac{\mathrm{d}\,\theta^T_{xxy}}{\mathrm{d}\,\pi}\right|_{\pi=1} - 2\left.\frac{\mathrm{d}\,\theta^T_{xyx}}{\mathrm{d}\,\pi}\right|_{\pi=1} = \frac{(\theta_H - \theta_L)^3(1+\alpha\theta_H)}{(1-\theta_H)^2\theta_H(1+\alpha\theta_L)} > 0.$$

Therefore, by the intermediate value theorem, there must exist $\pi^* \in (0,1)$ to make $\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xxy}) - 2\Phi(\theta^T_{xyx}) = 0$. It fulfills the sufficient condition for the existence of truthful equilibrium and if $\alpha > 0$, there is an interval around π^*. □

Proof of Proposition 2. Assume that the expert reports honestly in the first period. Without loss of generality, suppose $s_1 = r_1 = y$. If $s_2 = x$ (or $s_2 = y$), the requirement for the expert to be truthful corresponds to the inequality, the right (or left) side of (16). This criterion corresponds to the definition of $r^*_2(x, x; y)$ in (3) (or $r^*_2(x, y; y)$ in (4)). Hence, if there is a deviation strategy that is profitable, it has to be such that the expert lies in the first period for some signals. Arbitrarily fix such $\hat{s}_1 \in \{x, y\}$ that the expert report $\bar{r}_1 \neq \hat{s}_1$. Let $\bar{s}_2 \in \{x, y\}$ be the different with $\hat{s}_1$. Note that, $\bar{r}_1 = \bar{s}_2$.

If the expert received $s_2 = \hat{s}_1$ in the following period, it is the best response to report truthfully. This is because the left side of inequality of (18), the condition for the expert to do so, is strictly weaker than the left side of inequality of (5). Consider r' and r'' such that $r'_1(\hat{s}_1) = r''_1(\hat{s}_1) = \bar{r}_1$. Let

$$r'_2(\hat{s}_1, \bar{s}_2; \bar{r}_1) = \bar{s}_2$$

and

$$r''_2(\hat{s}_1, \bar{s}_2; \bar{r}_1) = \hat{s}_1.$$

Any deviation strategy would be a form of either r' and r''. However, r' after receiving $\hat{s}_1$, is such that the expert lies in the first period; and tell the truth in the second period. It has been already shown in the proof of Lemma 1 that telling the truth is a best response for the expert as long as the right inequality of (5) holds.

r'', on the other hand, after receiving $\hat{s}_1$ is such that the expert lies in the first period; tell the truth if the second signal is consistent with the first one; and keep lying if it is inconsistent. However, this gives the same consequence with giving different recommendations. Suppose $s_1 = x$. If the signal history is $(s_1, s_2) = (x, x)$, then the report history will be $(r_1, r_2) = (y, x)$. If the signal history is $(s_1, s_2) = (x, y)$, then the report history still will be $(r_1, r_2) = (y, x)$. The expert gets better off by playing r^* than to playing r'' if

$$\begin{aligned} &q_{cc}\Phi(\theta^T_{xxx}) + q_{ii}\Phi(\theta^T_{xxy}) + q_{ci}\Phi(\theta^T_{xyx}) + q_{ic}\Phi(\theta^T_{xyx}) \\ \geq &\Phi(\theta^T_{xyx}) \equiv q_{cc}\Phi(\theta^T_{xyx}) + q_{ii}\Phi(\theta^T_{xyx}) + q_{ci}\Phi(\theta^T_{xyx}) + q_{ic}\Phi(\theta^T_{xyx}), \end{aligned} \tag{23}$$

which is equivalent to

$$\frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx})} \leq \frac{q_{cc}}{q_{ii}}. \tag{24}$$

From (11) and (6), $q_{cc}/q_{ii} = p_{xx}/(1 - p_{xx})$. Then it is equivalent to the left inequality of (5). The last part is already proven in the proof of Lemma 1.

The discussion above shows that (5) is a sufficient condition for r^* to form a PBE. As to the reverse, the first paragraph of this proposition implies that it is a necessary condition. □

Proof of Proposition 3. The expected accuracy corresponding to waiting strategy and expected reputational payoffs are given by

$$\begin{aligned}\theta^W_{\emptyset xx} &= \frac{\theta_H^2(1+\alpha\theta_L)\pi + \theta_L^2(1+\alpha\theta_H)(1-\pi)}{\theta_H(1+\alpha\theta_L)\pi + \theta_L(1+\alpha\theta_H)(1-\pi)} \\ \theta^W_{\emptyset xy} &= \frac{\theta_H(1-\theta_H)(1+\alpha\theta_L)\pi + \theta_L(1-\theta_L)(1+\alpha\theta_H)(1-\pi)}{(1-\theta_H)(1+\alpha\theta_L)\pi + (1-\theta_L)(1+\alpha\theta_H)(1-\pi)} \\ \theta^W_{r_1r_2\omega} &= \theta_L \text{ if } r_1 \neq \emptyset\end{aligned} \tag{25}$$

and

$$F^W(\alpha,\Phi) = \left(\frac{(1+\alpha)\theta_H}{1+\alpha\theta_H}\pi + \frac{(1+\alpha)\theta_L}{1+\alpha\theta_L}(1-\pi)\right)\Phi(\theta^W_{\emptyset xx}) + \left(\frac{1-\theta_H}{1+\alpha\theta_H}\pi + \frac{1-\theta_L}{1+\alpha\theta_L}(1-\pi)\right)\Phi(\theta^W_{\emptyset xy}) \tag{26}$$

The expected reputational payoffs of truthful recommendation strategy are

$$F^T(\alpha,\Phi) = q_{cc}\Phi(\theta_{xxx}) + q_{ii}\Phi(\theta_{xxy}) + (q_{ic}+q_{ci})\Phi(\theta_{xyx}))$$

Suppose $\alpha = 0$. Then $\theta^W_{r_1r_2\omega}|_{\alpha=0}$ and $\theta^T_{r_1r_2\omega}|_{\alpha=0}$ will be used to calculate the difference of the payoffs compared to waiting strategy.

$$\begin{aligned}F^T(0,\Phi) - F^W(0,\Phi) =&(\pi(1-\theta_H)^2 + (1-\pi)(1-\theta_L)^2)\Phi(\theta^T_{xxy}|_{\alpha=0}) \\ &+ 2(\pi\theta_H(1-\theta_H) + (1-\pi)\theta_L(1-\theta_L))\Phi(\theta^T_{xyx}|_{\alpha=0}) \\ &+ (\pi\theta_H^2 + (1-\pi)\theta_L^2)\Phi(\theta^T_{xxx}|_{\alpha=0}) \\ &- (\pi\theta_H + (1-\pi)\theta_L)\Phi(\theta^W_{\emptyset xx}|_{\alpha=0}) \\ &- (\pi(1-\theta_H) + (1-\pi)(1-\theta_L))\Phi(\theta^W_{\emptyset xy}|_{\alpha=0}).\end{aligned} \tag{27}$$

As the ex post beliefs are more dispersed under truthful recommendation than waiting strategy, $\theta^W_{\emptyset xx}$ and $\theta^W_{\emptyset xy}$ can be expressed by a convex combination of θ^T_{xxy}, θ^T_{xyx}, and θ^T_{xxx}. They can be rewritten as

$$\theta^W_{\emptyset xx}|_{\alpha=0} = \frac{\theta_H^2\pi + \theta_L^2(1-\pi)}{\theta_H\pi + \theta_L(1-\pi)}\theta^T_{xxx}|_{\alpha=0} + \left(1 - \frac{\theta_H^2\pi + \theta_L^2(1-\pi)}{\theta_H\pi + \theta_L(1-\pi)}\right)\theta^T_{xyx}|_{\alpha=0} \tag{28}$$

$$\theta^W_{\emptyset xy}|_{\alpha=0} = \frac{\theta_H(1-\theta_H)\pi + \theta_L(1-\theta_L)(1-\pi)}{(1-\theta_H)\pi + (1-\theta_L)(1-\pi)}\theta^T_{xyx}|_{\alpha=0} \tag{29}$$

$$+ \left(1 - \frac{\theta_H(1-\theta_H)\pi + \theta_L(1-\theta_L)(1-\pi)}{(1-\theta_H)\pi + (1-\theta_L)(1-\pi)}\right)\theta^T_{xxy}|_{\alpha=0} \tag{30}$$

By Jensen's inequality, the followings hold if Φ is strictly convex:

$$\begin{aligned}\Phi(\theta^W_{\emptyset xx}|_{\alpha=0}) &< \frac{\theta_H^2\pi + \theta_L^2(1-\pi)}{\theta_H\pi + \theta_L(1-\pi)}\Phi(\theta^T_{xxx}|_{\alpha=0}) + \left(1 - \frac{\theta_H^2\pi + \theta_L^2(1-\pi)}{\theta_H\pi + \theta_L(1-\pi)}\right)\Phi(\theta^T_{xyx}|_{\alpha=0}) \\ \Phi(\theta^W_{\emptyset xy}|_{\alpha=0}) &< \frac{\theta_H(1-\theta_H)\pi + \theta_L(1-\theta_L)(1-\pi)}{(1-\theta_H)\pi + (1-\theta_L)(1-\pi)}\Phi(\theta^T_{xyx}|_{\alpha=0}) \\ &+ \left(1 - \frac{\theta_H(1-\theta_H)\pi + \theta_L(1-\theta_L)(1-\pi)}{(1-\theta_H)\pi + (1-\theta_L)(1-\pi)}\right)\Phi(\theta^T_{xxy}|_{\alpha=0})\end{aligned} \tag{31}$$

Hence, if Φ is strictly convex,

$$\begin{aligned}
&F^T(0,\Phi) - F^W(0,\Phi)\\
&= F^T(0,\Phi) - ((\pi\theta_H + (1-\pi)\theta_L)\Phi(\theta^W_{\emptyset xx}|_{\alpha=0}) + (\pi(1-\theta_H) + (1-\pi)(1-\theta_L))\Phi(\theta^W_{\emptyset xy}|_{\alpha=0}))\\
&> F^T(0,\Phi)\\
&\quad - (\pi\theta_H + (1-\pi)\theta_L)\left\{\frac{\theta_H^2\pi + \theta_L^2(1-\pi)}{\theta_H\pi + \theta_L(1-\pi)}\Phi(\theta^T_{xxx}|_{\alpha=0}) + (1 - \frac{\theta_H^2\pi + \theta_L^2(1-\pi)}{\theta_H\pi + \theta_L(1-\pi)})\Phi(\theta^T_{xyx}|_{\alpha=0})\right\}\\
&\quad - (\pi(1-\theta_H) + (1-\pi)(1-\theta_L)) \times \left\{\frac{\theta_H(1-\theta_H)\pi + \theta_L(1-\theta_L)(1-\pi)}{(1-\theta_H)\pi + (1-\theta_L)(1-\pi)}\Phi(\theta^T_{xyx}|_{\alpha=0})\right.\\
&\qquad\qquad\qquad \left. + \left(1 - \frac{\theta_H(1-\theta_H)\pi + \theta_L(1-\theta_L)(1-\pi)}{(1-\theta_H)\pi + (1-\theta_L)(1-\pi)}\right)\Phi(\theta^T_{xxy}|_{\alpha=0})\right\}
\end{aligned} \tag{32}$$

After calculation, the (RHS) equals to 0. By continuity of $F^T(\alpha,\cdot)$ and $F^W(\alpha,\cdot)$ in α, the inequality holds when α is sufficiently small as well. □

Proof of Proposition 4. Note that consistent recommendation is essentially equal to recommending only once in the first period. If α is negligibly small, the expert receives signals of the same quality, hence, the evaluator's beliefs would not depend on the timing of recommendation. If $\alpha = 0$, $\theta^C_{xxx} = \theta^W_{\emptyset xx}$ and $\theta^C_{xxy} = \theta^W_{\emptyset xy}$. Suppose $\alpha = 0$. The expected reputational payoff under consistent recommendation is given by

$$\begin{aligned}
F^C(0,\Phi) &= (\theta_H\pi + \theta_L(1-\pi))\,\Phi(\theta^C_{xxx}|_{\alpha=0}) + ((1-\theta_H)\pi + (1-\theta_L)(1-\pi))\,\Phi(\theta^C_{xxy}|_{\alpha=0})\\
&= F^W(0,\Phi)
\end{aligned} \tag{33}$$

Thus, $F^T(0,\Phi) - F^C(0,\Phi) = F^T(0,\Phi) - F^W(0,\Phi)$. In the proof of Proposition 3, it is shown that the RHS is positive when Φ is strictly convex. Continuity concludes the proof. □

Proof of Proposition 5. As the monetary payoff only depends on the recommendation in the final period, the first recommendation only affects the reputation term. Therefore there is no deviation incentive in the first period when the continuation strategy is fixed as truth-telling, as seen in the proof of Lemma 1. In the second period, it can be readily shown that the incentive compatibility conditions in the same proof, (16) and (18), are the special cases with $K = 0$. With $K \geq 0$, they are re-written to

$$\frac{p_{xx}}{1-p_{xx}} \geq \frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy}) + K}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx}) + K} \geq \frac{1}{1+\alpha} \tag{34}$$

and

$$\frac{p_{xx}}{1-p_{xx}} \geq \left(\frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy}) + K}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx}) + K}\right)^{-1} \geq \frac{1}{1+\alpha}, \tag{35}$$

resulting in (7). □

Proof of Proposition 6. Start with the last claim. Re-arrange (9) to

$$(q_c - q_i)\frac{K}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx}) + K} < q_{cc} - (1+\alpha)q_{ii}. \tag{36}$$

The fact that $q_{cc}/q_{ii} = p_{xx}/(1-p_{xx})$ is used to derive (36). Using the definition of q_c and q_i, and the fact that $\Phi(\theta^T_{xxx}) > \Phi(\theta^T_{xyx})$, (36) holds if the following inequality holds which is always true.

$$\alpha q_{ii} < q_{ic} - q_{ci} \equiv \alpha q_{ci} \tag{37}$$

Move on to the first claim. The optimality of truth-telling after previous truth-telling recommendation is guaranteed if

$$\frac{1}{1+\alpha} \leq \frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy}) + K}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx}) + K} \leq \frac{p_{xx}}{1-p_{xx}}, \tag{38}$$

which is satisfied if (8) holds. Hence, provided the last claim into account, if there exists a deviation strategy, there must be a signal $s_1 = \hat{s}_1$ after which the expert tells lie. Re-using the same notation in the proof of Proposition 2, it must be a form of either r' or r''; and r' is ruled out by the same logic. In r'', the incentive condition in the proof of the proposition, (23), is re-written to

$$\begin{aligned} &q_{cc}\Phi(\theta^T_{xxx}) + q_{ii}\Phi(\theta^T_{xxy}) + q_{ci}\Phi(\theta^T_{xyx}) + q_{ic}\Phi(\theta^T_{xyx}) + (q_{cc} + q_{ic})K \\ \geq &\Phi(\theta^T_{xyx}) + (q_{cc} + q_{ci})K \\ \equiv &q_{cc}\Phi(\theta^T_{xyx}) + q_{ii}\Phi(\theta^T_{xyx}) + q_{ci}\Phi(\theta^T_{xyx}) + q_{ic}\Phi(\theta^T_{xyx}) + (q_{cc} + q_{ci})K, \end{aligned} \tag{39}$$

which is equivalent to the right inequality of (8). □